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Citation: AIP Advances **7**, 015211 (2017); doi: 10.1063/1.4975406
View online: http://dx.doi.org/10.1063/1.4975406
View Table of Contents: http://aip.scitation.org/toc/adv/7/1
Published by the American Institute of Physics

AIP ADVANCES 7, 015211 (2017)

Characterization of blowup for the Navier-Stokes equations using vector potentials

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(Received 28 April 2016; accepted 22 January 2017; published online 31 January 2017)

We characterize a possible blowup for the 3D Navier-Stokes on the basis of dynamical equations for vector potentials $\boldsymbol{A}$. This is motivated by a known interpolation $\|\boldsymbol{A}\|_{\mathrm{BMO}} \le \|\boldsymbol{u}\|_{L^3}$, together with recent mathematical results. First, by working out an inversion formula for singular integrals that appear in the governing equations, we derive a criterion using the nonlinear term of $\boldsymbol{A}$ as $\int_0^{t_*} \|\frac{\partial \boldsymbol{A}}{\partial t} - \nu \triangle \boldsymbol{A}\|_{L^\infty} dt = \infty$ for a blowup at t_*. Second, for a particular form of a scale-invariant singularity of the nonlinear term we show that the vector potential becomes unbounded in its L^∞ and BMO norms. Using the stream function, we also consider the 2D Navier-Stokes equations to seek an alternative proof of their known global regularity. It is not yet proven that the BMO norm of vector potentials in 3D (or, the stream function in 2D) serve as a blow up criterion in more general cases. [http://dx.doi.org/10.1063/1.4975406]

I. INTRODUCTION

We consider the fundamental problems of the incompressible Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \triangle \boldsymbol{u}, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{2}$$

with smooth initial data of finite energy in $\mathbb{R}^3$. The study of the system (1) and (2) was pioneered in Ref. 18. Lots of progress has been made since then, including[2,3,5–12,15,19,21,23–26,28] and various kinds of blowup or regularity criteria have been developed.

We define the vector potential $\boldsymbol{A}$ by $\boldsymbol{u} = \nabla \times \boldsymbol{A}$ in three dimensions, where $\nabla \cdot \boldsymbol{A} = 0$ and the stream function in two dimensions by $\boldsymbol{u} = (\partial_2 \psi, -\partial_1 \psi)$. Both $\boldsymbol{A}$ and $\boldsymbol{\psi}$ have the same physical dimensions as that of ν and they are critical. Our motivation is to characterize possible blowup in term of those critical dependent variables, that is, the vector potential, or the stream function. This is inspired by a number of recent mathematical results.

For example, a critical norm $\|\boldsymbol{u}\|_{L^3}$ is shown to be a blowup criterion in three-dimensions.[9,27] Its proof is based an a sophisticated use of a contradiction argument based on backward uniqueness of the heat equation. Also, a work by Koch and Tataru[16] has shown global regularity provided that $\|\boldsymbol{u}\|_{\mathrm{BMO}^{-1}}$, which by definition $\simeq \|\boldsymbol{A}\|_{\mathrm{BMO}}$, is sufficiently small initially. (Hereafter $f \simeq g$ means that they are comparable in the sense $c_1 g < f < c_2 g$.) Let us clarify the relationship of that result to the problems we consider in this paper. There are two kinds of partial regularity results; (a) global existence existence for small initial data, and (b) blowup criterion for general (large) initial data. The statement (a) is achieved by using a critical norm and often the same norm appears in the statement of (b). Norms such as $\|\boldsymbol{u}\|_{\dot{H}^{1/2}}$ and $\|\boldsymbol{u}\|_{L^3}$ are examples.[29] As for $\|\boldsymbol{u}\|_{\mathrm{BMO}^{-1}} (\simeq \|\boldsymbol{A}\|_{\mathrm{BMO}})$, (a) is true by Ref. 16 as mentioned, but it is noteworthy that (b) is open.

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Furthermore, a recent work[4] has investigated a possibility of a Besov norm $\|\boldsymbol{u}\|_{B^{-1}_{\infty,\infty}}$ serving as a blowup criterion. Because

$$\|\boldsymbol{u}\|_{B^{-1}_{\infty,\infty}} < \|\boldsymbol{u}\|_{\mathrm{BMO}^{-1}} (\simeq \|\boldsymbol{A}\|_{\mathrm{BMO}}),$$

their motivation is more ambitious than ours. They have obtained a dichotomy-type result: upon a possible singularity at $t = t_*$, either (i) $\|\boldsymbol{u}\|_{B^{-1}_{\infty,\infty}}$ becomes unbounded, or (ii) it is bounded but there is a gap in the norm near the critical time $\left| \|\boldsymbol{u}(t_*)\|_{B^{-1}_{\infty,\infty}} - \lim_{t\to t_*} \|\boldsymbol{u}(t)\|_{B^{-1}_{\infty,\infty}} \right| > C\nu$. It is not known whether $\|\boldsymbol{u}\|_{B^{-1}_{\infty,\infty}}$ becomes unbounded or not, but those results do show a growing interest in characterizing possible singularities in term of weaker norms related with $\boldsymbol{A}$. Hence it does make sense to seek a possibility of $\|\boldsymbol{u}\|_{\mathrm{BMO}^{-1}} (\simeq \|\boldsymbol{A}\|_{\mathrm{BMO}})$ serving as a blowup criterion.

Now, in view of embedding e.g.[13]

$$\|\boldsymbol{A}\|_{\mathrm{BMO}} \le C\|\boldsymbol{u}\|_{L^3} \text{ in 3D}, \tag{3}$$

and

$$\|\psi\|_{\mathrm{BMO}} \le C\|\boldsymbol{u}\|_{L^2} \text{ in 2D}, \tag{4}$$

we are led to consider whether $\|\boldsymbol{A}\|_{\mathrm{BMO}}$ serves as a blowup criterion in 3D and $\|\psi\|_{\mathrm{BMO}}$ in 2D. In other words, we ask whether the following statements

$$\text{blowup at } t = t_* \text{ in 3D } \Rightarrow \|\boldsymbol{A}\|_{\mathrm{BMO}} \to \infty \text{ as } t \to t_* \tag{5}$$

and

$$\text{blowup at } t = t_* \text{ in 2D } \Rightarrow \|\boldsymbol{\psi}\|_{\mathrm{BMO}} \to \infty \text{ as } t \to t_* \tag{6}$$

hold true or not.

Indeed, by inserting the Leray bound

$$\sup_{\boldsymbol{x}} |\boldsymbol{u}(\boldsymbol{x},t)| \ge c\frac{\nu^{1/2}}{\sqrt{t_* - t}}$$

for a possible singularity into (12) below, because a principal-value integral has no regularizing effects, we expect on a heuristic basis

$$\frac{\partial \boldsymbol{A}}{\partial t} \simeq C\frac{\nu}{t_* - t},$$

which suggests

$$\boldsymbol{A} \simeq \nu \log \frac{1}{t_* - t}.$$

This paper discusses what we can tell about such a possibility on a more solid ground.

There are at least two reasons for studying 2D Navier-Stokes equations, for which global regularity is already known. One is to introduce ideas in preparation for handling the 3D cases. The other one is to try giving an alternative proof of the regularity. Note that if (6) holds, then by (4) we get a contradiction immediately if a 2D Navier-Stokes solution develops a singularity, as the total kinetic energy decreases in time.

The significance of the criterion with $T[\nabla \boldsymbol{A}]$ that we will derive is as follows. It will fix the blowup rate under the assumption of a power-law singularity so that we can study as a first step what will happen to $\boldsymbol{A}$ itself, by choosing a typical form of possible singularities which realises that rate.

The rest of the paper is organised as follows. We derive the conditions for blowup for the 2D and 3D Navier-Stokes equations in Sections II and III, respectively. In Section IV, we compute L^∞ and BMO norms for simple scale-invariant singularities in both dimensions. Section V is devoted to summary.

II. 2D NAVIER-STOKES EQUATIONS

While global regularity is well-established in two dimensions, we describe the 2D Navier-Stokes equations here to show that the argument runs equally well, independent of spatial dimensions when we use the critical dependent variables. Using the stream function ψ, the Navier-Stokes equation can be written[20]

$$\frac{\partial \psi}{\partial t} - \nu \triangle \psi = \frac{1}{\pi}\mathrm{P.V.}\int_{\mathbb{R}^2} \frac{[(\boldsymbol{x}-\boldsymbol{x}')\times \nabla\psi(\boldsymbol{x}')]\,(\boldsymbol{x}-\boldsymbol{x}')\cdot\nabla\psi(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^4}\,\mathrm{d}\boldsymbol{x}', \tag{7}$$

or, equivalently

$$\frac{\partial \psi}{\partial t} - \nu\triangle\psi = \epsilon_{jk} R_i R_j \partial_k \psi \partial_i \psi, \tag{8}$$

where R_i denotes the Riesz transforms defined by $R_j = -(-\triangle)^{-1/2}\partial_j$, $j = 1, 2$ and ϵ_{ij} the 2D Eddington tensor (i.e. $\epsilon_{12} = -\epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0$). The right-hand side of (7) will be denoted by $T[\nabla\psi]$ hereafter.

We start considering the viscous term which is straightforward. The Beale-Kato-Majda blowup criterion[1] for the Euler equations is also valid for the Navier-Stokes equations, in fact even with a weaker BMO norm.[17] Combining this fact with the definition $-\triangle\psi = \omega$, the viscous term becomes singular in the sense that

$$\int_0^{t_*} \|\omega\|_{\mathrm{BMO}} dt = \int_0^{t_*} \|\triangle\psi\|_{\mathrm{BMO}} dt = \infty.$$

To handle the nonlinear term, we need the following inversion formula for singular integrals.

Proposition 1. *The singular integrals on the right-hand side of* (8) *can be inverted as*

$$\partial_i\psi\partial_k\psi - \frac{1}{2}|\nabla\psi|^2\delta_{ik} = -\epsilon_{kl}R_iR_l(\psi_t - \nu\triangle\psi) + \frac{1}{2}(\psi_t - \nu\triangle\psi)\epsilon_{ik} \text{ for } i, k = 1, 2. \tag{9}$$

Proof. i) We begin with a trial ansatz

$$\partial_i\psi\partial_k\psi = -\epsilon_{kl}R_iR_l(\psi_t - \nu\triangle\psi). \tag{10}$$

By inserting (10) into the right-hand side of (8), we find

$$\begin{aligned} -\epsilon_{jk}R_iR_j\epsilon_{kl}R_iR_l(\psi_t - \nu\triangle\psi) &= \epsilon_{jk}\epsilon_{kl}R_jR_l(\psi_t - \nu\triangle\psi) \\ &= -R_jR_j(\psi_t - \nu\triangle\psi) = \psi_t - \nu\triangle\psi, \end{aligned}$$

which reproduces the right-hand side of (8). However, the trial ansatz (10) is incomplete as its left-hand side vanishes when $i = k$.

ii) To compensate for the inconsistency, we add a diagonal element

$$\partial_i\psi\partial_k\psi = -\epsilon_{kl}R_iR_l(\psi_t - \nu\triangle\psi) + A\delta_{ik}.$$

Taking $i = k$, A can be fixed as $|\nabla\psi|^2 = 2A$ or $A = \frac{1}{2}|\nabla\psi|^2$. Now, the two of the resultant expressions read

$$\partial_1\psi\partial_2\psi = R_1R_1(\psi_t - \nu\triangle\psi),$$
$$\partial_2\psi\partial_1\psi = -R_2R_2(\psi_t - \nu\triangle\psi),$$

from which we find

$$-R_jR_j(\psi_t - \nu\triangle\psi) = \psi_t - \nu\triangle\psi = 0,$$

which is still inconsistent.

iii) To complete the inversion formula, we need to add a skew-symmetric part as well

$$\partial_i\psi\partial_k\psi = -\epsilon_{kl}R_iR_l(\psi_t - \nu\triangle\psi) + A\delta_{ik} + B\epsilon_{ik}.$$

The coefficient B can be readily fixed as $B = \frac{1}{2}(\psi_t - \nu\triangle\psi)$ by taking e.g. $i = 1$, $k = 2$ and (9) follows. □

As mentioned above, global regularity of the 2D Navier-Stokes equation has been well-known, but here we seek characterizations of blowup pretending that a singularity exists as a preparation for the similar analysis for the 3D case. We first note the following.

Proposition 2. *IF a solution of 2D Navier-Stokes equations breaks down, the nonlinear term becomes singular in the sense that*

$$\int_0^{t_*} \|T[\nabla\psi]\|_{L^\infty} dt = \infty. \tag{11}$$

Proof. Taking the BMO norm of (9), (actually $c = 1/\sqrt{2}$ suffices[30]) we have

$$c\|\nabla\psi\|_{\mathrm{BMO}}^2 \le \|RR(\psi_t - \nu\triangle\psi)\|_{\mathrm{BMO}} + \frac{1}{2}\|\psi_t - \nu\triangle\psi\|_{\mathrm{BMO}}$$
$$\le \|\psi_t - \nu\triangle\psi\|_{L^\infty} + \frac{1}{2}\|\psi_t - \nu\triangle\psi\|_{\mathrm{BMO}} \le \frac{3}{2}\|\psi_t - \nu\triangle\psi\|_{L^\infty},$$

where indices are suppressed for simplicity. Thus we find

$$c\|\boldsymbol{u}\|_{\mathrm{BMO}}^2 \le \|T[\nabla\psi]\|_{L^\infty},$$

with an updated constant c. Applying the Serrin's criterion generalized by Ref. 17

$$\int_0^{t_*} \|\boldsymbol{u}\|_{\mathrm{BMO}}^2 dt = \infty,$$

we obtain (11). □

We note that if a power-law behaviour is assumed $\|T[\nabla\psi]\|_{L^\infty} = O\left(\frac{1}{(t_*-t)^\alpha}\right)$, then α should satisfy $\alpha \ge 1$.

The minimum blowup rates of both the viscous and the nonlinear terms are the same and the time derivative term might not become unbounded because of cancellations between them. To shed some light on this issue, in Section IV we will see how the stream function behaves for a particular form of singularities.

III. 3D NAVIER-STOKES EQUATIONS

With the vector potentials $\boldsymbol{A}$, the Navier-Stokes equations can be written[22]

$$\frac{\partial \boldsymbol{A}}{\partial t} - \nu\triangle\boldsymbol{A} = \frac{3}{4\pi}\mathrm{P.V.}\int_{\mathbb{R}^3} \frac{\boldsymbol{r}\times(\nabla\times\boldsymbol{A}(\boldsymbol{x}'))\,\boldsymbol{r}\cdot(\nabla\times\boldsymbol{A}(\boldsymbol{x}'))}{|\boldsymbol{r}|^5}\,\mathrm{d}\boldsymbol{x}', \tag{12}$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$. The nonlinear term on right-hand side (12) is denoted by $\boldsymbol{T}[\nabla\boldsymbol{A}]$. Or, in components we have

$$\frac{\partial A_i}{\partial t} - \nu\triangle A_i = \epsilon_{kpq}R_jR_k\partial_pA_q(\partial_jA_i - \partial_iA_j), \tag{13}$$

where R_j here denotes the 3D Riesz transforms. As in two dimensions, if there is blowup at $t = t_*$, the viscous term must obey

$$\int_0^{t_*} \|\triangle\boldsymbol{A}\|_{\mathrm{BMO}} dt = \int_0^{t_*} \|\boldsymbol{\omega}\|_{\mathrm{BMO}} dt = \infty,$$

because of $-\triangle\boldsymbol{A} = \boldsymbol{\omega}$ and Refs. 1 and 17. For the singular integrals on the right-hand side of (13), a similar inversion formula is available.

Proposition 3. *The singular integral on the right-hand side of* (13) *can be inverted as*

$$(\partial_pA_q - \partial_qA_p)(\partial_jA_i - \partial_iA_j)$$
$$-\frac{\delta_{jp}}{2}(\partial_{p'}A_q - \partial_qA_{p'})(\partial_{p'}A_i - \partial_iA_{p'}) - \frac{\delta_{jq}}{2}(\partial_pA_{q'} - \partial_{q'}A_p)(\partial_{q'}A_i - \partial_iA_{q'})$$
$$= \epsilon_{lpq}R_lR_j\left(\frac{\partial A_i}{\partial t} - \nu\triangle A_i\right) + \frac{1}{3}\epsilon_{pqj}\left(\frac{\partial A_i}{\partial t} - \nu\triangle A_i\right), \ \textit{for } i,j,p,q = 1,2,3. \tag{14}$$

Proof. We first symmetrise (13) as

$$\frac{\partial A_i}{\partial t} - \nu\triangle A_i = \frac{1}{2}\epsilon_{kpq}R_jR_k(\partial_pA_q - \partial_qA_p)(\partial_jA_i - \partial_iA_j).$$

i) We try an ansatz of the form

$$(\partial_pA_q - \partial_qA_p)(\partial_jA_i - \partial_iA_j) = \epsilon_{lpq}R_lR_j(\partial_tA_i - \nu\triangle A_i),$$

to find that the right-hand side equals

$$\frac{1}{2}\epsilon_{kpq}R_jR_k\epsilon_{lpq}R_lR_j(\partial_t A_i - \nu\triangle A_i) = \delta_{kl}R_jR_jR_kR_l(\partial_t A_i - \nu\triangle A_i) = \partial_t A_i - \nu\triangle A_i.$$

ii) The left-hand side have residuals

$$(\partial_{p'}A_q - \partial_q A_{p'})(\partial_{p'}A_i - \partial_i A_{p'}) \text{ when } j = p(= p', \text{ say})$$

and

$$(\partial_p A_{q'} - \partial_{q'}A_p)(\partial_{q'}A_i - \partial_i A_{q'}) \text{ when } j = q(= q', \text{ say}),$$

whereas the right-hand side vanishes for these cases. To compensate for the discrepancies, we put

$$(\partial_p A_q - \partial_q A_p)(\partial_j A_i - \partial_i A_j) - a\delta_{jp}(\partial_{p'}A_q - \partial_q A_{p'})(\partial_{p'}A_i - \partial_i A_{p'}) - a\delta_{jq}(\partial_p A_{q'} - \partial_{q'}A_p)(\partial_{q'}A_i - \partial_i A_{q'}) = \epsilon_{lpq}R_lR_j(\partial_t A_i - \nu\triangle A_i).$$

When $j = p$, the left-hand side equals

$$(\partial_p A_q - \partial_q A_p)(\partial_j A_i - \partial_i A_j) - 3a(\partial_{p'}A_q - \partial_q A_{p'})(\partial_{p'}A_i - \partial_i A_{p'}) - a(\partial_q A_{q'} - \partial_{q'}A_q)(\partial_{q'}A_i - \partial_i A_{q'}) = (1 - 2a)(\partial_p A_q - \partial_q A_p)(\partial_j A_i - \partial_i A_j),$$

hence we find that $a = 1/2$.

iii) Finally, we add the anti-symmetric part

$$\epsilon_{lpq}R_lR_j(\partial_t A_i - \nu\triangle A_i) = (\partial_p A_q - \partial_q A_p)(\partial_j A_i - \partial_i A_j) - \frac{\delta_{jp}}{2}(\partial_{p'}A_q - \partial_q A_{p'})(\partial_{p'}A_i - \partial_i A_{p'}) - \frac{\delta_{jq}}{2}(\partial_p A_{q'} - \partial_{q'}A_p)(\partial_{q'}A_i - \partial_i A_{q'}) + \epsilon_{pqj}B_i.$$

Choosing $j = 3,1,2$ respectively for $(p, q) = (1, 2), (2, 3), (3, 1)$, its left-hand side equals

$$(R_3R_3 + R_1R_1 + R_2R_2)(\partial_t - \nu\triangle)A_i = -(\partial_t - \nu\triangle)A_i.$$

The right-hand side is given by

$$(\partial_1 A_2 - \partial_2 A_1)(\partial_3 A_i - \partial_i A_3) + (\partial_2 A_3 - \partial_3 A_2)(\partial_1 A_i - \partial_i A_1) + (\partial_3 A_1 - \partial_1 A_3)(\partial_2 A_i - \partial_i A_2) + 3B_i.$$

Taking e.g. $i = 1$ we find

$$-(\partial_t - \nu\triangle)A_1 = 3B_1,$$

which fixes $\boldsymbol{B}$. □

On this basis, we show the main result.

Proposition 4. *If a solution of 3D Navier-Stokes equations breaks down, the nonlinear term must become singular in the sense that*

$$\int_0^{t_*} \|\boldsymbol{T}[\nabla\boldsymbol{A}]\|_{L^\infty} dt = \infty. \tag{15}$$

Proof. By evaluating the both sides of (14) in BMO norm, we find

$$c\|\boldsymbol{u}\|^2_{\mathrm{BMO}} \le \|(\partial_t - \nu\triangle)\boldsymbol{A}\|_{L^\infty},$$

where $c = 2$ will be fine.[31] Hence, if there is a singularity the nonlinear term becomes unbounded as

$$c\int_0^{t_*} \|\boldsymbol{u}\|^2_{\mathrm{BMO}} dt \le \int_0^{t_*} \|\boldsymbol{T}[\nabla\boldsymbol{A}]\|_{L^\infty} dt = \infty,$$

by applying Refs. 1 and 17. □

Remark. An application of a simple maximum principle to (12) only implies that

$$\|\boldsymbol{A}(t)\|_{L^\infty} \le \|\boldsymbol{A}(0)\|_{L^\infty} + \int_0^t \|\boldsymbol{T}[\nabla\boldsymbol{A}]\|_{L^\infty} dt.$$

Therefore it follows from the condition $\int_0^{t_*} \|\boldsymbol{T}[\nabla\boldsymbol{A}]\|_{L^\infty} dt < \infty$ that $\|\boldsymbol{A}(t)\|_{L^\infty} < \infty$ only. This is far from sufficient to claim regularity up to t_*, showing the above analysis contains more substance. Again, if a power-law behaviour $\|T[\nabla\boldsymbol{A}]\|_{L^\infty} = O\left(\frac{1}{(t_*-t)^\alpha}\right)$ is assumed then α should satisfy $\alpha \ge 1$.

If a solution to the Navier-Stokes equations breaks down, both the viscous term and the nonlinear term must become unbounded. Under the power-law assumption, the minimum rates are basically the same, apart from the slight difference in the norms. Therefore, substantial cancellations can take place to drop the time derivative term to the sub-leading order so that the vector potential might stay bounded. Hence it makes sense to see how the vector potential behaves for a particular form of singularity. This will be studied in the next section.

IV. POSSIBLE SCALE-INVARIANT SINGULARITIES

In this section we study how ψ or $\boldsymbol{A}$ would behave if a solution to the Navier-Stokes equations breaks down in finite time by considering specific forms of singularities.

We recall the Duhamel principle for this purpose. The governing equations

$$\left(\frac{\partial}{\partial t} - \nu\triangle\right)\boldsymbol{A} = \boldsymbol{T}[\nabla\boldsymbol{A}]$$

can be recast as

$$e^{\nu t\triangle}\frac{\partial}{\partial t}\left(e^{-\nu t\triangle}\boldsymbol{A}\right) = \boldsymbol{T}[\nabla\boldsymbol{A}],$$

or

$$\boldsymbol{A}(t) = e^{\nu t\triangle}\boldsymbol{A}(0) + \int_0^t e^{\nu(t-s)\triangle}\boldsymbol{T}[\nabla\boldsymbol{A}](s)ds.$$

More explicitly, we have

$$\begin{aligned}\boldsymbol{A} = &\frac{1}{(4\pi\nu t)^{n/2}}\int_{\mathbb{R}^n}\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\nu t}\right)\boldsymbol{A}_0(\boldsymbol{y})d\boldsymbol{y}\\ &+\int_0^t ds\int_{\mathbb{R}^n}\frac{1}{(4\pi\nu(t-s))^{n/2}}\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\nu(t-s)}\right)\boldsymbol{T}[\nabla\boldsymbol{A}](\boldsymbol{y},s)d\boldsymbol{y},\end{aligned}$$

where $n = 2$ and 3. For $n = 2$, $\boldsymbol{A}$ should be replaced by the stream function ψ.

A. Two-dimensional case

Let us denote the spatial integral above by

$$I_{2\mathrm{D}}(\boldsymbol{x},\tau) \equiv \frac{1}{4\pi\nu\tau}\int_{\mathbb{R}^2} T[\nabla\psi](\boldsymbol{y},s)\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\nu\tau}\right)d\boldsymbol{y}, \quad (16)$$

where $\tau = t - s$ with t the time of a fictitious blowup. As a first step, we consider a scale-invariant singularity which was motivated by Refs. 2 and 14 and assume the following form

$$\boldsymbol{T}[\nabla\psi](\boldsymbol{y},s) \simeq \frac{\nu^2}{|\boldsymbol{y}|^2 + \nu(t-s)}.$$

(See Subsection B. for details.) We have

$$I_{2\mathrm{D}}(\boldsymbol{x},\tau) \gtrsim \frac{\nu}{4\pi\tau}\int_{\mathbb{R}^2}\frac{\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\nu\tau}\right)}{|\boldsymbol{y}|^2+\nu\tau}d\boldsymbol{y}.$$

Then the maximum value of the lowerbound (assumed to take place at $\boldsymbol{x} = 0$) becomes

$$M_{2\mathrm{D}}(\tau) = \frac{\nu}{2\tau}\int_0^\infty \exp\left(-\frac{r^2}{4\nu\tau}\right)\frac{rdr}{r^2+\nu\tau} = \frac{\nu}{4\tau}e^{1/4}\mathrm{E}_1\left(\frac{1}{4}\right) \simeq 0.34\frac{\nu}{\tau}, \quad (17)$$

where use has been made of a formula

$$\int_0^\infty \frac{\exp(-ax)}{x+b}dx = \exp(ab)\,E_1(ab),\ \ (a,b>0)$$

and $E_1(x) \equiv \int_x^\infty \frac{e^{-u}}{u}du\ (x>0)$ is a kind of exponential integrals. Therefore we confirm that $\|\psi\|_{L^\infty} \to \infty$ as $t \to t_*$, so long as this special example is concerned. More generally, by carrying out angular integration in (16) we find

$$I_{2D}(\boldsymbol{x},\tau) \geq \frac{\nu \exp\left(-\frac{|\boldsymbol{x}|^2}{4\tau}\right)}{2\tau}\int_0^\infty \frac{\exp\left(-\frac{r^2}{4\nu\tau}\right)}{r^2+\nu\tau} I_0\left(\frac{|\boldsymbol{x}|r}{2\nu\tau}\right) rdr,$$
$$= \frac{\nu \exp\left(-\frac{|\boldsymbol{\xi}|^2}{2}\right)}{2\tau}\int_0^\infty \frac{ue^{-u^2/2}}{u^2+1/2} I_0(|\boldsymbol{\xi}|u)du,$$

where $\boldsymbol{\xi} = \frac{\boldsymbol{x}}{\sqrt{2\nu\tau}}, u = \frac{r}{\sqrt{2\nu\tau}}$ and

$$I_0(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}d\theta$$

is the modified Bessel function of the 0-th order. Because $I_0(x) \to 1$ as $x \to 0$, the u-integral is convergent for small $\boldsymbol{\xi}$, we find that $I_{2D} \propto \frac{\nu}{\tau}\exp\left(-\frac{|\boldsymbol{\xi}|^2}{2}\right)$. It follows that

$$\|\psi\|_{\mathrm{BMO}} = \infty \ \text{as}\ t \to t_*$$

for the scale-invariant singularity.

B. Three-dimensional case

In three dimensions, the corresponding spatial integral is bounded by

$$I_{3D}(\boldsymbol{x},t) = \frac{1}{(4\pi\nu\tau)^{3/2}}\int_{\mathbb{R}^3} \boldsymbol{T}[\nabla \boldsymbol{A}](\boldsymbol{y},s)\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\nu\tau}\right)d\boldsymbol{y}.$$

We consider a special case of a scale-invariant singularity.[2,14] This ansatz of a scale-invariant singularity on a parabolic cylinder is a rather strong assumption, but not entirely unnatural. Actually, this is the motivation used in partial regularity theory for the Navier-Stokes equation, e.g. on p.776 of Ref. 2. There the classical Leray bound

$$\|\boldsymbol{u}\|_\infty \simeq \frac{c\nu^{1/2}}{\sqrt{t_*-t}}$$

has been generalised to

$$|\boldsymbol{u}(\boldsymbol{x},t)| \simeq \frac{C\nu}{\sqrt{|\boldsymbol{x}|^2+\nu(t_*-t)}},$$

as $|\boldsymbol{x}|^2 + \nu(t_* - t) \to 0$. A straightforward extension of the scale-invariant form to the nonlinear term of the $\boldsymbol{A}$ equation gives

$$\boldsymbol{T}[\nabla \boldsymbol{A}](\boldsymbol{y},s) \simeq \frac{\nu^2}{|\boldsymbol{y}|^2+\nu(t-s)}, \tag{18}$$

if a singularity dominates the whole flow field. Note that the physical dimension of $\boldsymbol{T}[\nabla \boldsymbol{A}]$ is the same as that of $|\boldsymbol{u}|^2$. This leads to a logarithmic singularity if we discard the viscous term in (12). When the viscous term is taken into account, we need to consider the above spatial integral

$$I_{3D}(\boldsymbol{x},\tau) \gtrsim \frac{\nu^{1/2}}{(4\pi\tau)^{3/2}}\int_{\mathbb{R}^3} \frac{\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\nu\tau}\right)}{|\boldsymbol{y}|^2+\nu\tau}d\boldsymbol{y}. \tag{19}$$

We first compute the time evolution of the maximum value, assumed at $\boldsymbol{x} = 0$ for simplicity, $M_{3D}(\tau)$ of the lowerbound

$$M_{3D}(\tau) = \frac{\nu^{1/2}}{(4\pi)^{1/2}\tau^{3/2}} \int_0^\infty \frac{r^2 \exp\left(-\frac{r^2}{4\nu\tau}\right)}{r^2 + \nu\tau} dr \tag{20}$$

$$= \frac{\nu}{2\tau}\left(1 - e^{1/4}\frac{\sqrt{\pi}}{2}\mathrm{Erfc}\left(\frac{1}{2}\right)\right) \simeq 0.23\frac{\nu}{\tau}, \tag{21}$$

where we have made use of a formula

$$\int_0^\infty \frac{\exp\left(-a^2r^2\right)}{r^2+b^2}dr = \frac{\pi}{2b}\exp\left(a^2b^2\right)\mathrm{Erfc}(ab)\ (a, b > 0)$$

and the definition $\mathrm{Erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-u^2}du$. For this special case, just like in 2D we confirm that $\|\boldsymbol{A}\|_{L^\infty} \to \infty$ as $t \to t_*$. Note that the prefactor 0.23 in (21) is smaller than the two-dimensional counterpart 0.34 in (17). This means that diffusion has more significant regularization effects in 3D than in 2D, which is consistent with intuition.

To take into account the spatial structure of the singularity, we carry out the angle integration in (19) to find

$$I_{3D}(\boldsymbol{x}, t) \gtrsim \frac{\nu^{1/2}\exp\left(-\frac{|\boldsymbol{x}|^2}{4\nu\tau}\right)}{(4\pi)^{1/2}\tau^{3/2}}\int_0^\infty \frac{r\exp\left(-\frac{r^2}{4\nu\tau}\right)}{r^2+\nu\tau}\frac{\sinh\frac{|\boldsymbol{x}|r}{2\nu\tau}}{\frac{|\boldsymbol{x}|}{2\nu\tau}}dr$$
$$= \frac{\nu^{1/2}\exp\left(-\frac{|\boldsymbol{\xi}|^2}{2}\right)}{(4\pi)^{1/2}\tau^{3/2}}\int_0^\infty \frac{ue^{-u^2/2}}{u^2+1/2}\frac{\sinh(|\boldsymbol{\xi}|u)}{|\boldsymbol{\xi}|}du,$$

where $\boldsymbol{\xi} = \frac{\boldsymbol{x}}{\sqrt{2\nu\tau}}, u = \frac{r}{\sqrt{2\nu\tau}}$. It is easily checked that the it reduces to the above result (20) in the limit $\boldsymbol{\xi} \to 0$. For fixed $\boldsymbol{\xi}$, the u-integral is convergent because of $\sinh(|\boldsymbol{\xi}|u) < \frac{1}{2}\exp(|\boldsymbol{\xi}|u)$, which means that $I_{3D} \propto \frac{\nu}{\tau|\boldsymbol{\xi}|}\exp\left(-\frac{|\boldsymbol{\xi}|^2}{2}\right)$ for finite but small $\boldsymbol{\xi}$. Hence we find that

$$\|\boldsymbol{A}\|_{\mathrm{BMO}} = \infty \text{ as } t \to t_*.$$

It is in order to study what happens to singularities with a different structure. A kink case

$$\boldsymbol{T}[\nabla\boldsymbol{A}](\boldsymbol{y}, s) \simeq \frac{\nu^{3/2}|\boldsymbol{y}|}{\sqrt{t-s}\,(|\boldsymbol{y}|^2+\nu(t-s))}$$

is of interest.[32] In this case, we compute that

$$I_{3D}^{\mathrm{kink}}(\boldsymbol{x}, \tau) = \frac{\nu\exp\left(-\frac{|\boldsymbol{\xi}|^2}{2}\right)}{\pi^{1/2}\tau}\int_0^\infty \frac{u^2e^{-u^2/2}}{u^2+1/2}\frac{\sinh(|\boldsymbol{\xi}|u)}{|\boldsymbol{\xi}|}du,$$

from which it follows that $I_{3D}^{\mathrm{kink}} \propto \frac{\nu}{\tau|\boldsymbol{\xi}|}\exp\left(-\frac{|\boldsymbol{\xi}|^2}{2}\right)$ for small $\boldsymbol{\xi}$. This again shows that $\|\boldsymbol{A}\|_{\mathrm{BMO}} \to \infty$ and the maximum value is given by

$$M_{3D}^{\mathrm{kink}}(t) = \frac{1}{(4\pi)^{1/2}\tau^2}\int_0^\infty \frac{r^3\exp\left(-\frac{r^2}{4\nu\tau}\right)}{r^2+\nu\tau}dr$$
$$= \frac{\nu}{\sqrt{\pi}\tau}\left(1 - \frac{1}{4}e^{1/4}\mathrm{Erfc}\left(\frac{1}{4}\right)\right) \simeq 0.375\frac{\nu}{\tau},$$

showing a stronger blowup than (21).

V. SUMMARY

In this paper, we have explored characterizations of putative blowup for the Navier-Stokes equations using the stream function in 2D and vector potentials in 3D, centering on their possible role as blowup criteria.

We have discussed how we may justify such an expectation in two different ways. First, we derive the conditions that nonlinear terms must satisfy if a singularity takes place. Second, for the simple case of scale-invariant singularities in space-time, we compute how the stream function in 2D and vector potential in 3D actually behave in time. These show that not only the L^∞-norm, but also the BMO become divergent at the time of breakdown. It should be mentioned that even though the $1/\tau$ behaviour can be expected on dimensional grounds, it is necessary to carry out integration to determine the prefactors to estimate their strength.

These results show the advantages of working with the critical dependent variables and further studies are under way in that direction. As noted above, it has been shown that global regularity follows if $\boldsymbol{u}$ is sufficiently small in BMO^{-1} norm initially.[16] This means that no blowup can take place if $\|\boldsymbol{A}\|_{\mathrm{BMO}}$ is sufficiently small initially. It is of interest to try giving an alternative proof based on (12).

ACKNOWLEDGMENTS

The author would like to thank Chae Dongho for helpful discussion. This work has been partially supported by an EPSRC grant EP/N022548/1.

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[29] For the $\dot{H}^{1/2}$ norm, (a) was proven in Refs. 11 and 15 and (b) is true by the L^3-criterion[9] and the embedding $\|\boldsymbol{u}\|_{L^3} \le c\|\boldsymbol{u}\|_{\dot{H}^{1/2}}$. For the L^3 norm, (a) is true by Ref. 16 and $\|\boldsymbol{u}\|^{-1}_{\mathrm{BMO}} \le c\|\boldsymbol{u}\|_{L^3}$ and (b) is true by Ref. 9.
[30] Denoting the left-hand side of (9) by L_{ik}, it is easily checked $\sqrt{\sum_{i,k=1}^{2} L_{ik}^2} = \frac{|\nabla\psi|^2}{\sqrt{2}}$.
[31] Denoting the left-hand side of (14) by M_{ijpq}, it can be checked after some algebra that $\sqrt{\sum_{i,j,p,q=1}^{3} M_{ijpq}^2} = 2|\nabla \boldsymbol{A}|^2$.
[32] This is motivated by $H[\frac{1}{x^2+a^2}] = -\frac{x}{a(x^2+a^2)}$, where $H[\cdot]$ denotes the Hilbert transform a a constant.