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Focused Compressive Sensing for Underdetermined Wideband DOA Estimation Exploiting High-Order Difference Co-Arrays

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***Abstract*—Group sparsity based method is applied to the $2q$-th order difference co-array for underdetermined wideband direction of arrival (DOA) estimation. For complexity reduction, a focused compressive sensing based approach is proposed, without sacrificing its performance. Different from the conventional focusing approach, in the proposed one, focusing is applied to the virtual arrays and no preliminary DOA estimation is required. Simulation results are provided to demonstrate the effectiveness of the proposed methods.**

***Index Terms*—High-order difference co-array, direction of arrival estimation, compressive sensing, focusing, group sparsity.**

I. Introduction

Recently, underdetermined DOA estimation has received considerable attention [1]–[5]. Sparse arrays based on the second-order difference co-array (SODC) equivalence have been proposed to increase the number of degrees of freedom (DOFs) beyond those offered by the physical array. Commonly used sparse configurations include nested [6] and co-prime arrays [7], [8]. For the narrowband case, both subspace methods and compressive sensing (CS) techniques have been applied for DOA estimation, irrespective of the employed array configuration [6]–[11]. CS and sparse reconstruction techniques exploit all DOFs of the SODC, leading to improved performance [12]–[16]. However, the CS framework can entail a high computational complexity, specifically, under large virtual array extent and increased number of search grid points.

The CS-based DOA estimation for narrowband arrays can be extended to wideband. In [17], sparse reconstruction of the source DOAs is performed for SODC with group sparsity (GS) applied across the different signal frequencies. However, in such a case, computational complexity increases significantly with the number of employed frequencies. On the other hand, to further increase the DOFs of the system, high-order cumulant-based methods were developed as part of the $2q$-th ($q \geq 2$) order difference co-array concept, which have the additional advantage of improved robustness against strong background Gaussian noise [5], [18]. Moreover, the fourth-order difference co-array based estimation method for quasi-stationary sources was proposed in [19], where the source signals can be either Gaussian or non-Gaussian. In general, for the $2q$-th ($q \geq 2$) order difference co-arrays [20], $O(N^{2q})$ DOFs are provided with only N physical sensors.

This paper considers the underdetermined wideband DOA estimation problem for high-order difference co-arrays. We propose a focusing based method within the CS framework, in lieu of GS, which achieves significant reduction in computations without a noticeable sacrifice in performance. However, unlike conventional focusing algorithms for physical arrays [21]–[23], we apply focusing to the virtual array corresponding to the $2q$-th order difference co-arrays. The proposed CS-focusing method does not require a preliminary estimate of the DOAs, and avoids the accumulated error caused by a mismatched focusing matrix.

II. Wideband Signal Model

Consider a general linear array structure with N physical sensors distributed in the set of positions S,

$$S = \left\{ \alpha_n, 0 \leq n \leq N-1 \bigcap n \in \mathbf{Z} \right\} , \tag{1}$$

where $\mathbf{Z}$ is the integer set and α_n the n-th sensor position.

Denote $s_k(t)$ with incident angles $\theta_k, k = 1, \ldots, K$, as K independent far-field zero-mean wideband signals impinging on the array [24]. After sampling with a frequency f_s, an L-point discrete Fourier transform (DFT) is applied and the output signal model in the frequency domain is expressed as

$$\mathbf{X}[l,p] = \mathbf{A}(l,\boldsymbol{\theta})\mathbf{S}[l,p] + \overline{\mathbf{N}}[l,p] , \tag{2}$$

where $\mathbf{X}[l,p]$ is the $N \times 1$ observed signal vector at the p-th discrete-time segment and the l-th frequency bin, $\mathbf{S}[l,p] = [S_1[l,p], \ldots, S_K[l,p]]^T$ with $S_k[l,p]$ as the DFT of the signals $s_k[i]$ and $\{\cdot\}^T$ denotes transpose. $\overline{\mathbf{N}}[l,p]$ represents the corresponding column noise vector in the frequency domain, whose elements are uncorrelated zero-mean Gaussian. $\mathbf{A}(l,\boldsymbol{\theta}) = [\mathbf{a}(l,\theta_1), \ldots, \mathbf{a}(l,\theta_K)]$ is the steering matrix, with each column vector $\mathbf{a}(l,\theta_k)$ representing the steering vector at frequency f_l and angle θ_k, expressed as

$$\mathbf{a}(l,\theta_k) = \left[e^{-j\frac{2\pi\alpha_0}{\lambda_l}\sin(\theta_k)}, \ldots, e^{-j\frac{2\pi\alpha_{N-1}}{\lambda_l}\sin(\theta_k)} \right]^T , \tag{3}$$

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where $\lambda_l = c/f_l$ and c is the propagation velocity of the signal.

For each frequency bin, the $2q$-th order circular auto-cumulant of $S_k[l,p]$ is given by [25]

$$\begin{aligned} &c_{2q,S_k}[l] = \\ &\mathrm{Cum}\left\{S_{k_1}[l,p],\ldots,S_{k_q}[l,p],S^*_{k_{q+1}}[l,p],\ldots,S^*_{k_{2q}}[l,p]\right\}, \end{aligned} \tag{4}$$

where $k_m = k$, $1 \le m \le 2q$ and $1 \le k \le K$. $\mathrm{Cum}\{\cdot\}$ is the cumulant operator and $\{\cdot\}^*$ the conjugate operation.

Define $N^\mu \times 1$ column vector $\mathbf{a}(l,\theta_k)^{\otimes\mu} \triangleq \mathbf{a}(l,\theta_k) \otimes \mathbf{a}(l,\theta_k)\otimes\ldots\otimes\mathbf{a}(l,\theta_k)$ as the Kronecker product of μ vectors of $\mathbf{a}(l,\theta_k)$. Then, the $2q$-th order circular cumulant of $X_n[l,p]$ at the l-th frequency bin for the arrangement indexed by μ $(0 \le \mu \le q)$ is given by [4], [20]

$$\begin{aligned} \mathbf{C}_{2q,\mathbf{x}}[l,\mu] &= \sum\nolimits_{k=1}^{K} c_{2q,S_k}[l]\left[\mathbf{a}(l,\theta_k)^{\otimes\mu}\otimes\mathbf{a}(l,\theta_k)^{*\otimes(q-\mu)}\right] \\ &\times\left[\mathbf{a}(l,\theta_k)^{\otimes\mu}\otimes\mathbf{a}(l,\theta_k)^{*\otimes(q-\mu)}\right]^H + \sigma_{\bar{n}}^2[l]\mathbf{I}_{N^q}\cdot\delta(q-1), \end{aligned} \tag{5}$$

where $\{\cdot\}^H$ denotes Hermitian transpose, $\sigma_{\bar{n}}^2[l]$ is the noise power at the l-th frequency bin, $\mathbf{I}_{N^q}$ is the $N^q \times N^q$ identity matrix, and $\delta(\cdot)$ is the Kronecker delta function. For zero-mean white Gaussian noise, its $2q$-th order cumulant ($q \ge 2$) is zero. Therefore, the noise term is equal to zero for $q \ge 2$. For the sensor positions given in (1), the set of the $2q$-th order difference co-array is defined as [20]

$$\Phi_{2q} = \left\{\sum_{m=1}^{q}\alpha_{n_m} - \sum_{m=q+1}^{2q}\alpha_{n_m}, 0 \le n_m \le N-1\right\}\ . \tag{6}$$

According to Theorem 1 in [20], the model obtained by vectorizing $\mathbf{C}_{2q,\mathbf{x}}[l,\mu]$ is independent of μ and its equivalent steering matrix behaves like the manifold of virtual sensors corresponding to the $2q$-th order difference co-array with a large number of DOFs provided. This virtual model is expressed as

$$\begin{aligned} \mathbf{z}[l] &= \mathrm{vec}\left\{\mathbf{C}_{2q,\mathbf{x}}[l,\mu]\right\} \\ &= \mathbf{B}(l,\boldsymbol{\theta})\mathbf{u}[l] + \sigma_{\bar{n}}^2[l]\tilde{\mathbf{i}}_{N^{2q}}\cdot\delta(q-1)\ , \end{aligned} \tag{7}$$

where $\mathbf{B}(l,\boldsymbol{\theta}) = [\mathbf{b}(l,\theta_1),\ldots,\mathbf{b}(l,\theta_K)]$ with each $N^{2q}\times 1$ column vector $\mathbf{b}(l,\theta_k) = \left[\mathbf{a}(l,\theta_k)^{\otimes\mu}\otimes\mathbf{a}(l,\theta_k)^{*\otimes(q-\mu)}\right]^*\otimes\left[\mathbf{a}(l,\theta_k)^{\otimes\mu}\otimes\mathbf{a}(l,\theta_k)^{*\otimes(q-\mu)}\right]$, and the equivalent signal vector $\mathbf{u}[l] = [c_{2q,S_1}[l],\ldots,c_{2q,S_K}[l]]^T$. In addition, $\tilde{\mathbf{i}}_{N^{2q}} = \mathrm{vec}\left\{\mathbf{I}_{N^q}\right\}$ is an $N^{2q}\times 1$ column vector. For $q=1$, i.e., the more familiar SODC, the above formulation becomes

$$\mathbf{z}[l] = \mathrm{vec}\left\{\mathbf{C}_{2,\mathbf{x}}[l,\mu]\right\} = \mathbf{B}(l,\boldsymbol{\theta})\mathbf{u}[l] + \sigma_{\bar{n}}^2[l]\tilde{\mathbf{i}}_{N^2}\ . \tag{8}$$

III. DOA Estimation Under the CS Framework Based on the $2q$-th Order Difference Co-Array

A. Group sparsity based wideband extension

For single-frequency signals, a CS-based method can be applied directly to exploit the increased DOFs provided by the high-order difference co-array. First, we rewrite (7) as

$$\mathbf{z}[l] = \mathbf{B}^\circ(l,\boldsymbol{\theta})\mathbf{u}^\circ[l]\ , \tag{9}$$

where $\mathbf{B}^\circ(l,\boldsymbol{\theta}) = \left[\mathbf{B}(l,\boldsymbol{\theta}),\delta(q-1)\tilde{\mathbf{i}}_{N^{2q}}\right]$, and $\mathbf{u}^\circ[l] = \left[\mathbf{u}^T[l],\sigma_{\bar{n}}^2[l]\right]^T$. After generating a search grid of K_g potential incident angles $\theta_{g,0},\ldots,\theta_{g,K_g-1}$, we construct an $N^{2q}\times K_g$ steering matrix over the search grid as $\mathbf{B}_{\mathbf{g}}(l,\boldsymbol{\theta}_{\mathbf{g}}) = \left[\mathbf{b}(l,\theta_{g,0}),\ldots,\mathbf{b}(l,\theta_{g,K_g-1})\right]$ with $\boldsymbol{\theta}_{\mathbf{g}} = \left[\theta_{g,0},\ldots,\theta_{g,K_g-1}\right]$, where the subscript $\{\cdot\}_g$ is used to describe matrices, vectors or elements related to the generated search grid. Denote

$$\begin{aligned} \mathbf{B}^\circ_{\mathbf{g}}(l,\boldsymbol{\theta}_{\mathbf{g}}) &= \left[\mathbf{B}_{\mathbf{g}}(l,\boldsymbol{\theta}_{\mathbf{g}}),\delta(q-1)\tilde{\mathbf{i}}_{N^{2q}}\right]\ , \\ \mathbf{u}^\circ_{\mathbf{g}}[l] &= \left[\mathbf{u}^T_{\mathbf{g}}[l],\sigma_{\bar{n}}^2[l]\right]^T\ , \end{aligned} \tag{10}$$

where $\mathbf{u}_{\mathbf{g}}[l]$ is a $K_g\times 1$ column vector with its entries representing the potential source signals over the generated search grids, and $\mathbf{u}^\circ_{\mathbf{g}}[l]$ includes the unknown noise power to be estimated. Then, the CS-based DOA estimation for a single-frequency case is formulated as

$$\min\left\|\mathbf{u}^\circ_{\mathbf{g}}[l]\right\|_1 \text{ subject to } \left\|\mathbf{z}[l]-\mathbf{B}^\circ_{\mathbf{g}}(l,\boldsymbol{\theta}_{\mathbf{g}})\mathbf{u}^\circ_{\mathbf{g}}[l]\right\|_2 \le \varepsilon, \tag{11}$$

where ε is the allowable error bound, $\|\cdot\|_1$ is the ℓ_1 norm and $\|\cdot\|_2$ the ℓ_2 norm. The first K_g entries of $\mathbf{u}^\circ_{\mathbf{g}}[l]$ represent the DOA results over the generated K_g search grids.

Now, we extend it to multiple frequencies through the application of GS as follows. Assume that there are $M \le L$ DFT frequency bins within the bandwidth of signals, with their indexes described as l_m, $0 \le m \le M-1$. These indexes may or may not be consecutive. The choice of the bandwidth is made to include all or part of the spectrum of each source. In this respect, sources could be fully, partially or non-overlapping in frequency, and not all sources need to be present at each frequency. However, the advantages of GS become more pronounced when more sources share the same frequencies. With the same K_g search grids, an $MN^{2q}\times M(K_g+1)$ block diagonal matrix $\widetilde{\mathbf{B}}^\circ_{\mathbf{g}}$ is formulated as

$$\widetilde{\mathbf{B}}^\circ_{\mathbf{g}}(\boldsymbol{\theta}_{\mathbf{g}}) = \mathrm{blkdiag}\left\{\mathbf{B}^\circ_{\mathbf{g}}(l_0,\boldsymbol{\theta}_{\mathbf{g}}),\ldots,\mathbf{B}^\circ_{\mathbf{g}}(l_{M-1},\boldsymbol{\theta}_{\mathbf{g}})\right\}\ . \tag{12}$$

We then construct a $(K_g+1)\times M$ matrix $\mathbf{U}^\circ_{\mathbf{g}} = \left[\mathbf{u}^\circ_{\mathbf{g}}[l_0],\ldots,\mathbf{u}^\circ_{\mathbf{g}}[l_{M-1}]\right]$, whose (k_g,m)-th entry, $0 \le k_g \le K_g-1$, represents the estimate of the equivalent source signal from the incident angle θ_{k_g} at the l_m-th frequency bin, while the (K_g,m)-th entry represents the corresponding circular auto-cumulant of the noise.

With $\mathbf{u}^\circ_{\mathbf{g},k_g}$, $0 \le k_g \le K_g$, representing the k_g-th row of $\mathbf{U}^\circ_{\mathbf{g}}$, a new $(K_g+1)\times 1$ column vector is generated based on the ℓ_2 norm of each row vector $\mathbf{u}^\circ_{\mathbf{g},k_g}$, expressed as

$$\hat{\mathbf{u}}^\circ_{\mathbf{g}} = \left[\left\|\mathbf{u}^\circ_{\mathbf{g},0}\right\|_2,\left\|\mathbf{u}^\circ_{\mathbf{g},1}\right\|_2,\ldots,\left\|\mathbf{u}^\circ_{\mathbf{g},K_g}\right\|_2\right]^T\ . \tag{13}$$

Finally, our extended GS-based estimation method employing the $2q$-th order difference co-array concept is formulated as

$$\min_{\mathbf{U}^\circ_{\mathbf{g}}}\left\|\hat{\mathbf{u}}^\circ_{\mathbf{g}}\right\|_1 \text{ subject to } \left\|\widetilde{\mathbf{z}}-\widetilde{\mathbf{B}}^\circ_{\mathbf{g}}(\boldsymbol{\theta}_{\mathbf{g}})\tilde{\mathbf{u}}^\circ_{\mathbf{g}}\right\|_2 \le \varepsilon, \tag{14}$$

where $\widetilde{\mathbf{z}} = \left[\mathbf{z}^T[l_0],\ldots,\mathbf{z}^T[l_{M-1}]\right]^T$, $\tilde{\mathbf{u}}^\circ_{\mathbf{g}} = \mathrm{vec}\{\mathbf{U}^\circ_{\mathbf{g}}\}$ is a $(K_g+1)M\times 1$ column vector, and the first K_g elements in $\hat{\mathbf{u}}^\circ_{\mathbf{g}}$ are the corresponding wideband DOA estimation results over the K_g search grids.

B. Focused compressive sensing based DOA estimation

The main drawback of the GS-based method is its high computational complexity. For complexity reduction, a novel focusing based approach under the CS framework, referred to as CS-focusing, is proposed in this section.

Traditional focusing algorithms construct focusing matrices for the physical array based on the correlation matrix, provided that a preliminary estimate of the DOAs is given [21], [22]. Then, the focussing process aligns the signal sub-spaces of the narrowband components within the frequency band of interest to a reference frequency using the generated focusing matrices [21]. Their performance is sensitive to the estimation error of the initial DOAs. In our method, we apply focusing to the virtual array structure, in lieu of the physical array. Further, the search grid generated for sparse signal representation is utilized as the initial DOA estimate. As a result, preliminary DOA estimates are no longer required.

Consider f_r as the reference frequency corresponding to the l_r-th frequency bin. The focusing algorithm of rotational signal-subspace (RSS) [26] is applied to the virtual array structure in (7) with the vector $\boldsymbol{\theta}_{\mathbf{g}}$ containing all the K_g search grids as the initial DOAs. Then, the RSS focusing matrix $\mathbf{T}[l]$ with size of $N^{2q} \times N^{2q}$ at the l-th frequency bin can be obtained by solving the following optimization problem:

$$\begin{aligned} \min \quad & \|\mathbf{B}_{\mathbf{g}}(l_r, \boldsymbol{\theta}_{\mathbf{g}}) - \mathbf{T}[l]\mathbf{B}_{\mathbf{g}}(l, \boldsymbol{\theta}_{\mathbf{g}})\|_F \\ \text{subject to} \quad & \mathbf{T}^H[l]\mathbf{T}[l] = \mathbf{I}_{N^{2q}} \ , \end{aligned} \tag{15}$$

where $\|\cdot\|_F$ is the Frobenius norm. Its solution is given by $\mathbf{T}[l] = \mathbf{V}[l]\mathbf{U}^H[l]$ [26], where the column vectors in $\mathbf{U}[l]$ and $\mathbf{V}[l]$ are the left and right singular vectors of $\mathbf{B}_{\mathbf{g}}(l, \boldsymbol{\theta}_{\mathbf{g}})\mathbf{B}_{\mathbf{g}}^H(l_r, \boldsymbol{\theta}_{\mathbf{g}})$, respectively.

The output model after focusing can be expressed as

$$\begin{aligned} \mathbf{y}[l] &= \mathbf{T}[l]\mathbf{z}[l] \\ &= \mathbf{T}[l]\mathbf{B}_{\mathbf{g}}(l, \boldsymbol{\theta}_{\mathbf{g}})\mathbf{u}_{\mathbf{g}}[l] + \sigma_{\bar{n}}^2[l]\mathbf{T}[l]\tilde{\mathbf{i}}_{N^{2q}} \cdot \delta(q-1) \\ &\approx \mathbf{B}_{\mathbf{g}}(l_r, \boldsymbol{\theta}_{\mathbf{g}})\mathbf{u}_{\mathbf{g}}[l] + \sigma_{\bar{n}}^2[l]\mathbf{T}[l]\tilde{\mathbf{i}}_{N^{2q}} \cdot \delta(q-1) \ . \end{aligned} \tag{16}$$

After focusing, the same equivalent steering matrix $\mathbf{B}_{\mathbf{g}}(l_r, \boldsymbol{\theta}_{\mathbf{g}})$ is shared among all frequencies of interest. Such transform requires that the focused model at the reference frequency is aliasing-free. By averaging the signal at each frequency bin, a single wideband model can be obtained as

$$\bar{\mathbf{y}} = \frac{1}{M}\sum_{l_m=0}^{M-1}\mathbf{y}[l_m] = \mathbf{B}_{\mathbf{g}}(l_r, \boldsymbol{\theta}_{\mathbf{g}})\bar{\mathbf{u}}_{\mathbf{g}} + \overline{\mathbf{T}}\tilde{\mathbf{i}}_{N^{2q}}\delta(q-1), \tag{17}$$

where $\bar{\mathbf{u}}_{\mathbf{g}} = \frac{1}{M}\sum_{l_m=0}^{M-1}\mathbf{u}_{\mathbf{g}}[l_m]$ is the $K_g \times 1$ column vector corresponding to the potential signals to be estimated under the CS framework, and $\overline{\mathbf{T}} = \frac{1}{M}\sum_{l_m=0}^{M-1}\{\mathbf{T}[l_m]\sigma_{\bar{n}}^2[l_m]\}$. For Gaussian white noise, we have $\sigma_{\bar{n}}^2[l_m] = \sigma_{\bar{n}}^2$, $0 \le m \le M-1$, and as such, $\overline{\mathbf{T}} = \frac{1}{M}\sum_{l_m=0}^{M-1}\mathbf{T}[l_m]\sigma_{\bar{n}}^2$.

Finally, the proposed CS-focusing method for wideband DOA estimation can be formulated as follows

$$\min_{\bar{\mathbf{u}}_{\mathbf{g}}^{\circ}} \left\|\bar{\mathbf{u}}_{\mathbf{g}}^{\circ}\right\|_1 \text{ subject to } \left\|\bar{\mathbf{y}} - \mathbf{B}_{\mathbf{g}}^{\circ}(l_r, \boldsymbol{\theta}_{\mathbf{g}})\bar{\mathbf{u}}_{\mathbf{g}}^{\circ}\right\|_2 \le \varepsilon, \tag{18}$$

where $\bar{\mathbf{u}}_{\mathbf{g}}^{\circ} = \left[\bar{\mathbf{u}}_{\mathbf{g}}^T, \sigma_{\bar{n}}^2\right]^T$, and $\mathbf{B}_{\mathbf{g}}^{\circ}(l_r, \boldsymbol{\theta}_{\mathbf{g}}) = \left[\mathbf{B}_{\mathbf{g}}(l_r, \boldsymbol{\theta}_{\mathbf{g}}), \frac{1}{M}\sum_{l_m=0}^{M-1}\mathbf{T}[l_m]\tilde{\mathbf{i}}_{N^{2q}}\delta(q-1)\right]$.

An important property that permits the development of the proposed method is that, in the underlying $2q$-th order difference co-array, the $\mathbf{u}_{\mathbf{g}}[l_m]$ terms have the same sign across all frequency bins and, as such, their summation leads to coherent signal combining for improved performance. This property is verified below:

1). For $q = 1$, each entry in the row vector $\mathbf{u}_{\mathbf{g},k_g}^{\circ}$ represents a positive power value at a potential angle. Then we can rewrite $\bar{\mathbf{u}}_{\mathbf{g}}$ as $\bar{\mathbf{u}}_{\mathbf{g}} = \frac{1}{M}\sum_{l_m=0}^{M-1}\mathbf{u}_{\mathbf{g}}[l_m] = \frac{1}{M}\left[\left\|\mathbf{u}_{\mathbf{g},0}^{\circ}\right\|_1, \ldots, \left\|\mathbf{u}_{\mathbf{g},K_g-1}^{\circ}\right\|_1\right]^T$.

2). For $q > 1$, assume that each source $s_k[i]$ is generated by an autoregressive moving average (ARMA) process (with impulse response $\tilde{h}_k[i]$) driven by an independent and identically distributed (i.i.d.) input non-Gaussian signal $w_k[i]$, i.e., $s_k[i] = w_k[i] * \tilde{h}_k[i]$, where $*$ represents convolution. For the l-th frequency bin output $S_k[l,p]$ of the DFT at instant p, the operation applied to $s_k[i]$ can also be considered as a convolution with $h_D[i,l]$ representing its impulse response. $S_k[l,p]$ can, therefore, be considered as a convolution between $w_k[i]$ and the equivalent impulse response $h_k[i,l] = \tilde{h}_k[i] * h_D[i,l]$. By a straightforward extension of the properties given in [25] to complex random processes, and with $k_m = k$, $1 \le m \le 2q$ and $1 \le k \le K$, (4) can be updated to

$$\begin{aligned} & c_{2q,S_k}[l] \\ &= \mathrm{Cum}\left\{w_{k_1}[i] * h_{k_1}[i,l], \ldots, w_{k_q}[i] * h_{k_q}[i,l],\right. \\ &\quad \left. w_{k_{q+1}}^*[i] * h_{k_{q+1}}^*[i,l], \ldots, w_{k_{2q}}^*[i] * h_{k_{2q}}^*[i,l]\right\} \\ &= c_{2q,w_k}\sum_{\mu=-\infty}^{+\infty}\underbrace{h_k[\mu,l]\cdots h_k[\mu,l]}_{q\text{ times}}\overbrace{h_k^*[\mu,l]\cdots h_k^*[\mu,l]}^{q\text{ times}} \\ &= c_{2q,w_k}\sum_{\mu=-\infty}^{+\infty}\left|h_k[\mu,l]\right|^{2q} \ , \end{aligned} \tag{19}$$

where c_{2q,w_k} is the $2q$-th order auto-cumulant of $w_k[i]$, and $|\cdot|$ is the absolute value operator. Clearly, $c_{2q,S_k}[l]$ shares the same sign with c_{2q,w_k} since $\sum_{\mu=-\infty}^{+\infty}\left|h_k[\mu,l]\right|^{2q}$ is positive.

The CS-focusing method is proposed based on a general linear array. As illustrated, for the SODC ($q = 1$), the sources are assumed uncorrelated, while for $q > 1$, they are assumed to be non-Gaussian and independent. For the off-grid problem (also known as dictionary mismatch problem) caused by off-grid sources [27], [28], a straightforward solution adopted here is to construct a more accurate model by predefining a denser search grid with an increased number of angles, which may lead to further increased computational complexity. However, under a coarser search grid, there are other methods available to deal with this mismatch problem [29]–[32]. Moreover, due to the DFT operation employed, to maintain a given level of performance, with the increase of the DFT points, the total number of snapshots required also increases.

The procedure of our proposed CS-focusing method is summarized as follows:

1. Generation of the virtual array at each frequency bin based on the $2q$-th order difference co-arrays in (7).

2. Focusing is applied to the virtual array and then, a single wideband model is obtained by averaging the signal model at each frequency bin as given in (17).

TABLE I
NUMBER OF ENTRIES IN VECTORS/MATRICES

Vector / Matrix	GS-Based Method	CS-focusing Method
$\tilde{\mathbf{u}}_{\mathbf{g}}^{\circ}$ / $\bar{\mathbf{u}}_{\mathbf{g}}^{\circ}$	$(K_g+1)M$	K_g+1
$\tilde{\mathbf{z}}$ / $\bar{\mathbf{y}}$	$N^{2q}M$	N^{2q}
$\widetilde{\mathbf{B}}_{\mathbf{g}}^{\circ}(\boldsymbol{\theta}_{\mathbf{g}})$ / $\mathbf{B}_{\mathbf{g}}^{\circ}(l_r,\boldsymbol{\theta}_{\mathbf{g}})$	$(K_g+1)N^{2q}M^2$	$(K_g+1)N^{2q}$
Number of Entries and Computation Time for the Example		
Vector / Matrix	GS-Based Method	CS-focusing Method
$\tilde{\mathbf{u}}_{\mathbf{g}}^{\circ}$ / $\bar{\mathbf{u}}_{\mathbf{g}}^{\circ}$	10830	722
$\tilde{\mathbf{z}}$ / $\bar{\mathbf{y}}$	9375	625
$\widetilde{\mathbf{B}}_{\mathbf{g}}^{\circ}(\boldsymbol{\theta}_{\mathbf{g}})$ / $\mathbf{B}_{\mathbf{g}}^{\circ}(l_r,\boldsymbol{\theta}_{\mathbf{g}})$	101531250	451250
Computation Time	700.146s	89.251s

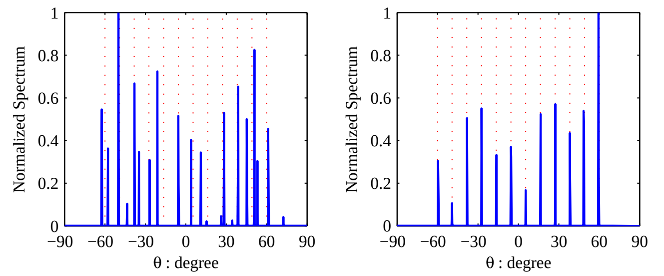
3. Results are obtained by solving the problem in (18).

IV. SIMULATION RESULTS

In the simulations, the signals have a normalized frequency range $[0.5\pi;\pi]$, and $f_r = 0.75\pi$, where the normalized frequency is defined as $\omega = \frac{2f}{f_s}\pi$ with f being the frequency of interest. For the l-th frequency bin, the corresponding normalized frequency is $\omega_l = \frac{2f_l}{f_s}\pi = \frac{2l}{L}\pi$. We first consider an example with signals at each frequency bin sharing the same distribution (i.e., uniformly distributed phase on $[0, 2\pi]$ with the same magnitude). Then, the inverse DFT is utilized to generate the signals in the time domain. A four-level nested array ($q = 2$) with 5 sensors and a position set $S = \{1d, 2d, 4d, 8d, 16d\}$ is adopted, where $d = \lambda_r/2$ with $\lambda_r = c/f_r$, and its fourth-order difference co-array contains a virtual ULA of 31 sensors [20]. $L = 64$-point DFT is applied, and the frequency band of interest covers $M = 15$ frequency bins with $17 \leq l_m \leq 31$. There are K sources with incident angles uniformly distributed between $-60°$ and $60°$. A search grid of $K_g = 721$ angles with a step size of $0.25°$ is generated from $-90°$ to $90°$. The allowable error bound ε is chosen to give the best result through trial-and-error in every experiment. All the optimisation problems in (11), (14), and (18) are solved using CVX, a software package for specifying and solving convex optimization problems [33], [34].

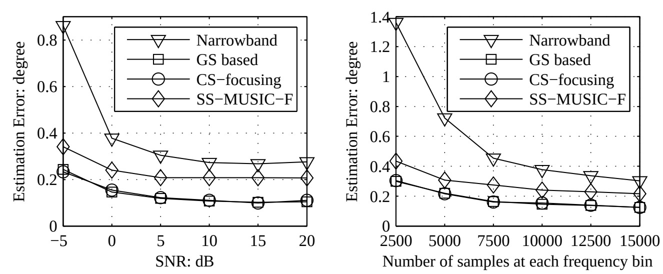
For comparison of computational complexity, the number of entries in the vectors/matrices involved is shown in Table I, where the computation time under the environment of Intel CPU I5-3470 (3.20 GHz) and 12 GB RAM is also listed. Clearly, a significant complexity reduction has been achieved by the CS-focusing method.

Fig. 1 compares the DOA estimation performance. The total signal power and noise power within the entire frequency band of interest are used to calculate the signal-to-noise ratio (SNR). The input SNR is 0 dB, and $K = 12$. The number of samples used to calculate the cumulant matrix at each frequency bin is fixed at 10000. Fig. 1(a) shows the results by applying the focusing algorithm to the physical array with perfect initial DOAs (i.e., focusing followed by virtual array generation). Clearly, this method fails to resolve all the sources due to the increased model mismatch error in generating the virtual array model with a much larger dimension from the focused physical array, whereas for both the GS-based method and the CS-focusing method, all the sources are successfully resolved.



(a) (b)

Fig. 1. DOA estimation results obtained by different methods. (a) Focusing on the physical array with perfect initial DOAs. (b) Group sparsity based method (RMSE: $0.4257°$). The CS-focusing method (RMSE: $0.4294°$) has a similar figure as in (b) and it is not shown here to save space.



(a) (b)

Fig. 2. RMSE results of different wideband DOA estimation methods.

Next we show the root mean square error (RMSE) results to compare the estimation accuracy of different methods through Monte Carlo simulations of 500 trials. The narrowband method uses only the 24-th frequency bin as it corresponds to the largest inter-element spacing without aliasing. With $K = 6$, the RMSE results obtained from the different methods with respect to a varied input SNR are shown in Fig. 2(a), whereas Fig. 2(b) gives the RMSE results versus the number of samples in the frequency domain. The SS-MUSIC-F refers to an approach by applying SS-MUSIC to the virtual model (17) after the proposed focusing operation. It is evident that the performances of the GS-based and the CS-focusing based methods are nearly the same, and both outperform the narrowband method by a large margin. Furthermore, the CS-focusing method consistently outperforms the SS-MUSIC-F method as it exploits all unique difference co-arrays lags, whereas SS-MUSIC-F only exploits the consecutive ones.

V. CONCLUSION

The CS-based wideband DOA estimation problem for the $2q$-th order difference co-array was considered. We first formulated the problem on a single frequency and then extended it to multiple frequencies based on the GS concept. It is effective, but has a very high computational complexity. To tackle this problem, a low-complexity CS-focusing method was introduced, with almost the same performance, as verified by simulation results.

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