# RATIONAL TORUS-EQUIVARIANT STABLE HOMOTOPY III: COMPARISON OF MODELS.

J.P.C.GREENLEES

ABSTRACT. We give details of models for rational torus equivariant homotopy theory (a) based on all subgroups, connected subgroups or dimensions of subgroups and (b) based on pairs of subgroups or general flags of subgroups. We provide comparison functors and show the models are equivalent.

## CONTENTS

## 1. BACKGROUND

The author's long standing project aims to give algebraic models for rational stable equivariant homotopy categories. More precisely, the aim is to establish a Quillen equivalence between the category of rational $G$-spectra and the category of differential graded objects in an abelian category $\mathcal{A}(G)$.

The most complicated results to date have been in the case when $G$ is a torus. As the project has developed, the technical details of different parts of the argument have made it convenient to formulate the definition of the category $\mathcal{A}(G)$ in several different ways. The narrow purpose of this paper is to give a systematic explanation of why these different formulations give equivalent categories.

The very purpose of this paper is to provide the proper language to describe the comparison, so we cannot fully describe our results until we have introduced a considerable amount

1

of infrastructure. For the purposes of the introduction we content ourselves with an informal account.

1.A. **History.** The category $\mathcal{A}(G)$ is assembled from isotropical information associated to the various closed subgroups of $G$, and it is principally the way this information is put together that has evolved.

When $G$ is the circle group, the models are simple enough for comparisons to be side comments in [4], but already there is a distinction between whether the data is indexed by *connected* closed subgroups ($c$) or *all* closed subgroups ($a$). Only the $c$ version was made explicit in [4], despite numerous motivations implicitly using the $a$ version.

When it comes to higher tori, the distinction between $a$ and $c$ versions is just as important, but for simplicity we just discuss the $c$ version for now. The paper [7] gives an account of a model $\mathcal{A}(G)$. However it became important in [13] to make some of the details more explicit and a variant was introduced in [8]. In the present paper, this version from [8] is called $\mathcal{A}^p_c(G)$, where $p$ refers to the fact that the data is assembled from *pairs* of subgroups. The category $\mathcal{A}^p_c(G)$ is the algebraically simplest formulation, and will remain the essential basis for calculation.

On the other hand, the proof in [13] is homotopical, so it is essential to explicitly include information about how pairs of subgroups fit together to make longer chains, and it is convenient to treat all subgroups of the same dimension together. Altogether we therefore have a structure based on dimensions of flags, and it is shown that the sphere spectrum is a homotopy pullback of a diagram of ring spectra indexed on this diagram. This leads to an algebraic model $\mathcal{A}^f_d(G)$ based on *dimensions* of *flags* of subgroups, and this model is essential for comparison with homotopy.

Our narrow purpose is therefore to show explicitly that $\mathcal{A}^p_c(G)$ is equivalent to $\mathcal{A}^f_d(G)$.

1.B. **Going further.** We have described the motivation explicit in past treatments, but the present paper also completes the unfinished business of relating the $a$ and $c$ versions. It has always been clear that the data in the $c$ model is assembled from that in the $a$ model, and that the data in the $a$ model can be recovered from the $c$ model. We identify here precisely what additional structure is required in the $a$ model to give an equivalence between the models. It turns out that the additional data should be thought of as a continuity condition on the subgroups with a given identity component; the continuity condition in the $a$ models is reflected purely algebraically in the $c$ models, and both languages are useful.

There are at least four further benefits from understanding this type of algebraic model and for being able to move routinely between them.

Firstly, it is becoming clear that when groups other than the torus are concerned, the partially ordered set structures considered here will need to be augmented in two ways (a) by adding an action of a finite group, just as representation theory works with the action of the Weyl group on the maximal torus (see [9]) and (b) the diagrams of rings and modules considered here can be viewed as sheaves over *discrete* topological categories; in the more general case we should consider sheaves over the topological category whose *space* of objects consists of the closed subgroups with a topology taking into account proximity for subgroups of finite index in their normalizers (see [2, 3]). It is essential to have a flexible formal framework as described in the present paper; this has already been useful in the construction of the toral part of the model for an arbitrary group [9].

Secondly, the algebra of these models needs to be well understood to construct and work with the Quillen model structures on the categories of differential graded modules. Thirdly, a detailed understanding of the models is extremely valuable when dealing with the additional subtleties of modelling change of groups.

Finally, the structures we introduce here will be described in terms of diagrams of rings and modules over them, and these same diagrams can be viewed as descriptions of rings of functions on algebraic varieties and subvarieties. At the crudest level, these give an algebraic geometric view of what we are doing, but more significantly it lets us sysematically construct objects in $\mathcal{A}(G)$ from purely geometric data as in [6, 1] and [10]. We plan to return to this in [12].

1.C. **Increasing precision.** As alluded to in Subsection 1.A above, the algebraic model $\mathcal{A}(G)$ is based on considering categories of modules over diagrams of rings. The diagrams will be rather simple in the sense that they are functors $\mathbb{R} : \Sigma \longrightarrow \textbf{Rings}$ which have the shape of a poset $\Sigma$. Although several different posets will be involved, the functor $\mathbb{R}$ will be essentially the same throughout. The three variables are

> **The poset $\Sigma$ describing the shape of the diagrams:** this is indicated by a letter from $\{a, c, d\}$, corresponding to **a**ll closed subgroups, **c**onnected subgroups or **d**imensions of subgroups.
> **The type of diagram we build from $\Sigma$:** this is indicated by by a letter from $\{s, p, f\}$, corresponding to whether it is based on **s**ingle subgroups, **p**airs of subgroups, or **f**lags of subgroups.
> **The conditions placed on the modules:** Here this means a binary choice for each of $\{qc, e, p, cts\}$ namely quasicoherence ($qc$), extendedness ($e$), product decompositions ($p$) or continuity ($cts$).

Not all of these $3 \times 3 \times 2^4$ combinations are relevant, but even for a single group $G$ it is clear we need a systematic framework for discussing them. The categories are connected by a web of adjoint pairs of functors. Since the functor $\mathbb{R}$ is the same throughout we will indicate the domain by a subscript ($a, c$ or $d$) and the type of poset by a superscript ($s, p$ or $f$). Omitting the conditions on the modules which may be necessary to define the functors, we will describe adjoint pairs

$$\mathbb{R}_a^p\text{-modules} \xrightleftharpoons{} \mathbb{R}_c^p\text{-modules} \xrightleftharpoons{} \mathbb{R}_c^f\text{-modules} \xrightleftharpoons{} \mathbb{R}_d^f\text{-modules}.$$

The point is that the pullback square of ring spectra from [13] delivers a coefficient system $\mathbb{R}_d^f$ on the punctured cube of non-empty subsets of $\{0, 1, \ldots, r\}$ so we are committed to the use of the right hand end. On the other hand, the essential part of the structure is the localization theorem, which (when $G$ is a torus) delivers a diagram based on pairs of subgroups. The idempotents of the Burnside ring allow us to separate subgroups with the same identity component, so we may represent that information in $\mathbb{R}_c^p$-modules or $\mathbb{R}_a^p$-modules at the left hand end.

1.D. **The plan.** For the rest of the paper we will steadily introduce language to give a general treatment of the structures that concern us. The particular examples from this motivational section will be introduced properly at the appropriate point in the discussion. Once the machinery is introduced, in Section 10 we give details of the comparison of the

models of rational $G$-spectra, with some results about torsion functors adapted from [8] deferred to Section 11.

## 2. Splitting systems and Euler classes

2.A. **Flags.** We suppose given a countable partially ordered set $\Sigma$. The prime example is that $\Sigma$ consists of the connected subgroups of a torus $G$, and the notation is chosen accordingly. The order relation is $G \supseteq H \supseteq K \supseteq L$ with $G$ denoting the top element. We do not want to insist on a bottom element. The maximal elements (i.e., $H$ so that $H' \supset H$ implies $H' = G$) will play a special role.

The motivating examples are as follows.

**Example 2.1.** (i) The partially ordered set $\Sigma_c = \mathrm{ConnSub}(G)$ of connected subgroups of a torus $G$ under containment.

(ii) The poset $\mathrm{Sub}(G)$ of all closed subgroups of a compact Lie group $G$, under containment. In fact this example will not be very relevant to us, but a certain non-full poset will be.

(iii) The poset $\Sigma_a = \mathcal{TC}(G)$ of all closed subgroups of a compact Lie group $G$, with $L \subseteq K$ if $L$ is normal in $K$ with a torus quotient [2]. We emphasize that this has many fewer morphisms than the poset with all inclusions. In the applications it is this toral-chain poset that is relevant.

(iv) The set $\Sigma_d = [0, r] := \{i \in \mathbb{Z} \mid 0 \le i \le r\}$ with the usual ordering of integers.

A sequence of elements
$$F = (H_0 \supset H_1 \supset \cdots \supset H_s)$$
is called an *$s$-flag*, and we write $|F| = s$. We call $H_0 = f(F)$ the *first* element of $F$ and $H_s = l(F)$ the *last* element of $F$. It is worth emphasizing that the biggest element of the flag is first (this is to take notational advantage of standard bracketing conventions in one of the applications).

We write
$$\mathrm{flag}_s(\Sigma) = \{F \mid |F| = s\},$$
and
$$\mathrm{flag}(\Sigma) = \bigcup_s \mathrm{flag}_s(\Sigma).$$

We note that for $s \ge 1$ we have maps $\partial_i : \mathrm{flag}_s(\Sigma) \longrightarrow \mathrm{flag}_{s-1}(\Sigma)$ for $i = 0, \ldots, s$ by omitting the $i$th term. If we permitted degenerate flags (i.e., containing equalities) we would obtain a simplicial set, but instead we simply view $\mathrm{flag}(\Sigma)$ as poset.

Finally, we will need to consider various maps of posets, such as the dimension function $d : \Sigma \longrightarrow I = [0, r] = \{0, 1, \ldots, r\}$ with $d(H) = \dim(H)$.

2.B. **$\Sigma$-diagrams of rings.**

**Definition 2.2.** (i) A *$\Sigma$-splitting diagram* is a diagram $R : \Sigma^{op} \longrightarrow \mathbf{Rings}$ of rings. We may write $R(G/L)$ for the value at $L$. If $K \supseteq L$ we write $\inf_{G/K}^{G/L} : R(G/K) \longrightarrow R(G/L)$ and call it *inflation* from $G/K$ to $G/L$.

(ii) A *system of Euler classes* for a splitting diagram $R$ is a collection of functors $\mathcal{E}_{/L} : \Sigma_{\supseteq L} \longrightarrow Mult(R(G/L))$ from elements above $L$ to multiplicatively closed subsets of $R(G/L)$; the functoriality is the statement that $\mathcal{E}_{K/K} = \{1\}$ and that if $L \subseteq K \subseteq H$ then
$$\mathcal{E}_{K/L} \subseteq \mathcal{E}_{H/L}.$$

These functors are said to be a *transitive system* if whenever $H \supseteq K \supseteq L$ the multiplicative system $\mathcal{E}_{H/L}$ is generated by $\mathcal{E}_{K/L}$ and the inflation of the one for $\mathcal{E}_{H/K}$:

$$\mathcal{E}_{H/L} = \langle \inf_{G/K}^{G/L} \mathcal{E}_{H/K}, \mathcal{E}_{K/L} \rangle$$

**Remark 2.3.** The notation $G/K$ has no meaning in itself. However $R(G/K)$ is supposed to suggest that information for objects above $K$ is being captured.

**Definition 2.4.** The systems of Euler classes we are most concerned with will all be *maximally generated* in the following sense. For each maximal element $H$ in $\Sigma$ we are given elements $e_i^H \in R(G/H)$ for $i \in I(H)$. We then obtain a transitive system by defining

$$\mathcal{E}_{K/L} = \langle \inf_{G/H}^{G/L}(e_i^H) \mid L \subseteq H, K \nsubseteq H \rangle.$$

In our main example there is just one element $e^H \in R(G/H)$ for each maximal $H$.

**Example 2.5.** Given a torus $G$ we may let $\Sigma = \mathrm{Sub}(G)$ and take $d(H) = \dim(H)$. The most important splitting diagram for us is the diagram $\mathbb{R}$ of polynomial rings defined by

$$\mathbb{R}(G/L) := H^*(BG/L; \mathbb{Q}).$$

This is the diagram referred to in Section 1, and the notation $\mathbb{R}$ will be reserved for this use throughout.

More generally, given a cohomology theory $E$, we obtain an splitting diagram $\mathbb{E}$ by taking

$$\mathbb{E}(G/L) := E^*(BG/L),$$

and where inflation has its usual meaning. The main example $\mathbb{R}$ is the one corresponding to rational ordinary cohomology: $\mathbb{R} = \mathbb{HQ}$.

If in addition $E$ is complex orientable, Euler classes $e_G(V)$ are defined for complex representations $V$ of $G$. These are compatible with inflation in the sense that if $W$ is a representation of $G/L$ then $e_G(\inf_{G/L}^{G/1} W) = \inf_{G/L}^{G/1} e_{G/L}(W)$, so we may omit the subscript $G$ without confusion.

Now take

$$\mathcal{E}_{K/L} = \{ e(V) \mid V \text{ is a representation of } G/L \text{ with } V^{K/L} = 0 \}.$$

This evidently gives a system of Euler classes and it is transitive since

$$e(V) = e(V^{K/L}) e(V/V^{K/L}).$$

Now we note that we have inclusions $\mathrm{ConnSub}(G) \subset \mathcal{TC}(G) \subset \mathrm{Sub}(G)$. The poset $\mathrm{Sub}(G)$ does not have maximal elements, but the maximal elements in both $\mathcal{TC}(G)$ and $\mathrm{ConnSub}(G)$ are the codimension 1 subgroups they contain.

For $\Sigma = \mathcal{TC}(G)$ or for $\Sigma = \mathrm{ConnSub}(G)$, this example is maximally generated; for each maximal subgroup $H$ we choose one of the two faithful representations of $G/H$ and call it $\hat{H}$. The system of Euler classes is maximally generated by

$$e(\hat{H}) \in E^*(BG/H) = E^*[[e(\hat{H})]].$$

**Example 2.6.** As a slight generalization of Example 2.5, we may suppose given any global equivariant theory $E$, and define

$$\mathbb{E}(G/L) = E^*_{G/L}.$$

If the cohomology theory is globally complex stable (i.e., all equivariant theories are complex stable, and the Thom isomorphisms are compatible with inflation), we define a system of Euler classes as before. Again this is maximally generated by the Euler classes $e(\hat{H}) \in E^*_{G/H}$.

This cohomological example explains the terminology, since one says that a cohomology theory is split if there is a ring map $\inf^{G/1}_{G/G} E \longrightarrow E$ which is a non-equivariant equivalence. Taking fixed points we obtain a map $E^* \longrightarrow E^*_G$. For global equivariant theories, we have ring maps $\inf^{G/1}_{G/K}(E_{G/K}) \longrightarrow E_G$, showing they are split.

2.C. **Coefficient systems on the flag complex.** Given a splitting system $R$ on $\Sigma$ with Euler classes there is an associated *coefficient system* on flag($\Sigma$). When helpful for emphasis, we write $R^s$ for the original splitting system and $R^f$ for the associated coefficient system.

First we note that flag($\Sigma$) is itself a poset where $E \leq F$ if $E$ is obtained by omitting some terms from $F$. We may define the (covariant) functor

$$R^f : \mathrm{flag}(\Sigma) \longrightarrow \mathbf{Rings}$$

by

$$R^f(F) := \mathcal{E}^{-1}_{H_0/H_1} \mathcal{E}^{-1}_{H_1/H_2} \cdots \mathcal{E}^{-1}_{H_{s-1}/H_s} R(G/H_s) = \mathcal{E}^{-1}_{H_0/H_s} R(G/H_s),$$

where the second equality uses the fact that the system is transitive.

We note that $R^f(F)$ is *middle-independent* in the sense that the values are unaffected by omitting middle vertices:

$$R^f(\partial_i F) = R^f(F) \text{ if } 0 < i < s.$$

On the other hand, omitting the first element we have a localization map

$$R^f(\partial_0 F) = \mathcal{E}^{-1}_{H_1/H_s} R(G/H_s) \longrightarrow \mathcal{E}^{-1}_{H_0/H_s} R(G/H_s) = R^f(F),$$

and omitting the last element we have an inflation map

$$R^f(\partial_s F) = \mathcal{E}^{-1}_{H_0/H_{s-1}} R(G/H_{s-1}) \longrightarrow \mathcal{E}^{-1}_{H_0/H_s} R(G/H_s) = R^f(F).$$

**Remark 2.7.** The splitting system $R^s$ should not be confused with the coefficient system $R^f$. The notational distinction between $R(G/H)$ (value of the splitting system at $H$) and $R(H)$ (value of the coefficient system at the flag $H$ of length 0) should help.

The point to bear in mind is that the coefficient system $R^f$ includes the *values* of the splitting system as the values on length 0 flags: $R^f(K) = R^s(G/K)$. However the *maps* of the splitting system are not included. If $H \supset K$ there is an inflation map $R(G/H) \longrightarrow R(G/K)$, but in flag($\Sigma$) there is no direct relation between the flag $(H)$ and the flag $(K)$. The flag $(H \supset K)$ gives inclusions

$$(H) \longrightarrow (H \supset K) \longleftarrow (K)$$

and hence ring maps

$$R^f(H) = R^f(\partial_1(H \supset K)) \longrightarrow R^f(H \supset K) \longleftarrow R^f(\partial_0(H \supset K)) = R^f(K).$$

In our case these become

$$R(G/H) \longrightarrow \mathcal{E}^{-1}_{H/K} R(G/K) \longleftarrow R(G/K).$$

2.D. **Modules over the coefficient system.** Note that the coefficient system $R$ is a flag($\Sigma$)-diagram of rings, and we may consider modules over it. Explicitly, $M(F)$ is an $R(F)$-module, and if $E \leq F$ there is a map $M(E) \longrightarrow M(F)$ over the structure map $R(E) \longrightarrow R(F)$.

**Definition 2.8.** We say that $M$ is a *qce-module* if, for all inclusions $E \subseteq F$, the map $M(E) \longrightarrow M(F)$ induces an isomorphism

$$R(F) \otimes_{R(E)} M(E) \xrightarrow{\cong} M(F).$$

**Remark 2.9.** By associativity of the tensor product, the value of a *qce*-module is determined by the structure maps and the values on length 0 flags.

In particular, it is *last-determined* in the sense that for any flag $F = (L_0 \supset \cdots \supset L_t)$, we have

$$M(F) = R(F) \otimes_{R(L_t)} M(L_t) = \mathcal{E}_{L_0/L_t}^{-1} M(L_t).$$

In view of the resulting formula we call such last-determined modules *quasicoherent (qc)*, explaining the *qc* in the definition.

It is also *first-determined* in the sense that for any flag $F$ as above,

$$M(F) = R(F) \otimes_{R(L_0)} M(L_0).$$

We call such first-determined modules *extended (e)*, explaining the $e$ in the definition.

Because of middle independence, we only need names for first-determined and last-determined modules.

**Remark 2.10.** As in Remark 2.7, if $H \supset K$ we have maps

$$M(H) \longrightarrow M(H \supset K) \longleftarrow M(K)$$

but there is no direct map $M(H) \longrightarrow M(K)$.

**Remark 2.11.** (i) By quasicoherence, the values on single element flags determine all values and therefore play a special role. Accordingly, we sometimes use the notation

$$\phi^K M = M(K)$$

to emphasize this.

(ii) The coefficient system $R$ is a module over itself, and as such we acquire a third notation: $\phi^K R = R(K) = R(G/K)$. The notations $\phi^K R$ and $R(G/K)$ both have connotations in the equivariant setting, and the notation here is consistent with [7, 8, 13]. Indeed, considering the flag $(G \supset 1)$, we obtain

$$
\begin{array}{ccccc}
M(1) & \longrightarrow & M(G \supset 1) & \longleftarrow & M(G) \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
M(1) & \longrightarrow & \mathcal{E}_G^{-1} M(1) & \longleftarrow & \phi^G M
\end{array}
$$

Thus if $G$ is a torus we may consider the poset $\Sigma_a$ as in Example 2.5 above. Now if $X$ is a finite $G$-space and the module $M$ is given by Borel cohomology of fixed points

$$\phi^K M = H_{G/K}^*(X^K),$$

then $M$ is qce by the Borel-Hsiang-Quillen localization theorem; for example,

$$\mathcal{E}_G^{-1} H_G^*(X) \cong \mathcal{E}_G^{-1} H^*(BG) \otimes_{\mathbb{Q}} H^*(X^G).$$

The definition was originally designed precisely to capture the Localization Theorem.

## 3. The category of pairs

Suppose we have a splitting system $R^s$ with Euler classes and consider the associated coefficient system $R^f : \mathrm{flag}(\Sigma) \longrightarrow \mathbf{Rings}$. In view of the fact that the value $R(F)$ depends only on the first and last term of the flag $F$, most of the coefficient system is rather redundant, at least when we consider $qc$-modules or $e$-modules.

Accordingly we may introduce a more economical category to capture this.

**Definition 3.1.** (i) The category of *pairs* $P(\Sigma)$, which is a partially ordered set with objects the pairs $(K \supseteq L)$. The order is given by $(K \supseteq L) \leq (H \supseteq M)$ if $H \supseteq K \supseteq L \supseteq M$, so that there is a morphism if we increase the first term and decrease the last. We will use the letter $p$ to indicate the use of pairs.

(ii) The morphisms are composites of the *horizontal* morphisms $h : (K \supseteq L) \longrightarrow (H \supseteq L)$ increasing the first term and *vertical* morphisms $v : (H \supseteq K) \longrightarrow (H \supseteq L)$ decreasing the last, where $H \supseteq K \supseteq L$.

**Remark 3.2.** In the terminology of [8], $P(\Sigma)$ would be called the category of 'quotient pairs' and $(K \supseteq L)$ would be written $(G/K)_{G/L}$; it embodies the $G/L$-equivariant information in the $L$-fixed points, namely the part that the localization theorem says should give the $(G/L)/(K/L)$-equivariant $K/L$-fixed point information. In any case, the value at $(K \supseteq L)$ only considers information above $L$, and concentrates on the part coming from above $K$.

Note that $P(\Sigma)$ is not simply related to $\mathrm{flag}(\Sigma)$ since there are no morphisms between two 2-flags. Nonetheless, because $R$ is a splitting system with Euler classes, it does define a $P(\Sigma)$-diagram $R^p$ of rings. Indeed, when $H \supseteq K \supseteq L \supseteq M$ we have a commutative square

$$
\begin{array}{ccc}
\mathcal{E}_{K/L}^{-1}R(G/L) = R(K \supseteq L) & \longrightarrow & R(H \supseteq L) = \mathcal{E}_{H/L}^{-1}R(G/L) \\
\downarrow & & \downarrow \\
\mathcal{E}_{K/L}^{-1}R(G/K) = R(K \supseteq M) & \longrightarrow & R(H \supseteq M) = \mathcal{E}_{H/M}^{-1}R(G/M)
\end{array}
$$

**Definition 3.3.** (i) The category of $R^p$-modules is the category of modules over the $P(\Sigma)$-diagram $R^p$ of rings.

(ii) A module $M$ is *quasi-coherent (qc)* if the horizontal maps are given by extensions of scalars, so that if $H \supseteq K \supseteq L$ then the horizontal structure map induces an isomorphism

$$
R(H \supseteq L) \otimes_{R(K \supseteq L)} M(K \supseteq L) = \mathcal{E}_{H/L}^{-1}M(K \supseteq L) \overset{\cong}{\longrightarrow} M(H \supseteq L).
$$

(ii) A module $M$ is *extended (e)* if the vertical maps are given by extensions of scalars, so that if $H \supseteq K \supseteq L$ then the vertical structure map induces an isomorphism

$$
R(H \supseteq L) \otimes_{R(H \supseteq K)} M(H \supseteq K) \overset{\cong}{\longrightarrow} M(H \supseteq L).
$$

We may define a functor

$$
f : R^p\text{-modules} \longrightarrow R^f\text{-modules}
$$

by selecting just the first and last term of the flag:

$$
(fN)(F) := N(f(F) \supseteq l(F)).
$$

For the structure maps we assume $E$ is a subflag of $F = (L_0 \supset \cdots \supset L_t)$ obtained by omitting the single term $L_i$. If $0 < i < t$ the structure map is the identity. If $i = t$ we use the vertical morphism and if $i = 0$ we use the horizontal morphism.

**Lemma 3.4.** *The functor $f$ identifies $R^p$-modules as the $R^f$-modules which are middle-independent in the sense that inclusions of flags $E \longrightarrow F$ induce isomorphisms if $E$ and $F$ have the same first and last entry.*

*In particular, it induces equivalences on the subcategories of $e$, $qc$ and $qce$ modules:*

$$e\text{-}R^p\text{-}modules \simeq e\text{-}R^f\text{-}modules$$

$$qc\text{-}R^p\text{-}modules \simeq qc\text{-}R^f\text{-}modules$$

$$qce\text{-}R^p\text{-}modules \simeq qce\text{-}R^f\text{-}modules$$

**Proof:** One may define a functor in the opposite direction

$$p : \text{middle-independent-}R^f\text{-modules} \longrightarrow R^p\text{-modules}$$

on the category of middle-independent modules. On objects, we simply take $(pM)(K \supseteq L) := M(K \supseteq L)$. For $H \supseteq K \supseteq L$ the horizontal and vertical morphisms are obtained from

$$M(K \supseteq L) \longrightarrow M(H \supseteq K \supseteq L) \xleftarrow{\cong} M(H \supseteq L)$$

and

$$M(H \supseteq K) \longrightarrow M(H \supseteq K \supseteq L) \xleftarrow{\cong} M(H \supseteq L)$$

by inverting the second map. To see this respects compositions, we compare to higher flags involving all objects involved in the composition. It is clear that $f$ and $p$ are quasi-inverse.

Quasi-coherent $R^f$-modules are last-determined in the sense of the formula $M(F) = R(F) \otimes_{R(l(F))} M(l(F))$; since $R(F)$ is middle-independent, the quasi-coherent modules are middle-independent. Similarly, extended $R^f$-modules are first-determined and a dual argument applies. $\qquad\square$

## 4. Multiplicities

On some occasions we want to artificially increase the size of our poset $\Sigma$, constructing a new poset $\widetilde{\Sigma}$ in rather a trivial way. We will use this to bring the rings occurring in our coefficient systems under control.

**Definition 4.1.** A *system of multiplicities* is a covariant functor $\mathcal{F}/ : \Sigma \longrightarrow \mathbf{Sets}$ so that if $i : L \subset K$ then $i_* : \mathcal{F}/L \longrightarrow \mathcal{F}/K$ is surjective. We also require that $\mathcal{F}/G$ is a singleton (also denoted $G$).

**Example 4.2.** (i) If $\Sigma = \mathrm{ConnSub}(G)$ there is a system of multiplicities given by specifying the set of subgroups

$$\mathcal{F}/K := \{\tilde{K} \mid \text{ the identity component of } \tilde{K} \text{ is } K \}.$$

If $i : L \subset K$ then the map $i_* : \mathcal{F}/L \longrightarrow \mathcal{F}/K$ is given by $i_*(\tilde{L}) := \tilde{L} \cdot K$. Note that $i_*(\tilde{L}) = \tilde{L} \cdot K$ has identity component $K$ and it has $\tilde{L}$ as a cotoral subgroup; it is the unique subgroup with these two properties. To see the map is surjective, note that any subgroup $\tilde{K}$ is an internal direct product of $K$ and a finite group $F$, and so $\tilde{K} = i_*(L.F)$.

(ii) A surjective map of posets $q : \widetilde{\Sigma} \longrightarrow \Sigma$ has fibres $\mathcal{F}/K = q^{-1}(K)$, but these do not generally form a system of multiplicities. If we require that the elements of $\mathcal{F}/K$ are incomparable for each $K$ then the condition is that given $K \supset L$, and $\tilde{L}$ with $q(\tilde{L}) = L$, then there is a unique $\tilde{K} \supset \tilde{L}$ with $q(\tilde{K}) = K$, and we write $i_*(\tilde{L}) = \tilde{K}$. (This is a very degenerate case of the requirement that $q$ is a Grothendieck opfibration with cleavage). This defines a functor $\mathcal{F}/ : \Sigma \longrightarrow \mathbf{Sets}$, and we require in addition that the morphisms are surjective.

Given a poset $\Sigma$ and a system of multiplicities $\mathcal{F}/$ we may form a new poset $\widetilde{\Sigma} = \Sigma\mathcal{F}$ with a surjective poset map $q : \Sigma\mathcal{F} \longrightarrow \Sigma$ preserving the top and maximal elements. Its objects are pairs $(K, \tilde{K})$ where $K \in \Sigma$ and $\tilde{K} \in \mathcal{F}/K$. The order relation is given by $(L, \tilde{L}) \subset (K, \tilde{K})$ if (a) $L \subset K$ and (b) $i_*\tilde{L} = \tilde{K}$. Where $K$ can be inferred from $\tilde{K}$ (as in the subgroup example), we may abbreviate $(K, \tilde{K})$ to $\tilde{K}$. Note in particular that for a specified $K$, the elements of $\mathcal{F}/K$ are incomparable.

We note that this gives an alternative approach to a familiar example.

**Example 4.3.** If we take $\Sigma_c = \mathrm{ConnSub}(G)$ and $\mathcal{F}$ to be the system of subgroups with a given identity component, we recover the toral chain poset:

$$\Sigma_a = \mathcal{TC}(G) = \mathrm{ConnSub}(G)\mathcal{F} = \Sigma_c\mathcal{F}.$$

4.A. **Splitting systems with multiplicities.** Given a splitting system $R$ and a system of multiplicities $\mathcal{F}/$, we may introduce mutiplicities into $R$.

First we note that any map $q : \widetilde{\Sigma} \longrightarrow \Sigma$ lets us define a $\widetilde{\Sigma}$-splitting system $q^*R$ by $(q^*R)(\tilde{K}) = R(q(\tilde{K}))$. We may apply this to $\widetilde{\Sigma} = \Sigma\mathcal{F}$ and the map $q : \Sigma\mathcal{F} \longrightarrow \Sigma$ defined by $q(K, \tilde{K}) = K$ to obtain a $\Sigma\mathcal{F}$ splitting system by taking

$$R(G/(K, \tilde{K})) := R(G/K),$$

and using the original inflation maps as structure maps.

We may define a new $\Sigma$-splitting system $R\mathcal{F}$ by taking products over the fibres of $q$. Explicitly, we take

$$R\mathcal{F}(G/K) = (R(G/K))^{\mathcal{F}/K}.$$

If $i : L \subseteq K$ the inflation map

$$(R\mathcal{F})(G/K) = (R(G/K))^{\mathcal{F}/K} \longrightarrow (R(G/L))^{\mathcal{F}/L} = (R\mathcal{F})(G/L)$$

is defined as a product of the diagonal inflation maps. To explain, the map is a product indexed by $\mathcal{F}/K$. The factor corresponding to $\tilde{K} \in \mathcal{F}/K$ is the map

$$(R(G/K))^{\{\tilde{K}\}} \longrightarrow (R(G/L))^{i_*^{-1}(\tilde{K})}$$

whose components are all inflation. This is where we use the surjectivity in the system of multiplicities.

**Remark 4.4.** It is natural to use the notation $(R\Sigma)^s = q_! q^* R^s$, and we will justify this in due course. However, some care is necessary, since the two coefficient systems $(q_! R^s)^f$ and $q_!(R^f)$ are usually different.

10

4.B. **Euler classes on $R\mathcal{F}$.** We note that once we define a set of Euler classes, the splitting system $R\mathcal{F}$ gives rise to a flag$(\Sigma)$-coefficient system $(R\mathcal{F})^f$. All such coefficient systems take the value $\prod_{\tilde{K}\in\mathcal{F}/K} R(G/K)$ at $K$, but the values elsewhere will depend on Euler classes.

To define Euler classes it is natural to assume $q$ takes maximal elements to maximal elements, and use a suitable induced system of maximally generated Euler classes. We illustrate this in the topological examples of [7, 8, 13]. At present, there are several candidate constructions corresponding to that for the sphere. The purpose of the present subsection is to make these explicit, explain their differences and identify the topologically relevant one.

**Example 4.5.** We consider various examples with $\Sigma_a = \mathcal{TC}(G)$ and $\Sigma_c = \mathrm{ConnSub}(G)$. We have maps of posets

$$\Sigma_c \xrightarrow{i} \Sigma_a \xrightarrow{q} \Sigma_c,$$

so that $\Sigma_c$ is a retract of $\Sigma_a$ and $\Sigma_a = \Sigma_c\mathcal{F}$.

We start with the ordinary Borel splitting system $\mathbb{R}$ of Example 2.5, now introducing decorations so we can introduce the diagrams from Section 1. To start with, we have the basic splitting system

$$\mathbb{R}_a^s(G/K) = H^*(BG/K)$$

on $\Sigma_a$. From this we form $\mathbb{R}_c^s = q_!\mathbb{R}_a^s$ so that

$$\mathbb{R}_c(G/K) = \prod_{\tilde{K}\in\mathcal{F}/K} H^*(BG/\tilde{K}) = \mathcal{O}_{\mathcal{F}/K}$$

(where the notation $\mathcal{O}_{\mathcal{F}/K}$ is that used in [7, 8, 13]).

We note that we could also form $i^*\mathbb{R}_a^s$ on $\Sigma_c$ and introduce multiplicities to form $\overline{\mathbb{R}}_c^s = (i^*\mathbb{R}_a^s)\mathcal{F}$, where we have

$$\overline{\mathbb{R}}_c^s(G/K) = (i^*\mathbb{R}_a^s)\mathcal{F}(G/K) = \prod_{\tilde{K}\in\mathcal{F}/K} H^*(BG/K).$$

We note that since we are working over the rationals, termwise inflation gives an isomorphism

$$\mathbb{R}_c^s \xrightarrow{\cong} \overline{\mathbb{R}}_c^s$$

of splitting systems.

We now consider several choices of maximally generated Euler classes.

**Example 4.6.** On a codimension 1 connected subgroup $H$, the canonical example $\mathbb{R}_c$ has the value

$$\mathbb{R}_c(G/H) = \prod_{\tilde{H}\in\mathcal{F}/H} H^*(BG/\tilde{H}) = \mathcal{O}_{\mathcal{F}/H}.$$

It has Euler classes $c(\alpha)(\tilde{H}) = c_1(\alpha^{\tilde{H}})$, where $\alpha$ runs through all non-trivial one dimensional representations of $G$. In particular, if $\alpha$ is a faithful representation of $G/H$ then

$$c(\alpha^n) = \begin{cases} nc_1(\alpha) & \text{if } |\tilde{H}/H| \mid n \\ 1 & \text{if } |\tilde{H}/H| \nmid n \end{cases}.$$

This is different to the diagram used in the construction onf $\mathcal{A}_c^f(G)$.

11

**Example 4.7.** Now consider $\overline{\mathbb{R}}_c$; on a codimension 1 connected subgroup $H$ it has the value
$$\overline{\mathbb{R}}_c(G/H) = \prod_{\tilde{H} \in \mathcal{F}/H} H^*(BG/H)$$
and we need to consider how to define Euler classes $c(\alpha) \in \overline{\mathbb{R}}_c(G/H)$.

(i) Diagonal maps give a map of $\Sigma_c$-splitting systems $i^*\mathbb{R}_a \longrightarrow (i^*\mathbb{R}_a)\mathcal{F}$. We may use the system of Euler classes from $\mathbb{R}_a$ to give a system on $(i^*\mathbb{R}_a)\mathcal{F}$.

This would mean that we use generating Euler classes defined by $c(\alpha)(\tilde{H}) = c_1(\alpha^H)$, independent of $\tilde{H}$. In particular, if $\alpha$ is a faithful representation of $G/H$ then $c(\alpha^n) = nc(\alpha)$. In this case we would only need to use Euler classes of characters with connected kernel.

(ii) If we forget the diagonal is available, for each $\tilde{H} \in \mathcal{F}/H$ we have an Euler class $c(\alpha)_{\tilde{H}}$ whose value at $\tilde{H}'$ is 1 if $\tilde{H}' \neq \tilde{H}$ and is $c_1(\alpha^H)$ if $\tilde{H}' = \tilde{H}$.

Under the isomorphism $\mathbb{R}_c \cong \overline{\mathbb{R}}_c$ described in Example 4.5, this second collection of Euler classes gives the same localization as the natural Euler classes of $\mathbb{R}_c$ so we have an isomorphism of coefficient systems
$$\mathbb{R}_c^f \cong \overline{\mathbb{R}}_c^f.$$
Thus we have two slightly different approaches to the diagram of rings used in constructing $\mathcal{A}_c^f(G)$.

## 5. Change of poset

In organizing the information in categories of modules, there is a balance between the information put into the poset $\Sigma$ and the information put into the rings. We aim to show that on the categories of modules of interest to us, we can move between these easily. However there are a number of different functors that will all be important. In this section we give an overview. It is easy to break apart modules over product rings with idempotents, giving a functorial construction $e$. Depending on the domain and codomain categories, the functor $e$ has a number of left and right adjoints. In this section we construct the most obvious adjoint to $e$. In Sections 6 to 8 we construct other functors and establish adjunctions that let us work with them.

5.A. **Change of poset for coefficient systems.** We start with a surjective poset map $\pi : \Sigma \longrightarrow \overline{\Sigma}$ which takes the top element $G$ of $\Sigma$ to the top element $\overline{G}$ of $\overline{\Sigma}$, and also takes the set of maximal elements of $\Sigma$ onto the set of maximal elements of $\overline{\Sigma}$.

**Example 5.1.** (i) One example of importance is when we have a dimension function $d : \Sigma \longrightarrow I$ where $I = [0, r] = \{0, 1, \ldots, r\}$.

(ii) A second example is the map $q : \overline{\Sigma\mathcal{F}} \longrightarrow \overline{\Sigma}$ arising from a system of multiplicites $\overline{\mathcal{F}}$ on $\overline{\Sigma}$. This example has special features.

We first note that given a splitting system $\overline{R}$ on $\overline{\Sigma}$ we may define a splitting system $\pi^*\overline{R}$ on $\Sigma$ by
$$(\pi^*\overline{R})(K) = \overline{R}(\pi K).$$
One naturally expects a right adjoint to this construction to be given on objects by the formula
$$(\pi_! R)(\overline{K}) = \prod_{\pi K = \overline{K}} R(K),$$

but $\pi$ needs to satisfy additional properties before we may define structure maps. Fortunately, $\pi$ induces a map on flags, and it is straightforward to observe this has the property we require.

**Lemma 5.2.** *The map $\pi : flag(\Sigma) \longrightarrow flag(\overline{\Sigma})$ is a Grothendieck fibration with cleavage in the sense that given an inclusion $\overline{E} \longrightarrow \overline{F}$ of $\overline{\Sigma}$-flags and $F$ with $\pi F = \overline{F}$, there is a unique $\Sigma$-subflag $E$ of $F$ with $\pi E = \overline{E}$.* $\qquad\square$

In this section we deal with the general framework, and in Section 7 we look at the Euler adapted context which is more directly relevant.

**Definition 5.3.** Given a surjective map $\pi : \Sigma \longrightarrow \overline{\Sigma}$ and a coefficient system $R$ on $flag(\Sigma)$ we may define a coefficient system $\pi_! R$ on $flag(\overline{\Sigma})$ on flags by

$$(\pi_! R)(\overline{F}) = \prod_{\pi F = \overline{F}} R(F).$$

Given a map $\overline{E} \longrightarrow \overline{F}$ of flags, the map

$$(\pi_! R)(\overline{E}) = \prod_{\pi E = \overline{E}} R(E) \longrightarrow \prod_{\pi F = \overline{F}} R(F) = (\pi_! R)(\overline{F})$$

is a product indexed by $E$ with $\pi E = \overline{E}$ of the maps

$$R(E) \longrightarrow \prod_{F \supset E, \pi F = \overline{F}} R(F)$$

with components coming from the structure maps of $R$.

5.B. **Flag idempotents.** First we describe how we may obtain $R$-modules from $\pi_! R$-modules.

Recall that $lF$ means the last (or smallest) term in the flag $F$. The key is to note that there is a canonical choice of idempotent

$$e_F \in \prod_{\pi F = \overline{F}} R(lF),$$

and if $E$ is a subflag of $F$ then $e_F$ is a refinement of the image of $e_E$ in $\prod_{\pi F = \overline{F}} R(lF)$. This gives compatible idempotents for all systems of Euler classes.

**Lemma 5.4.** *If $E$ is a subflag of $F$ with $\pi E = \overline{E}, \pi F = \overline{F}$ then $e_F (\pi_! R)(\overline{F}) = R(F)$ and the map*

$$R(E) = e_E (\pi_! R)(\overline{E}) \longrightarrow e_F (\pi_! R)(\overline{F}) = R(F)$$

*coincides with the original structure map of $R$.* $\qquad\square$

**Lemma 5.5.** *Applying idempotents gives a functor*

$$e : \pi_! R^f\text{-modules} \longrightarrow R^f\text{-modules}.$$

*defined by*

$$(e\overline{M})(F) = e_F \left[ \overline{M}(\pi F) \right],$$

*where $\overline{M}$ is a $\pi_! R^f$-module and $e_F \in R(\pi F)$ is the idempotent corresponding to $F$.*

13

**Proof:** First, we need to describe the structure maps associated to an inclusion $E \longrightarrow F$ of flags. We have an inclusion $\pi E \longrightarrow \pi F$ giving $\overline{M}(\pi E) \longrightarrow \overline{M}(\pi F)$. Since the idempotent the image of $e_E$ in $R(\pi F)$ refines $e_F$ we have an induced map

$$(e\overline{M})(E) = e_E \overline{M}(\pi E) \longrightarrow e_E \overline{M}(\pi F) \longrightarrow e_F \overline{M}(\pi F) = (e\overline{M})(F).$$

These are compatible with the module structure. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

5.C. **The various adjoints.** This subsection is designed as a guide to the following sections where a number of different adjoints to $e$ are described. The point is that the functor $e$ can be viewed as a functor between several different pairs of categories, and in each case it may have left or right adjoints.

We start by assuming that the flag($\Sigma$)-diagram of rings $R$ is given, and we have formed flag($\overline{\Sigma}$)-diagram $\pi_! R$.

- The functor $e : \pi_! R$-modules $\longrightarrow R$-modules has a right adjoint $\pi_!$ consistent with the notation $\pi_! R$ for coefficient systems (see Subsection 5.D).
- Given a flag($\overline{\Sigma}$) diagram $\pi_!' R$ with a map $\pi_!' R \longrightarrow \pi_! R$ inducing an isomorphism $e\pi_!' R \cong e\pi_! R$, the functor $e : \pi_!' R$-modules $\longrightarrow R$-modules has a left adjoint $\pi_*$ (see Section 6).
- If $R$ has a system of maximally generated Euler classes, there is a $\overline{\Sigma}$-diagram $\pi_!^e R$ of rings with a map $\pi_!^e R \longrightarrow \pi_! R$ which induces an isomorphism $e\pi_!^e R \cong e\pi_! R$. The functor $e : iqc\text{-}\pi_!^e R$-modules $\longrightarrow \pi\text{-cts-}qc\text{-}R$-modules has a right adjoint $\pi_!^e$, where $iqc$ modules are those $M$ for which $eM$ is $qc$, and where $\pi$ continuity is a notion to be defined below (see Section 7).
- A version of the previous right adjoint with flags replaced by pairs (see Section 8).

We attempt to use notation that suggests the category of origin. For example, $M$ is a module based on a $\Sigma$-diagram of rings, $\overline{M}$ is a module based on a $\overline{\Sigma}$-diagram of rings.

5.D. **Modules over $R$ and $\pi_! R$.** To obtain $\pi_! R$ modules from $R$-modules, we extend the functor $\pi_!$ to modules.

**Definition 5.6.** (i) For a module $M$ over $R$ we take

$$(\pi_! M)(\overline{F}) = \prod_{\pi F = \overline{F}} M(F)$$

with structure maps given by Lemma 5.2 as for $\pi_! R$.

(ii) A flag($\overline{\Sigma}$)-$\pi_! R$-module $\overline{M}$ is said to be a *p-module* (or *product module*) if the natural map

$$\overline{M}(\overline{F}) \longrightarrow \prod_{\pi F = \overline{F}} e_F \overline{M}(\overline{F})$$

is an isomorphism for all flags $\overline{F}$.

These constructions give the relationship we need between flag($\Sigma$)-$R$-modules and flag($\overline{\Sigma}$)-$\pi_! R$-modules.

**Lemma 5.7.** *The constructions $e$ and $\pi_!$ above give an adjunction*

$$e : \pi_! R^f\text{-modules} \rightleftarrows R^f\text{-modules} : \pi_! \; .$$

*We find $e\pi_! = 1$ and the adjunction gives an equivalence*

$$p\text{-flag}(\overline{\Sigma})\text{-}\pi_! R\text{-modules} \simeq flag(\Sigma)\text{-}R\text{-modules}. \quad \square$$

**Remark 5.8.** If $\overline{E}$ is a subflag of $\overline{F}$ then the structure map $(\pi_! R)(\overline{E}) \longrightarrow (\pi_! R)(\overline{F})$ induces a map

$$(\pi_! R)(\overline{F}) \otimes_{(\pi_! R)(\overline{E})} (\pi_! M)(\overline{E}) \longrightarrow (\pi_! M)(\overline{F}).$$

This is a product over flags $E$ with $\pi E = \overline{E}$ of terms

$$\left( \prod_{F \geq E, \pi F = \overline{F}} R(F) \right) \otimes_{R(E)} M(E) \longrightarrow M(F).$$

Note in particular that even if $R(F) \otimes_{R(E)} M(E) \cong M(F)$, the corresponding statement will usually not hold for the $\pi_! R$-module $\pi_! M$.

## 6. A left adjoint to $e$

In this subsection we again consider a surjective map $\pi : \Sigma \longrightarrow \overline{\Sigma}$. Given a $\Sigma$-diagram of rings $R$, we form the flag($\Sigma$)-diagram $R^f$. We suppose given a flag($\overline{\Sigma}$)-diagram $\overline{R}^f$ with a map $\overline{R}^f \longrightarrow \pi_! R$ which becomes an isomorphism with $e$ applied, so that $e\overline{R}^f = R^f$.

Using idempotents as in Subsection 5.B, and using the fact that $e\overline{R}^f = R^f$, we have a functor

$$e : \overline{R}^f\text{-modules} \longrightarrow R^f\text{-modules}.$$

We have already constructed a right adjoint $\pi_!$ to $e$, and in this section we construct a left adjoint

$$\pi_* : R^f\text{-modules} \longrightarrow \overline{R}^f\text{-modules}.$$

We do not display the dependence of this functor on $\overline{R}^f$ in the notation.

6.A. **Definition of $\pi_*$.** Notationally, we consider flags $E = (K_0 \supset K_1 \supset \cdots \supset K_s)$ and $F = (L_0 \supset L_1 \supset \cdots \supset L_t)$ in $\Sigma$, and flags in $\overline{\Sigma}$ will use corresponding barred notation so that $\overline{E} = (\overline{K}_0 \supset \overline{K}_1 \supset \cdots \supset \overline{K}_s)$ and $\overline{F} = (\overline{L}_0 \supset \overline{L}_1 \supset \cdots \supset \overline{L}_t)$.

We first recall that the *right* adjoint $\pi_!$ was defined as follows: for an $R^f$-module $X$, the module $\pi_! X$ is defined on the flag $\overline{F}$ using products

$$(\pi_! X)(\overline{F}) = \prod_{\pi F = \overline{F}} X(F).$$

We needed the fact that the map of flags was a Grothendieck fibration as stated in Lemma 5.2 to define the structure maps. The first guess about how to construct a *left* adjoint would be to replace the product with a sum. This works if $\Sigma$ is finite, but in general the structure map including a length 0 flag in a length 1 flag cannot be defined because the map $X(L) \longrightarrow \prod_{K \supset L} X(K \supset L)$ usually fails to factor through the sum.

15

**Definition 6.1.** For each $R^f$-module $X$, we define $\pi_* X$ in steps. First, on the flag $\overline{F} = (\overline{K}_0 \supset \overline{K}_1 \supset \cdots \supset \overline{K}_s)$ we define $(\pi'_* X)(\overline{F})$ to be the sum:

$$(\pi'_* X)(\overline{F}) = \bigoplus_{\pi F = \overline{F}} X(F) \subseteq \prod_{\pi F = \overline{F}} X(F) = (\pi_! X)(\overline{F}).$$

The value we want $(\pi_* X)(\overline{F})$ lies between the sum and the product

$$(\pi'_* X)(\overline{F}) = \bigoplus_{\pi F = \overline{F}} X(F) \subseteq (\pi_* X)(\overline{F}) \subseteq \prod_{\pi F = \overline{F}} X(F) = (\pi_! X)(\overline{F}).$$

We take $(\pi_* X)(\overline{F})$ to be the $\overline{R}^f(\overline{F})$-submodule spanned by the sum $(\pi'_* X)(\overline{F})$ together with the images of the singleton flags:

$$(\pi_* X)(\overline{F}) = (\pi'_* X)(\overline{F}) + \sum_{j=0}^{t} (\pi_* X)(\overline{F}, \overline{L}_j)$$

where

$$(\pi_* X)(\overline{F}, \overline{L}_j) = \overline{R}^f(\overline{F}) \cdot \pi_!(\overline{L}_j \longrightarrow \overline{F})(X(\overline{L}_j)).$$

**Remark 6.2.** It is important that we have not taken the image of $\pi_!$ but rather the $\overline{R}^f(\overline{F})$-submodule it generates.

**Lemma 6.3.** *The structure maps of $\pi_! X$ respect the submodules $(\pi_* X)(\overline{F})$, and hence $\pi_* X$ is an $\overline{R}^f$-module functorially associated to $X$.*

**Proof:** The additional generators in $\pi_* X$ beyond $\pi'_* X$ all come from singleton flags, so that the image of any subflag $\overline{E}$ of $\overline{F}$ is contained in the sum of the images of its terms. $\qquad\square$

**Proposition 6.4.** *The functor $\pi_*$ is left adjoint to e:*

$$\pi_* : R^f\text{-modules} \;\underset{\longleftarrow}{\overset{\longrightarrow}{\rule{2cm}{0pt}}}\; \overline{R}^f\text{-modules} : e$$

**Proof:** To define the unit $X \longrightarrow e\pi_* X$ we need only note that since each $(\pi_* X)(\overline{F})$ is between the sum and the product, we have equality $e\pi_* X = X$.

The counit $\pi_* e\overline{X} \longrightarrow \overline{X}$ is taken to be the inclusion, since by definition, for each flag $\overline{F}$ $(\pi_* e\overline{X})(\overline{F})$ is a submodule of $\overline{X}(\overline{F})$.

The triangular identities are readily verified. $\qquad\square$

6.B. **The functor $\pi_*$ on qce-modules.** In this section we suppose given a $qce$-$R^f$-module $M$, and we consider the behaviour of $\pi_*$ on $X = \iota M$, where $\iota$ is the functor including $qce$-modules in all modules.

**Lemma 6.5.** *If $X = \iota M$ for a qce-module $M$, then the submodule $(\pi_* X)(\overline{F}, \overline{L}_j)$ contains (as a retract) each of the submodules $X(F)$ with $\pi F = \overline{F}$.*

16

**Proof:** Suppose $F = (L_0 \supset L_1 \supset \cdots \supset L_t)$. Note that we have an idempotent $e_F$ in $\overline{R}^f(\overline{F})$ so that for any $\overline{R}^f$-module $X$, the image of $X(\overline{L}_j)$ in $X(\overline{F})$ contains the image of $e_{L_j} X(\overline{L}_j)$ in $e_F X(\overline{F})$.

Now observe that if $M$ is *qce* then the image of any $M(L_j)$ in $M(F)$ generates $M(F)$ as an $R^f(F)$-module. $\qquad\square$

The submodule $(\pi_* X)(\overline{F}) \subseteq (\pi_! X)(\overline{F})$ is obtained by permitting elements with infinitely many non-zero terms when they occur along certain specific diagonals. However, as we saw in the previous lemma, the diagonal elements automatically lead to the inclusion of elements with only finitely many terms. To get the combinatorics under control, we consider intersections of the submodules $(\pi_* X)(\overline{F}, \overline{L}_j)$. For the subflag, $\overline{E} = (\overline{K}_0 \supset \cdots \supset \overline{K}_s) \subseteq \overline{F}$ we take

$$(\pi_* X)(\overline{F}, \overline{E}) = \bigcap_i (\pi_* X)(\overline{F}, \overline{K}_i).$$

**Remark 6.6.** In applications, we need to consider differential graded objects $X$, and the Mayer-Vietoris spectral sequence gives a means of calculating the homology of a complex $(\pi_* X)(\overline{F})$ from those of the intersections.

**Lemma 6.7.** *If $X = \iota M$ for a qce-module $M$ then for flags $\overline{E} \subseteq \overline{F}$ of $\overline{\Sigma}$,*

$$(\pi_* X)(\overline{F}, \overline{E}) = \bigoplus_{\pi E = \overline{E}} (\pi_* X)(\overline{F}, \Delta E),$$

*where*

$$(\pi_* X)(\overline{F}, \Delta E) = \overline{R}^f(\overline{F}) \cdot \mathrm{im}\left[ X(E) \xrightarrow{\Delta} (\pi_! X)(\overline{F}) \right].$$

**Proof:** In view of the intersection result, it suffices to prove the result for the singleton subflags $\overline{E} = \overline{L}_j$.

Note that since $(\pi_* X)(\overline{F}, \overline{L}_j)$ is the image of a map from a sum of the terms $X(L_j)$ with $\pi L_j = \overline{L}_j$ the image is a corresponding sum. This gives the first equality

$$(\pi_* X)(\overline{F}, \overline{L}_j) = \sum_{\pi L_j = \overline{L}_j} (\pi_* X)(\overline{F}, L_j) = \bigoplus_{\pi L_j = \overline{L}_j} (\pi_* X)(\overline{F}, L_j);$$

the sum is direct, since the term $(\pi_* X)(\overline{F}, L_j)$ is only non-zero in the $F$-components if the flag $F$ contains $L_j$. $\qquad\square$

## 7. Euler-adapted change of poset for coefficient systems

We continue with the notation of Section 5 with a splitting system $R^s$ on $\Sigma$ giving a coefficient system $R^f$ on flag($\Sigma$) and a map $p : \Sigma \longrightarrow \overline{\Sigma}$. We now suppose that $R$ is equipped with maximally generated Euler classes, and that $\pi : \Sigma \longrightarrow \overline{\Sigma}$ takes top and maximal elements to top and maximal elements.

In Subsection 5.A we constructed a right adjoint functor $\pi_!$ to $e$ on coefficient systems and on modules, and in this subsection we describe a variant $\pi_!^e$ suitable for quasi-coherent modules in which the Euler classes are taken from $\Sigma$.

7.A. **The Euler-adapted construction.** To start with, the coefficient system agrees with $\pi_! R$ on vertices

$$(\pi_!^e R)(\overline{K}) = \prod_{\pi K = \overline{K}} R(K).$$

**Definition 7.1.** (i) In a maximally generated system of Euler classes, any $e \in \mathcal{E}_K$ may be written as a finite product $e = \prod_i e_i$ where $e_i = \inf_{G/H_i}^{G/K} e_i'$ with $H_i$ maximal and $K \not\subseteq H_i$. We may then write $e_L$ for the product of those $e_i$ with $H_i \supset L$.

If we have some set of subgroups $L \subseteq K$ then we may define

$$\mathcal{E}_K^{-1} \prod_L M(L) = \lim_{\substack{\to \\ e \in \mathcal{E}_K}} \prod_L \left[ M(L) \xrightarrow{e_L} M(L) \right].$$

(ii) We define a coefficient system $\pi_!^e R$ on $\mathrm{flag}(\overline{\Sigma})$ as follows. If $\overline{F} = (\overline{L}_0 \supset \overline{L}_1 \supset \cdots \supset \overline{L}_t)$ we take

$$(\pi_!^e R)(\overline{F}) = \prod_{p(L_0) = \overline{L}_0} \mathcal{E}_{L_0}^{-1} \prod_{\pi(L_1) = \overline{L}_1, L_1 \subseteq L_0} \cdots \mathcal{E}_{L_{t-1}}^{-1} \prod_{\pi(L_t) = \overline{L}_t, L_t \subseteq L_{t-1}} R(G/L_t).$$

If we have an inclusion $a : \overline{E} \longrightarrow \overline{F}$ of flags we need to describe the induced map

$$(\pi_!^e R)(a) : (\pi_!^e R)(\overline{E}) \longrightarrow (\pi_!^e R)(\overline{F}).$$

It suffices to do this when $\overline{E}$ is obtained by omitting one factor, so we suppose $\overline{F} = (\overline{L}_0 \supset \cdots \supset \overline{L}_t)$ and that $\overline{E}$ omits $\overline{L}_j$.

If $j = t$ then we first describe $(\pi_!^e R)(\overline{L}_{t-1}) \longrightarrow (\pi_!^e R)(\overline{L}_{t-1} \supset \overline{L}_t)$. We take the product of factors indexed by $L_{t-1}$ with $\pi L_{t-1} = \overline{L}_{t-1}$; the $L_{t-1}$ factor is the map

$$R(G/L_{t-1}) \longrightarrow \prod_{\pi L_t = \overline{L}_t, L_t \subset L_{t-1}} R(G/L_t) \longrightarrow \mathcal{E}_{L_{t-1}}^{-1} \prod_{\pi L_t = \overline{L}_t, L_t \subset L_{t-1}} R(G/L_t)$$

where the first map has components which are the inflations from $G/L_{t-1}$ to $G/L_t$ and the second map inverts $\mathcal{E}_{L_{t-1}}$. To obtain the map for $a : \overline{E} \longrightarrow \overline{F}$ we apply the sequence of localizations and products to each term.

If $j < t$ then we apply an operation to $R^\dagger = (\pi_!^e R)(\overline{L}_{j+1} \supset \cdots \supset \overline{L}_t)$. Indeed, adding $\overline{L}_j$ to the flag on the codomain we need a map

$$R^\dagger \longrightarrow \prod_{\pi L_j = \overline{L}_j} \mathcal{E}_{L_j}^{-1} R^\dagger = (\pi_!^e R)(\overline{L}_j \supset \overline{L}_{j+1} \supset \cdots \supset \overline{L}_t),$$

and we use the map whose components are the localizations. To obtain the map for $a : \overline{E} \longrightarrow \overline{F}$ we apply the sequence of localizations and products to each term.

**Remark 7.2.** (i) If the localizations all involved inverting only units, we could omit $\mathcal{E}^{-1}$ everywhere and find $R(F) = R(G/L_t)$, and $(\pi_!^e R)(\overline{F}) = \prod_{\pi F = \overline{F}} R(F)$. When we invert non-units, the localizations for the $\mathrm{flag}(\Sigma)$ system just accumulate, but those for the $\mathrm{flag}(\overline{\Sigma})$ impose a continuity condition related to the finiteness of the fibres of $\pi$. The statement of Lemma 7.3 below is a stronger variant of this.

(ii) We are assuming that the number of maximal elements of $\Sigma$ and the number of maximal generators are countable. To calculate the direct limit in the first part of the definition we may choose an ordering on the maximal elements $H$ not containing $K$ and the maximal

18

generators and then order the elements $e$ accordingly. The colimit is independent of this ordering.

(iii) The coefficient system on flag$(\overline{\Sigma})$ differs from the coefficient system on flag$(\Sigma)$ in that the maps $\partial_i F \subseteq F$ will usually not induce the identity. This is partly because the number flags $E$ over a subflag $\overline{E}$ of $\overline{F}$ will depend on $\overline{E}$, and partly because of the relationship between the localization of a product and the product of localizations.

7.B. **Relationship between $\pi_!$ and $\pi_!^e$.** First we note that the coefficient rings $\pi_! R$ and $\pi_!^e R$ are closely related.

**Lemma 7.3.** *There is a map $\pi_!^e R \longrightarrow \pi_! R$ of coefficient systems on flag$(\overline{\Sigma})$ which is the identity on flags of length 0.*

**Proof:** Writing $\mathcal{N}_L$ as a typographical placeholder for the identity functor, we see the universal properties of localization give maps

$$
\begin{aligned}
(\pi_!^e R)(\overline{F}) &= \prod_{\pi(L_0)=\overline{L}_0} \mathcal{E}_{L_0}^{-1} \prod_{\pi(L_1)=\overline{L}_1, L_1 \subseteq L_0} \cdots \mathcal{E}_{L_{t-1}}^{-1} \prod_{\pi(L_t)=\overline{L}_t, L_t \subseteq L_{t-1}} R(G/L_t) \\
&\quad \downarrow \qquad\quad \downarrow \\
(\pi_! R)(\overline{F}) &= \prod_{\pi(L_0)=\overline{L}_0} \mathcal{N}_{L_0} \prod_{\pi(L_1)=\overline{L}_1, L_1 \subseteq L_0} \cdots \mathcal{N}_{L_{t-1}} \prod_{\pi(L_t)=\overline{L}_t, L_t \subseteq L_{t-1}} \mathcal{E}_{L_0/L_t}^{-1} R(G/L_t)
\end{aligned}
$$

$\square$

The fact that the idempotents came from an unlocalized product means $\pi_!^e$ inherits the idempotent properties of $\pi_!$.

**Lemma 7.4.** *The map of Lemma 7.3 is compatible with idempotents; indeed $e_F(\pi_! R)(\overline{F}) = R(F) = e_F(\pi_!^e R)(\overline{F})$ so that applying $e$ to $\pi_!^e R \longrightarrow \pi_! R$ we obtain the identity.* $\square$

Using the idempotents introduced in Lemma 8.3 we may define a functor as follows.

**Lemma 7.5.** *Extending scalars to $\pi_! R$ and applying idempotents gives a functor*

$$ e : \pi_!^e R^f\text{-modules} \longrightarrow R^f\text{-modules}. $$

*given by*

$$ (e\overline{M})(F) = e_F\left[ \overline{M}(\pi F) \right], $$

*where $e_F \in R(\pi F)$ is the idempotent corresponding to $F$.* $\square$

7.C. **Relative continuity.** We will define an Euler-compatible right adjoint to $e$. This involves retricting the $R$-modules to be compatible with Euler classes in the sense that they are quasi-coherent (or last-determined), and so that they are compatible with $p$. In effect, the poset structure on $\overline{\Sigma}$ specifies a topology on $\overline{\Sigma}$ (open sets generated by the sets $V(L) := \{K \mid K \supseteq L\}$ of elements above an element), and we may imagine that the fibres of $\pi$ specify infinitesimal neighbourhoods of points of $\overline{\Sigma}$. This 'topological' structure is then inherited by flag$(\Sigma)$ and flag$(\overline{\Sigma})$. The additional continuity condition explains how the points of the infinitesimal neighbourhoods approach the limit point.

19

**Definition 7.6.** If we are given an $R$-module $M$ over flag$(\Sigma)$ we may consider its $\pi$-*continuous* sections over a flag in $\overline{\Sigma}$. Indeed, if $\overline{F} = (\overline{L}_0 \supset \cdots \supset \overline{L}_t)$, we take

$$(\pi_!^e M)(\overline{F}) = M(\overline{F})_c = \prod_{\pi(L_0)=\overline{L}_0} \mathcal{E}_{L_0}^{-1} \prod_{\pi(L_1)=\overline{L}_1, L_1 \subseteq L_0} \cdots \mathcal{E}_{L_{t-1}}^{-1} \prod_{\pi(L_t)=\overline{L}_t, L_t \subseteq L_{t-1}} M(L_t).$$

This is evidently a module over

$$(\pi_!^e R)(\overline{F}) = R(\overline{F})_c = \prod_{\pi(L_0)=\overline{L}_0} \mathcal{E}_{L_0}^{-1} \prod_{\pi(L_1)=\overline{L}_1, L_1 \subseteq L_0} \cdots \mathcal{E}_{L_{t-1}}^{-1} \prod_{\pi(L_t)=\overline{L}_t, L_t \subseteq L_{t-1}} R(L_t).$$

As things stand, there is no reason why the structure maps of $M$ should take continuous sections to continuous sections if the last term in the flag changes.

**Definition 7.7.** A $\pi$-*structure* on an $R^f$-module $M$ is a transitive choice of liftings for each $\overline{K} \supset \overline{L}$ and $K$ with $\pi K = \overline{K}$:

$$
\begin{array}{ccc}
& & \mathcal{E}_K^{-1} \prod_{\pi L = \overline{L}, L \subset K} M(L) \\
& \nearrow & \downarrow \\
M(K) & \longrightarrow & \prod_{\pi L = \overline{L}, L \subset K} \mathcal{E}_{K/L}^{-1} M(L)
\end{array}
$$

Using the language of continuous sections, this can be written in the form

$$
\begin{array}{ccc}
& & e_K M(\overline{K} \supset \overline{L})_c \\
& \nearrow & \downarrow \\
M(K) & \longrightarrow & \prod_{\pi L = \overline{L}, L \subset K} e_{K \supset L} M(\overline{K} \supset \overline{L})_c
\end{array}
$$

A map of $R^f$-modules is compatible with $\pi$-structure if it commutes with the chosen liftings. We write $\pi$-cts-$R$-modules for the category of these.

**Remark 7.8.** Perhaps the best way to formalize this structure and to make the statement of transitivity clear is to say that a $\pi$-structure is a map $M \longrightarrow \pi_!^e e M$. This turns out to be the unit of an adjunction, which then explains the role of $\pi$-structures.

It seems that a $\pi$-structure is quite subtle in general, but there is a simple source of $\pi$-structures important in our applications.

**Lemma 7.9.** *If $\Sigma$ has a bottom element 1 then any quasi-coherent flag$(\Sigma)$ module $M$ has a canonical $\pi$-structure.*

**Proof:** In the following diagram, $K$ is fixed as $\overline{L} \subseteq \overline{K} = \pi K$. The subgroups $L$ in the products run through subgroups $L \subset K$ with $\pi L = \overline{L}$. Apart from the two diagonal maps, all maps come by extension of scalars from the structure maps of $M$. The isomorphisms come from quasicoherence

$$\begin{array}{ccc}
R(K \supset 1) \otimes_{R(K)} M(K) & \xrightarrow{\hspace{5cm}} & M(K \supset 1) \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\
\prod_L R(K \supset 1) \otimes_{R(K)} M(K) & \xrightarrow{\hspace{5cm}} & \prod_L M(K \supset 1) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\prod_L R(L \supset 1) \otimes_{R(L)} R(K \supset L) \otimes_{R(K)} M(K) & & \prod_L R(K \supset L) \otimes_{R(L)} M(L \supset 1) \\
& & \uparrow \\
& & \prod_L R(K \supset L) \otimes_{R(L)} R(L \supset 1) \otimes_{R(L)} M(L) \\
& & \downarrow{\scriptstyle \cong} \\
\prod_L R(L \supset 1) \otimes_{R(L)} M(K \supset L) & \xleftarrow[\cong]{\hspace{2cm}} & \prod_L R(L \supset 1) \otimes_{R(L)} R(K \supset L) \otimes_{R(L)} M(L)
\end{array}$$

The required lift arises since the top left vertical takes values in $R(K \supset 1) \otimes_{R(K)} \prod_L M(K)$.
$\hspace{14cm}\square$

Reassured by the fact $\pi$-structures can arise naturally, we may proceed.

**Lemma 7.10.** *There is a functor*

$$\pi_!^e : qc\text{-}\pi\text{-}cts\text{-}R^f\text{-}modules \longrightarrow \pi_!^e R^f\text{-}modules,$$

*defined on vertices by*

$$(\pi_!^e M)(\overline{K}) = (\pi_! M)(\overline{K}) = \prod_{\pi K = \overline{K}} M(K).$$

**Proof:** Since $qc$-modules are last-determined, it is reasonable to extend the definition on vertices to the entire flag complex by concentrating on the last term in the flag; Thus if $\overline{F} = (\overline{L}_0 \supset \cdots \supset \overline{L}_t)$, we take

$$(\pi_!^e M)(\overline{F}) = \prod_{\pi(L_0)=\overline{L}_0} \mathcal{E}_{L_0}^{-1} \prod_{\pi(L_1)=\overline{L}_1, L_1 \subseteq L_0} \cdots \mathcal{E}_{L_{t-1}}^{-1} \prod_{\pi(L_t)=\overline{L}_t, L_t \subseteq L_{t-1}} M(L_t).$$

This is evidently a module over

$$(\pi_!^e R)(\overline{F}) = \prod_{\pi(L_0)=\overline{L}_0} \mathcal{E}_{L_0}^{-1} \prod_{\pi(L_1)=\overline{L}_1, L_1 \subseteq L_0} \cdots \mathcal{E}_{L_{t-1}}^{-1} \prod_{\pi(L_t)=\overline{L}_t, L_t \subseteq L_{t-1}} R(L_t).$$

If we have an inclusion $i : \overline{E} \longrightarrow \overline{F}$ of flags we need to describe the induced map $(\pi_!^e M)(i) : (\pi_!^e M)(\overline{E}) \longrightarrow (\pi_!^e M)(\overline{F})$. It suffices to do this when $\overline{E}$ is obtained by omitting one factor, so we suppose $\overline{F} = (\overline{L}_0 \supset \cdots \supset \overline{L}_t)$ and that $\overline{E}$ omits $\overline{L}_i$.

If $i = t$ then we first describe $(\pi_!^e M)(\overline{L}_{t-1}) \longrightarrow (\pi_!^e M)(\overline{L}_{t-1} \supset \overline{L}_t)$. This is

$$\prod_{\pi L_{t-1}=L'_{t-1}} M(L_{t-1}) \longrightarrow \prod_{\pi L_{t-1}=L'_{t-1}} \mathcal{E}_{L_{t-1}}^{-1} \prod_{\pi L_t=\overline{L}_t, L_t \subset L_{t-1}} M(L_t),$$

and is a product of factors indexed by $L_{t-1}$ with $\pi L_{t-1} = \overline{L}_{t-1}$. The $L_{t-1}$ factor is the map

$$M(L_{t-1}) \longrightarrow \mathcal{E}_{L_{t-1}}^{-1} \prod_{\pi L_t = \overline{L}_t, L_t \subset L_{t-1}} M(L_t)$$

given by the $\pi$-structure.

To obtain the map for the full length flags $\overline{E} \longrightarrow \overline{F}$ we apply the sequence of localizations and products to the above shortened flags.

If $i < t$ then we apply an operation to $M^{\dagger} = (\pi_!^e M)(\overline{L}_{i+1} \supset \cdots \supset \overline{L}_t)$. Indeed, adding $\overline{L}_i$ on the codomain we have

$$M^{\dagger} \longrightarrow \prod_{\pi L_i = \overline{L}_i} \mathcal{E}_{L_i}^{-1} M^{\dagger} = (\pi_!^e M)(\overline{L}_i \supset \overline{L}_{i+1} \supset \cdots \supset \overline{L}_t)$$

where we use the map whose components are the localizations. To obtain the map for $\overline{E} \longrightarrow \overline{F}$ we apply the sequence of localizations and products to each term. $\qquad \square$

**Remark 7.11.** The functor $\pi_!^e$ takes values in the $\pi_!^e R$-modules $\overline{M}$ which are themselves $i$-quasi-coherent in the sense that $e\overline{M}$ is quasi-coherent ($i$ for idempotent).

Indeed,

$$(e\pi_!^e M)(F) = e_F(\pi_!^e M)(\pi F) = \mathcal{E}_{L_0/L_1}^{-1} \mathcal{E}_{L_1/L_2}^{-1} \ldots \mathcal{E}_{L_{t-1}/L_t}^{-1} M(L_t) = \mathcal{E}_{L_0/L_t}^{-1} M(L_t) = M(F).$$

We will seek a right adjoint on the restricted category of $i$-quasi-coherent modules.

7.D. **An Euler adapted right adjoint extending modules from flag$(\Sigma)$ to flag$(\overline{\Sigma})$.** We are ready to explain the universal property of $\pi_!^e$.

**Lemma 7.12.** *There is an adjunction*

$$e : iqc\text{-}\pi_!^e R^f\text{-}modules \underset{\longleftarrow}{\overset{\longrightarrow}{\rule{2cm}{0pt}}} qc\text{-}\pi\text{-}cts\text{-}R^f\text{-}modules : \pi_!^e$$

**Proof:** The counit $e\pi_!^e M \longrightarrow M$ is described and seen to be an isomorphism in Remark 7.11.

On the other hand we obtain a natural map $\overline{M} \longrightarrow \pi_!^e e\overline{M}$ which at $\overline{F}$ is

$$\overline{M}(\overline{F}) \longrightarrow \prod_{\pi F = \overline{F}} e_F \overline{M}(\overline{F}) \longrightarrow \prod_{\pi F = \overline{F}} e_F R(F) \otimes_{R(\overline{L}_t)} \overline{M}(\overline{L}_t)$$

$$\longrightarrow \prod_{\pi(L_0) = \overline{L}_0} \mathcal{E}_{L_0}^{-1} \prod_{\pi(L_1) = \overline{L}_1, L_1 \subseteq L_0} \cdots \mathcal{E}_{L_{t-1}}^{-1} \prod_{\pi(L_t) = \overline{L}_t, L_t \subseteq L_{t-1}} e_{L_t} \overline{M}(\overline{L}_t)$$

$$= (\pi_!^e e\overline{M})(\overline{F})$$

These together satisfy the triangular identities and give an adjunction. $\qquad \square$

7.E. $p$-**modules.** The adjunction in Lemma 7.12 shows that the category of $qc$-$\pi$-cts-$R^f$-modules is a retract of the category of $qc$-$R^f$-modules.

**Definition 7.13.** We say that a module over $\pi_!^e R$ is a pqc-module if the natural map $\overline{M} \longrightarrow \pi_!^e e\overline{M}$ made explicit in Lemma 7.12 gives an isomorphism

$$\overline{M}(\overline{F}) \cong \prod_{\pi(L_0)=\overline{L}_0} \mathcal{E}_{L_0}^{-1} \prod_{\pi(L_1)=\overline{L}_1, L_1 \subseteq L_0} \cdots \mathcal{E}_{L_{t-1}}^{-1} \prod_{\pi(L_t)=\overline{L}_t, L_t \subseteq L_{t-1}} e_{L_t}\overline{M}(\overline{L}_t)$$

**Remark 7.14.** (i) For length 0 flags, the condition states that the values on are simply the products of the values of $e\overline{M}$ on the length 0 flags of $\Sigma$:

$$\overline{M}(\overline{K}) = \prod_{\pi K=\overline{K}} e_K \overline{M}(\overline{K}).$$

This accounts for the letter $p$ in $pqc$.

(ii) We have already noted that $\pi_!^e e\overline{M}$ is an $iqc$-module. This means that if $\overline{E}$ and $\overline{F}$ have the same last terms, and $E \leq F$ with $\pi E = \overline{E}, \pi F = \overline{F}$ then for any $pqc$-module $\overline{M}$, writing $f$ for the first (largest) term in a flag, we have

$$e_F \overline{M}(\overline{F}) = \mathcal{E}_{fF/fE}^{-1} e_E \overline{M}(\overline{E}).$$

This accounts for the letters $qc$ in $pqc$.

**Definition 7.15.** We say that a $\pi_!^e R$-module $\overline{M}$ is $i$-*extended* (or an $ie$-module) if $e\overline{M}$ is extended as an $R$-module.

More explicitly, the inclusion of the flag $\overline{E}$ into $\overline{F}$ induces $\overline{M}(\overline{E}) \longrightarrow \overline{M}(\overline{F})$ and hence $R(\overline{F}) \otimes_{R(\overline{E})} \overline{M}(\overline{E}) \longrightarrow \overline{M}(\overline{F})$. If $\overline{M}$ is extended and we take $\overline{E} = (\overline{K}), \overline{F} = (\overline{K} \supset \overline{L})$ and choose $K \supset L$ with $\pi K = \overline{K}, pL = \overline{L}$ then we obtain an isomorphism

$$e_{K \supset L}\overline{M}(\overline{K} \supset \overline{L}) = R(K \supset L) \otimes_{R(K)} e_K \overline{M}(\overline{K}).$$

For brevity we write $pqc$ of product modules which are $i - qc$, and $pqce$ for product modules that are $i - qce$, since in the presence of the $p$ condition, the $i$ requirement is the only appropriate choice.

**Corollary 7.16.** *The adjunction of Lemma 7.12 gives an equivalence*

$$pqc\text{-}\pi_!^e R^f\text{-modules} \simeq qc\text{-}\pi\text{-}cts\text{-}R^f\text{-modules},$$

*and this restricts to an equivalence*

$$pqce\text{-}\pi_!^e R^f\text{-modules} \simeq qce\text{-}\pi\text{-}cts\text{-}R^f\text{-modules.} \quad \square$$

## 8. Euler-adapted change of poset for pair systems

The purpose of this section is to record the Euler-adapted change of poset for systems of pairs. The proofs are essentially specializations of those for flags, so we will not give full details.

8.A. **The Euler-adapted construction.** As before the Euler-adapted construction is a product on vertices.

**Definition 8.1.** We define a coefficient system $\pi_!^e R$ on $P(\overline{\Sigma})$ as follows:

$$(\pi_!^e R)(\overline{K} \supseteq \overline{L}) = \prod_{\pi(K)=\overline{K}} \mathcal{E}_K^{-1} \prod_{\pi(L)=\overline{L}, L \subseteq K} R(G/L).$$

We need to describe the induced maps, and it suffices to do this for the horizontal and vertical cases. If we have $\overline{H} \supseteq \overline{K} \supseteq \overline{L}$, we have the horizontal inclusion $h : (\overline{K} \supseteq \overline{L}) \longrightarrow (\overline{H} \supseteq \overline{L})$ and the vertical inclusion $v : (\overline{H} \supseteq \overline{K}) \longrightarrow (\overline{H} \supseteq \overline{L})$.

Starting with the horizontal map, $(\pi_!^e R)(h) : (\pi_!^e R)(\overline{K} \supseteq \overline{L}) \longrightarrow (\pi_!^e R)(\overline{H} \supseteq \overline{L})$, we need to define

$$\prod_{\pi(K)=\overline{K}} \mathcal{E}_K^{-1} \prod_{\pi(L)=\overline{L}, L \subseteq K} R(G/L) \longrightarrow \prod_{\pi(H)=\overline{H}} \mathcal{E}_H^{-1} \prod_{\pi(L)=\overline{L}, L \subseteq H} R(G/L).$$

Using Lemma 5.2, for each particular $K$ with $\pi K = \overline{K}$, there is a unique $H$ with $\pi H = \overline{H}$ and $H \supseteq K$. Accordingly we may take a product indexed by $K$ of maps

$$\mathcal{E}_K^{-1} \prod_{\pi(L)=\overline{L}, L \subseteq K} R(G/L) \longrightarrow \mathcal{E}_H^{-1} \prod_{\pi(L)=\overline{L}, L \subseteq H} R(G/L);$$

this includes the smaller product (over $L \subseteq K$) in the larger one (over $L \subseteq H$) and localizes.

Moving on to the vertical map, $(\pi_!^e R)(v) : (\pi_!^e R)(H \supseteq K) \longrightarrow (\pi_!^e R)(H \supseteq L)$, we need to define

$$\prod_{\pi(H)=\overline{H}} \mathcal{E}_H^{-1} \prod_{\pi(K)=\overline{K}, K \subseteq H} R(G/K) \longrightarrow \prod_{\pi(H)=\overline{H}} \mathcal{E}_H^{-1} \prod_{\pi(L)=\overline{L}, L \subseteq H} R(G/L).$$

This is a product over $H$ of localizations of

$$\prod_{\pi(K)=\overline{K}, K \subseteq H} R(G/K) \longrightarrow \prod_{\pi(L)=\overline{L}, L \subseteq H} R(G/L).$$

Using Lemma 5.2, for each particular $L$ with $\pi L = \overline{L}$, there is a unique $K$ with $\pi K = \overline{K}$ and $L \supseteq K$. Accordingly we may take a product indexed by $K$ of maps

$$R(G/K) \longrightarrow \prod_{\pi(L)=\overline{L}, L \subseteq H} R(G/L),$$

whose components are inflations.

8.B. **Relationship between $\pi_!$ and $\pi_!^e$.** The following three lemmas are precisely like the case of flags.

**Lemma 8.2.** *There is a map $\pi_!^e R \longrightarrow \pi_! R$ of coefficient systems on $P(\overline{\Sigma})$ which is the identity on flags of length 0.* $\square$

**Lemma 8.3.** *The map of Lemma 8.2 is compatible with idempotents; indeed $e_{K \supseteq L}(\pi_! R)(\overline{K} \supseteq \overline{L}) = R(K \supseteq L) = e_{K \supseteq L}(\pi_!^e R)(\overline{K} \supseteq \overline{L})$ so that applying $e$ to $\pi_!^e R \longrightarrow \pi_! R$ we obtain the identity.* $\square$

**Lemma 8.4.** *Extending scalars to $\pi_! R$ and applying idempotents gives a functor*

$$e : \pi_!^e R^p\text{-modules} \longrightarrow R^p\text{-modules}.$$

*given by*

$$(e\overline{M})(K \supseteq L) = e_{K \supseteq L}\left[\overline{M}(\pi(K \supseteq L))\right],$$

*where $e_{K \supseteq L} \in R(\pi(K \supseteq L))$ is the idempotent corresponding to $(K \supseteq L)$.* $\square$

8.C. **Relative continuity.** The relative continuity condition for coefficient systems is already formulated for pairs, so we refer the reader to Subsection 7.C.

**Lemma 8.5.** *There is a functor*

$$\pi_!^e : qc\text{-}\pi\text{-}cts\text{-}R^p\text{-}modules \longrightarrow \pi_!^e R^a\text{-}modules,$$

*defined on vertices by*

$$(\pi_!^e M)(\overline{K}) = (\pi_! M)(\overline{K}) = \prod_{\pi K = \overline{K}} M(K).$$

**Proof:** Since *qc*-modules are last-determined, it is reasonable to extend the definition on vertices to the entire flag complex by concentrating on the last term in the flag; Thus we take

$$(\pi_!^e M)(\overline{K} \subseteq \overline{L}) = \prod_{\pi(K) = \overline{K}} \mathcal{E}_K^{-1} \prod_{\pi(L) = \overline{L}, L \subseteq K} M(L).$$

This is evidently a module over

$$(\pi_!^e R)(\overline{K} \subseteq \overline{L}) = \prod_{\pi(K) = \overline{K}} \mathcal{E}_K^{-1} \prod_{\pi(L) = \overline{L}, L \subseteq K} R(L).$$

We need to describe the induced maps, and it suffices to do this for the horizontal and vertical cases. If we have $\overline{H} \supseteq \overline{K} \supseteq \overline{L}$, we have the horizontal inclusion $h : (\overline{K} \supseteq \overline{L}) \longrightarrow (\overline{H} \supseteq \overline{L})$ and the vertical inclusion $v : (\overline{H} \supseteq \overline{K}) \longrightarrow (\overline{H} \supseteq \overline{L})$.

Starting with the horizontal map, $(\pi_!^e M)(h) : (\pi_!^e M)(\overline{K} \supseteq \overline{L}) \longrightarrow (\pi_!^e M)(\overline{H} \supseteq \overline{L})$, we need to define a map

$$\prod_{\pi(K) = \overline{K}} \mathcal{E}_K^{-1} \prod_{\pi(L) = \overline{L}, L \subseteq K} M(L) \longrightarrow \prod_{\pi(H) = \overline{H}} \mathcal{E}_H^{-1} \prod_{\pi(L) = \overline{L}, L \subseteq H} M(L).$$

Using Lemma 5.2, for each particular $K$ with $\pi K = \overline{K}$, there is a unique $H$ with $\pi H = \overline{H}$ and $H \supseteq K$. Accordingly we may take a product indexed by $K$ of maps

$$\mathcal{E}_K^{-1} \prod_{\pi(L) = \overline{L}, L \subseteq K} M(L) \longrightarrow \mathcal{E}_H^{-1} \prod_{\pi(L) = \overline{L}, L \subseteq H} M(L);$$

this includes the smaller product (over $L \subseteq K$) in the larger one (over $L \subseteq H$) and localizes.

Moving on to the vertical map, $(\pi_!^e M)(v) : (\pi_!^e M)(\overline{H} \supseteq \overline{K}) \longrightarrow (\pi_!^e M)(\overline{H} \supseteq \overline{L})$, we need to define

$$\prod_{\pi(H) = \overline{H}} \mathcal{E}_H^{-1} \prod_{\pi(K) = \overline{K}, K \subseteq H} M(K) \longrightarrow \prod_{\pi(H) = \overline{H}} \mathcal{E}_H^{-1} \prod_{\pi(L) = \overline{L}, L \subseteq H} M(L).$$

This is a product over $H$, and it suffices to construct

$$\prod_{\pi(K) = \overline{K}, K \subseteq H} M(K) \longrightarrow \mathcal{E}_H^{-1} \prod_{\pi(L) = \overline{L}, L \subseteq H} M(L).$$

Using Lemma 5.2, for each particular $L$ with $\pi L = \overline{L}$, there is a unique $K$ with $\pi K = \overline{K}$ and $L \supseteq K$. Accordingly we may use the $\pi$-structure to obtain a map

$$M(K) \longrightarrow \mathcal{E}_K^{-1} \prod_{\pi(L) = \overline{L}, L \subseteq K} M(L).$$

25

Now take products, localize further to replace $\mathcal{E}_K^{-1}$ by $\mathcal{E}_H^{-1}$ in every factor, and include the smaller product (over $L \subseteq K$) into the larger product (over $L \subseteq H$). $\qquad\square$

**Remark 8.6.** The functor $\pi_!^e$ takes values in the $\pi_!^e R$-modules $\overline{M}$ which are themselves i-quasi-coherent in the sense that $e\overline{M}$ is quasi-coherent.

Indeed,

$$(e\pi_!^e M)(K \supseteq L) = e_{K \supseteq L}(\pi_!^e M)(\pi K \supseteq \pi L) = \mathcal{E}_{K/L}^{-1} M(L) = M(K \supseteq L).$$

We will seek a right adjoint on the restricted category of i-quasi-coherent modules.

8.D. **An Euler adapted right adjoint extending modules from $P(\Sigma)$ to $P(\overline{\Sigma})$.** We are ready to explain the universal property of $\pi_!^e$.

**Lemma 8.7.** *There is an adjunction*

$$e : qc\text{-}\pi_!^e R^p\text{-}modules \; \underset{\longleftarrow}{\overset{\longrightarrow}{\rule{3cm}{0pt}}} \; qc\text{-}\pi\text{-}cts\text{-}R^p\text{-}modules : \pi_!^e$$

**Proof:** The counit $e\pi_!^e M \longrightarrow M$ is described and seen to be an isomorphism in Remark 8.6.

On the other hand we obtain a natural map $\overline{M} \longrightarrow \pi_!^e e\overline{M}$ which at $\overline{K} \supseteq \overline{L}$ is

$$\overline{M}(\overline{K} \supseteq \overline{L}) \longrightarrow \prod_{\pi K = \overline{K}, \pi L = \overline{L}} e_{K \supseteq L} \overline{M}(\overline{K} \supseteq \overline{L}) \longrightarrow \prod_{\pi K = \overline{K}, \pi L = \overline{L}} e_{K \supseteq L} R(K \supseteq L) \otimes_{R(\overline{L})} \overline{M}(\overline{L})$$

$$\longrightarrow \prod_{\pi(K) = \overline{K}} \mathcal{E}_K^{-1} \prod_{\pi(L) = \overline{L}, L \subseteq K} e_L \overline{M}(\overline{L})$$

$$= (\pi_!^e e\overline{M})(\overline{K} \supseteq \overline{L})$$

These together satisfy the triangular identities and give an adjunction. $\qquad\square$

8.E. **$p$-modules.** The adjunction in Lemma 8.7 shows that the category of qc-$\pi$-cts-$R^p$-modules is a retract of the category of i-qc-$R^p$-modules.

**Definition 8.8.** We say that a module over $\pi_!^e R^p$ is a *pqc*-module if the natural map $\overline{M} \longrightarrow \pi_!^e e\overline{M}$ made explicit in Lemma 8.7 gives an isomorphism

$$\overline{M}(\overline{K} \supseteq \overline{L}) \cong \prod_{\pi(K) = \overline{K}} \mathcal{E}_K^{-1} \prod_{\pi(L) = \overline{L}, L \subseteq K} e_L \overline{M}(\overline{L}).$$

For brevity, we write *pqc* for *p*-modules which are *iqc* and *pqce* for *p*-modules which are *iqc* and *ie* on the grounds that these are the appropriate notions for *p*-modules.

**Remark 8.9.** (i) For length 0 flags, the condition states that the values on are simply the products of the values of $e\overline{M}$ on the length 0 flags of $\Sigma$:

$$\overline{M}(\overline{K}) = \prod_{\pi K = \overline{K}} e_K \overline{M}(\overline{K}).$$

This accounts for the letter $p$ in *pqc*.

(ii) We have already noted that $\pi_!^e e\overline{M}$ is a $qc$-module. Thus, the horizontal map $L \longrightarrow (K \supseteq L)$ induces

$$e_{K \supseteq L} \overline{M}(\overline{K} \supseteq \overline{L}) = \mathcal{E}_{K/L}^{-1} e_L \overline{M}(\overline{L}).$$

This accounts for the letters $qc$ in $pqce$.

**Definition 8.10.** We say that a $\pi_!^e R$-module $\overline{M}$ is is *i-extended* (or an *ie*-module) if $e\overline{M}$ is extended as an $R$-module.

More explicitly, the vertical inclusion $K \longrightarrow (K \supseteq L)$ induces $\overline{M}(\overline{K}) \longrightarrow \overline{M}(\overline{K} \supseteq \overline{L})$ and hence $R(\overline{K} \supseteq \overline{L}) \otimes_{R(\overline{K})} \overline{M}(\overline{K}) \longrightarrow \overline{M}(\overline{K} \supseteq \overline{L})$. If $\overline{M}$ is extended and we choose $K \supset L$ with $\pi K = \overline{K}, \pi L = \overline{L}$ then we obtain an isomorphism

$$e_{K \supseteq L} \overline{M}(\overline{K} \supseteq \overline{L}) = R(K \supseteq L) \otimes_{R(K)} e_K \overline{M}(\overline{K}).$$

**Corollary 8.11.** *The adjunction of Lemma 7.12 gives an equivalence*

$$pqc\text{-}\pi_!^e R^p\text{-}modules \simeq qc\text{-}\pi\text{-}cts\text{-}R^p\text{-}modules,$$

*and this restricts to an equivalence*

$$pqce\text{-}\pi_!^e R^p\text{-}modules \simeq qce\text{-}\pi\text{-}cts\text{-}R^p\text{-}modules. \quad \square$$

## 9. $R^p$-MODULES AND $(R\mathcal{F})^p$-MODULES

For any surjective map $q : \widetilde{\Sigma} \longrightarrow \Sigma$, we have explained how to compare $R^p$-modules and $q_!^e R^p$-modules. However when $q$ itself is an opfibration there is the alternative of forming the splitting system $q_! R^s$, and comparing $R^p$-modules and $(q_! R^s)^p$-modules. We restrict attention to the case that $\widetilde{\Sigma} = \Sigma \mathcal{F}$ is formed by introducing multiplicities, and we have $q : \Sigma \mathcal{F} \longrightarrow \Sigma$; accordingly we write $R\mathcal{F} = q_! R$. The case of flags follows from the case of pairs, so we will restrict to pairs.

**Example 9.1.** If we take $\Sigma = \Sigma_c$ then $\Sigma \mathcal{F} \cong \Sigma_a$ and $q : \Sigma_a \longrightarrow \Sigma_c$. We then take $R^s = \mathbb{R}_a^s$, so that $q_! R^s = \mathbb{R}_a \mathcal{F} = \mathbb{R}_c^s$. Thus this section is exactly designed to consider the relationship between $\mathbb{R}_a$-modules and $\mathbb{R}_c$-modules.

One method is to observe that there is a map $\lambda : (R\mathcal{F})^p \longrightarrow q_!^e R^p$ and then use restriction, extension and coextension of scalars. Since the map $\lambda$ is an isomorphism on idempotent pieces, this allows one to construct a right adjoint to $e : (R\mathcal{F})^p\text{-modules} \longrightarrow R^p\text{-modules}$, namely $\lambda^* q_!^e$. However this takes values in the *iqc*-modules (i.e., ones which are $qc$ after $e$ is applied), which are different from straightforwardly $qc$-modules and so quasi-coherification would be necessary. Instead it seems better to work directly.

Because of the special nature of $q : \Sigma \mathcal{F} \longrightarrow \Sigma$ we may formulate a stronger continuity condition on sections, requiring a continuity condition on $\mathcal{F}/K$ as well as on inclusions. We therefore refer to this as $\mathcal{F}$-continuity.

**Definition 9.2.** An $\mathcal{F}$-$q$-*structure* on $qc$-$R^p$-module is a transitive system of lifts for all pairs $K \supseteq L$

$$\mathcal{E}_K^{-1} \prod_{\tilde{K}} \prod_{\tilde{L} \subseteq \tilde{K}} \tilde{M}(\tilde{L}) = \tilde{M}(K \supseteq L)_{\mathcal{F}c}$$

$$\prod_{\tilde{K}} \mathcal{E}_{\tilde{K}}^{-1} \prod_{\tilde{L} \subseteq \tilde{K}} \tilde{M}(\tilde{L}) = \tilde{M}(K \supseteq L)_c$$

$$\prod_{\tilde{K}} \tilde{M}(\tilde{K}) \longrightarrow \prod_{\tilde{K}} \prod_{\tilde{L} \subseteq \tilde{K}} \mathcal{E}_{\tilde{K}/\tilde{L}}^{-1} \tilde{M}(\tilde{L}) = \tilde{M}(K \supseteq L)$$

**Definition 9.3.** We define a functor

$$q_!^d : q\text{-}\mathcal{F}\text{cts-}qc\text{-}R^p\text{-modules} \longrightarrow qc\text{-}(R\mathcal{F})^p\text{-modules}$$

by

$$(q_!^d \tilde{M})(K \supseteq L) = \mathcal{E}_K^{-1} \prod_{\tilde{L}} \tilde{M}(\tilde{L}) = \tilde{M}(K \supseteq L)_{\mathcal{F}c}.$$

The horizontal structure maps are simply localizations, and the vertical structure map for $v : (K) \longrightarrow (K \supseteq L)$ is the map

$$\prod_{\tilde{K}} \tilde{M}(\tilde{K}) \longrightarrow \mathcal{E}_K^{-1} \prod_{\tilde{L} \subseteq \tilde{K}} \tilde{M}(\tilde{L})$$

given by the $\mathcal{F}$-$q$-structure.

**Remark 9.4.** The definition of an $\mathcal{F}$-$q$-structure on $\tilde{M}$ may now be rephrased as saying that the values $\tilde{M}(K \supseteq L)_{\mathcal{F}c}$ fit together to make an $(R\mathcal{F})^p$-module $q_!^d \tilde{M}$, equipped with a map $q_!^d e \tilde{M} \longrightarrow q_! e \tilde{M}$, and an $\mathcal{F}$-$q$-structure is a map $\tilde{M} \longrightarrow q_!^d e \tilde{M}$.

**Definition 9.5.** A $qc$-$(R\mathcal{F})^p$-module $M$ is a *p-module* if the unit $M \longrightarrow q_!^d e M$ gives an isomorphism

$$N(K \supseteq L) \overset{\cong}{\longrightarrow} \mathcal{E}_K^{-1} \prod_{\tilde{L}} e_{\tilde{L}} M(L).$$

**Lemma 9.6.** *There is an adjunction*

$$e : qc\text{-}(R\mathcal{F})^p\text{-modules} \rightleftarrows q\text{-}\mathcal{F}cts\text{-}qc\text{-}R^p\text{-modules} : q_!^d$$

*which restricts to an equivalence*

$$pqc\text{-}(R\mathcal{F})^p\text{-modules} \simeq q\text{-}\mathcal{F}cts\text{-}qc\text{-}R^p\text{-modules}. \quad \square$$

The functor $e$ takes extended modules to extended modules, but $q_!^d$ does not. Instead, if we compose with the associated extended module functor $\Gamma_v$ from [8] we obtain the following.

**Corollary 9.7.** *If $\Sigma$ is finite, there is an equivalence*

$$qce\text{-}(R\mathcal{F})^p\text{-modules} \simeq q\text{-}\mathcal{F}cts\text{-}qce\text{-}R^p\text{-modules} \quad \square$$

28

# 10. Applications to models for rational torus-equivariant spectra

The purpose of this section is to record the consequences of the general theory for the special case relevant to rational $G$-spectra where $G$ is an $r$-torus. In this subsection we introduce the diagrams of rings and the modules over them, in Subsection 10.A we will display the categories and functors, and then in a series of subsections we describe how our general results establish the equivalences we need.

We consider $\Sigma_c = \mathrm{ConnSub}(G)$ ($c$ stands for 'connected'), and the standard system of multiplicities so that $\Sigma_a = \Sigma\mathcal{F} = \mathcal{TC}(G)$ is the toral chain category ($a$ stands for 'all') and the dimension poset $\Sigma_d = \{0, 1, \ldots, r\}$ ($d$ stands for 'dimension'). We consider the $\Sigma_a$-splitting system $\mathbb{R}_a$ defined by

$$\mathbb{R}_a(G/K) = H^*(BG/K)$$

equipped with its standard system of Euler classes, which is maximally generated. We could then introduce multiplicities and hence get a $\Sigma_c$-splitting system $\overline{\mathbb{R}}_c = (i^*\mathbb{R}_a)\mathcal{F}$, but as shown in Example 4.7, this is isomorphic to the $\Sigma_c$-system $\mathbb{R}_c$ defined by

$$\mathbb{R}_c(G/K) = \mathbb{R}_a\mathcal{F}(G/K) = \prod_{\tilde{K} \in \mathcal{F}/K} H^*(BG/\tilde{K}) = \mathcal{O}_{\mathcal{F}/K}.$$

Note that the associated coefficient system of this splitting system is middle independent so is quite different from $q_!^e\mathbb{R}_a$.

The other important coefficient system is $\mathbb{R}_d = d_!^e\mathbb{R}_c$. This is defined on the subdivided $r$-simplex $\mathrm{flag}([0, r])$, and takes the following values on vertices:

$$\mathbb{R}_d(m) = d_!^e\mathbb{R}_c(m) \cong \prod_{\dim(K)=m} \mathcal{O}_{\mathcal{F}/K} \cong \prod_{\dim(\tilde{K})=m} H^*(BG/\tilde{K}).$$

The purpose of this section is to assemble all the work we have done to give adjoint equivalences between categories of $\mathbb{R}_d^f$-modules (i.e., modules over a coefficient system on the subdivided $r$-simplex, as comes out of the topology), categories of $\mathbb{R}_c^p$-modules (i.e., modules over pairs as in [7, 8]) and categories of $\mathbb{R}_a^p$-modules (i.e., modules over pairs encoding the localization theorem).

More precisely, we show the following four versions of the category $\mathcal{A}(G)$ are equivalent:

- $\mathcal{A}_a^p(G) := q\text{-}\mathcal{F}\text{cts-}qce\text{-}\mathbb{R}_a^p\text{-mod}$ (the model most clearly embodying the localization theorem)
- $\mathcal{A}_c^p(G) := qce\text{-}\mathbb{R}_c^p\text{-mod}$ (the model used in previous work from [7] onwards)
- $\mathcal{A}_c^f(G) := qce\text{-}\mathbb{R}_c^f\text{-mod}$ (the model used to compare with $\mathcal{A}_d^f(G)$ below)
- $\mathcal{A}_d^f(G) := pqce\text{-}\mathbb{R}_d^f\text{-mod}$ (the model coming out of the proof in [13])

We note that there are numerous other variants that could be discussed (for example (i) replace $\mathbb{R}_a^p$ by $\mathbb{R}_a^f$, (ii) replace $\mathbb{R}_c^p$ by $q_!^d\mathbb{R}_a^p$, (iii) replace $\mathbb{R}_c^f$ by $q_!^e\mathbb{R}_a^f$, (iv) $\mathbb{R}_d^f$ by $(dq)_!^e\mathbb{R}_a^f$). Our general results do give models based on each of these alternatives, but we will focus on the four listed.

## 10.A. Some diagrams.

In the following diagrams, a number of functors are used, and their definitions may be found as follows: $e$ in 5.5, $q_*$ in 6.1, $q_!$ in 5.6, $q_!^d$ in 9.3, $d_!^e$ in 7.6, $p$ and $f$ in Section 3. The definitions of the categories are recalled near those of the appropriate functors.

To start with, we display the module categories of concern, together with the functors that exist on the whole module categories.

$$
\begin{array}{cccc}
 & a & c & d \\[1em]
s & \mathbb{R}_a^s\text{-modules} \underset{q_!}{\overset{e}{\leftrightarrows}} \mathbb{R}_c^s\text{-modules} & & \times \\[2em]
p & \mathbb{R}_a^p\text{-modules} \overset{e}{\longleftarrow} \mathbb{R}_c^p\text{-modules} & & \times \\
 & \quad\Big\downarrow{\scriptstyle f} \qquad\qquad \Big\downarrow{\scriptstyle f} & & \\[1em]
f & \mathbb{R}_a^f\text{-modules} \overset{e}{\longleftarrow} \mathbb{R}_c^f\text{-modules} \overset{e}{\longleftarrow} \mathbb{R}_d^f\text{-modules} &
\end{array}
$$

Restricting to categories of $qc$-modules on which the Euler-adapted right adjoints exist, we have the diagram

$$
\begin{array}{cccc}
 & a & c & d \\[1em]
s & \mathbb{R}_a^s\text{-modules} \underset{q_!}{\overset{e}{\leftrightarrows}} \mathbb{R}_c^s\text{-modules} & & \times \\[2em]
p & qc\text{-}q\text{-}\mathcal{F}\text{cts-}\mathbb{R}_a^p\text{-modules} \underset{q_!^d}{\overset{e}{\leftrightarrows}} qc\text{-}\mathbb{R}_c^p\text{-modules} & & \times \\
 & \qquad\qquad\qquad {\scriptstyle p}\Big\uparrow\Big\downarrow{\scriptstyle f} & & \\[1em]
f & \times \qquad qc\text{-}\mathbb{R}_c^f\text{-modules} \underset{d_!^e}{\overset{e}{\leftrightarrows}} qc\text{-}\mathbb{R}_d^f\text{-modules} &
\end{array}
$$

Restricting further to categories on which we have equivalences, and using the torsion functor $\Gamma$ right adjoint to inclusion of $qce\text{-}\mathbb{R}_c^p$-modules into all $\mathbb{R}_c^p$-modules defined in [8] (see also Section 11 below)

$$
\begin{array}{cccc}
 & a & c & d \\[1em]
s & \times & \times & \times \\[2em]
p & qce\text{-}q\text{-}\mathcal{F}\text{cts-}\mathbb{R}_a^p\text{-modules} \underset{\Gamma q_!^d}{\overset{e}{\underset{\sim}{\leftrightarrows}}} qce\text{-}\mathbb{R}_c^p\text{-modules} & & \times \\
 & \qquad\qquad\qquad {\scriptstyle p}\Big\uparrow{\scriptstyle\simeq}\Big\downarrow{\scriptstyle f} & & \\[1em]
f & \times \qquad qce\text{-}\mathbb{R}_c^f\text{-modules} \underset{d_!^e}{\overset{e}{\underset{\sim}{\leftrightarrows}}} pqce\text{-}\mathbb{R}_d^f\text{-modules} &
\end{array}
$$

More succinctly

$$
\begin{array}{ccccc}
 & a & c & d \\[4pt]
s & \times & \times & \times \\[10pt]
p & \mathcal{A}_a^p(G) \underset{\Gamma q_!^d}{\overset{e}{\underset{\simeq}{\rightleftarrows}}} \mathcal{A}_c^p(G) & & \times \\[6pt]
 & & p \downarrow\!\uparrow\!\simeq\, f & \\[6pt]
f & \times & \mathcal{A}_c^f(G) \underset{d_!^e}{\overset{e}{\underset{\simeq}{\rightleftarrows}}} \mathcal{A}_d^f(G)
\end{array}
$$

**10.B. Flags and pairs.** For the two posets $\Sigma_a$ and $\Sigma_c$, we have splitting systems and we may therefore define categories of pairs. For each of these Lemma 3.4 gives an equivalence between a pair of algebraic models of rational $G$-spectra.

**Corollary 10.1.** *(i) There is an equivalence of categories*

$$
\mathcal{A}_a^p(G) = qce\text{-}q\text{-}\mathcal{F}cts\text{-}\mathbb{R}_a^p\text{-}modules \simeq qce\text{-}q\text{-}\mathcal{F}cts\text{-}\mathbb{R}_a^f\text{-}modules = \mathcal{A}_a^f(G)
$$

*(ii) There is an equivalence of categories*

$$
\mathcal{A}_c^p(G) = qce\text{-}\mathbb{R}_c^p\text{-}modules \simeq qce\text{-}\mathbb{R}_c^f\text{-}modules = \mathcal{A}_c^f(G). \quad \square
$$

**10.C. $\mathbb{R}_a^p$-modules and $\mathbb{R}_c^p$-modules.** We apply the results of Section 9 to compare $\mathbb{R}_a^p$-modules and $\mathbb{R}_c^p$-modules. Special cases of Lemma 9.6 and Corollary 9.7 give the following.

**Lemma 10.2.** *There is an adjunction*

$$
e : qc\text{-}\mathbb{R}_c^p\text{-}modules \rightleftarrows q\text{-}\mathcal{F}cts\text{-}qc\text{-}\mathbb{R}_a^p\text{-}modules : q_!^d
$$

*which restricts to an equivalence*

$$
pqc\text{-}\mathbb{R}_c^p\text{-}modules \simeq q\text{-}\mathcal{F}cts\text{-}qc\text{-}\mathbb{R}_a^p\text{-}modules
$$

*If we compose with the functor $\Gamma$ from [8] we obtain an equivalence*

$$
\mathcal{A}_c^p(G) = qce\text{-}\mathbb{R}_c^p\text{-}modules \simeq q\text{-}\mathcal{F}cts\text{-}qce\text{-}\mathbb{R}_a^p\text{-}modules = \mathcal{A}_a^p(G). \quad \square
$$

**10.D. Collecting subgroups of the same dimension.** In this subsection we consider the dimension function $d : \Sigma_c \longrightarrow \Sigma_d = [0, r]$. Since $\Sigma_c$ has a minimal element, by Lemma 7.9 we do not need to mention $d$-continuity, and we prove that the category of $qce$-modules over the flag complex of all connected subgroups (i.e., $qce\text{-}\mathbb{R}_c^f$-modules) is equivalent to the category of pqce-modules over the subdivided $r$-simplex (i.e., $pqce\text{-}\mathbb{R}_d^f$-modules). Corollary 7.16 has the following special case.

**Corollary 10.3.** *With*

$$
\mathbb{R}_c(G/K) = \prod_{\tilde{K} \in \mathcal{F}/K} H^*(BG/\tilde{K}) = \mathcal{O}_{\mathcal{F}/K} \ \text{and} \ \mathbb{R}_d(m) = \prod_{\dim(K)=m} \mathcal{O}_{\mathcal{F}/K}
$$

31

*as above, there is an adjunction*

$$e : qc\text{-}\mathbb{R}_d^f\text{-}mod \rightleftarrows qc\text{-}\mathbb{R}_c^f\text{-}mod : d_!^e$$

*This induces an equivalence*

$$qc\text{-}\mathbb{R}_c^f\text{-}mod \simeq pqc\text{-}\mathbb{R}_d^f\text{-}mod.$$

*Furthermore, it respects extended modules, and induces an equivalence*

$$\mathcal{A}_c^f(G) = qce\text{-}\mathbb{R}_c^f\text{-}mod \simeq pqce\text{-}\mathbb{R}_d^f\text{-}mod = \mathcal{A}_d^f(G). \quad \square$$

Note that composing our equivalences does not give a simple comparison of $\mathbb{R}_a^f$-modules and $\mathbb{R}_d^f$-modules. The continuity conditions mean that $(dq)_!^e\mathbb{R}_a^f \not\simeq \mathbb{R}_d^f$ (even though they have the same values on vertices). Instead we have the following equivalence.

**Corollary 10.4.** *We have an equivalence*

$$dq\text{-}cts\text{-}qc\text{-}\mathbb{R}_a^f\text{-}mod \simeq pqc\text{-}((dq)_!^e\mathbb{R}_a^f)\text{-}mod.$$

*Furthermore, it respects extended modules, and induces an equivalence*

$$qce\text{-}dq\text{-}cts\text{-}\mathbb{R}_a^f\text{-}mod \simeq ppqce\text{-}(dq)_!^e\mathbb{R}_a^f\text{-}mod. \quad \square$$

**Remark 10.5.** We could presumably also obtain this equivalence as a composite.

10.E. **The case of rank 1.** The case of the circle group is too simple to be representative. To start with $\Sigma_c = (1 \longrightarrow T)$ is finite, avoiding one layer of continuity conditions. The fact that the chains are of length $\leq 1$ means there is no distinction between pairs (p) and flags (f). Finally, since all subgroups of dimension 0 have the same identity component $\Sigma_d = \Sigma_c$. In short, the only two models that really need comparision are $\mathcal{A}_c^p(T)$ and $\mathcal{A}_a^p(T)$. This has been discussed in [4], but it is helpful to relate it to the present framework and notation. The recollections here will also be useful preparation for Section 11.

In any case

$$\Sigma_a = \{i \longrightarrow T \mid i \geq 1\},$$

where $i$ is short for the cyclic subgroup of order $i$. The map $q : \Sigma_a \longrightarrow \Sigma_c$ is defined by $q(i) = 1$ and $q(T) = T$. The diagram of rings is specified by the values on $i \longrightarrow T$, which we abbreviate to

$$R_i \longleftarrow k$$

In the original case

$$R_i = H^*(B(T/C_i)) = \mathbb{Q}[c_i] \text{ and } k = \mathbb{Q},$$

where $c_i$ is of cohomological degree 2. We then find that on the diagram $\Sigma_c$ we have

$$R = \prod_i R_i \longleftarrow k.$$

Next,

$$\text{flag}(\Sigma_c) = \{(1) \longrightarrow (T \supset 1) \longleftarrow (T)\}$$

and

$$\text{flag}(\Sigma_a) = \{(i) \longrightarrow (T \supset i) \longleftarrow (T) \mid i \geq 1\}.$$

The system of Euler classes is given by choosing an element $c_i \in R_i$. The value on the $i$th flag from $\mathrm{flag}(\Sigma_a)$ is

$$R_i \longrightarrow R_i[\frac{1}{c_i}] \longleftarrow k$$

and the value on the flag from $\mathrm{flag}(\Sigma_c)$ is

$$R \longrightarrow \mathcal{E}^{-1}R \longleftarrow k$$

where

$$\mathcal{E}^{-1}R = \varinjlim(R \overset{(c_1,1,1,1,1\cdots)}{\longrightarrow} R \overset{(c_1,c_2,1,1,1\cdots)}{\longrightarrow} R \overset{(c_1,c_2,1c_3,1,1\cdots)}{\longrightarrow} R \overset{(c_1,c_2,1c_3,c_4,1\cdots)}{\longrightarrow} \cdots).$$

Now $\mathcal{A}_c^p(T)$ is a certain category of $\mathbb{R}_c$-modules, namely

$$\mathcal{A}_c^p(T) = \{N \longrightarrow P \longleftarrow V \mid \mathcal{E}^{-1}N \cong P \cong \mathcal{E}^{-1}R \otimes_k V\}$$

(where the first isomorphism is quasicoherence, and the second is extendedness).

Next, $\mathcal{A}_a^p(T)$ is a certain category of $\mathbb{R}_a$-modules, namely

$$\mathcal{A}_a^p(T) = \{N_i \longrightarrow P_i \longleftarrow V \mid N_i[\frac{1}{c_i}] \cong P_i \cong R[\frac{1}{c_i}] \otimes_k V, \quad \begin{array}{c} \mathcal{E}^{-1}\prod_i N_i \\ {}^{\kappa}\nearrow \quad \downarrow \\ V \longrightarrow \prod_i(N_i[\frac{1}{c_i}]) \end{array} \}$$

(where the first isomorphism is quasicoherence, the second is extendedness, and $\kappa$ is the continuity structure).

Finally, the equivalence between these two categories is induced by $e$ and $\Gamma q_!^d$. In the easier direction

$$e : \mathcal{A}_c^p(T) \longrightarrow \mathcal{A}_a^p(T)$$

takes $N \longrightarrow P \longleftarrow V$ to the object with $N_i = e_i N, P_i = e_i P$ and the continuity condition $\kappa$ is the composite

$$V \longrightarrow P \cong \mathcal{E}^{-1}N \longrightarrow \mathcal{E}^{-1}\prod_i e_i N.$$

In the other direction,

$$\Gamma q_!^d : \mathcal{A}_a^p(T) \longrightarrow \mathcal{A}_c^p(T)$$

takes an object $\{N_i \longrightarrow P_i \longleftarrow V, \kappa\}$ first by $q_!^d$ to

$$(\prod_i N_i \longrightarrow \prod_i P_i \longleftarrow V)$$

and then by $\Gamma$ to $(N \longrightarrow P \longleftarrow V)$ where $N$ is the pullback

$$\begin{array}{ccc} N & \longrightarrow & \mathcal{E}^{-1}R \otimes_k V \\ \downarrow & & \downarrow \\ \prod_i N_i & \longrightarrow & \mathcal{E}^{-1}\prod_i N_i \end{array}$$

and $P = \mathcal{E}^{-1}N$.

The fact that these are inverse equivalences follows from the fact that the square for $N = R$ is a pullback (i.e., that $\mathbb{R}_c = \Gamma q_!^d e \mathbb{R}_c$), together with the quasicoherence condition.

# 11. Torsion functors

Various obvious constructions on qce-modules give objects which are not *qce*. It is therefore convenient to have a right adjoint $\Gamma$ to the inclusion of *qce*-modules in all modules. There are three cases where we need this: for $\mathbb{R}_c^p$-modules, $\mathbb{R}_c^f$-modules and $\mathbb{R}_d^f$-modules. The case of $\mathbb{R}_c^p$ was dealt with in [8]. It does not seem possible to deduce the case of $\mathbb{R}_c^f$ directly, because one can only construct pair modules from flag modules in the middle-independent case. Nonetheless, the methods of [8] remain effective. The case of $\mathbb{R}_d^f$ is a little different, because the coproduct of $p$-modules is not the coproduct of the underlying modules; this will be discussed in Subsection 11.C.

The strategy for the two cases involving $\mathbb{R}_c$ is the same for pairs and flags. As in [8], we factorize the inclusion

$$\mathbb{R}_c\text{-modules} \xrightarrow{k} e\text{-}\mathbb{R}_c\text{-modules} \xrightarrow{j} qce\text{-}\mathbb{R}_c\text{-modules}.$$

The right adjoint to the first is called $\Gamma_v$ (the associated extended module construction) and to the second $\Gamma_h$ (quasi-coherification); the letters $v$ and $h$ refer to horizontal and vertical structure maps.

The functor $\Gamma_v$ does not use very much about our particular context: it would apply to any poset $\Sigma$ with ranks (i.e., with a dimension function $d : \Sigma \longrightarrow [0, r]$ for some $r$, so that $K \supset L$ implies $d(K) > d(L)$). However the construction of $\Gamma_h$ needs finiteness conditions on the rings as well, and we will be content to cover our immediate applications.

**Theorem 11.1.** *There are right adjoints to inclusions as follows*

(1) $\Gamma_{cv}^p$ *to*

$$k : \mathbb{R}_c^p\text{-modules} \longrightarrow e\text{-}\mathbb{R}_c^p\text{-modules}$$

(2) $\Gamma_{ch}^p$ *to*

$$j : e\text{-}\mathbb{R}_c^p\text{-modules} \longrightarrow qce\text{-}\mathbb{R}_c^p\text{-modules}$$

(3) $\Gamma_c^p = \Gamma_{ch}^p \Gamma_{cv}^p$ *to*

$$i = jk : \mathbb{R}_c^p\text{-modules} \longrightarrow qce\text{-}\mathbb{R}_c^p\text{-modules}$$

(4) $\Gamma_{cv}^f$ *to*

$$k : \mathbb{R}_c^f\text{-modules} \longrightarrow e\text{-}\mathbb{R}_c^f\text{-modules}$$

(5) $\Gamma_{ch}^f$ *to*

$$j : e\text{-}\mathbb{R}_c^f\text{-modules} \longrightarrow qce\text{-}\mathbb{R}_c^f\text{-modules}$$

(6) $\Gamma_c^f = \Gamma_{ch}^f \Gamma_{cv}^f$ *to*

$$i = jk : \mathbb{R}_c^f\text{-modules} \longrightarrow qce\text{-}\mathbb{R}_c^f\text{-modules}$$

We note that Part 1 is [8, Theorem 7.1] and Part 2 is [8, Theorem 8.1]. Part 3 follows from Parts 1 and 2 and Part 6 follows from Parts 4 and 5.

We will prove Part 4 in Subsection 11.A. In Subsection 11.B we show how to deduce Part 5 from Part 2. Finally in Subsection 11.C we will discuss the category of $\mathbb{R}_d^f$ modules and construct a right adjoint from $\mathbb{R}_d^f$-modules to $qce\text{-}\mathbb{R}_c^p$-modules.

11.A. **The associated extended functor.** The purpose of this section is to give a construction of a functor $\Gamma_v$ replacing an $\mathbb{R}_c^f$-module by an extended $\mathbb{R}_c^p$-module, so that its vertical structure maps become extensions of scalars. The proof is a direct adaption of [8, Theorem 7.1]. In fact it applies whenever $\Sigma$ has a dimension function as described above.

**Theorem 11.2.** *There is a right adjoint $\Gamma_v = k^!$ to the inclusion*

$$e\text{-}\mathbb{R}_c^f\text{-}mod \xrightarrow{k} \mathbb{R}_c^f\text{-}mod.$$

We will give an explicit construction of the functor $k^!$, referring the reader to [8, Section 7] for motivating discussion.

**Definition 11.3.** (i) The *codimension* of a flag $F$ is the codimension of the largest element, $fF$.

(ii) Given an $\mathbb{R}_c^f$-module $M$, we describe the construction of the associated extended module $k^!M$. We will describe how to define $k^!M$ on singleton flags, $(k^!M)(L)$ and then use extendedness to determine the values on other flags with first element $L$:

$$(k^!M)(E) := R(E) \otimes_{R(L)} (k^!M)(L).$$

We proceed in order of increasing codimension, starting in codimension 0 by taking $(k^!M)(G) = M(G)$. Assume $k^!M$ has been defined on flags of codimension $\leq n$ in such a way that the vertical maps for flags of codimension $\leq n$ from each point are extensions of scalars. Now suppose $L$ is of codimension $n + 1$.

The value at $L$ is determined as an inverse limit of a diagram with two rows, the zeroth given by existing values of $k^!M$ on flags of codimension $n$ and the first by the values of $M$ itself. The diagram takes the form

$$
\begin{array}{ccc}
\bullet & \longrightarrow & (K \supset L), 0) \\
\downarrow & & \downarrow \\
((L), 1) & \longrightarrow & ((K \supset L), 1)
\end{array}
$$

where $K$ runs through the codimension $\leq n$ subgroups containing $L$. Since we are defining a middle-independent module, it is not necessary to mention longer flags. More precisely,

$$
k^!(L) = \varprojlim \left(
\begin{array}{ccc}
& & (k^!M)(K \supset L) \\
& & \downarrow \\
M(L) & \longrightarrow & M(K \supset L)
\end{array}
\right)
$$

**Lemma 11.4.** *The maps $\lambda : k^!M \longrightarrow M$ induce isomorphisms*

$$\lambda_* : \mathrm{Hom}(T, k^!M) \longrightarrow \mathrm{Hom}(T, M)$$

*for any extended $\mathbb{R}_c^f$-module $T$. In particular $k^!$ is right adjoint to the inclusion*

$$k : e\text{-}\mathbb{R}_c^f\text{-}mod \longrightarrow \mathbb{R}_c^f\text{-}mod.$$

**Proof:** To see $\lambda_*$ is an epimorphism, suppose $f : T \longrightarrow M$ is a map, and we attempt to lift it to a map $f' : T \longrightarrow k^!M$. We start with $f'(G) = f(G)$, and proceed by induction on the codimension of the flag. Once we have defined on a singleton flag $f'(L)$ we are forced to use extendedness to define $f'(E) = \mathbb{R}_c(E) \otimes_{\mathbb{R}_c(L)} f'(L)$ on flags $E$ with first term $L$.

35

Now suppose $L$ is of codimension $n+1$ and that $f'$ is defined on flags of lower codimension.

$$
\begin{array}{ccc}
T(L) & \longrightarrow & (k^!M)(K \supset L) \\
\downarrow & & \downarrow \\
M(L) & \longrightarrow & M(K \supset L)
\end{array}
$$

Since $(k^!M)(L)$ is defined as an inverse limit, we use its universal property to give $f'(L) : T(L) \longrightarrow k^!M(L)$.

To see $\lambda_*$ is a monomorphism, suppose the two maps $f_1, f_2 : L \longrightarrow M$ give the same map to $k^!M$. Evidently $f_1(G) = f_2(G)$, so we may consider the minimum codimension in which they disagree, perhaps codimension $n+1$; by extendedness they must therefore disagree on a subgroup $L$ of codimension $n+1$. However the defining diagram for $k^!M(L)$ involves only the values in $M$ and in flags of lower codimension so the universal property shows $f_1(L) = f_2(L)$, contradicting the choice of $L$. $\qquad\square$

### 11.B. The horizontal torsion functor $\Gamma_h$. Consider the diagram

$$
\begin{array}{ccc}
qce\text{-}\mathbb{R}_c^p\text{-modules} & \underset{p}{\overset{f}{\rightleftarrows}} & qce\text{-}\mathbb{R}_c^f\text{-modules} \\
j \downarrow \uparrow \Gamma_{ch}^p & & j \downarrow \uparrow \Gamma_{ch}^f \\
e\text{-}\mathbb{R}_c^p\text{-modules} & \underset{p}{\overset{f}{\rightleftarrows}} & e\text{-}\mathbb{R}_c^f\text{-modules} \\
k \downarrow \uparrow \Gamma_{cv}^p & & k \downarrow \uparrow \Gamma_{cv}^f \\
\mathbb{R}_c^p\text{-modules} & \overset{f}{\longrightarrow} & \mathbb{R}_c^f\text{-modules}
\end{array}
$$

The two $\Gamma^p$ functors on the left were constructed in [8]. The functor $\Gamma_{cv}^f$ at the bottom right was constructed in Subsection 11.A. The right adjoint to $j^f = fj^p p$ is the composite $\Gamma_{cv}^f := f\Gamma_{ch}^p p$. The point is that $f$ and $p$ are quasi-inverse and hence both left and right adjoint to each other.

### 11.C. The dimensional torsion functor $\Gamma_d$. A special case of the results of Section 6 shows that the functor $e$ from $\mathbb{R}_d^f$-modules to $\mathbb{R}_c^f$-modules has a *left* adjoint, $d_*$. This gives a diagram

$$
\begin{array}{ccc}
qce\text{-}\mathbb{R}_c^f\text{-modules} & \underset{d_!^e}{\overset{e}{\rightleftarrows}} & pqce\text{-}\mathbb{R}_d^f\text{-modules} \\
i \downarrow \uparrow \Gamma_c^f & & d_*ie \downarrow \uparrow \Gamma_d^f \\
\mathbb{R}_c^f\text{-modules} & \underset{e}{\overset{d_*}{\rightleftarrows}} & \mathbb{R}_d^f\text{-modules}
\end{array}
$$

in which $\Gamma_d^f = d_!^e \Gamma_c^f e$. This is right adjoint to $d_*ie$, and not to $i$ (typically $d_*ieM$ will not be a $p$-module). This reflects the fact that $i$ is not a left adjoint (coproducts in the category of $p$-modules are not coproducts in the ambient category of $\mathbb{R}_d^f$-modules). In our applications [13] we will actually use $p\Gamma_c^f e : \mathbb{R}_d^f\text{-modules} \longrightarrow qce\text{-}\mathbb{R}_c^p\text{-modules}$, which in any case takes us to the category $\mathcal{A}_c^p(G)$ that we want to work with.

## References

[1] D.Gepner and J.P.C.Greenlees "Comparing rational $U(1)$-equivariant elliptic cohomology theories" In preparation (15pp)

[2] J.P.C.Greenlees "Rational Mackey functors for compact Lie groups I" Proc. London Math. Soc **76** (1998) 549-578

[3] J.P.C.Greenlees "Rational $O(2)$-equivariant cohomology theories." Fields Institute Communications **19** (1998) 103-110

[4] J.P.C.Greenlees "Rational $S^1$-equivariant stable homotopy theory." Mem. American Math. Soc. (1999) xii+289 pp.

[5] J.P.C.Greenlees "Rational $SO(3)$-equivariant cohomology theories." Contemporary Maths. **271**, American Math. Soc. (2001) 99-125

[6] J.P.C.Greenlees "Rational $S^1$-equivariant elliptic cohomology" Topology **44** (2005) 1213-1279

[7] J.P.C.Greenlees "Rational torus-equivariant stable homotopy I: calculating groups of stable maps." JPAA **212** (2008) 72-98 (http://dx.doi.org/10.1016/j.jpaa.2007.05.010)

[8] J.P.C.Greenlees "Rational torus-equivariant stable homotopy II: the algebra of localization and inflation." JPAA **216** (2012) 2141-2158

[9] J.P.C.Greenlees "Rational equivariant cohomology theories with toral support." Algebraic and Geometric Topology (to appear), 52 pp, arXiv:1501.06167

[10] J.P.C.Greenlees "An $S^1$-equivariant cohomology theory associated to a curve of higher genus." (2003) 12pp

[11] J.P.C.Greenlees "Equivariant cohomology theories from algebraic geometry" In preparation (31pp)

[12] J.P.C.Greenlees "Rational torus-equivariant stable homotopy IV: geometry of flags." (in preparation)

[13] J.P.C.Greenlees and B.E.Shipley "An algebraic model for rational torus-equivariant spectra." Preprint 76pp Submitted for publication, arXiv:1101.2511

[14] B.E.Shipley, "An algebraic model for rational $S^1$-equivariant stable homotopy theory." Q. J. Math. 53 (2002), no. 1, 87–110.

School of Mathematics and Statistics, Hicks Building, Sheffield S3 7RH. UK.
*E-mail address*: j.greenlees@sheffield.ac.uk