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Study of the Navier-Stokes regularity problem with critical norms

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Abstract. We study the basic problems of regularity of the Navier-Stokes equations. The blowup criteria on the basis of the critical $H^{1/2}$-norm, is bounded from above by a logarithmic function, Robinson, Sadowski and Silva (2012). Assuming that the Cauchy-Schwarz inequality for the $H^{1/2}$-norm is not an overestimate, we replace it by a square-root of a product of the energy and the enstrophy. We carry out a simple asymptotic analysis to determine the time evolution of the energy. This generalises the (already ruled-out) self-similar blowup ansatz. Some numerical results are also presented, which support the above-mentioned replacement. We carry out a similar analysis for the four-dimensional Navier-Stokes equations.

Keywords: Navier-Stokes equations, Leray equations, blowup, critical norms

1. Introduction

In this paper we will be concerned with the fundamental problems of the incompressible Navier-Stokes equations, which read in standard notations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \triangle \boldsymbol{u}, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{2}$$

with smooth initial data with finite total kinetic energy in $\mathbb{R}^n$, where $n = 3$ or 4.

The seminal paper of Leray, which initiated mathematical study of the Navier-Stokes equations [23], has established global existence of weak solutions and local existence of classical (i.e. smooth) solutions. It has also provided some criteria that monitor a possible onset of singularity. In spite of lots of effort made since then, we do not know whether smoothness persists for long time for large initial data.

There are a number of criteria known for possible blowup of solutions at $t = t_*$, e.g. by the maximum of velocity magnitude

$$\sup_{\boldsymbol{x}} |\boldsymbol{u}(\boldsymbol{x}, t)| \ge c \frac{\nu^{1/2}}{\sqrt{t_* - t}}$$

or, by the total enstrophy

$$\int_{\mathbb{R}^3} |\boldsymbol{\omega}|^2 d\boldsymbol{x} \ge C \frac{\nu^{3/2}}{\sqrt{t_* - t}}.$$

Hereafter, c, C and so on denote positive constants, which may take different values. These are subcritical (see below for definition) and specific time-dependence is known for their lowerbounds, which happens to be the same time-dependence $(t_* - t)^{-1/2}$. Thanks to recent progress, blowup criteria are also known for critical norms, such as the L^3-norm and the $H^{1/2}$-norm. For those critical norms, no specific power-law lowerbounds are specified, but transcendental (typically, logarithmic) time-dependence is expected.

A possible blowup which satisfies

$$\int_{\mathbb{R}^3} |\boldsymbol{\omega}|^2 d\boldsymbol{x} \le C' \frac{\nu^{3/2}}{\sqrt{t_* - t}}$$

is called Type I, e.g. [31]. In other words, this class of singularity blows up on the same order as the ordinary differential inequality predicts, which we denote by

$$\int_{\mathbb{R}^3} |\boldsymbol{\omega}|^2 d\boldsymbol{x} \simeq \frac{\nu^{3/2}}{\sqrt{t_* - t}}.$$

Any blow-up other than Type I is called Type II. These terminologies have come from study of Ricci curvature equations. Type I singularities have been ruled out for axis-symmetric Navier-Stokes equations [31], however, they have not been excluded for general settings.

The rest of this paper is organised as follows. In Section 2, we review scale-invariance and known criteria of blowup for the Navier-Stokes equations. In Section 3, we recast the well-known enstrophy bound in a form suitable for an asymptotic analysis. In Section 4, a simple asymptotic theory is presented under the above assumption for the three-dimensional case. In Section 5, a similar analysis is done for the four-dimensional case. In Section 6, we give some numerical results in the three-dimensional case. Section 7 is devoted to a summary.

There is a large amount of literature on the mathematical problems of the Navier-Stokes equations: articles include [21, 15, 1, 20, 17, 8, 28, 16, 26, 22, 35] and books [36, 5, 9, 14, 34]. For applied mathematical interests, see e.g. [12, 27, 30].

2. Scale-invariance and criticality

2.1. Scale-invariance

We begin by reviewing concepts of scale-invariance and criticality. It is well-known that the Navier-Stokes equations are invariant under the scaling transforms

$$\boldsymbol{x} \to \boldsymbol{x}/\lambda, t \to t/\lambda^2,$$

$$\boldsymbol{u}(\boldsymbol{x},t) \to \boldsymbol{U}(\boldsymbol{\xi},\tau) \equiv \lambda^{-1}\boldsymbol{u}(\boldsymbol{x}/\lambda, t/\lambda^2),$$
$$p(\boldsymbol{x},t) \to P(\boldsymbol{\xi},\tau) \equiv \lambda^{-2}p(\boldsymbol{x}/\lambda, t/\lambda^2),$$

where λ is a positive parameter. Under the transformations

$$\boldsymbol{x} = \lambda\boldsymbol{\xi}, \boldsymbol{t} = \lambda^2\boldsymbol{\tau}, \boldsymbol{u} = \lambda^{-1}\boldsymbol{U},$$

the L^q-norm is transformed as

$$\int |\boldsymbol{u}|^q d\boldsymbol{x} = \lambda^{n-q}\int |\boldsymbol{U}|^q d\boldsymbol{\xi}.$$

The norm is called supercritical if $q < 3$, subcritical if $q > 3$ and critical if $q = 3$ in three-spatial dimensions ($n = 3$). The L^3-norm has the physical dimension of ν^3.

In three dimensions, the (squared) $H^{1/2}$-norm

$$H = \int |\Lambda^{1/2}\boldsymbol{u}|^2 d\boldsymbol{x} \equiv \int \boldsymbol{u}\cdot\Lambda\boldsymbol{u} d\boldsymbol{x} \le \left(\int |\boldsymbol{u}|^2 d\boldsymbol{x}\int |\boldsymbol{\omega}|^2 d\boldsymbol{x}\right)^{1/2}$$

is also critical, whose physical dimension is the same as that of ν^2. Here $\Lambda = (-\triangle)^{1/2}$ denotes the Zygmund operator with its Fourier transform defined by $|\boldsymbol{k}|$, where $\boldsymbol{k}$ is wavenumber. We note that an inequality of the form

$$\boldsymbol{u}\cdot\Lambda\boldsymbol{u} \ge \Lambda\frac{|\boldsymbol{u}|^2}{2} \tag{3}$$

holds in $\mathbb{R}^3$ and $\mathbb{T}^3$ [7]. We also note that the above integral H has the same physical dimension as that of the helicity.

We will use following notations for the energy

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x}$$

and the enstrophy

$$Q(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x},$$

where $n = 3$ or 4.

By assuming that $H(t) \simeq \sqrt{E(t)Q(t)}$, we will derive an asymptotic equation for the evolution of energy. Under this assumption we may regard $\sqrt{E(t)Q(t)}$ as a surrogate for $H(t)$.

2.2. Self-similar blowup

Leray considered a self-similar blowup solution [23] of the form

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{[2a(t_* - t)]^{1/2}}\boldsymbol{U}(\boldsymbol{\xi}), \quad \boldsymbol{\xi} = \frac{\boldsymbol{x}}{[2a(t_* - t)]^{1/2}}$$

and derived the so-called Leray equations

$$\boldsymbol{U}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U} + a(\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U} + \boldsymbol{U}) = -\nabla_{\boldsymbol{\xi}}P + \nu\triangle_{\boldsymbol{\xi}}\boldsymbol{U},$$

$$\nabla_{\boldsymbol{\xi}}\cdot\boldsymbol{U} = 0,$$

where $a(> 0)$ is a parameter whose physical dimension is the same as ν. This kind of solution, based on Leray's scaling ansatz, is of Type I, but there can be other types of singularities for the Navier-Stokes equations that do not obey such an ansatz.

If such a solution exists on $0 \le t < t_*$, it satisfies the following identities

$$\int_{\mathbb{R}^3} |\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x} = \sqrt{2a(t_* - t)}\int_{\mathbb{R}^3} |\boldsymbol{U}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi},$$

$$\int_{\mathbb{R}^3} |\Lambda^{1/2}\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x} = \int_{\mathbb{R}^3} |\Lambda_{\boldsymbol{\xi}}^{1/2}\boldsymbol{U}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi},$$

and

$$\int_{\mathbb{R}^3} |\nabla\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x}\frac{1}{\sqrt{2a(t_* - t)}}\int_{\mathbb{R}^3} |\nabla_{\boldsymbol{\xi}}\boldsymbol{U}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}.$$

We note that for a self-similar blowup, the total energy converges to zero as $t \to t_*$. We also note that the product of the energy and the enstrophy is a constant

$$\int_{\mathbb{R}^3} |\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x}\int_{\mathbb{R}^3} |\nabla\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x} = \int_{\mathbb{R}^3} |\boldsymbol{U}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}\int_{\mathbb{R}^3} |\nabla_{\boldsymbol{\xi}}\boldsymbol{U}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi},$$

because the right-hand side is independent of t due to cancellation of the temporal factors. This possibility of self-similar blowup has been ruled out: if $\boldsymbol{U} \in L^3(\mathbb{R}^3)$ then $\boldsymbol{U} \equiv 0$ [25], see also [37]. We note that asymptotically self-similar blowup has also been ruled out [2, 19, 3, 4].

Progress has been made recently regarding blowup criteria using critical norms, such as the L^3-norm [13]. It was proved in [13] that for a possible singularity at $t = t_*$ we have

$$\|\boldsymbol{u}\|_{L^3(\mathbb{R}^3)} \to \infty \ \text{ as } \ t \to t_*.$$

By a standard embedding

$$\|\boldsymbol{u}\|_{H^{1/2}} \geq C\|\boldsymbol{u}\|_{L^3},$$

it follows that

$$\|\boldsymbol{u}\|_{H^{1/2}(\mathbb{R}^3)} \to \infty \ \text{ as } \ t \to t_*.$$

See [32, 33, 29] for this criterion. There is a similar result for the n-dimensional Navier-Stokes equations [10].

3. Enstrophy bound

By applying standard energy methods to the enstrophy equation

$$\frac{dQ}{dt} = \int_{\mathbb{R}^3} \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{\omega} d\boldsymbol{x} - \nu \int_{\mathbb{R}^3} |\nabla \times \boldsymbol{\omega}|^2,$$

we obtain the well-known enstrophy bound

$$\frac{dQ}{dt} \leq C\frac{Q^3}{\nu^3} - \frac{5\nu}{4}\frac{Q^2}{E}, \tag{4}$$

e.g. [24] and references cited therein. The first term of the above bound is predictable on dimensional grounds. We note that the instantaneous growth rate (4) is proven to be optimal and that it is associated with a vortex ring [24].

We make some observations on the bound (4). An inequality expressing the evolution of the critical norm H, solely in terms of H and ν, is not known. However, it is possible to write down an inequality for the surrogate EQ, which bounds $H \leq \sqrt{EQ}$ by Cauchy-Schwarz inequality. In fact, by coupling the energy balance equation

$$\frac{dE}{dt} = -2\nu Q \tag{5}$$

with the above bound (4), we obtain

$$\frac{d}{dt}\log(EQ) \leq C\frac{Q^2}{\nu^3} \tag{6}$$

and hence find

$$E(t)Q(t) \leq E(0)Q(0)\exp\left(\frac{C}{\nu^3}\int_0^t Q(t')^2 dt'\right).$$

It is of interest to note that Type I singularity

$$Q(t) \leq \frac{c\nu^{3/2}}{\sqrt{t_* - t}}$$

gives rise to a power-law upperbound for the product $E(t)Q(t)$. In order to obtain a logarithmic upperbound on $E(t)Q(t)$, an upperbound which blows up more slowly

$$Q(t) \le \frac{c\nu^{3/2}}{\sqrt{(t_* - t)\log\dfrac{1}{t_* - t}}}$$

would be required. (In fact, this *cannot* takes place because $\sqrt{t_* - t}\,Q(t) \to 0$ as $t \to t_*$.) Also, by recasting (6) as

$$\frac{d}{dt}\log(EQ) \le C\frac{(EQ)^2}{\nu^3 E^2},$$

we observe that we need the energy E to close the bound for EQ, on top of EQ and ν.

Multiplying (4) by $\frac{1}{C\nu^5}\frac{E^2}{Q}$, we recast the bound (4) as

$$\frac{1}{C\nu^5}\frac{E^2}{Q}\frac{dQ}{dt} \le f^2 - \frac{5}{4C}f, \tag{7}$$

where $f(t) \equiv \frac{E(t)Q(t)}{\nu^4}$ is a non-dimensionalised critical criterion. Solving the above quadratic inequality for f, we get

$$f \ge \frac{1}{2}\left(\frac{5}{4C} + \sqrt{\left(\frac{5}{4C}\right)^2 + \frac{4E^2}{C\nu^5 Q}\frac{dQ}{dt}}\right),$$

which essentially states that

$$f \gtrsim c\frac{E}{\nu^{5/2}}\sqrt{\frac{d}{dt}\log Q}.$$

We know that if initial $E(0)Q(0)$ is small in the sense that $E(0)Q(0) = O(\nu^4)$, then global regularity follows. Then we may ask how the solution behaves in time, if EQ is not small, but goes singular mildly.

4. Asymptotic analysis

In the Cauchy-Schwarz inequality

$$\frac{1}{2}\|\boldsymbol{u}(t)\|_{H^{1/2}}^2 \equiv H(t) \le \sqrt{E(t)Q(t)},$$

we distinguish two cases: i) $H(t) \simeq \sqrt{E(t)Q(t)}$ and ii) $H(t) \ll \sqrt{E(t)Q(t)}$. In view of some numerical supports below, we consider the case i) and will carry out an asymptotic analysis.

In view of an upperbound [29] derived under the assumption of blowup

$$\|\boldsymbol{u}(t)\|_{H^{1/2}} \le C\log\frac{t_*}{t_* - t} + \|\boldsymbol{u}(0)\|_{H^{1/2}},$$

we have

$$\sqrt{E(t)Q(t)} \le C\left(\log\frac{t_*}{t_* - t}\right)^2.$$

If we denote weak (i.e. transcendental) singularities by‡ $f(t) \leq$ '$\log \dfrac{1}{t-t_*}$', we find from (7)

$$\frac{dQ}{dt} \leq C\nu^5 \frac{Q}{E^2} \text{'}\log \frac{1}{t-t_*}\text{'}.$$

We will carry out a leading-order analysis and interpret its outcome *modulo* transcendental factors. To leading-order, we have

$$\begin{cases} \dfrac{dQ}{dt} \lesssim C\nu^5 \dfrac{Q}{E^2}, \\[2ex] \dfrac{dE}{dt} = -2\nu Q, \end{cases} \tag{8}$$

because the energy equality holds prior to the first singularity. It seems impossible to solve for the energy as a function of time, but it is possible to represent time as a function of the energy. Actually, by defining $e(t) \equiv \dfrac{E(t)}{C^{1/2}\nu^{5/2}}$, we find

$$\log|1 - c\,e(t)| + c\,e(t) \gtrsim c^2\,(t-t_*),$$

where t_* and c are constants.

Proof

By eliminating Q from (8), we have

$$\ddot{E} \geq C\nu^5 \frac{\dot{E}}{E^2},$$

or, upon normalisation we get

$$\ddot{e} \geq \frac{\dot{e}}{e^2} \equiv F(e, \dot{e}).$$

Let $W \equiv \dot{e}$, then by

$$\ddot{e} = \frac{dW}{dt} = \frac{dW}{de}\dot{e},$$

we find

$$W\frac{dW}{de} \geq F(e, \dot{e}).$$

Because $W < 0$, we have

$$\frac{dW}{de} \leq \frac{F(e,\dot{e})}{W} = \frac{1}{e^2},$$

which is integrated to give

$$\frac{de}{dt} = W \leq -\frac{1}{e} + c. \tag{9}$$

As we will see below, the physically relevant branch satisfies $ce(t) - 1 < 0$. Hence we find

$$\frac{1}{c}\left(\int \frac{de}{ce-1} + \int de\right) \geq t + c_1,$$

‡ More generally, any function f such that $\lim_{t\to t_*}(t_* - t)^\epsilon f(t) = 0$ with $\epsilon > 0$.

or

$$\log|1 - c\,e(t)| + c\,e(t) \geq c^2\,(t + c_1).$$

The constant c_1 is fixed as $c_1 = -t_*$ by the condition $e(t_*) = 0$ and we arrive at

$$\log|1 - c\,e(t)| + c\,e(t) \geq c^2\,(t - t_*). \tag{10}$$

Replacing $\geq$ with $\gtrsim$ to allow for a logarithmic factor, we obtain the desired result. □

By Taylor expanding around $e = 0$, we confirm that $e(t) \lesssim \sqrt{2(t_* - t)}$, which shows that its final behaviour agrees with what the self-similar evolution predicts. However, this does not necessarily imply that the evolution is self-similar, hence it remains a non-trivial open problem to rule out the possibility of this asymptotic behaviour .

We define the scaled enstrophy by $q(t) = -\dot{e}(t)$, that is, $q(t) = 2Q(t)/(c^{1/2}\nu^{3/2})$. For the borderline behaviour, by (9) we have $e(t) \simeq \dfrac{1}{q(t) + c}$, and hence

$$\log\left(\frac{q(t)}{q(t) + c}\right) + \frac{c}{q(t) + c} \simeq c^2(t_* - t).$$

This expresses time t as a function of the enstrophy $q(t)$.

In Fig.1 we show time as a function of energy. There are two branches, but only one of them is physically relevant, because the other one increases energy monotonically. The realisable region is on or below the lower branch (solid). We observe that it asymptotes to the self-similar evolution near $t = t_*$, and that $e(t)$ apparently shows a decay of the energy over a finite time interval. However, this does not necessarily mean that such an evolution is actually realised. The corresponding enstrophy is also inserted for a comparison.

Because the analysis is performed to leading-order, formally the same result is obtained for the more stringent assumption $E(t)Q(t) \leq C$, where C is a constant. In this case we know that there is no blowup. But it seems hard to deduce the non-existence from (10).

5. Four-dimensional Navier-Stokes equations

In four dimensions the enstrophy is critical and we can carry out a similar analysis in a parallel manner as in three dimensions. There are not many papers in this case, but see [11, 18] and references cited therein.

We need a (squared) second-order Sobolev norm, palinstrophy,

$$P(t) = \frac{1}{2}\int_{\mathbb{R}^4} |\nabla \times \boldsymbol{\omega}|^2 \mathrm{d}\boldsymbol{x} = \frac{1}{2}\|\boldsymbol{u}\|_{H^2}^2$$

in this case. The four-dimensional Navier-Stokes equations in vorticity form read

$$\frac{\partial \omega_{ij}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\omega_{ij} = \omega_{jk}\frac{\partial u_k}{\partial x_i} - \omega_{ik}\frac{\partial u_k}{\partial x_j} + \nu\triangle\omega_{ij}, \tag{11}$$

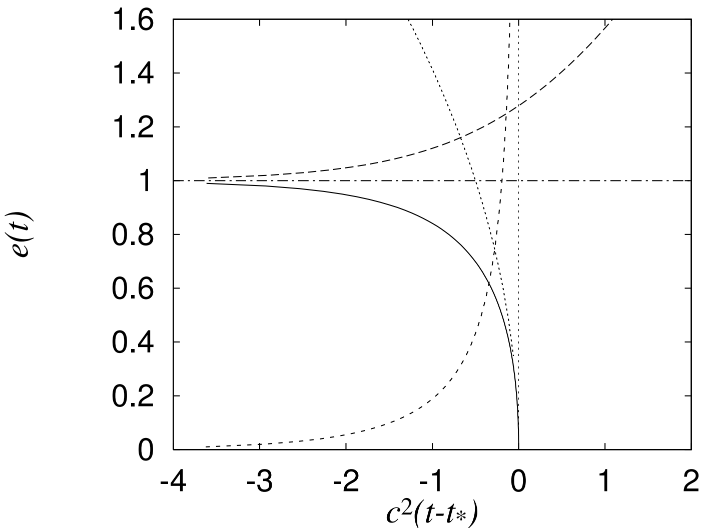

Figure 1. The Evolution of the energy (solid) for the 3D case. The self-similar asymptote (dotted) is inserted to the lower physical branch. The upper branch (dashed) is unphysical. The enstrophy is also shown (short-dashed). The horizontal and vertical lines are included as a guide.

where $\omega_{ij} = \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}$ is the vorticity tensor $(i, j = 1, 2, 3, 4)$. Note that the velocity and the vorticity are related by the 'Biot-Savart' law

$$u_i(\boldsymbol{x}) = -\frac{1}{2\pi}\int_{\mathbb{R}^4} \frac{(x_j - y_j)\omega_{ij}(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^4} d\boldsymbol{y},$$

which results from $\triangle \dfrac{1}{4\pi r^2} = -\delta(\boldsymbol{x})$. By (11) it is straightforward to derive the following enstrophy inequality

$$\frac{dQ}{dt} \leq CPQ^{1/2} - 2\nu P.$$

Unlike the three-dimensional case, it is impossible to split the product $PQ^{1/2}$ as a sum of two terms by using the Young inequality, because of the exponent 1 of P.

Defining a non-dimensional function $g(t) = E(t)P(t)/\nu^4$ and using Cauchy-Schwarz inequality

$$Q(t)^2 \leq E(t)P(t),$$

we find

$$\frac{E}{\nu^5}\frac{dQ}{dt} \leq Cg(t)\left(g(t)^{1/4} - 2\right). \tag{12}$$

By assuming that

$$g(t) \leq C\text{'} \log \frac{1}{t - t_*}, \tag{13}$$

we obtain

$$Q(t) \lesssim Q(0) + C\nu^5 \int_0^t \frac{dt'}{E(t')},$$

which shows that

$$\lim_{t \to t_*} \int_0^t \frac{dt'}{E(t')} = \infty$$

is necessary for a possible blow-up. This shows that $E(t) \to 0$ when $t \to t_*$ at least as fast as $(t_* - t)$ to leading-order.

We determine the time evolution of the energy in more detail below. Let us assume $Q(t)/\nu^2 \geq 1$, because we are not interested in small initial data, for which global existence is known. Multiplying (12) by this factor, we have

$$\frac{E}{\nu^5}\frac{dQ}{dt} \leq Cg(t)\left(g(t)^{1/4} - 2\right)\frac{Q(t)}{\nu^2},$$

or

$$\frac{E}{\nu^3}\frac{d\log Q}{dt} \leq Cg(t)\left(g(t)^{1/4} - 2\right).$$

Under the assumption of (13), we deduce

$$\begin{cases} \dfrac{dQ}{dt} \lesssim C\nu^3\dfrac{Q}{E}, \\[2ex] \dfrac{dE}{dt} = -2\nu Q. \end{cases} \tag{14}$$

This system can be solved as before. Introducing a normalisation $e(t) \equiv \dfrac{E(t)}{C\nu^3}$, we obtain

$$\mathrm{Li}(ce(t)) \gtrsim c(t - t_*), \tag{15}$$

where Li denotes the Logarithmic integral defined by

$$\mathrm{Li}(x) \equiv \begin{cases} \int_0^x \dfrac{du}{\log u}, & (x < 1), \\[2ex] ⨍_0^x \dfrac{du}{\log u}, & (x > 1). \end{cases}$$

Proof

By elimination of Q from (14) and normalisation, we find that $e(t)$ satisfies

$$\ddot{e} \geq \frac{\dot{e}}{e} \equiv F(e, \dot{e}).$$

By $W \equiv \dot{e}$, we have

$$\frac{dW}{de} \leq \frac{F(e,\dot{e})}{W} = \frac{1}{e}$$

and

$$\frac{de}{dt} = W \leq \log e + c_1 = \log(ce),$$

where $c = e^{c_1}$. For the physically relevant case of $ce(t) < 1$ it follows that

$$\frac{de}{\log(ce(t))} \geq dt,$$

or

$$\frac{1}{c}\mathrm{Li}(ce(t)) \geq t + c_2.$$

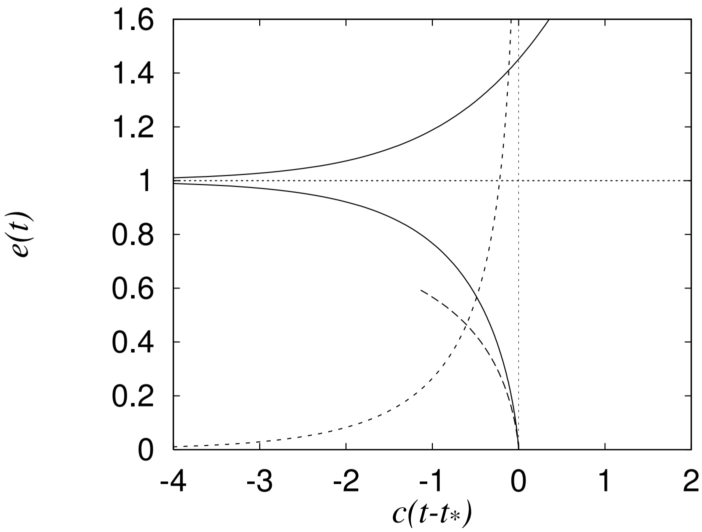

Figure 2. The Evolution of the energy (solid) for the 4D case. The self-similar asymptote (short-dashed) is inserted to the lower physical branch. The upper branch is unphysical. The enstrophy is also shown (dashed). The horizontal and vertical lines are included as a guide.

Fixing c_2 by the condition $e(t_*) = 0$, we obtain

$$\mathrm{Li}(ce(t)) \gtrsim c(t - t_*). \ \square$$

The enstrophy in the borderline case is defined by $q(t) = -\dot{e}(t)$, it is related by

$$\mathrm{Li}(\exp(-q(t))) \simeq c(t - t_*).$$

In Fig.2 we show time as a function of energy. Again, there are two branches and only one of them is physically relevant, because the other one increases energy monotonically. The realisable region is either on or below the lower branch (solid). We again observe that it asymptotes to the self-similar evolution near t_*, but that it covers longer time evolution. The corresponding enstrophy is also plotted. At the moment it is not known whether we can exclude this scenario or not.

6. Numerical experiments

In order to examine the validity of the assumption used in the asymptotic analysis we present the numerical results. We solve the three-dimensional Navier-Stokes equations under periodic boundary conditions. Nonlinear terms are evaluated by the pseudo-spectral method with 2/3-dealiasing and time-marching was done the fourth-order Runge-Kutta method. The number of grid points used is $N = 256^3$ with kinematic viscosity $\nu = 1 \times 10^{-3}$ and time increment $\Delta t = 2 \times 10^{-3}$. We use two kinds of initial conditions.

Case 1. Random initial data

The initial energy spectrum is prescribed by

$$E(k) = ck^4 \exp(-k^2),$$

where c is chosen to have unit enstrophy and the phase of Fourier components are randomised.

Case 2. the Taylor-Green vortex

$$\boldsymbol{u} = \begin{pmatrix} \cos x \sin y \sin z \\ -\sin x \cos y \sin z \\ 0 \end{pmatrix}.$$

In this case we have $\Lambda \boldsymbol{u} = \sqrt{3}\boldsymbol{u}$ and $\boldsymbol{u} \cdot \Lambda \boldsymbol{u} = \sqrt{3}|\boldsymbol{u}|^2 \geq 0$, initially.

In Fig.3 we show the time evolution of the energy, the enstrophy and the $H^{1/2}$-norm. The energy decays monotonically and the enstrophy increases in the early stage and attains a maximum around $t = 8$ and starts to decay. The $H(t)$ shows a similar growth, but it is much smaller. In Fig.4, we compare the time evolution of $H(t)$ with that of $\sqrt{E(t)Q(t)}$. The ratio $r(t)$ defined by

$$r(t) = \frac{H(t)}{\sqrt{E(t)Q(t)}}$$

reaches a minimum value around $t = 4$, but overall it remains on the order of 0.75, or larger. In Fig.5, we show the PDF of the angle θ between $\boldsymbol{u}$ and $\Lambda \boldsymbol{u}$ at several different times. The PDF is strongly skewed toward positive values even at $t = 0$ and this feature persists at later times.

We now turn our attention to Case 2. The time evolution of the energy, the enstrophy and the $H^{1/2}$-norm shown in Fig.6 is similar to Case 1. A comparison of $\sqrt{E(t)Q(t)}$ and $H(t)$ is made in Fig.7, which shows that the ratio $r(t)$ remains above 0.7. This again indicates a good performance of the Cauchy-Schwarz inequality. As long as these numerical experiments with moderate Reynolds numbers, where flows remains smooth, the ratio $r(t)$ remains of $O(1)$.

In Fig.8, we show the PDF of the angle θ. The PDF at $t = 0$ is not shown because it is a Dirac delta function at $\cos\theta = 1$. Initially $\boldsymbol{u} \cdot \Lambda \boldsymbol{u}$ is positive-definite everywhere, but it acquires negative values under short time evolution of the Navier-Stokes equations. It is again strongly skewed toward positive values throughout the computation.

7. Summary

In three dimensions, starting from the Leray bound of the form

$$\frac{E^2}{\nu^5}\frac{d \log Q}{dt} \leq C f^2 - \frac{5}{4} f, \quad \text{where} \quad f(t) \equiv \frac{E(t)Q(t)}{\nu^4},$$

we have carried out an asymptotic analysis for the time evolution of the energy by assuming $H(t) \simeq \sqrt{E(t)Q(t)}$. We note in passing that the time scale which appears on the left-hand side $\dfrac{E(0)^2}{\nu^5}$ taken initially, gives the time beyond which no singularities can form. By assuming a weak singularity in $f(t)$, we determine the evolution of the energy. In the final stage, it behaves just as the self-similarity predicts.

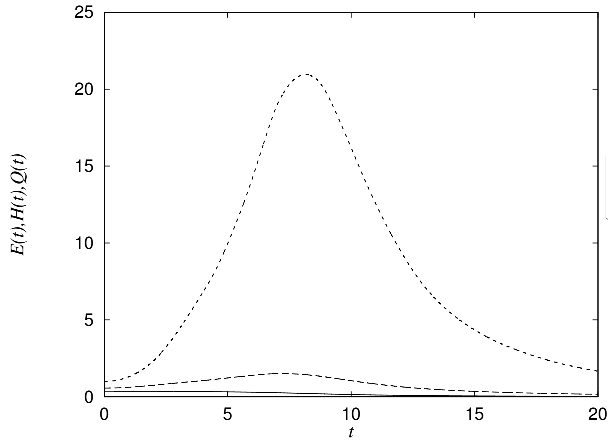

Figure 3. Time evolution of the $E(t)$ (solid), $Q(t)$ (short-dashed) and $H(t)$ (dashed) for Case 1.

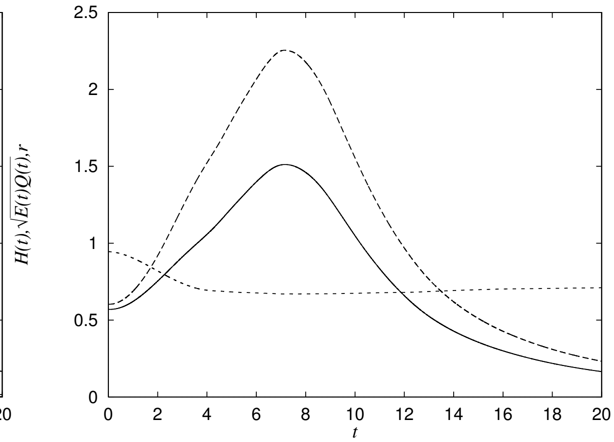

Figure 4. Time evolution of the $H(t)$ (solid), $\sqrt{E(t)Q(t)}$ (dashed) and $r(t)$ (short-dashed) for Case 1.

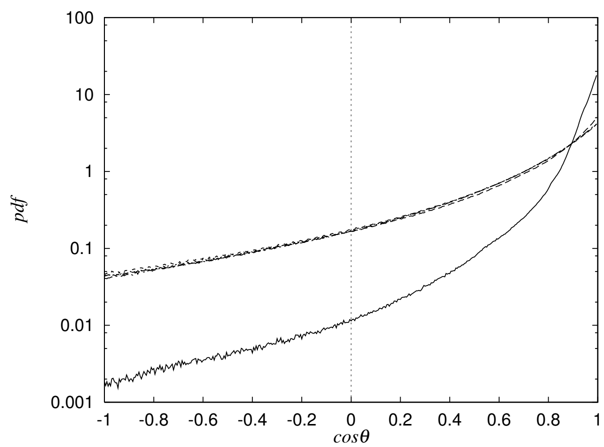

Figure 5. The PDF of $\cos\theta$ at $t = 0, 5, 10, 15, 20$ for Case 1. The one at the bottom is for $t = 0$ and the rest basically collapses.

We have given a similar analysis for the four-dimensional Navier-Stokes equations, starting from

$$\frac{E}{\nu^3}\frac{d\log Q}{dt} \le Cg(t)\left(g(t)^{1/4} - 2\right), \quad \text{where} \;\; g(t) \equiv \frac{E(t)P(t)}{\nu^4}$$

and assuming that $Q(t) \simeq \sqrt{E(t)P(t)}$. We note that the initial time scale which appears on the left-hand side $\dfrac{E(0)}{\nu^3}$ is the time beyond which no singularities can form. In five or higher dimensional spaces, the enstrophy is supercritical and this kind of asymptotic analysis would not be available.

In both cases, specific forms of time evolution of the energy is determined, which apparently generalise the self-similar ansatz. It is left for future study to see if and how we may rule out this kind of singularity formation.

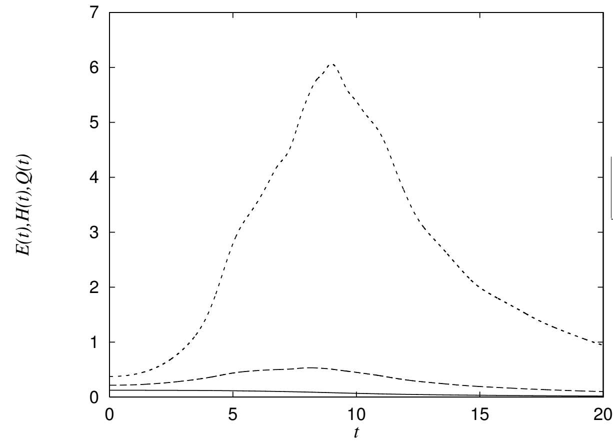

Figure 6. Time evolution of the $E(t)$ (solid), $Q(t)$ (short-dashed) and $H(t)$ (dashed) for Case 2.

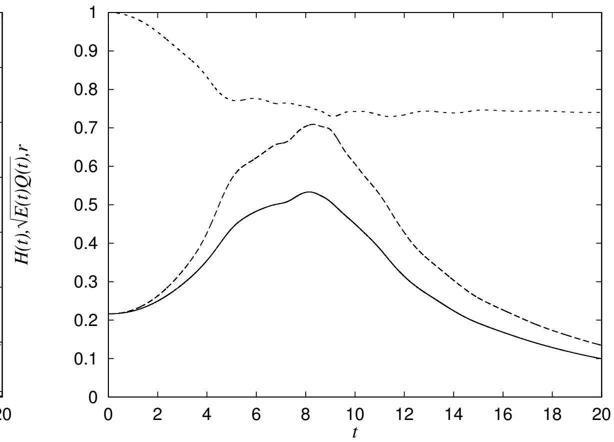

Figure 7. Time evolution of the $H(t)$ (solid), $\sqrt{E(t)Q(t)}$ (dashed) and $r(t)$ (short-dashed) for Case 2.

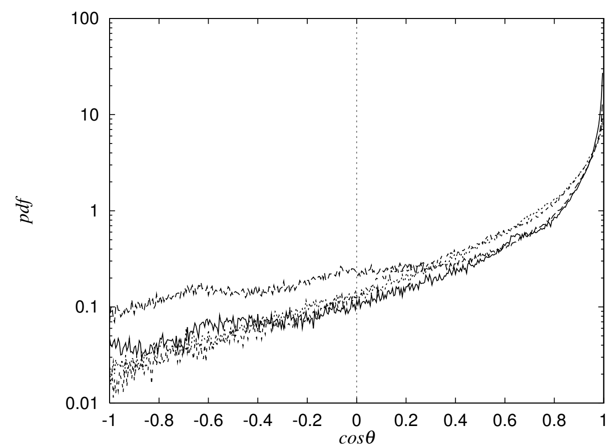

Figure 8. The PDF of $\cos\theta$ at $t = 5, 10, 15, 20$ for Case 2. The one at the top is for $t = 10$ and the rest basically collapses. The PDF for $t = 0$ is a delta function at 1, which is omitted here.

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