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# Beta lives* - Some Statistical Perspectives on the Capital Asset Pricing Model

by

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## Abstract

This note summarises some technical issues relevant to the use of the idea of excess return in empirical modelling. We cover the case where the aim is to construct a measure of expected return on an asset and a model of the CAPM type is used. We review some of the problems and show examples where the basic CAPM may be used to develop other results which relate the expected returns on assets both to the expected return on the market and other factors.

* Apologies to the late Charlie Parker

Key Words:. Arbitrage pricing theory, ARCH models, Beta, Capital asset pricing model, Conditional distributions, Multi-factor models, Non-central Chi-squared

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## 1. INTRODUCTION

In this note we are concerned with the excess return on a risky asset, $R_i - R_f$ say, where $R_i$ is the return on asset i over a specified investment horizon and $R_f$ is the corresponding risk free rate. The importance of this concept stems from the Capital Asset Pricing Model of Sharpe(1964) and others. The traditional CAPM is a theorem which relates the expected return on an asset only to the expected return on the market and to the risk free rate. Excess return is widely used in a broader class of models which depart from the CAPM, either because their structure is different, or because they include other variables, or both. The most common extensions of the CAPM are the multi-factor models, henceforth MFM, originally introduced by King(1966) and summarised elegantly by Jacobs & Levy(1988), the equilibrium factor models of Rosenberg et al(1973) and the arbitrage pricing theory(APT) of Ross(1976). These classes of model preserve the linear structure of the CAPM and the assumption of normality. However, as Black(1993a) points out, they are different from the CAPM because they are really extensions of the market model. The key feature of them all is that they introduce additional variables to which the expected return on the asset is related. More than thirty years after Sharpe's paper, the CAPM is still one of the main motivating forces behind much recent work.

The aim of this short paper is to summarise some technical issues relevant to the use of excess return in empirical modelling. We look mainly at two general themes: the recent debate on the death of beta initiated by Fama & French(1992) and technical developments of the CAPM, for example the ARCH approach to time series analysis introduced by Engle(1982). We cover the case where the aim is to construct a measure of expected return on an asset and a linear model of the CAPM type is used. We also show examples where the basic CAPM may be used to develop other results which relate the expected returns on assets to the expected return on the market and other factors. This note represents a personal view and we describe some technical issues which we have found important in our own work. We make no claims to have covered the subject exhaustively.

The structure of this paper is as follows. Section two comments on the CAPM theorem itself. Section three is concerned with the recent debate on the death of beta. Sections four and five look at different aspects of empirical work based on the CAPM. Section six outlines an ARCH version of the CAPM and section seven concludes.

Except where stated otherwise in the text, we assume that returns follow a multivariate normal distribution and we employ the standard vector notation R ~ N(μ, V), where R is the n-vector of realised returns, μ is the corresponding n-vector of expected values and V is the variance-covariance(VC) matrix, which we assume to be of full rank. The realised return on a portfolio is denoted by $R_{port}$ and this is given by the inner product $R_{port} = w^T R$ where w is an n-vector of portfolio weights. The corresponding realised return on the market portfolio is denoted by $R_M$ and this has variance $\sigma_M^2$. Where it is necessary to refer to the return on an individual asset, we call it $R_i$. We use E[] , V[] and C[] to denote expectation, variance and covariance respectively. Other notation is that in common use.

## 2. THE CAPM THEOREM

The CAPM is a powerful model (a) because it relates the expected return on an asset only to the expected return on the market and the risk free rate and (b) because it results in a very simple linear equation. In the usual notation, the CAPM is:

$$E[R_i] = R_f + \beta_i\{ E[R_M] - R_f \} = \mu_i, \text{ say} \qquad (1.)$$

This result, first presented in Sharpe(1964) but also attributed to Mossin(1966) and Lintner(1965), may be derived by assuming that investors espouse a quadratic utility function of the form:

$$U(R_{port}) = R_{port} - \rho(R_{port} - \mu_{port})^2;$$

where $\mu_{port}$ is the expected value of $R_{port}$ and $\rho$ ($\geq 0$) represents risk aversion, and build a portfolio by finding the maximum of the expected utility function $E[U(R_{port})]$ subject only to specified restrictions on the vector of portfolio weights.

When returns follow a multivariate normal distribution, maximisation of $E[U(R_{port})]$, subject to convex constraints, gives the same result as maximising the expected value of the exponential utility function:

$$U(R_{port}) = 1 - \exp(-\rho R_{port})$$

This result, which is its general form is due to Kallberg & Ziemba(1983), deals with the frequently heard criticism that the quadratic function of $R_{port}$ above is not a proper utility function.

In addition, the result at (1.) holds in more general conditions than those often implicitly assumed and so in some ways is more powerful. As pointed out by Roll(1977), an equation of the form (1.) holds when $R_M$ represents the return on any portfolio which is Markowitz efficient. It also holds if maximising the expected value of the utility function is replaced by either of the two more common criteria;

$$\max E[R_{port}] \text{ such that } V[R_{port}] \leq \tau_1$$

or (2.)

$$\min V[R_{port}] \text{ such that } E[R_{port}] \geq \tau_2$$

where $\tau_1$ and $\tau_1$ are specified constants. In all of these circumstances, the linear relationship in (1.) holds regardless of the distribution of returns as long as the vector of mean returns m and the VC matrix V both exist.

However, there are theoretical limitations to the CAPM. For example, it is widely believed that returns have fat tailed distributions. There are many ways of modelling these, but one simple approach would be to assume that returns come from a multivariate Student distribution, see Johnson & Kotz(1972, page 132 et sec) for general background or Zhou(1993), who considers elliptically symmetric distributions for asset pricing. However, there is a theoretical limitation to the use of the multivariate Student distribution, because the expected value of the exponential utility function does not exist. Obviously this is not a practical restriction since either of the criteria at (2.) may be used, but it is nonetheless a brief illustration of the need for care if one wishes to move from the standard assumptions of the CAPM.

When the return vector R is $N(\mu, V)$, the connection of the CAPM to its empirical form is a particularly strong one. The return on the market may be written as $R_M = w^T R$ for some vector of weights w. If we use the conditional distribution of the vector of asset returns R given the market return $R_M$, a standard result in distribution theory (Anderson(1958, page29)) gives us:

$$E[R_i \mid R_M] = \mu_i + \beta_i\{E[R_M] - \mu_M\} \tag{3a.}$$

Substitution from (1.) then gives us the basis of empirical work:

$$E[R_i \mid R_M] = R_f + \beta_i\{R_M - R_f\} \tag{3b.}$$

The joint VC matrix of the vector R and $R_M$ is:

$$V[R, R_M] = \begin{bmatrix} V & Vw \\ w^T V & w^T V w \end{bmatrix} \tag{4a}$$

and so the VC matrix of R conditional on the value of market return $R_M$ is:

$$V[R \mid R_M] = V - \frac{1}{w^T V w}(Vw)(w^T V) = V - \frac{1}{\sigma_M^2}(Vw)(w^T V) \tag{4b.}$$

This result shows that the empirical form of the CAPM which is commonly used is based on a subtle mathematical fallacy. Standard texts (Elton & Gruber(1995) and Haugen(1993) to give just two examples) suggest use of the regression equation;

$$R_i = R_f + \beta_i\{R_M - R_f\} + \varepsilon_i$$

where the error terms $\varepsilon_i$ are assumed to be independently normally distributed. This model leads to the overall VC matrix of the returns $R_i$ being given by:

$$V[R_i] = \beta_i^2\sigma_M^2 + \sigma_i^2, \ C[R_i, R_j] = \beta_i\beta_j\sigma_M^2$$

That is, the overall VC matrix of the return vector R is given by the special structure:

$$V[R] = \beta\beta^T\sigma_M^2 + \Lambda \tag{5.}$$

where $\beta$ is the n-vector of betas and $\Lambda$ is a diagonal matrix containing the stock specific variances $\sigma_i^2$. If we now use (4.) we find that the conditional VC matrix of asset returns given the market return, $V[R \mid R_M]$ say, can be diagonal if and only if equation (5.) holds. However, there is also a contradiction. This may also be shown by evaluating the variance of a portfolio with arbitrary weights w. According to (5.) this is:

$$w^TVw = (w^T\beta)^2\sigma_M^2 + w^T\Lambda w. \tag{6.}$$

If w is the market portfolio and $(w^T\beta) = 1$, then $w^TVw = \sigma_M^2$ if and only if $\Lambda = 0$. Equally, the requirement that the vector of covariance of each asset with $R_M$ should equal $\sigma_M^2\beta$ when w is the market imposes the restriction:

$$\Lambda w = 0;$$

which implies that the specific variances $\sigma_i^2$ must be zero for all assets with non-zero weight.

Two points emerge from this treatment. First, although returns are still correlated after conditioning on the market return, it is correct to estimate the betas using the familiar regression formula as long as the underlying distribution is multivariate normal. Secondly, for purposes of portfolio construction it is, strictly speaking, necessary to use the full VC matrix and not the reduced form.

If the market portfolio is well diversified in the sense that the number of assets is large and the weight of any asset is small then the practical consequences are slight. However, if a particular market proxy is based on a small number of assets and some have large weights, then the reduced form of the VC matrix may be less accurate. This might arise in markets where the local index contains only a small number of names. Of course, it is recognised that errors in the estimates of variances and covariances are less important than errors in the estimates of mean returns, see Chopra & Ziemba(1993) for example, and use of the reduced form VC matrix brings substantial benefits in computation. We continue to use (5.) because of its simplicity. However, the treatment above suggests that may is a case for using other reduced forms, perhaps based on factor analysis of V.

## 3. THE DEATH OF BETA

We take the view that the CAPM is a theorem which relates expected values. It is a statement only about the parameters of a multivariate probability distribution. Viewed in this way, beta is immortal. Equation (1.) would cease to have any interest if and only if all betas equalled zero. Beta may be zero for some assets but this cannot happen for them all, except when the variance-covariance matrix is singular. Furthermore the CAPM is a cross-sectional theorem which pertains to expected values over the same time period. It is often inferred that the underlying parameters $\mu$ and V are constant. but in reality the CAPM makes no explicit statement about them. The real debate about the demise of beta is whether the expected value equation:

$$E[R_i] = R_f + \beta_i\{ E[R_M] - R_f \}$$

has the ability to explain the variation in realised values of $R_i$ and, specifically, to generate forecasts of future expected returns.

The cross-sectional nature of the CAPM has two implications. First, it means that the theorem can accommodate time varying parameters. Secondly, forecasts based on estimates made at time t will be valid if the parameters at time t+1 are the same, or if we can specify a second model to link them. The debate about the death of beta is really a debate about the model for the distribution of returns, which should be written as $R \sim N(\mu_t, V_t)$, and that we do not know the underlying dynamics of $\mu_t$, and/or $V_t$. This has been recognised implicitly in the professional investment community for a long time. All quantitative analysts and managers are familiar with the regular cycle that involves updating the models. This is often done using simple methods such as a rolling window, but it recognises explicitly both the need to incorporate new information and the fact the parameters $\mu$ and V change over time.

News of the death of beta, reported by Fama & French(1992), reflects the fact that the expected value theorem does not always lead to empirical results that are useful or conform to prior expectations. In particular, the familiar cross sectional regression(CSR) model of asset returns on estimated betas (in the usual notation):

$$R_i = \gamma b_i + \eta_i \text{ , } V[\eta_i] = \sigma_i^2$$

often does not lead to a significant fitted regression. In addition to Roll(1977), there have been several recent articles, most notably Roll & Ross(1994) and Kandel & Stambaugh(1995), which examine some of the deeper geometrical and statistical properties of the CAPM. We note in particular Kandel & Stambaugh's key finding: if the market index is inefficient, even by an arbitrarily small amount, then the cross sectional regression of R on a vector of estimated betas can have an arbitrary fit. That is, the R-squared can be anywhere between zero and one.

However, lack of significance in the cross-sectional regression may not be surprising. If we consider the case where the $b_i$ and $\sigma_i^2$ are assumed known, the GLS estimator of $\gamma$ is g which is given by:

$$g = \frac{\sum_i R_i b_i \,/\, \sigma_i^2}{\sum_i b_i^2 \,/ \sigma_i^2}$$

If the number of assets is large, say greater than 100, the regression is effectively tested by the sum of squares due to fitting which is:

$$\text{REGss} = \frac{(\sum_i R_i b_i \,/\, \sigma_i^2)^2}{\sum_i b_i^2 \,/ \sigma_i^2}$$

This quantity is distributed as a non-central Chi-squared variable with one degree of freedom and non-centrality parameter $\lambda$ given by:

$$\lambda = \gamma^2 \sum_i b_i^2 / \sigma_i^2$$

Noting that the expected excess return on asset i satisfies:

$$\mu_i = R_f + \gamma\beta_i$$

the significance of the regression therefore depends only on (a) the number of assets and (b) the average value of the squared Sharpe ratio of each asset. that is:

$$\lambda = \sum_i (\mu_i - R_f)^2 / \sigma_i^2 = n\omega^2, \text{ say}$$

where $\omega^2$ is the average of the square of the Sharpe ratio of each asset.

As a simple numerical example, we used weekly returns on the securities that have been members of the FTSE100 during the period 1st January 1978 through 20th October 1995. We took the risk free rate to be equivalent to 7% per annum and used the sample mean return and variance for each stock. The computed value of the non-centrality parameter is $\lambda' = 0.274$. This makes a negligible difference to the significance levels of the test statistic $REG_{ss}$, which has a central Chi-squared distribution under the null hypothesis $H_0$: $\gamma = 0$. In other words, the chances of getting a significant regression and rejecting the null hypothesis against a one or two sided alternative are low. The same arguments apply if we use a small number of assets; the non-central Chi-squared distribution is replaced by the non-central F. In each case, the power of the test is low.

Black(1993b) comments on the downward bias in the estimator g. If the $b_i$ and $\sigma_i^2$ used in the cross sectional regression are now treated as estimates based on T observations from time series regressions, then it may be shown that, as $T \to \infty$ and subject to a number of regularity assumptions, the estimator g is biased downwards. Specifically:

$$E[g] \to \gamma\, [1 - \omega_M^2 / T\omega^2]$$

where $\omega^2$ is as defined above and

$$\omega_M^2 = \frac{\gamma^2}{\sigma_M^2}$$

is the square of the Sharpe ratio of the market.

There are many papers which employ the CAPM, or more accurately the market model, as a starting point for the development of models which are used for empirical purposes. It is important to note that there may be theoretical dangers present when doing this. Briefly, acceptance of an empirical model may undermine the assumptions on which the CAPM is based. This may be a matter of relatively small importance if the objective is to forecast future expected returns, but it may have implications for the method of portfolio selection. For example, Pettengill et al(1995) propose a model of the form:

$$R_i = \gamma_0 + \gamma_1 \, db_i + \gamma_2 \, (1-d)b_i + \eta_i$$

where $d = 1$ if $R_M - R_f > 0$ and $d = 0$ otherwise. According to them, this model gives significant results and so it is to be welcomed. However, a consequence of the model is that returns do not follow an exact multivariate normal distribution even if the residuals $\{\eta_i\}$ are normally distributed. The implications of this for portfolio selection and for the CAPM are matters for further study.

## 4. A CAPM FOR FACTOR MODELS

Extensions of the CAPM are often intuitive. For example, models of the general form:

$$R_i = \alpha_i + \beta_i(R_M - R_f) + \Sigma_j \, \delta_{ij}X_j + \varepsilon_i;$$

where $X = \{ X_j \}$ is a vector of independent variables or factors, are common. These are essentially regression models and are empirical extensions of the CAPM. If we consider the joint probability distribution of the vector of returns R and the vector of factors X and assume that this is multivariate normal, then the regression model arises formally by conditioning on X, ie by assuming that X is known. It is easy to show that in these circumstances the CAPM becomes, in the usual notation for conditional expected returns:

$$E[R_i \mid X] = R_f + \beta_i^X \{ E[R_M \mid X] - R_f \} \qquad (7.)$$

where $\beta_i^X$ is the beta computed using the VC matrix of the returns conditional on the X's. That is:

$$\beta_i^X = C[R_i, R_M \mid X]/V[R_M \mid X]$$

The conditional expected value of the market return $R_M$ given X is of the form;

$$E[R_M \mid X] = \mu_M + c^T V_{XX}^{-1} (X - \mu_X) \qquad (8.)$$

where the vector c contains the unconditional covariance of each $X_j$ with the market return and $V_{XX}$ is the unconditional VC matrix of the X factors.

If we therefore elect to condition on a set of factors X, then there is a version of the CAPM which corresponds to (1.) and which is a more general expected value theorem. In addition, the resulting regression equation of $R_i$ on $R_M$ and X has a precisely defined structure, as given by (7.) and (8.). The CAPM at (7.) and (8.) is similar in spirit to that described in Rosenberg at al(1973). Their paper, however, begins with a regression equation and does not employ the explicit properties of the multivariate normal distribution which link the conditional distribution of R with a given vector of factors X.

An implication of this result is that we discord with Roll & Ross(1994) who state: "no variable other than beta can explain any part of the true cross-section of expected returns". If we have no information apart from μ and V, then Roll & Ross are correct. However, if we have a set of known factors X then equations (7.) and (8.) show two things. First, the betas will be different because they are based on a conditional VC matrix. Secondly, the cross-section of expected returns depends on a second set of covariances, which we call c, as well as the VC matrix of the factors themselves. In formulating this version of the CAPM, we are following the dictum of Dennis Lindley, (Lindley(1983)) who implies that one should always be able to use information that is available about a system of interest.

## 5. FORECASTING EXPECTED RETURNS FROM BETAS

To be an effective tool for portfolio management, the CAPM must be used to generate forecasts of future expected returns. At minimum, this requires the estimation of beta and specific risk for a number of assets. The forecasts are then used to construct an optimised portfolio. The practical value of the CAPM for this purpose suffers from the necessity to use a proxy portfolio to represent the market. Above we summarise some of the various problems that are encountered when using the CAPM in practice. In our own work, we have been particularly concerned with its use for international diversification where, according to Solnik(1988), market portfolios seem to be far from efficient. Typically, the out-of-sample performance of an optimised portfolio built using betas based on an international market proxy degenerates considerably and can often be dominated by a naive portfolio. To our knowledge Clark(1991, 1995) has developed the only market index where out-of-sample performance is better than that of naive construction strategies.

## 6. AN ARCH VERSION OF THE CAPM

The empirical use of the CAPM has given rise to many derivative models. The multi-factor and arbitrage pricing theory models are but two. There is widespread use of statistical techniques such as Kalman filters (Harvey(1992)) or shrinkage methods(Jorion(1985). Since its introduction by Engle(1982) there has been extensive use and development of auto-regressive conditionally heteroscedastic or ARCH models. It is possible to formulate ARCH and/or GARCH versions of the CAPM in different ways. Two approaches are described in Bollerslev et al(1988) and Ng(1991). More recently Giannopoulos(1995) proposed a bivariate GARCH approach for the single index model. There is a short summary of some other methods in Kroner & Ng(1998).

If we ignore the theoretical difficulties with the reduced form VC matrix described in section 2, the following ARCH form of the CAPM offers an interesting insight to systematic and specific risk. The starting point is the vector model:

$$R_t = \mu + \varepsilon_t \tag{9.}$$

where $R_t$ is the vector of actual returns for the period ending at time t, $\mu$ is the corresponding vector of expected values and $\varepsilon_t$ is the vector of unobserved residual returns. Conditional on information available up to and including time (t-1), we assume that the vector $\varepsilon_t$ has a multivariate normal distribution with zero mean vector and variance covariance matrix $h_t$, ie $\varepsilon_t \sim N(0, h_t)$, where the matrix $h_t$ satisfies:

$$ht = A_0 + A_1 \varepsilon_{t-1} \varepsilon_{t-1}{}^T A_1^T \tag{10.}$$

This is a simplified form of the so-called factor ARCH, or F-ARCH, model described in Engle et al(1990). When it exists, we can compute the unconditional VC matrix of $R_t$ in the usual way by taking expectations successively over $\varepsilon_{t-1}$ , $\varepsilon_{t-2}$ and so on. Thus we can write the unconditional VC matrix formally as:

$$V = E[h_t] = A_0 + \sum_{j=1}^{\infty} A_1^j A_0 (A_1^j)^T \tag{11.}$$

where $A_1^j$ denotes the product of $A_1$ multiplied together j times. We specify that for some value $\sigma_0^2$:

$$A_0 = \Lambda + \beta\beta^T\sigma_0^2$$

This means that if there is no ARCH effect and $A_1 = 0$, we get the standard model with $\sigma_0^2 = \sigma_M^2$. Secondly, equation (5.) imposes a structure on the matrix $A_1$. If we equate equation (5.) to (11.) and re-arrange, we get:

$$\sum_{j=1}^{\infty} A_1^j A_0 (A_1^j)^T = \beta\beta^T(\sigma_M^2 - \sigma_0^2)$$

Since $A_0$ is, by construction, a matrix of rank n, this equation imposes the restriction that $A_1$ must be a matrix of rank 1. Manipulation of the above equation shows that $A_1$ satisfies:

$$A_1 = \sigma_0 \beta \upsilon^T$$

for some vector $\upsilon$ and that for the unconditional VC matrix to exist, it is necessary that:

$$|\sigma_0 \beta^T \upsilon| < 1 \tag{11.}$$

Ignoring unnecessary constants, the log-likelihood function of the $R_{t,i}$ and $R_{t,M}$ at time t, denoted by the conventional *l*, is given by:

$$\begin{aligned} l = \; & -\frac{1}{2}\sum_{i=1}^{N} \ln \sigma_i^{\,2} - \frac{1}{2}\ln(\sigma_0^{\,2}) - \frac{1}{2}\ln(1+(\nu^T \varepsilon_{t-1})^2) \; - \\ & -\frac{1}{2}\sum_{i=1}^{N}(R_{t,i} - \alpha_i - \beta_i R_{t,M})^2 / \sigma_i^{\,2} - \frac{1}{2}\frac{(R_{t,M} - \mu_M)^2}{\sigma_0^{\,2}(1+(\nu^T \varepsilon_{t-1})^2)} \end{aligned} \tag{12.}$$

As (12.) shows, the basic ARCH model defined at (9.) and (10.) possesses the interesting property that the specific volatility of each asset always remains constant and that it is only the market volatility that changes. In other words we can write the ARCH-CAPM market model as:

$$R_{i,t} = \alpha_i + \beta_i R_{t.M} + \varepsilon_{i,t}, \text{ where } \varepsilon_{i,t} \sim IN(0, \sigma_i^2)$$

and where, conditional on information up to time (t-1) the market return has the time varying distribution

$$N(\mu_M, \sigma_0^2[1+(\nu^T \varepsilon_{t-1})^2]).$$

For the reasons above, strictly speaking this is a mathematical fiction, but it does give rise to a simple model which includes ARCH effects and which preserves the simple structure of the market model.

This model is originally due to Ng et al(1992), in which the single factor in their F-ARCH model is taken to be market return and a GARCH term is included in the variance covariance matrix. There is also a condition on the parameters of the model, as given by the converse of (11.), under which the unconditional variance of asset returns does not exist.

## 7. SUMMARY

The CAPM is very important because it is a theorem which relates expected returns on assets to the expected return on the market. The VC matrix of the empirical CAPM does not actually exist in the form usually quoted, but this is a technical problem which can usually be overlooked. The important point is that the model is cross sectional and so forecasting requires us to model the dynamics of the parameters. The CAPM may be used as a basis to develop other expected value results, which in turn lead to other models.

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