Dynamical equations for the vector potential and the velocity potential in incompressible irrotational Euler flows: A refined Bernoulli theorem

Koji Ohkitani*

School of Mathematics and Statistics, University of Sheffield, Hicks Building, Hounsfield Road, Sheffield S3 7RH, United Kingdom

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We consider incompressible Euler flows in terms of the stream function in two dimensions and the vector potential in three dimensions. We pay special attention to the case with singular distributions of the vorticity, e.g., point vortices in two dimensions. An explicit equation governing the velocity potentials is derived in two steps. (i) Starting from the equation for the stream function [Ohkitani, Nonlinearity **21**, T255 (2009)], which is valid for smooth flows as well, we derive an equation for the complex velocity potential. (ii) Taking a real part of this equation, we find a dynamical equation for the velocity potential, which may be regarded as a refinement of Bernoulli theorem. In three-dimensional incompressible flows, we first derive dynamical equations for the vector potentials which are valid for smooth fields and then recast them in hypercomplex form. The equation for the velocity potential is identified as its real part and is valid, for example, flows with vortex layers. As an application, the Kelvin-Helmholtz problem has been worked out on the basis the current formalism. A connection to the Navier-Stokes regularity problem is addressed as a physical application of the equations for the vector potentials for smooth fields.

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I. INTRODUCTION

The motion of inviscid fluids with unit density governed by the incompressible Euler equations reads

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p, \tag{1}$$

and

$$\nabla\cdot\boldsymbol{u} = 0, \tag{2}$$

where $\boldsymbol{u}$ denotes the velocity and p the pressure field. By using a simple vector identity, we may write (1), together with (2), as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla\left(p + \frac{|\boldsymbol{u}|^2}{2}\right) = \boldsymbol{u}\times\boldsymbol{\omega}, \tag{3}$$

where $\boldsymbol{\omega}$ denotes the vorticity. The Euler and Navier-Stokes equations for incompressible fluids are inherently nonlocal in nature. This is because of the presence of the pressure term, which makes fluid motions incompressible through a potential problem. Actually, the nonlocal character is a serious obstacle in mathematical analyses of the fluid equations; it spoils notably a maximum-principle type argument for the Navier-Stokes equations and methods of characteristics for the Euler equations.

On top of its nonlocal character, the pressure in incompressible flows is cumbersome to handle, because no closed equation is known for its dynamical evolution. One approach is to simply to remove this notorious term, say by solenoidal projection. As an alternative approach, we may attach importance to the pressure. Indeed, *in two dimensions*, given p, we can in principle recover ψ by solving the Monge-Ampere equation

$$\psi_{xx}\psi_{yy} - \psi_{xy}^2 = \tfrac{1}{2}\triangle p,$$

which relates the stream function ψ and the pressure p. It is worth seeking a dynamical equation governing the evolution of the pressure. In this paper, as a first step, dynamical equations are given for a velocity potential with singular vorticity distributions in two and three dimensions.

We first consider two-dimensional flows with singular vorticity distributions (e.g., point vortices) and derive a governing equation for the stream function. This is achieved by using complex function theory. Second, dynamical equations are given for the vector potentials in three-dimensional incompressible flows under similar conditions and their hypercomplex counterpart. These may be regarded as refined versions of the Bernoulli theorem.

The rest of the paper is organized as follows. We describe fundamentals of the two-dimensional flows including point vortices in Sec. II. In Sec. III, a dynamical equation for velocity potential is derived in two dimensions. In Sec. IV, the three-dimensional problem is considered, where dynamical equations for the vector potential and its hypercomplex counterpart are derived. In Sec. V, a linear stability analysis of the Kelvin-Helmholtz problem is carried out on the basis of the new form of equations to justify its usefulness. An application to the Navier-Stokes regularity issue is also addressed. Section VI is devoted to a summary and outlook.

II. BERNOULLI THEOREM FOR POINT VORTICES

We consider a system of N point vortices ($N \geqslant 2$)

$$\frac{dx_i}{dt} = -\frac{1}{2\pi}\sum_{j=1}^{N}{}'\kappa_j\frac{y_i-y_j}{(x_i-x_j)^2+(y_i-y_j)^2},$$
$$\frac{dy_i}{dt} = \frac{1}{2\pi}\sum_{j=1}^{N}{}'\kappa_j\frac{x_i-x_j}{(x_i-x_j)^2+(y_i-y_j)^2}, \tag{4}$$

where (x_i,y_i) denotes the position of each vortex of strength κ_i for $i = 1,2,\ldots,N$ and $'$ denotes a summation excluding

*K.Ohkitani@sheffield.ac.uk

$i = j$. We refer to the book [1] for a comprehensible account of the point-vortex systems. The complex velocity potential associated with this system is given by

$$W = \phi + i\psi = \frac{1}{2\pi i}\sum_{n=1}^{N}\kappa_n \ln(z - z_n), \tag{5}$$

where

$$z = x + iy, \quad z_n = x_n + iy_n \ \ (n = 1, \ldots, N).$$

The imaginary part is the stream function

$$\psi(\boldsymbol{x},t) = -\frac{1}{4\pi}\sum_{n=1}^{N}\kappa_n \ln[(x - x_n)^2 + (y - y_n)^2]$$

and its real part is the velocity potential

$$\phi(\boldsymbol{x},t) = \frac{1}{2\pi}\sum_{n=1}^{N}\kappa_n \tan^{-1}\frac{y - y_n}{x - x_n}.$$

After some straightforward algebra we find

$$\frac{\partial\phi}{\partial t} = -\frac{1}{(2\pi)^2}\sum_{m,n=1}^{N}{}'\kappa_m\kappa_n\frac{(\boldsymbol{x} - \boldsymbol{x}_m)\cdot(\boldsymbol{x}_m - \boldsymbol{x}_n)}{|\boldsymbol{x} - \boldsymbol{x}_m|^2|\boldsymbol{x}_m - \boldsymbol{x}_n|^2}$$

and

$$\frac{\partial\psi}{\partial t} = -\frac{1}{(2\pi)^2}\sum_{m,n=1}^{N}{}'\kappa_m\kappa_n\frac{(\boldsymbol{x} - \boldsymbol{x}_m)\times(\boldsymbol{x}_m - \boldsymbol{x}_n)}{|\boldsymbol{x} - \boldsymbol{x}_m|^2|\boldsymbol{x}_m - \boldsymbol{x}_n|^2},$$

where $\boldsymbol{x} = (x,y)$, $\boldsymbol{x}_n = (x_n, y_n)$, and $'$ implies exclusion of $n = m$. It is noted that they make a nice pair of equations: dot vs cross products in the expressions of their dynamical evolution. This symmetry suggests that of the governing equations for ϕ and ψ in the case of a continuum, as we will confirm below.

If we consider a potential flow $\boldsymbol{u} = \nabla\phi$ induced by with point vortices, we have

$$\frac{\partial\boldsymbol{u}}{\partial t} + \nabla\left(p + \frac{|\boldsymbol{u}|^2}{2}\right) = 0 \text{ a.e.}, \tag{6}$$

where a.e. stands for almost everywhere; here we mean exclusion of positions of point vortices (and in three dimensions below, exclusion of, e.g., those of vortex layers). By the Bernoulli theorem we find

$$p(\boldsymbol{x},t) = \frac{1}{(2\pi)^2}\sum_{n,m=1}^{N}{}'\kappa_n\kappa_m\frac{(\boldsymbol{x} - \boldsymbol{x}_n)\cdot(\boldsymbol{x}_n - \boldsymbol{x}_m)}{|\boldsymbol{x} - \boldsymbol{x}_n|^2|\boldsymbol{x}_n - \boldsymbol{x}_m|^2} - \frac{1}{2(2\pi)^2}\left(\sum_{n=1}^{N}\kappa_n\frac{\boldsymbol{x} - \boldsymbol{x}_n}{|\boldsymbol{x} - \boldsymbol{x}_n|^2}\right)^2 + \text{const.}$$

Here ϕ and ψ are harmonic everywhere except for vortex positions. Hence the Liouville theorem does not apply so that a nontrivial behavior is possible.

It may be in order to make the following remark. If we define a manifold M by $\mathbb{R}^2\backslash\{\text{vortexpositions}\}$, then ψ, p, and $|\boldsymbol{u}|^2$ are well defined everywhere on M and have zero winding numbers around vortex positions. On the other hand, ϕ generally has a nonzero winding number.

III. TWO-DIMENSIONAL FLOWS

A. Dynamics of stream function

For smooth flows we recall a dynamical equation for the stream function [2],

$$\frac{\partial\psi}{\partial t} = \frac{1}{\pi}\text{P.V.} \times \int_{\mathbb{R}^2}\frac{[(\boldsymbol{x} - \boldsymbol{x}')\times\nabla\psi(\boldsymbol{x}')](\boldsymbol{x} - \boldsymbol{x}')\cdot\nabla\psi(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^4}\,d\boldsymbol{x}', \tag{7}$$

where P.V. denotes a principal-value integral.

Liouville theorem states that if a bounded function is harmonic everywhere in $\mathbb{R}^2$, then it must be a constant. In the case of point vortex systems, this does not apply. Hence it makes sense to seek a dynamical equation for the velocity potential.

We begin by noting that (7) still holds valid except for the locations of point vortices. We also note that there is a framework of "very weak" solutions for treating point vortices rigorously [3].

B. Dynamics of velocity potential

We now seek a dynamical equation for ϕ. To this end, we first derive an equation for the complex potential and then find its complex conjugate. The velocity is given by the complex velocity potential

$$\frac{dW}{dz} = \phi_x + i\psi_x = \psi_y + i\psi_x$$

and we find

$$z\frac{dW}{dz} = (x + iy)(\psi_y + i\psi_x) = (x\psi_y - y\psi_x) + i(x\psi_x + y\psi_y).$$

Noting that $x\psi_y - y\psi_x = x\phi_x + y\phi_y, x\psi_x + y\psi_y = -x\phi_y + y\phi_x$ by Cauchy-Riemann equations, we recast (7) as

$$\text{Im}\left(\frac{\partial W}{\partial t}\right) = \frac{1}{\pi}\text{P.V.}\int_{\mathbb{R}^2}\text{Re}\left((z - z')\frac{dW}{dz}\right) \times \text{Im}\left((z - z')\frac{dW}{dz}\right)\frac{d\boldsymbol{x}'}{|z - z'|^4} \text{ a.e.},$$

where Re and Im stand for the real and imaginary parts, respectively. In view of the identity

$$(a + bi)^2 = a^2 - b^2 + 2abi$$

for any real numbers a and b, we observe that (7) is the imaginary part of the following equation:

$$\frac{\partial W}{\partial t} = \frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}^2}\left((z - z')\frac{dW(z')}{dz'}\right)^2\frac{d\boldsymbol{x}'}{|z - z'|^4} \text{ a.e.}, \tag{8}$$

where $z = x + iy$, $z' = x' + iy'$. Note that $W(z)$ is meromorphic, that is, analytic $\frac{dW(z)}{d\bar{z}} = 0$ except for poles. Taking the real part we find

$$\frac{\partial\phi}{\partial t} = \frac{1}{2\pi}\text{P.V.} \times \int_{\mathbb{R}^2} \frac{[(\boldsymbol{x}-\boldsymbol{x}')\times\nabla\psi(\boldsymbol{x}')]^2-[(\boldsymbol{x}-\boldsymbol{x}')\cdot\nabla\psi(\boldsymbol{x}')]^2}{|\boldsymbol{x}-\boldsymbol{x}'|^4}\,d\boldsymbol{x}' \text{ a.e.},$$

or, equivalently

$$\frac{\partial\phi}{\partial t} = \frac{1}{2\pi}\text{P.V.} \times \int_{\mathbb{R}^2} \frac{[(\boldsymbol{x}-\boldsymbol{x}')\cdot\nabla\phi(\boldsymbol{x}')]^2-[(\boldsymbol{x}-\boldsymbol{x}')\times\nabla\phi(\boldsymbol{x}')]^2}{|\boldsymbol{x}-\boldsymbol{x}'|^4}d\boldsymbol{x}' \text{ a.e.} \tag{9}$$

This is the equation for the velocity potential we are after, which expresses the evolution of ϕ in terms of ϕ and its derivatives. We note that it makes a symmetric appearance in comparison with (7). In the complex notations it can be written as

$$\frac{\partial W}{\partial t} = \frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}^2}\left(\frac{dW(z')}{dz'}\right)^2\frac{d\boldsymbol{x}'}{(\overline{z}-\overline{z'})^2} \text{ a.e.} \tag{10}$$

It can be checked that analyticity persists under the dynamical evolution as follows:

$$\frac{\partial}{\partial t}\frac{\partial W}{\partial \overline{z}} = 0 \text{ a.e.}$$

Proof. By regularizing the above to keep the order of singularity of the kernel at $1/z$, we have

$$\frac{\partial W}{\partial t} = \frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}^2}\left[\left(\frac{dW(z')}{dz'}\right)^2-\left(\frac{dW(z)}{dz}\right)^2\right] \times \frac{d\boldsymbol{x}'}{(\overline{z}-\overline{z'})^2} \text{ a.e.}$$

Differentiating with respect to $\overline{z}$, we find

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial W}{\partial \overline{z}} &= \frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}^2}\left[\left(\frac{dW(z')}{dz'}\right)^2-\left(\frac{dW(z)}{dz}\right)^2\right] \times \frac{\partial}{\partial\overline{z}}\frac{1}{(\overline{z}-\overline{z'})^2}d\boldsymbol{x}' \text{ a.e.}\\ &= \frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}^2}\left[\frac{\partial}{\partial\overline{z'}}\left(\frac{dW(z')}{dz'}\right)^2\right]\frac{d\boldsymbol{x}'}{(\overline{z}-\overline{z'})^2} = 0 \text{ a.e.},\end{aligned}$$

where the last line follows by integration by parts. ■

One interpretation of Eq. (10) is as follows: Given the complex velocity potential initially

$$W(z,0) = \frac{1}{2\pi i}\sum_{n=1}^{N}\kappa_n \ln[z-z_n(0)],$$

if $z_n = x_n + iy_n$ $(n = 1,\ldots,N)$ follow the evolution of the point vortex system (4), then the corresponding (5) satisfies (10) and *vice versa*. In this sense, (10) describes the time evolution of the meromorphic complex velocity potential for the point-vortex system.

C. Dynamical equation for the head pressure

Because one of the motivations of this study is seeking the dynamical equation of the pressure, here we write down a governing equation for the head pressure defined by

$$\Pi(\boldsymbol{x},t) = p + \frac{|\boldsymbol{u}|^2}{2} = -\frac{\partial\phi}{\partial t}.$$

Differentiating (9) with respect to t, we have

$$\frac{\partial \Pi}{\partial t} = \frac{1}{\pi}\text{P.V.}\int_{\mathbb{R}^2}\{[(\boldsymbol{x}-\boldsymbol{x}')\cdot\nabla\phi(\boldsymbol{x}')](\boldsymbol{x}-\boldsymbol{x}')\cdot\nabla\Pi(\boldsymbol{x}') - [(\boldsymbol{x}-\boldsymbol{x}')\times\nabla\phi(\boldsymbol{x}')](\boldsymbol{x}-\boldsymbol{x}')\times\nabla\Pi(\boldsymbol{x}')\}\frac{d\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^4}, \text{ a.e.}, \tag{11}$$

where

$$\Pi(0) = -\left(p + \frac{|\boldsymbol{u}|^2}{2}\right)\bigg|_{t=0}.$$

It is noted that as (11) depends on $\nabla\phi$, it is not closed with respect to $\Pi(\boldsymbol{x},t)$ but should be coupled with (9).

IV. THREE-DIMENSIONAL FLOWS

A. Dynamics of vector potential

We first derive a set of equations for the vector potential. For the incompressible three-dimensional Euler equations, we define the vector potential $\boldsymbol{A}$ by $\boldsymbol{u} = \nabla\times\boldsymbol{A}$ and we assume that it satisfies Coulomb gauge condition $\nabla\cdot\boldsymbol{A} = 0$. The dynamical equations for $\boldsymbol{A}(\boldsymbol{x},t)$ can be written as

$$\frac{\partial \boldsymbol{A}}{\partial t} = \frac{3}{4\pi}\text{P.V.}\int_{\mathbb{R}^3}\frac{[\boldsymbol{r}\times(\nabla\times\boldsymbol{A}(\boldsymbol{x}'))]\boldsymbol{r}\cdot(\nabla\times\boldsymbol{A}(\boldsymbol{x}'))}{|\boldsymbol{r}|^5}\,d\boldsymbol{x}', \tag{12}$$

or equivalently,

$$\frac{\partial \boldsymbol{A}}{\partial t} = \frac{3}{4\pi}\text{P.V.}\int_{\mathbb{R}^3}\frac{[\boldsymbol{r}\times(\boldsymbol{u}(\boldsymbol{x}'))]\boldsymbol{r}\cdot\boldsymbol{u}(\boldsymbol{x}')}{|\boldsymbol{r}|^5}\,d\boldsymbol{x}', \tag{13}$$

where $\boldsymbol{r} = \boldsymbol{x}-\boldsymbol{x}'$. It should be noted that (12) is equivalent to the conventional Euler equations in three dimensions. Note that the equations for $\boldsymbol{A}$ in three dimensions and ψ in two dimensions hold generally for smooth flows. We need the assumption of singularities to consider the velocity potential ϕ. A similar formulation is also available for the Navier-Stokes equations of viscous fluids (see Sec. VI).

Proof. In the vorticity equations

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}), \tag{14}$$

if we write

$$\boldsymbol{u}\times\boldsymbol{\omega} = \nabla f + \nabla\times\boldsymbol{g}$$

with $\nabla\cdot\boldsymbol{g} = 0$, then we have

$$\frac{\partial\Delta\boldsymbol{A}}{\partial t} = \Delta\boldsymbol{g},$$

or

$$\frac{\partial\boldsymbol{A}}{\partial t} = \boldsymbol{g}.$$

Here we have set the integration constant to be zero, without loss of generality. The function $\boldsymbol{g}$ is given by

$$\boldsymbol{g} = -\Delta^{-1}\nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}).$$

Now by the identity

$$\boldsymbol{u}\times\boldsymbol{\omega} = \nabla\frac{|\boldsymbol{u}|^2}{2} - \nabla(\boldsymbol{u}\otimes\boldsymbol{u}),$$

we compute that

$$\begin{aligned} g_i &= -\Delta^{-1}\epsilon_{ipq}\partial_p(\boldsymbol{u}\times\boldsymbol{\omega})_q \\ &= -\Delta^{-1}\epsilon_{ipq}\partial_p\left(\partial_q\frac{|\boldsymbol{u}|^2}{2} - \partial_j(u_ju_q)\right) \\ &= \Delta^{-1}\epsilon_{ipq}\partial_p\partial_j(u_ju_q). \end{aligned}$$

By a formula [4]

$$\begin{aligned} \frac{\partial^2 a}{\partial x_i\partial x_j} &= \frac{f(\boldsymbol{x})}{3}\delta_{ij} + \frac{1}{4\pi}\text{P.V.} \\ &\times\int\left(\frac{\delta_{ij}}{|\boldsymbol{x}-\boldsymbol{y}|^3} - \frac{3(x_i-y_i)(x_j-y_j)}{|\boldsymbol{x}-\boldsymbol{y}|^5}\right)f(\boldsymbol{y})d\boldsymbol{y} \end{aligned}$$

for a solution a of $\triangle a(\boldsymbol{x}) = f(\boldsymbol{x})$, we find

$$\begin{aligned} g_i &= -\frac{\delta_{pj}}{3}\epsilon_{ipj}u_ju_q - \frac{\epsilon_{ipq}}{4\pi}\text{P.V.} \\ &\times\int\left(\frac{\delta_{pj}}{|\boldsymbol{x}-\boldsymbol{y}|^3} - \frac{3(x_p-y_p)(x_j-y_j)}{|\boldsymbol{x}-\boldsymbol{y}|^5}\right) \\ &\times u_j(\boldsymbol{y})u_q(\boldsymbol{y})d\boldsymbol{y} \end{aligned} \quad (15)$$

$$= \frac{3}{4\pi}\text{P.V.}\int\frac{\epsilon_{ipq}(x_p-y_p)(x_j-y_j)}{|\boldsymbol{x}-\boldsymbol{y}|^5}u_j(\boldsymbol{y})u_q(\boldsymbol{y})d\boldsymbol{y}, \quad (16)$$

which is (12). ■

We note that (12) can also be written as

$$\frac{\partial A_i}{\partial t} = \epsilon_{kpq}R_jR_k\partial_pA_q(\partial_jA_i - \partial_iA_j), \quad i = 1,2,3.$$

Here R_j denotes Riesz transform, defined by

$$R_j[f](\boldsymbol{x}) = c_n\text{P.V.}\int\frac{x_j-y_j}{|\boldsymbol{x}-\boldsymbol{y}|^{n+1}}f(\boldsymbol{y})d\boldsymbol{y}$$

for $j = 1,2,\ldots n$, $\boldsymbol{x}\in\mathbf{R}^n$ with $c_n = \Gamma(\frac{n+1}{2})/\pi^{(n+1)/2}$ (Γ is the γ function). For $n = 1,2,3$ the constants are $c_1 = 1/\pi$ (for the Hilbert transform), $c_2 = 1/(2\pi)$, and $c_3 = 1/\pi^2$. We recall that the Fourier transform of the Riesz transform is given by $\widehat{R}_j = -ik_j/|\boldsymbol{k}|$.

B. Hypercomplex representation

We assume that the velocity field has also a scalar potential $\boldsymbol{u} = \nabla\phi$ and seek to find the governing equations for ϕ by hypercomplexification. For nontrivial flow fields to exist, it is again necessary to assume that the flow field has singularities somewhere in the domain, such as vorticity layers, because, otherwise, Liouville theorem trivializes ϕ and $\boldsymbol{A}$ to be constants.

Consider a quaternion-valued velocity potential

$$W = \phi + iA_1 + jA_2 + kA_3,$$

and a differential operator

$$D = i\partial_1 + j\partial_2 + k\partial_3.$$

Here i,j,k denote a basis for a quaternion $z = ix_1 + jx_2 + kx_3$, whose fundamental properties are

$$i^2 = j^2 = k^2 = -1$$

and

$$ij = k, \quad jk = i, \quad \text{and} \quad ki = j.$$

The conjugate of each element is defined by

$$\bar{i} = -i, \quad \bar{j} = -j, \quad \bar{k} = -k,$$

and hence the conjugate of W is

$$\overline{W} = \phi - iA_1 - jA_2 - kA_3.$$

For applications of hypercomplex formulations in fluid dynamics, see, e.g., [5–7]. See also [8–11] for the methods of generalized analytic functions.

It is easily checked that, under $\nabla\cdot\boldsymbol{A} = 0$, the (generalized) Cauchy-Riemann equation [12]

$$D\overline{W} = 0, \quad \text{or,} \quad W\overline{D} = 0$$

is equivalent to [13]

$$\nabla\phi = \nabla\times\boldsymbol{A}.$$

Actually, we have

$$\begin{aligned} D\overline{W} &= (i\partial_1 + j\partial_2 + k\partial_3)(\phi - iA_1 - jA_2 - kA_3) \\ &= i\partial_1\phi + j\partial_2\phi + k\partial_3\phi + (\partial_1A_1 + \partial_2A_2 + \partial_3A_3) \\ &\quad - ij(\partial_1A_2 - \partial_2A_1) - jk(\partial_2A_3 - \partial_3A_2) \\ &\quad - ki(\partial_3A_1 - \partial_1A_3) \\ &= i[\partial_1\phi - (\partial_2A_3 - \partial_3A_2)] + j[\partial_2\phi - (\partial_3A_1 - \partial_1A_3)] \\ &\quad + k[\partial_3\phi - (\partial_1A_2 - \partial_2A_1)] \end{aligned} \quad (17)$$

using fundamental relationships for i,j,k.

We also compute

$$\begin{aligned} DW &= (i\partial_1 + j\partial_2 + k\partial_3)(\phi + iA_1 + jA_2 + kA_3) \\ &= i\partial_1\phi + j\partial_2\phi + k\partial_3\phi - (\partial_1A_1 + \partial_2A_2 + \partial_3A_3) \\ &\quad + ij(\partial_1A_2 - \partial_2A_1) + jk(\partial_2A_3 - \partial_3A_2) \\ &\quad + ki(\partial_3A_1 - \partial_1A_3) \\ &= 2(iu_1 + ju_2 + ku_3) \end{aligned}$$

and

$$\begin{aligned} zDW &= -2(x_1u_1 + x_2u_2 + x_3u_3) + 2i(x_2u_3 - x_3u_2) \\ &\quad + 2j(x_3u_1 - x_1u_3) + 2k(x_1u_2 - x_2u_1). \end{aligned}$$

Hence we have

$$\begin{aligned} (zDW)^2 &= -8i(x_2u_3 - x_3u_2)(x_1u_1 + x_2u_2 + x_3u_3) \\ &\quad - 8j(x_3u_1 - x_1u_3)(x_1u_1 + x_2u_2 + x_3u_3) \\ &\quad - 8k(x_1u_2 - x_2u_1)(x_1u_1 + x_2u_2 + x_3u_3) \end{aligned}$$

$$+4[(x_1u_1+x_2u_2+x_3u_3)^2-(x_2u_3-x_3u_2)^2 -(x_3u_1-x_1u_3)^2-(x_1u_2-x_2u_1)^2].$$

Now, similar to the two-dimensional case above, if we consider

$$\frac{\partial W}{\partial t}=-\frac{3}{32\pi}\text{P.V.}\int_{\mathbb{R}^3}[(z-z')DW(z')]^2\frac{d\boldsymbol{x}'}{|z-z'|^5}\ \text{a.e.,} \tag{18}$$

then we can confirm that its i,j,k components reproduce correct dynamical equations for A_1, A_2, and A_3. Here a.e. means exclusion of vortex singularities, e.g., vortex layers. The equation for the velocity potential is obtained by taking the real part as

$$\frac{\partial\phi}{\partial t}=\frac{3}{8\pi}\text{P.V.}\int_{\mathbb{R}^3}\frac{|\boldsymbol{r}\times\nabla\phi(\boldsymbol{x}')|^2-[\boldsymbol{r}\cdot\nabla\phi(\boldsymbol{x}')]^2}{|\boldsymbol{r}|^5}d\boldsymbol{x}'\ \text{a.e.} \tag{19}$$

This is the equation we are after. Equation (18) for the hypercomplex velocity potential is a three-dimensional counterpart of (10). Equivalently, we can write

$$\frac{\partial W}{\partial t}=-\frac{3}{32\pi}\text{P.V.}\int_{\mathbb{R}^3}\{[(z-z')DW(z')]^2-[(z-z')DW(z)]^2\}\times\frac{d\boldsymbol{x}'}{|z-z'|^5}\ \text{a.e.} \tag{20}$$

after regularization, by which we can prove that the analyticity $\overline{DW(z)}=W(z)\overline{D}=0$ persists as we have done in two dimensions.

Proof. Writing $D=D_z$ in the integrands to clarify the argument, we have

$$\begin{aligned}\frac{\partial}{\partial t}[W(z)\overline{D}]=&-\frac{3}{32\pi}\text{P.V.}\int_{\mathbb{R}^3}[([(z-z')DW(z')]^2-[(z-z')DW(z)]^2)\overline{D}_z]\frac{d\boldsymbol{x}'}{|z-z'|^5}\\&-\frac{3}{32\pi}\text{P.V.}\int_{\mathbb{R}^3}([(z-z')DW(z')]^2-[(z-z')DW(z)]^2)\left(\overline{D}_z\frac{1}{|z-z'|^5}\right)d\boldsymbol{x}',\ \text{a.e.}\end{aligned}$$

as the Leibniz formula holds because $1/|z-z'|^5$ is a scalar, [10]. Note that in the first line of the above equation, the operator $\overline{D}_z$ acts from the right. Replacing $\overline{D}_z$ with $-\overline{D}_{z'}$ in the second term, we find after integration by parts

$$\begin{aligned}\frac{\partial}{\partial t}[W(z)\overline{D}]=&-\frac{3}{32\pi}\text{P.V.}\int_{\mathbb{R}^3}[([(z-z')DW(z')]^2-[(z-z')DW(z)]^2)\overline{D}_z]\frac{d\boldsymbol{x}'}{|z-z'|^5}\\&-\frac{3}{32\pi}\text{P.V.}\int_{\mathbb{R}^3}[([(z-z')DW(z')]^2-[(z-z')DW(z)]^2)\overline{D}_{z'}]\frac{d\boldsymbol{x}'}{|z-z'|^5}=0\ \text{a.e.}\end{aligned}$$

because the integrands flip their sign when z and z' are interchanged. ■

V. APPLICATIONS

A. Kelvin-Helmholtz instability

As an application of the current approach, we work out the Kelvin-Helmholtz instability problem. We linearize (9) around a steady solution $\boldsymbol{U}(\boldsymbol{x})=\nabla\phi(\boldsymbol{x})$ to write

$$\frac{\partial}{\partial t}\delta\phi=\frac{1}{\pi}\text{P.V.}\int_{\mathbb{R}^2}\frac{[(\boldsymbol{x}-\boldsymbol{x}')\cdot\nabla\phi(\boldsymbol{x}')]\cdot[(\boldsymbol{x}-\boldsymbol{x}')\cdot\nabla\delta\phi(\boldsymbol{x}')]-[(\boldsymbol{x}-\boldsymbol{x}')\times\nabla\phi(\boldsymbol{x}')]\cdot[(\boldsymbol{x}-\boldsymbol{x}')\times\nabla\delta\phi(\boldsymbol{x}')]}{|\boldsymbol{x}-\boldsymbol{x}'|^4}d\boldsymbol{x}'\ \text{a.e.}$$

Consider a flow with $\boldsymbol{U}(\boldsymbol{x})=(U(\boldsymbol{x}),0)$, where

$$U(\boldsymbol{x})=\begin{cases}U_1, & \text{for } y<0,\\ U_2, & \text{for } y>0;\end{cases}$$

we then have

$$\begin{aligned}(\boldsymbol{x}-\boldsymbol{x}')\cdot\nabla\phi(\boldsymbol{x}')&=U(\boldsymbol{x}')(x-x'),\\(\boldsymbol{x}-\boldsymbol{x}')\times\nabla\phi(\boldsymbol{x}')&=-U(\boldsymbol{x}')(y-y').\end{aligned}$$

Therefore we find

$$\begin{aligned}\frac{\partial}{\partial t}\delta\phi&=\frac{1}{\pi}\text{P.V.}\int_{\mathbb{R}^2}\frac{U(\boldsymbol{x}')[(x-x')(\boldsymbol{x}-\boldsymbol{x}')\cdot\nabla\delta\phi(\boldsymbol{x}')+(y-y')(\boldsymbol{x}-\boldsymbol{x}')\times\nabla\delta\phi(\boldsymbol{x}')]}{|\boldsymbol{x}-\boldsymbol{x}'|^4}d\boldsymbol{x}'\ \text{a.e.,}\\&=\frac{1}{\pi}\text{P.V.}\int_{\mathbb{R}^2}\frac{U(\boldsymbol{x}')\{[(x-x')^2-(y-y')^2]\partial_{x'}\delta\phi(\boldsymbol{x}')+2(x-x')(y-y')\partial_{y'}\delta\phi(\boldsymbol{x}')\}}{|\boldsymbol{x}-\boldsymbol{x}'|^4}d\boldsymbol{x}'\ \text{a.e.}\end{aligned}$$

We note that if we linearize (7) around $(\boldsymbol{x}-\boldsymbol{x}')\cdot\nabla\psi(\boldsymbol{x}')=U(\boldsymbol{x}')(y-y')$, $(\boldsymbol{x}-\boldsymbol{x}')\times\nabla\psi(\boldsymbol{x}')=U(\boldsymbol{x}')(x-x')$, we consistently get the same form of equation for $\delta\psi$,

$$\frac{\partial}{\partial t}\delta\psi=\frac{1}{\pi}\mathrm{P.V.}\int_{\mathbb{R}^2}\frac{U(\boldsymbol{x}')\{[(x-x')^2-(y-y')^2]\partial_{x'}\delta\psi(\boldsymbol{x}')+2(x-x')(y-y')\partial_{y'}\delta\psi(\boldsymbol{x}')\}}{|\boldsymbol{x}-\boldsymbol{x}'|^4}d\boldsymbol{x}' \text{ a.e.}$$

Hence we can proceed with either equation. To carry out a linear stability analysis, it is convenient to recast it as

$$\frac{\partial}{\partial t}\delta\psi=(R_1R_1-R_2R_2)U(\boldsymbol{x})\partial_1\delta\psi+2R_1R_2U(\boldsymbol{x})\partial_2\delta\psi,$$

where $R_j,\ j=1,2$ denotes the two-dimensional Riesz transform. Note also that this equation can be derived directly from yet another form for (7)

$$\frac{\partial\psi}{\partial t}=\epsilon_{jk}R_iR_j\partial_k\psi\partial_i\psi;$$

see [2].

We write the steady flow

$$U(y)=\frac{U_1+U_2}{2}+\frac{U_1-U_2}{2}\mathrm{sgn}(y), \tag{21}$$

where $\mathrm{sgn}(y)=1$ for $y>0$, $=-1$ for $y<0$. If only the first constant term $\frac{U_1+U_2}{2}$ is retained, we would have

$$\widehat{\delta\psi}_t=\frac{U_1+U_2}{2}\left(-\frac{k_1^2-k_2^2}{|\boldsymbol{k}|^2}ik_1\widehat{\delta\psi}-2\frac{k_1k_2}{|\boldsymbol{k}|^2}ik_2\widehat{\delta\psi}\right)$$
$$=-\frac{U_1+U_2}{2}ik_1\widehat{\delta\psi}.$$

To handle the second term of (21), we need to evaluate Fourier transforms of $\mathrm{sgn}(y)\partial_1\delta\psi$ and $\mathrm{sgn}(y)\partial_2\delta\psi$. We find

$$\begin{aligned}\widehat{\mathrm{sgn}(y)\partial_1}\delta\psi&=\frac{1}{(2\pi)^2}\iint\mathrm{sgn}(y)\partial_1\delta\psi(x,y)e^{-i(k_1x+k_2y)}dxdy\\&=\frac{1}{2\pi}\int dy\,\mathrm{sgn}(y)e^{-ik_2y}\\&\quad\times\underbrace{\frac{1}{2\pi}\int\partial_1\delta\psi(x,y)e^{-ik_1x}dx}_{=ik_1\delta\psi(k_1,y)}\\&=\frac{ik_1}{2\pi}\int\delta\psi(k_1,y)\mathrm{sgn}(y)e^{-ik_2y}dy\\&=\frac{-ik_1}{\pi}\mathrm{P.V.}\int\frac{\delta\widehat{\psi}(k_1,p_2)}{k_2-p_2}dp_2\\&=-ik_1H[\widehat{\delta\psi}(k_1,\cdot)].\end{aligned}$$

Here use has been made of a fact that $\widehat{\mathrm{sgn}(y)}=-\frac{i}{\pi}\mathrm{P.V.}\frac{1}{k_2}$ and the definition of the Hilbert transform $H[f]=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(y)}{x-y}dx$. Similarly, noting that $\partial_y\mathrm{sgn}(y)=2\delta(y)$, we obtain

$$\widehat{U(y)\partial_2}\delta\psi=-ik_2H[\widehat{\delta\psi}(k_1,\cdot)]-\frac{1}{\pi}\delta\widehat{\psi}(k_1,0),$$

where the final term on the right-hand side of the above equation annihilates upon taking R_2. Hence, if only the second term of (21) is retained, we would have

$$\frac{U_1-U_2}{2}ik_1H[\widehat{\delta\psi}(k_1,\cdot)].$$

Altogether when both terms are present, we have

$$\frac{\partial}{\partial t}\widehat{\delta\psi}=ik_1\left(-\frac{U_1+U_2}{2}\widehat{\delta\psi}+\frac{U_1-U_2}{2}H[\widehat{\delta\psi}(k_1,\cdot)]\right). \tag{22}$$

This is the linearized equation for the Kelvin-Helmholtz problem. To solve it, we take the Hilbert transform with respect to k_2 to write

$$\begin{aligned}&\frac{\partial}{\partial t}H[\widehat{\delta\psi}(k_1,\cdot)]\\&\quad=ik_1\left(-\frac{U_1-U_2}{2}H[\widehat{\delta\psi}(k_1,\cdot)]-\frac{U_1+U_2}{2}\widehat{\delta\psi}\right).\end{aligned} \tag{23}$$

Defining $f=\widehat{\delta\psi}$, $g=H[\widehat{\delta\psi}(k_1,\cdot)]$, the governing equations become

$$\frac{\partial f}{\partial t}=ik_1\left(-\frac{U_1+U_2}{2}f+\frac{U_1-U_2}{2}g\right),$$
$$\frac{\partial g}{\partial t}=ik_1\left(-\frac{U_1-U_2}{2}f-\frac{U_1+U_2}{2}g\right).$$

Assuming $f=Ae^{\lambda t},g=Be^{\lambda t}$, we find

$$\begin{pmatrix}-\frac{U_1+U_2}{2}ik_1-\lambda & \frac{U_1-U_2}{2}ik_1\\ -\frac{U_1-U_2}{2}ik_1 & -\frac{U_1+U_2}{2}ik_1-\lambda\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$

from which

$$\lambda=-\frac{U_1+U_2}{2}ik_1\pm\frac{|U_1-U_2|}{2}|k_1|$$

is obtained as the eliminant. This completely agrees with the growth rates obtained by more conventional methods based on classical Bernoulli theorem [14,15]. Note also that this method is different from the analysis based on the Birkhoff-Rott equation. This demonstrates the performance of the current approach.

B. Navier-Stokes regularity issues

Finally, we note an application of the equations for the vector potentials to smooth fields. We write the Navier-Stokes equations as

$$\frac{\partial\boldsymbol{A}}{\partial t}=\frac{3}{4\pi}\mathrm{P.V.}\int_{\mathbb{R}^3}\frac{[\boldsymbol{r}\times(\nabla\times\boldsymbol{A}(\boldsymbol{x}'))]\boldsymbol{r}\cdot(\nabla\times\boldsymbol{A}(\boldsymbol{x}'))}{|\boldsymbol{r}|^5}\times d\boldsymbol{x}'+\nu\triangle\boldsymbol{A}, \tag{24}$$

where $\nabla\cdot\boldsymbol{A}=0$ and $\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{x}'$. This may be regarded as a nonlocal version of the Hamilton-Jacobi equations. This form of the Navier-Stokes equations has a distinctive property that the dependent variable $\boldsymbol{A}$ is critical in the sense that its physical dimension is identical to that of kinematic viscosity ν. This observation of physical nature led Cole to connect the Burgers equations with heat equations, resulting in a discovery of the Cole-Hopf transformation [16,17].

A blowup criterion is known in terms of velocity [18]; we have

$$\sup_{\boldsymbol{x}} |\boldsymbol{u}(\boldsymbol{x},t)| \geqslant c\frac{\nu^{1/2}}{\sqrt{t_* - t}}$$

for a blowup at $t = t_*$. Conversely, if $\sup_{\boldsymbol{x}} |\boldsymbol{u}(\boldsymbol{x},t)|$ is bounded, the flow remains smooth up to t. We may consider criteria using dependent variables with reduced order of spatial derivatives. In the limiting case, we arrive at $\sup_{\boldsymbol{x}} |\boldsymbol{A}|$. It is an open problem to decide whether $\sup_{\boldsymbol{x}} |\boldsymbol{A}|$ serves as a criterion for a possible singularity formation in Navier-Stokes flows.

The integral operator in (24) is of zeroth order and has no smoothing effect. If a solution to the Navier-Stokes equations blow up, the integrand is divergent and on heuristic grounds we expect

$$\frac{\partial \boldsymbol{A}}{\partial t} \simeq C\frac{\nu}{t_* - t},$$

leading to

$$\boldsymbol{A} \simeq \nu C \ln\frac{1}{t_* - t},$$

where C is a positive constant. If this is the case, $\sup_{\boldsymbol{x}} |\boldsymbol{A}|$ serves as a blowup criterion. Pursuing the problem using the tamer variable $\boldsymbol{A}$ may be useful.

VI. SUMMARY AND OUTLOOK

In order to establish a refined version of Bernoulli theorem, we have derived the evolution equations for the velocity potential for flows that allow singular vorticity distributions in two and three dimensions.

In two dimensions, by viewing the equation for the stream function as an imaginary part, we have derived the equation for the complex velocity potential. In three dimensions, first we have derived dynamical equations for the vector potential. By regarding it as (i,j,k) elements, we have derived an equation for hypercomplex velocity potential. The equation for the velocity potential has been identified as its real part.

By allowing singular distributions in the flow field, we obtain a closed equation for the velocity potential, which is directly connected to the pressure. In two dimensions, this equation offers an alternative governing equation in the form of a partial integrodifferential equation for the meromorphic complex velocity potential.

The newly obtained equations for complex (or hypercomplex) velocity potentials are valid when we have singularities, such as point vortices in two dimensions and vorticity layers in three dimensions. They have been applied to the Kelvin-Helmholtz instability problem in two dimensions.

Some comments on the potential applications of the current approach may be in order. For inviscid fluids, there are two quadratic invariants of motion the energy E and the helicity H [19]:

$$E = \frac{1}{2}\int |\boldsymbol{u}(\boldsymbol{x})|^2 d\boldsymbol{x} = \frac{1}{8\pi}\iint \frac{\boldsymbol{\omega}(\boldsymbol{x})\cdot\boldsymbol{\omega}(\boldsymbol{x}')}{r} d\boldsymbol{x}d\boldsymbol{x}',$$

$$H = \frac{1}{2}\int \boldsymbol{u}\cdot\boldsymbol{\omega} d\boldsymbol{x} = \frac{1}{8\pi}\iint \frac{\boldsymbol{r}\cdot(\boldsymbol{\omega}(\boldsymbol{x})\times\boldsymbol{\omega}(\boldsymbol{x}'))}{r^3} d\boldsymbol{x}d\boldsymbol{x}'.$$

If we consider vortex filaments with strength κ_i supported on closed circuits C_i, $i = 1,2,\ldots,N$, then the invariants take the following forms:

$$E = \frac{1}{2}\sum_{i,j}\kappa_i\kappa_j L_{ij}, \quad L_{ij} = \frac{1}{4\pi}\oint_{C_i}\oint_{C_j}\frac{d\boldsymbol{s}_i\cdot d\boldsymbol{s}_j}{r},$$

$$H = \frac{1}{2}\sum_{i,j}\kappa_i\kappa_j\alpha_{ij}, \quad \alpha_{ij} = \frac{1}{4\pi}\oint_{C_i}\oint_{C_j}\frac{\boldsymbol{r}\cdot(d\boldsymbol{s}_i\times d\boldsymbol{s}_j)}{r^3},$$

where $\boldsymbol{r} = \boldsymbol{x}_i - \boldsymbol{x}_j$, $\boldsymbol{x}_i \in C_i$, $\boldsymbol{x}_j \in C_j$, and $d\boldsymbol{s}_i$ is a line element along C_i. Note that α_{ij} denotes the Gauss linking number and L_{ij} the Neumann mutual inductance. It is of interest to study this system with the current approach.

There is a similarity between the current method with the so-called panel methods in aerodynamics, as a branch of the boundary element method. In panel methods in its original formulation, distant velocities for steady flow are represented by a layer of dipoles, e.g., see Sec. 14.1 of [20]. The current approach covers more general cases in that it describes nonsteady flows and the types of singularities are not restricted to dipoles.

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[12] For $f = \phi + i\psi$, it is well known in complex function theory that the Cauchy-Riemann equations (the simplest form of div-curl system) $\phi_x = \psi_y, \phi_y = -\psi_x$ can be written as $\frac{\partial f}{\partial \bar{z}} = 0$. Note that (17) is equivalent to $\nabla \cdot \boldsymbol{A} = \nabla \times \boldsymbol{A} = 0$ with $\phi = 0$.

[13] For $D = \sum_{l=1}^{3} e_l \partial_l$, $W = \sum_{m=0}^{3} e_m A_m$ with $e_1 = i$, $e_2 = j$, $e_3 = k$, and $A_0 = \phi$, we define $D\overline{W} = \sum_{l=1}^{3} \sum_{m=0}^{3} e_l \overline{e_m} (\partial_l A_m)$ and $W\overline{D} = \sum_{l=1}^{3} \sum_{m=0}^{3} e_m \overline{e_l} (\partial_l A_m)$.

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