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# Irrelevant Vertices for the Planar Disjoint Paths Problem[a]

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## Abstract

The DISJOINT PATHS PROBLEM asks, given a graph $C$ and a set of pairs of terminals $(s_1, t_1),\ldots,(s_k, t_k)$, whether there is a collection of k pairwise vertex-disjoint paths linking $s_i$ and $t_i$, for $i = 1,\ldots,k$. In their $f(k) \cdot n^3$ algorithm for this problem, Robertson and Seymour introduced the irrelevant vertex technique according to which in every instance of treewidth greater than $g(k)$ there is an "irrelevant" vertex whose removal creates an equivalent instance of the problem. This fact is based on the celebrated Unique Linkage Theorem, whose – very technical – proof gives a function $g(k)$ that is responsible for an immense parameter dependence in the running time of the algorithm. In this paper we give a new and self-contained proof of this result that strongly exploits the combinatorial properties of planar graphs and achieves $g(k) = O(k^{3/2} \cdot 2^k)$. Our bound is radically better than the bounds known for general graphs.

Keywords: Graph Minors, Treewidth, Disjoint Paths Problem

# 1 Introduction

One of the most studied problems in graph theory is the DISJOINT PATHS PROBLEM (DPP): Given a graph G and a set P of k pairs of terminals, $(s_1,t_1),...,(s_k,t_k)$, decide whether G contains k vertex-disjoint paths $P_1,...,P_k$ where $P_i$ has endpoints $s_i$ and $t_i$, i = 1,... , k. In addition to its numerous applications in areas such as network routing and VLSI layout, this problem has been the catalyst for extensive research in algorithms and combinatorics [27]. DPP is NP-complete, along with its edge-disjoint or directed variants, even when the input graph is planar [16–18,28]. The celebrated algorithm of Roberson and Seymour solves it however in $f(k) \cdot n$ steps, where f is some computable function [22]. This implies that, when we parameterize DPP by the number k of pairs of terminals, the problem is fixed-parameter tractable. The Robertson-Seymour algorithm is the central algorithmic result of the Graph Minors series of papers, one of the deepest and most influential bodies of work in graph theory.

The basis of the algorithm in [22] is the so-called irrelevant-vertex technique which can be summarized very roughly as follows. As long as the input graph G violates certain structural conditions, it is possible to find a vertex v that is solution-irrelevant: every collection of paths certifying a solution to the problem can be rerouted to an equivalent one, that links the same pairs of terminals, but in which the new paths avoid v. One then iteratively removes such irrelevant vertices until the structural conditions are met. By that point the graph has been simplified enough so that the problem can be attacked via dynamic programming.

The following two structural conditions are used by the algorithm in [22]: (i) G excludes a clique, whose size depends on k, as a minor and (ii) G has treewidth bounded by some function of k. When it comes to enforcing Condition (ii), the aim is to prove that in graphs without big clique-minors and with treewidth at least g(k) there is always a solution-irrelevant vertex. This is the most complicated part of the proof and it was postponed until the later papers in the series [23, 24]. The bad news is that the complicated proofs also imply an immense parametric dependence, as expressed by the function f, of the running time on the parameter k. This puts the algorithm outside the realm of feasibility even for elementary values of k.

The ideas above were powerful enough to be applicable also to problems outside the context of the Graph Minors series. During the last decade, they have been applied to many other combinatorial problems and now they constitute a basic paradigm in parameterized algorithm design (see, e.g., [6, 7,9, 12, 13, 15]). However, in most applications, the need for overcoming the high parameter dependence emerging from the structural theorems of the Graph Minors series, especially those in [23, 24], remains imperative. Hence two natural directions of research are: simplify parts of the original proof for the general case or focus on specific graph classes that may admit proofs with better parameter dependence. An important step in the first direction was taken recently by Kawarabayashi and Wollan in [14] who gave an easier and shorter proof of the results

in [23,24]. While the parameter dependence of the new proof is certainly much better than the previous, immense, function, it is still huge: a rough estimation from [14] gives a lower bound for g(k) of magnitude $2^{2^{2^{\Omega(k)}}}$ which in turn implies a lower bound for f(k) of magnitude $2^{2^{2^{2^{\Omega(k)}}}}$ .

In this paper we offer a solid advance in the second direction, focusing on planar graphs (see also [20,26] for previous results on planar graphs). We show that, for planar graphs, g(k) is single exponential. In particular we prove the following result.

Theorem 1. Every instance of DPP consisting of a planar graph C with treewidth at least $82 \cdot k^{3/2} \cdot 2^k$ and k pairs of terminals contains a vertex v such that every solution to DPP can be replaced by an equivalent one whose paths avoid v.

The proof of Theorem 1 is presented in Section 3 and deviates significantly from those in [14, 23, 24]. It is self-contained and exploits extensively the combinatorics of planar graphs. Given a DPP instance defined on a planar graph C, we prove that if C contains as a subgraph a subdivision of a sufficiently large (exponential in k) grid, whose "perimeter" does not enclose any terminal, then the "central" vertex v of the grid is solution-irrelevant for this instance. It follows that the "area" provided by the grid is big enough so that every solution that uses v can be rerouted to an equivalent one that does not go so deep in the grid and therefore avoids the vertex v.

Combining Theorem 1 with known algorithmic results, it is possible to reduce, in $2^{2^{O(k)}} \cdot n^2$ steps, a planar instance of DPP to an equivalent one whose graph has treewidth $2^{2^{O(k)}}$. Then, using standard dynamic programming on tree decompositions, a solution, if one exists, can be found in $2^{2^{O(k)}} \cdot n$ steps. The parametric dependence of this algorithm is a step forward in the study of the parameterized complexity of DPP on planar graphs. This algorithm is abstracted in the following theorem, whose proof is in Section 4.

Theorem 2. There exists an algorithm that, given an instance (C, 2) of DPP, where C is a planar n-vertex graph and |2| = k, either reports that (C, 2) is a NO-instance or outputs a solution of DPP for (C, 2). This algorithm runs in $2^{2^{O(k)}} \cdot n^2$ steps.

An extended abstract of this work, without any proofs, appeared in [2]. Some of our ideas have proved useful in the recent breakthrough result of Cygan et al. that establishes fixed-parameter tractability for k-disjoint paths on planar directed graphs [5].

## 2 Basic definitions

Throughout this paper, given a collection of sets C we denote by $\bigcup C$ the set $\bigcup_{x \in C} x$, i.e., the union of all sets in C.

All graphs that we consider are finite, undirected, and simple. We denote the vertex set of a graph C by V(C) and the edge set by E(C). Every edge is a two-element subset of V(C). A graph H is a subgraph of a graph C, denoted by H ⊂ C, if V(H) ⊂ V(C) and

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$E(H) \subseteq E(G)$. Given two graphs G and H, we define $G \cap H = (V(G) \cap V(H), E(G) \cap V(H))$ and $G \cup H = (V(G) \cup V(H), E(G) \cup V(H))$. Given a $S \subseteq V(G)$, we also denote by G[S] the subgraph of G induced by S.

A path in a graph G is a connected acyclic subgraph with at least one vertex whose vertices have degree at most 2. The length of a path P is equal to the number of its edges. The endpoints of a path P are its vertices of degree 1 (in the trivial case where there is only one endpoint x, we say that the endpoints of P are x and x). An (x, y)-path of G is any path of G whose endpoints are x and y.

A cycle of a graph G is a connected subgraph of G whose vertices have degree 2. For graphs G and H the cartesian product is the graph whose vertex set is $V(G) \times V(H)$ and whose edge set is $\{\{(v, v^0), (w, w^0)\} \mid (\{v, w\} \in E(G) \wedge v^0 = w^0) \vee (v = w \wedge \{v^0, w^0\} \in E(H))\}$.

The DISJOINT PATHS problem. The problem that we examine in this paper is the following.

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DISJOINT PATHS (DPP)

Input: A graph G, and a collection $P = \{(s_i, t_i) \in V(G)^2, i \in \{1, \ldots, k\}\}$ of pairs of 2k terminals of G.

Question: Are there k pairwise vertex-disjoint paths $P_1, \ldots, P_k$ in G such that for $i \in \{1, \ldots, k\}$, $P_i$ has endpoints $s_i$ and $t_i$?

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We call the k-pairwise vertex-disjoint paths certifying a YES-instance of DPP a solution of DPP for the input (G,P). Given an instance (G,P) of DPP, we say that a non-terminal vertex $v \in V(G)$ is irrelevant for (G, P), if (G, P) is a YES-instance if and only if $(G \setminus v, P)$ is a YES-instance. We denote by PDPP the restriction of DPP on instances (G,P) where G is a planar graph.

Minors. A graph H is a minor of a graph G, if there is a function $\varphi : V(H) \to 2^{V(G)}$, such that

i. For every two distinct vertices x and y of H, $G[\varphi(x)]$ and $G[\varphi(y)]$ are two vertex-disjoint connected subgraphs of G and

ii. for every two adjacent vertex x and y of H, $G[\varphi(x) \cup \varphi(y)]$ is a connected subgraph of G.

We call the function $\varphi$ minor model of H in G.

Grids. Let m, n > 1. The (m x n)-grid is the Cartesian product of a path of length m — 1 and a path of length n — 1. In the case of a square grid where m = n, we say that n is the size of the grid. Given that n, m > 2, the corners of an (m x n)-grid are

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its vertices of degree 2. When we refer to a (m x n)-grid we will always assume an orthogonal orientation of it that classifies its corners to the upper left, upper right, down right, and down left corner of it.

Given that Γ is an (m x n)-grid, we say that a vertex of C is one of its centers if its distance from the set of its corners is the maximum possible. Observe that a square grid of even size has exactly 4 centers. We also consider an (m x n)-grid embedded in the plane so that, if it has more than 2 faces then the infinite one is incident to more than 4 vertices. The outer cycle of an embedding of an (m x n)-grid is the one that is the boundary of its infinite face. We also refer to the horizontal and the vertical lines of an (m x n)-grid as its paths between vertices of degree smaller than 4 that are traversing it either "horizontally" or "vertically" respectively. We make the convention that an (m x n)-grid contains m vertical lines and n horizontal lines. The lower horizontal line and the higher horizontal line of Γ are defined in the obvious way (see Figure 1 for an example).
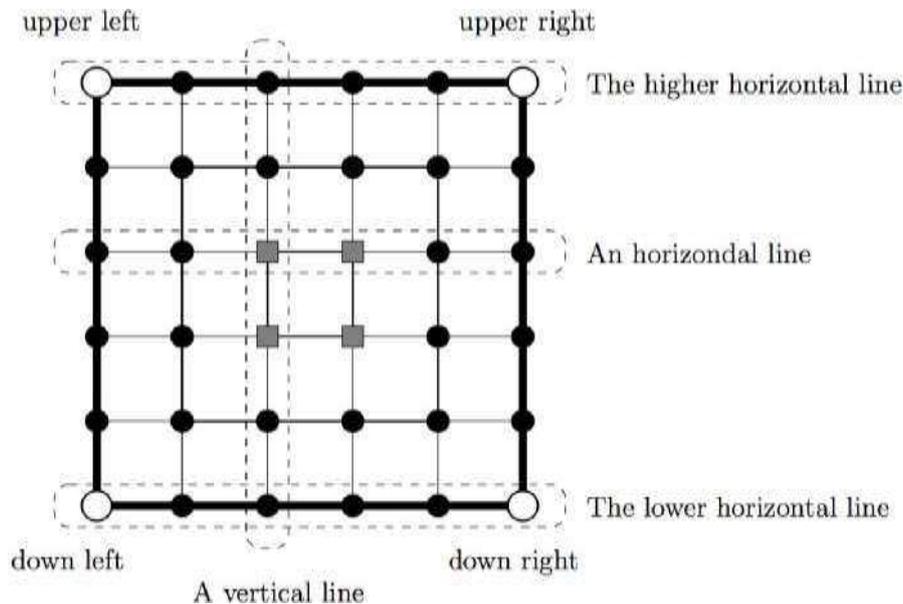


Figure 1: A drawing of the (6 x 6)-grid. The four white round vertices are its corners and the four grey square vertices are its centers. The cycle formed by the "fat" edges is the outer cycle.


Plane graphs Whenever we refer to a planar graph C we consider an embedding of C in the plane $E = R^2$. To simplify notation, we do not distinguish between a vertex of C and the point of E used in the drawing to represent the vertex or between an edge

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and the arc representing it. We also consider a plane graph G as the union of the points corresponding to its vertices and edges. That way, edges and faces are considered to be open sets of E. Moreover, a subgraph H of G can be seen as a graph H, where the points corresponding to H are a subset of the points corresponding to G.

Recall that $\Delta \subseteq E$ is an open (resp. closed) disc if it is homeomorphic to $\{(x, y) : x^2 + y^2 < 1\}$ (resp. $\{(x, y) : x^2 + y^2 < 1\}$). Given a cycle C of G we define its open-interior (resp. open-exterior) as the connected component of $E \setminus C$ that is disjoint from (resp. contains) the infinite face of G. The closed-interior (resp. closed-exterior) of C is the closure of its open-interior (resp. open-exterior). Given a set $A \subseteq E$, we denote its interior (resp. closure) by $\mathrm{int}(A)$ (resp. $\mathrm{clos}(A)$). An open (resp. closed) arc I in $R^2$ is any set homeomorphic to the set $\{(x, 0) \mid x \in (0, 1)\}$ (resp. $\{(x, 0) \mid x \in [0,1]\}$) and the endpoints of I are defined in the obvious way. We also define $\mathrm{trim}(I)$ as the set of all points of the arc I except for its endpoints.

Outerplanar graphs. An outerplanar graph is a plane graph whose vertices are all incident to the infinite face. If an edge of an outerplanar graph is incident to its infinite face then we call it external, otherwise we call it internal. The weak dual of an outerplanar graph G is the graph obtained from the dual of G after removing the vertex corresponding to the infinite face of the embedding. Notice that if the outerplanar graph G is biconnected, then its weak dual is a tree. We call a face of an outerplanar graph simplicial if it corresponds to a leaf of the graph's weak dual.

Treewidth. A tree decomposition of a graph G is a pair $(T, \chi)$, consisting of a rooted tree T and a mapping $\chi: V(T) \longrightarrow 2^{V(G)}$, such that for each $v \in V(G)$ there exists $t \in V(T)$ with $v \in \chi(t)$, for each edge $e \in E(G)$ there exists a node $t \in V(T)$ with $e \subseteq \chi(t)$, and for each $v \in V(G)$ the set $\{t \in V(T) \mid v \in \chi(t)\}$ is connected in T.

The width of $(T, \chi)$ is defined as $w(T, \chi) := \max \{ |\chi(t)| - 1 \mid t \in V(T)\}$. The tree-width of G is defined as

$$tw(G) := \min \{ w(T, \chi) \mid (T, \chi) \text{ is a tree decomposition of G}\}.$$

We need the next proposition that follows directly by combining the main result of [10] and (5.1) from [21].

Proposition 1. If G is a planar graph and $tw(G) > 4.5 \bullet k + 1$, then G contains a $(k \times k)$-grid as a mimor.

Our algorithmic results require the following proposition. It follows from the main result of [19] (see also Algorithm (3.3) in [22]). The parametric dependence of k in the running time follows because the algorithm in [19] uses as a subroutine the algorithm in [4] that runs in $2^{k^{O(1)}} \bullet n$ steps.

**Proposition 2.** There exists an algorithm that, given an n-vertex graph C and a positive integer k, either outputs a tree decomposition of C of width at most k or outputs a subgraph C' of C with treewidth greater than k and a tree decomposition of C' of width at most 2k, in $2^{kO(1)} \cdot n$ steps.

# 3 Irrelevant vertices in graphs of large treewidth

In this section we prove our main result, namely Theorem 1. We introduce the notion of cheap linkages and explore their structural properties in Subsections 3.1 and 3.4. In Subsection 3.7 we bring together the structural results to show the existence of an irrelevant vertex in a graph of large treewidth.

## 3.1 Configurations and cheap linkages

In this subsection we introduce some basic definitions on planar graphs that are necessary for our proof.

**Tight concentric cycles.** Let C be a plane graph and let D be a disk that is the closed interior of some cycle C of C. We say that D is internally chordless if there is no path in C whose endpoints are vertices of C and whose edges belong to the open interior of C.

Let $C = \{C_0, \ldots, C_r\}$, be a sequence of cycles in C. We denote by $D_i$ the closed-interior of $C_i$, $i \in \{0,\ldots, r\}$, and we say that $V = \{D_0,\ldots, D_r\}$ is the disc sequence of C. We call C concentric, if for all $i \in \{0,\ldots, r-1\}$, the cycle $C_i$ is contained in the open-interior of $D_{i+1}$. The sequence C of concentric cycles is tight in C, if, in addition,

- $D_0$ is internally chordless.

- For every $i \in \{0,\ldots, r-1\}$, there is no cycle of C that is contained in $D_{i+1} \setminus D_i$ and whose closed-interior D has the property $D_i \subsetneq D \subsetneq D_{i+1}$.

**Lemma 1.** There exists an algorithm that given a positive integer r, an n-vertex plane graph C, and a $T \subseteq V(C)$, either outputs a tree decomposition of C of width at most $9 \cdot (r+1) \cdot \lceil \sqrt{|T|} + 11 \rceil$) or an internally chordless cycle C of C such that there exists a tight sequence of cycles $C_0,\ldots, C_r$ in C where

- $C_0 = C$ and

- all vertices of T are in the open exterior of $C_r$.

Moreover, this algorithm runs in $2^{(r \cdot \sqrt{|T|})O(1)} \cdot n$ steps.
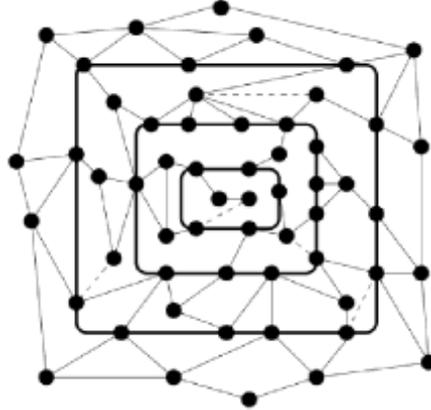
Figure 2: An example of a plane graph C and a tight sequence of 3 concentric cycles in it. Notice that the addition to C of any of the dashed edges makes this collection of cycles non-tight.

Proof. Let $x = |T|+1$ and $y = 2(r+1)\cdot d\sqrt{x}e$. From Proposition 1, if $tw(C) \geq 4.5 \cdot y+1$, then C contains as a minor a $(y \times y)$-grid $\Gamma$. We now observe that the grid $\Gamma$ contains as subgraphs $x$ pairwise disjoint $(2(r + 1) \times 2(r + 1))$-grids $\Gamma_1,... , \Gamma_x$. Note that each $\Gamma$, $i \in \{1, . . . , x\}$ contains a sequence of $r+1$ concentric cycles that, given a minor model $\varphi$ of $\Gamma$ in C, can be used to construct, in linear time, a sequence of $r +1$ concentric cycles $C = \{C_0, C_1, ... , C_r\}$ in C such that for every $i,j \in \{1,.. .,x\}$, where $i =6 j$, all cycles in $C_j$ are in the open exterior of $C_r$.

Note that at least one, say $C_r$, of the cycles in $\{C_{1r},.. ., C_{xr}\}$ should contain all the vertices of T in its open exterior. Let $e$ be any edge of $C_0$. Let also $f$ be the face of C that is contained in the open interior of $C_0$ and is incident to $e$. Let $J_f$ be the graph consisting of the vertices and the edges that are incident to $f$. It is easy to verify that, $J_f$ contains an internally chordless cycle C that contains the edge $e$. Given $C_0$, the cycle C can be found in linear time. Notice now that C contains a tight sequence of cycles $C_0, C_1, . . . ,C_r$ such that $C_0 = C$ and where, for $h \in \{0, . . . , r\}$, $C_h$ is in the closed interior of $C_h$. The result follows as the open exterior of $C_r$ contains the open exterior of $C_r$ and therefore contains all vertices in T.

The algorithm runs as follows: it first uses the algorithm of Proposition 2 for $k = 4.5 \cdot y$. If the algorithm outputs a tree decomposition of C of width at most $k$, then we are done. Otherwise it outputs a subgraph $C^0$ of C where $tw(C^0) > k$ and a tree decomposition of $C^0$ of width $\leq 2k$. We use this tree decomposition in order to find a minor model $\varphi$ of the $(y \times y)$-grid $\Gamma$ in $C^0$. This can be done in $2k_{O(1)} = 2(r\cdot\sqrt{|T|})_{O(1)} \cdot n$ steps using the algorithm in [1] (or, alternatively, the algorithm in [11]). Clearly, $\varphi$ is also

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a minor model of Γ in C. We may now use φ, as explained above, in order to identify, in linear time, the required internally chordless cycle C in C. □

Linkages. A linkage in a graph C is a non-empty subgraph L of C whose connected components are all paths. The paths of a linkage are its connected components and we denote them by 2(L). The terminals of a linkage L are the endpoints of the paths in 2(L), and the pattern of L is the set {{s,t} | 2(L) contains a path from s to t in C}. Two linkages are equivalent if they have the same pattern.

Segments. Let C be a plane graph and let C be a cycle in C whose closed-interior is D. Given a path P in C we say that a subpath $P_0$ of P is a D-segment of P, if $P_0$ is a non-empty (possibly edgeless) path obtained by intersecting P with D. For a linkage L of C we say that a path $P_0$ is a D-segment of L, if $P_0$ is a D-segment of some path P in 2(L).
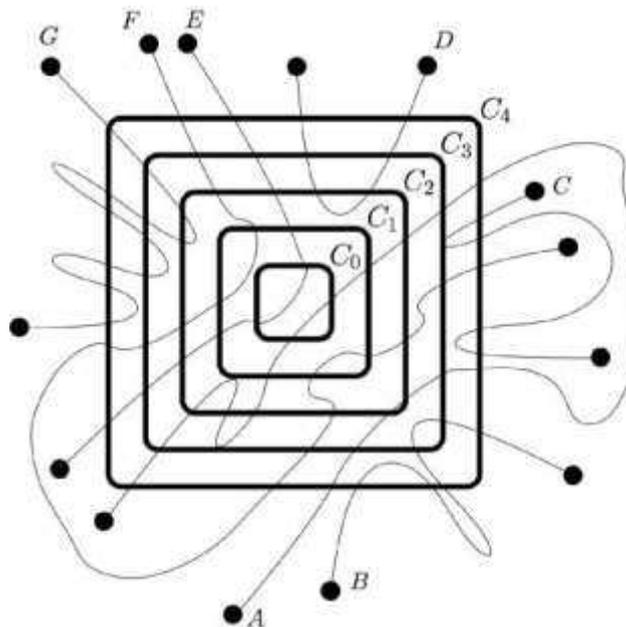


Figure 3: An example of a CL-configuration Q = (C, L) where C contains 5 cycles and L has 7 paths. Q has 13 segments. Linkage paths A, B, C, D, E, F, and C, contain 2, 2, 2, 1, 1, 2, 3 of these segments respectively. Also the eccentricities of the segments of A, are 0 and 2, of B are 3 and 4. Notice that one of the two segments of A has two 3-chords, each having 2 semi 3-chords.

CL-configurations. Given a plane graph $C$, we say that a pair $Q = (C, L)$ is a CL-configuration of $C$ of depth $r$ if $C = \{C_0, \ldots, C_r\}$ is a sequence of concentric cycles in $C$, $L$ is a linkage of $C$, and $D_r$ does not contain any terminals of $L$. A segment of $Q$ is any $D_r$-segment of $L$. The eccentricity of a segment $P$ of $Q$ is the minimum $i$ such that $V(C_i \cap P) \neq 0$. A segment of $Q$ is extremal if it is has eccentricity $r$. Observe that if $C$ is tight then any extremal segment is a subpath of $C_r$. Given a cycle $C_i \in C$ and a segment $P$ of $Q$ we define the $i$-chords of $P$ as the connected components of $P \cap \text{int}(D_i)$ (notice that $i$-chords are open arcs). For every $i$-chord $X$ of $P$, we define the $i$-semichords of
$P$ as the connected components of the set $X \setminus D_{i-1}$ (notice that $i$-semichords are open arcs). Given a segment $P$ that does not have any $0$-chord, we define its zone as the connected component of $D_r \setminus P$ that does not contain the open-interior of $D_0$ (a zone is an open set).
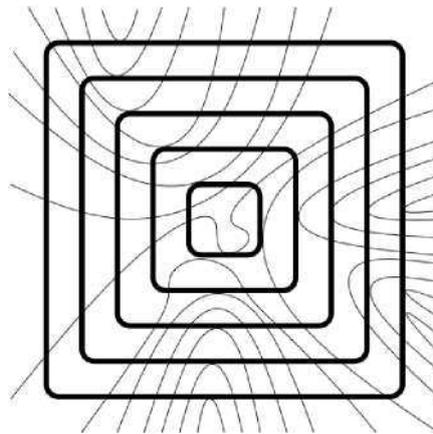


Figure 4: An example of a CL-configuration $(C, L)$ where the linkage $L$ is $C$-cheap. Only the 5 concentric cycles of $C$ and a cropped part of the linkage $L$ are depicted. Notice that the collection of concentric cycles $C$ is not tight.

A CL-configuration $Q = (C, L)$ is called reduced if the graph $L \cap \cup C$ is edgeless. Let $Q = (C, L)$ be a CL-configuration of $C$ and let $E$ be the set of all edges of the graph $L \cap \cup C$. We then define $C^*$ as the graph obtained if we contract in $C$ all edges in $E^*$. We also define $Q^*$ as the pair $(C^*, L^*)$ obtained if in $L$ and in the cycles of $C$ we contract all edges of $E^*$. Notice that $Q^*$ is a reduced CL-configuration of $C^*$. We call $(Q^*, C^*)$ the reduced pair of $C$ and $Q$.

Cheap linkages. Let G be a plane graph and Q = (C, L) be a CL-configuration of G of depth r. We define the function $c : \{L \mid L \text{ is a linkage of } G\} \rightarrow \mathbb{N}$ so that

$$c(L) = \left| E(L) \setminus \bigcup_{i \in \{0,\ldots,r\}} E(C_i) \right|.$$

A linkage L of G is C-cheap, if there is no other CL-configuration $Q' = (C, L')$ such that $L'$ has the same pattern as L and $c(L) > c(L')$. Intuitively, the function c defined above penalizes every edge of the linkage that does not lie on some cycle $C_i$.

Observation 1. Let Q = (C, L) be a CL-configuration and let $(G^*, Q^* = (C^*, L^*))$ be the reduced pair of G and Q. Then

- If L is C-cheap, then $L^*$ is $C^*$-cheap.

- If C is tight in G, then $C^*$ is tight in $G^*$.
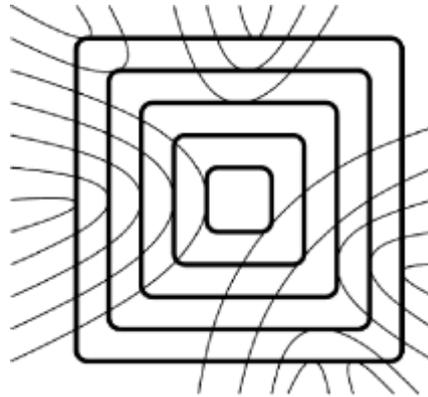


Figure 5: An example of a convex CL-configuration (C, L). In the picture, only the 5 cycles in C and a cropped portion of L is depicted.

## 3.2 Convex configurations

We introduce CL-configurations with particular characteristics that will be useful for the subsequent proofs. We then show that these characteristics are implied by tightness and cheapness.

Convex CL-configurations. A segment P of Q is convex if the following three conditions are satisfied:

  (i) it has no 0-chord and

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(ii) for every $i \in \{1, \ldots, r\}$, the following hold:

    a. $P$ has at most one $i$-chord

    b. if $P$ has an $i$-chord, then $P \cap G_{i-1} \neq 0$.

    c. Each $i$-chord of $P$ has exactly two $i$-semichords.

(iii) If $P$ has eccentricity $i < r$, there is another segment inside the zone of $P$ with eccentricity $i + 1$.

We say $Q$ is convex if all its segments are convex.

**Observation 2.** Let $Q = (C, L)$ be a CL-configuration and let $(G_*, Q_* = (C_*, L_*))$ be the reduced pair of $G$ and $Q$. Then $Q$ is convex if and only $Q_*$ is convex.
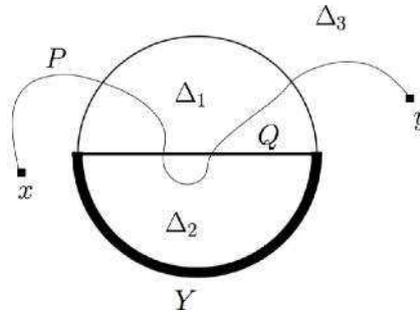


Figure 6: A visualization of the conditions of Lemma 2.

The proof of the following lemma uses elementary topological arguments.

**Lemma 2.** Let $A_1, A_2$ be closed disks of $R^2$ where $\text{int}(A_1) \cap \text{int}(A_2) = 0$ and such that $A_1 \cup A_2$ is also a closed disk. Let $A_3 = R^2 \setminus \text{int}(A_1 \cup A_2)$ and let $Y = \text{bnd}(A_3) \cap A_2$ and $Q = \text{trim}(A_1 \cap A_2)$. Let $P$ be a closed arc of $R^2$ whose endpoints are not in $A_1 \cup A_2$ and such that $Y \cap P = 0$ and $Q \cap P \neq 0$. Then $\text{int}(A_1) \cap P$ has at least two connected components.

Proof. Let $q$ be some point in $Q \cap P$. Let $Q^0$ be an open arc that is a subset of $\text{int}(A_1)$ and has the same endpoints as $Y$. Notice that $q$ and $x$ belong to different open disks defined by the cycle $Q^0 \cup Y$. Therefore $P$ should intersect $Q^0$ or $Y$. As $Y \cap P = 0$, $P$ intersects $Q^0$. As $Q^0 \subset \text{int}(A_1)$, $\text{int}(A_1) \cap P$ has at least one connected component.

Assume now that $\text{int}(A_1) \cap P$ has exactly one connected component. Clearly, this connected component will be an open arc $I$ such that at least one of the endpoints of $I$, say $q$, belongs to $Q$. Moreover, there is a subset $P^0$ of $P$ that is a closed arc where $P^0 \cap I = 0$ and whose endpoints are $q$ and one of $x$ and $y$, say $y$. As $\text{int}(A_1) \cap P$ has

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exactly one connected component, it holds that $P' \cap \text{int}(L_1) = 0$. Let $Q'$ be an open arc that is a subset of $\text{int}(L_1)$ and has the same endpoints as $Y$. Notice that $q$ and $y$ belong to different open disks defined by the cycle $Q' \cup Y$. Therefore $P'$ should intersect $\text{int}(L_1)$ or $Y$, a contradiction as $P' \subset P$ and $Y \cap P = 0$. $\qquad\square$

**Lemma 3.** Let $G$ be a plane graph and $Q = (C, L)$ be a CL-configuration of $G$ where $C$ is tight in $G$ and $L$ is $C$-cheap. Then $Q$ is convex.

Proof. By Observations 1 and 2, we may assume that $Q$ is reduced. Consider any segment of $Q$. We show that it satisfies the three conditions of convexity. Conditions (i) and (ii).b follow directly from the tightness of $C$. Condition (iii) follows from the fact that $L$ is $C$-cheap. In the rest of the proof we show Conditions (ii).a. and (ii).c. For this, we consider the minimum $i \in \{0, \ldots, r\}$ such that one of these two conditions is violated. From Condition (i), $i \geq 1$. Let $W$ be a segment of $Q$ containing an $i$-chord $X$ for which one of Conditions (ii).a, (ii).c is violated.

We now define the set $Q$ according to which of the two conditions is violated. We distinguish two cases:

Case 1. Condition (ii).c is violated. From Condition (ii).b, $X_{\backslash D_{i-1}}$ contains more than two $i$-semichords of $X$. Let $J_1$ be the biconnected outerplanar graph defined by the union of $C_{i-1}$ and the $i$-semichords of $X$ that do not have an endpoint on $C_i$. As there are at least three $i$-semichords in $X$, $J_1$ has at least one internal edge and therefore at least two simplicial faces. Moreover there are exactly two $i$-semichords of $X$, say $K_1$, $K_2$, that have an endpoint in $C_i$ and $K_1$ and $K_2$ belong to the same, say $F'$, face of $J_1$. Let $L_2$ be the closure of a simplicial face of $J_1$ that is not $F'$.

Case 2. Condition (ii).c holds while Condition (ii).a is violated. Let $J_2$ be the biconnected outerplanar graph defined by the union of $C_{i-1}$ and the connected components of $W \setminus D_{i-1}$ that do not contain endpoints of $W$ in their boundary. Notice that the rest of the connected components of $W \setminus D_{i-1}$ are exactly two, say $K_1$ and $K_2$. Notice that $K_1$ and $K_2$ are subsets of the same face, say $F'$, of $J_2$. As there are at least two $i$-chords in $W$, $J_2$ contains at least one internal edge and therefore at least two simplicial faces. Let $L_2$ be the closure of a simplicial face of $J_2$ that is not $F'$.

In both of the above cases, we set $L_1 = D_{i-1}$, $L_3 = R^2 \setminus \text{int}(L_1 \cup L_2)$, $Y = \text{bnd}(L_3) \cap L_2$, and $Q = \text{trim}(L_1 \cap L_2)$. Notice that $Y = \text{bnd}(L_2) \setminus Q$, therefore $Y \subset W$.

We claim that $L \cap Q = 0$. Suppose not. We consider $W'$ as the path in $W \cup Q$ that contains $Q$ as a subset and has the same endpoints as $W$. Then, $L' = (L \setminus W) \cup W'$ is a linkage, equivalent to $L$, where $c(L') < c(L)$, a contradiction to the fact that $L$ is $C$-cheap. We just proved that $L \cap Q = 0$ which in turn implies that $L$ contains a segment $P$ for which $P \cap Q = 0$. We distinguish two cases:

Case A. $W \neq P$. This implies that $W \cap P = 0$. As $Y \subset W$, it follows that $Y \cap P = 0$. Therefore, by Lemma 2, $\text{int}(L_1) \cap P$ has at least two connected components, therefore $P$
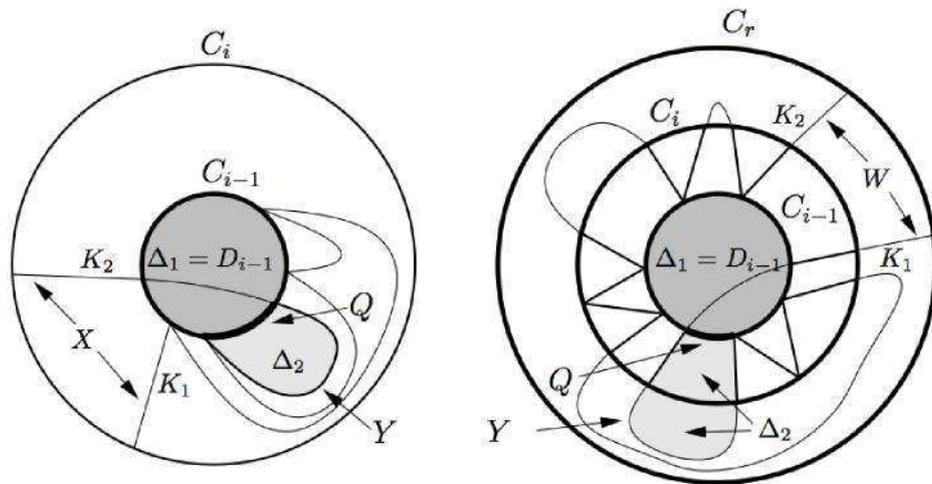
Figure 7: The two cases of the proof of Lemma 3. In the left part is depicted an $i$-chord $X$ that has 6 $i$-semichords and in the right part is depicted a segment $W$ and the way it crosses the cycles $D_i$ and $_{D_{i-1}}$. In the figure on the right the segment $W$ has 7 $i$-chords and 14 $i$-semichords.

has at least two $(i-1)$-chords. If $i > 1$, then Condition (ii).a is violated for $i-1$, which contradicts the choice of $i$. If $i = 1$, then $P$ has at least one 0-chord, which violates Condition (i), that, as explained at the beginning of the proof, holds for every segment of $Q$.

Case B. $W = P$. Recall that $Y \subset W$ therefore $Y \subset P$. Let $p_1$ and $p_2$ be the endpoints of $Q$. As $Q$ is reduced there exists two disjoint closed arcs $Z_1$ and $Z_2$ with endpoints $p_1$, $p^0_1$ and $p_2$, $p^0_2$ respectively, such that

- $p_i$ is an endpoint of $Z_i$, $i \in \{1, 2\}$.

- $Z_i \subset \text{clos}(Q)$, $i \in \{1,2\}$, and

- $P \cap Z_i = \{p_i\}$, $i \in \{1,2\}$.

Consider also a closed arc $Y^0$ that is a subset of $\text{int}(\Delta_2) \cup \{p^0_1, p^0_2\}$ that does not intersect $L$ and whose endpoints are $p^0_1$ and $p^0_2$. Let now $\Delta^0_1 = \Delta_1$, let $\Delta^0_2$ be the closed disk defined by the cycle $\text{clos}(Q \setminus (Z_1 \cup Z_2)) \cup Y^0$ that is a subset of $\Delta_2$. Let also $\Delta_{0\,3} = R^2 \setminus \text{int}(\Delta^0_1 \cup \Delta^0_2)$ and $Q^0 = \text{trim}(\Delta^0_1 \cap \Delta^0_2)$. As $Y^0$ does not intersect $L$, we obtain $Y^0 \cap P = 0$. Observe that $Z_1$, $Q^0$, $Z_2$ form a partition of $Q$. As $Q \cap P =6\,0$ and $(Z_i \setminus \{p_i\}) \cap P = 0, i \in \{1,2\}$, we conclude that $Q^0 \cap P =6\,0$.

By applying Lemma 2, $\text{int}(\Delta^0_1) \cap P$ has at least two connected components. Therefore $P$ has at least two $(i-1)$-chords. This yields a contradiction, as in Case A. $\qquad\square$

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## 3.3 Bounding the number of extremal segments

In this subsection we prove that the number of extremal segments is bounded by a linear function of the number of linkage paths.

**Out-segments, hairs, and flying hairs.** Let $G$ be a plane graph and $Q = (C, L)$ be a CL-configuration of $G$ of depth $r$. An out-segment of $L$ is a subpath $P'$ of a path in $P(L)$ such that the endpoints of $P'$ are in $C_r$ and the internal vertices of $P'$ are not in $D_r$. A hair of $L$ is a subpath $P'$ of a path in $P(L)$ such that one endpoint of $P'$ is in $C_r$, the other is a terminal of $L$, and the internal vertices of $P'$ are not in $D_r$. A flying hair of $L$ is a path in $P(L)$ that does not intersect $C_r$.

Given a linkage $L$ of $G$ and a closed disk $D$ of $\mathbb{R}^2$ whose boundary is a cycle of $G$, we define $\mathrm{out}_D(L)$ to be the graph obtained from the graph $(L \cup \mathrm{bnd}(D)) \setminus \mathrm{int}(D)$ after dissolving all vertices of degree 2. For example $\mathrm{out}_{D_r}(L)$ is a plane graph consisting of the out-segments, the hairs, the flying hairs of $L$, and what results from $C_r$ after dissolving its vertices of degree 2 that do not belong in $L$. Let $f$ be a face of $\mathrm{out}_{D_r}(L)$ that is different from $\mathrm{int}(D_r)$. We say that $f$ is a cave of $\mathrm{out}_{D_r}(L)$ if the union of the out-segments and extremal segments in the boundary of $f$ is a connected set. Recall that a segment of $Q$ is extremal if it is has eccentricity $r$, i.e., it is a subpath of $C_r$.

Given a plane graph $G$, we say that two edges $e_1$ and $e_1$ are cyclically adjacent if they have a common endpoint $x$ and appear consecutively in the cyclic ordering of the edges incident to $x$, as defined by the embedding of $G$. A subset $E$ of $E(G)$ is cyclically connected if for every two edges $e$ and $e'$ in $E$ there exists a sequence of edges $e_1, \ldots, e_r \in E$ where $e_1 = e$, $e_r = e'$ and for each $i \in \{1, \ldots, r - 1\}$ $e_i$ and $e_{i \pm 1}$ are cyclically adjacent.

Let $Q = (C, L)$ be a CL-configuration. We say that $Q$ is touch-free if for every path $P$ of $L$, the number of the connected components of $P \cap C_r$ is not 1.

**Lemma 4.** Let $G$ be a plane graph and $Q = (C, L)$ be a touch-free CL-configuration of $G$ where $C$ is tight in $G$ and $L$ is $C$-cheap. The number of extremal segments of $Q$ is at most $2 \cdot |P(L)| - 2$.

Proof. Let $(G_*, Q^* = (C^*, L^*))$ be the reduced pair of $G$ and $Q$. Notice that, by Observation 1, $C^*$ is tight in $G$ and $L^*$ is $C^*$-cheap. Moreover, it is easy to see that $Q^*$ is touch-free and $Q$ and $Q^*$ have the same number of extremal segments which are all trivial paths (i.e., paths consisting of only one vertex). Therefore, it is sufficient to prove that the lemma holds for $Q^*$. Let $\rho$ be the number of extremal segments of $Q^*$.

Let $J = \mathrm{out}_{D_*}(L^*)$ and $k = |P(L^*)|$. Notice that the number of extremal segments of $Q^*$ is equal to the number of vertices of degree 4 in $J$.

The terminals of $L^*$ are partitioned in three families

- flying terminals, $T_0$: endpoints of flying hairs.

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- invading terminals $T_1$: these are endpoints of hairs whose non terminal endpoint has degree 3 in J.

- bouncing terminals $T_2$: these are endpoints of hairs whose non terminal endpoint has degree 4 in J.

A hair containing an invading and bouncing terminal is called invading and bouncing hair respectively.

Recall that $|T_0| + |T_1| + |T_2| = 2k$.

Claim 1. The number of caves of J is at most the number of invading terminals.

Proof of claim 1. Clearly, a hair cannot be in the common boundary of two caves. Therefore it is enough to prove that the set obtained by the union of a cave f and its boundary contains at least one invading hair. Suppose this is not true. Consider the open arc R obtained if we remove from bnd(f) all the points that belong to out-segments. Clearly, R results from a subpath $R^+$ of $C_r^*$ after removing its endpoints, i.e., $R = \text{trim}(R^+)$.

Notice that because f is a cave, R is a non-empty connected subset of $C_r^*$. Moreover, $R \cap L^*$ is non-empty, otherwise $L^{*'} = (L^* \setminus (\text{bnd}(f)) \cup R$ is also a linkage with the same pattern as $L^*$ where $c(L^{*'}) < c(L^*)$, a contradiction to the fact that $L^*$ is $C$-cheap. Let Y be a connected component of $R \cap L^*$. As $Q^*$ is reduced, Y consists of a single vertex y in the open set R. Notice that Y is a subpath of a segment $Y'$ of $Q^*$. We claim that $Y'$ is not extremal. Suppose to the contrary that $Y'$ is extremal. Then $Y' = Y$ and there should be two distinct out-segments that have y as a common endpoint. This contradicts the fact that $y \in R$.

By Lemma 3, $Q^*$ is convex, therefore one of the endpoints of the non-extremal segment $Y'$ is y and thus is in R as well. This means that y is the endpoint of one out-segment which again contradicts the fact that $y \in R$. This completes the proof of Claim 1.

Let $J^-$ be the graph obtained from J by removing all hairs and notice that $J^-$ is a biconnected outerplanar graph. Let S be the set of vertices of $J^-$ that have degree 4. Notice that, because $Q^*$ is touch-free, $|S|$ is equal to the number of vertices of J that have degree 4 minus the number of bouncing terminals. Therefore,

$$\rho = |T_2| + |S|. \tag{1}$$

Notice that if we remove from $J^-$ all the edges of $C_r^*$, the resulting graph is a forest $\Psi$ whose connected components are paths. Observe that none of these paths is a trivial path because $Q^*$ is touch-free. We denote by $\kappa(\Psi)$ the number of connected components of $\Psi$. Let F be the set of faces of $J^-$ that are different from $D_r^*$. F is partitioned into the faces that are caves, namely $F_1$ and the non-cave faces, namely $F_0$. By the Claim 1, $|F_1| \quad |T_1|$.

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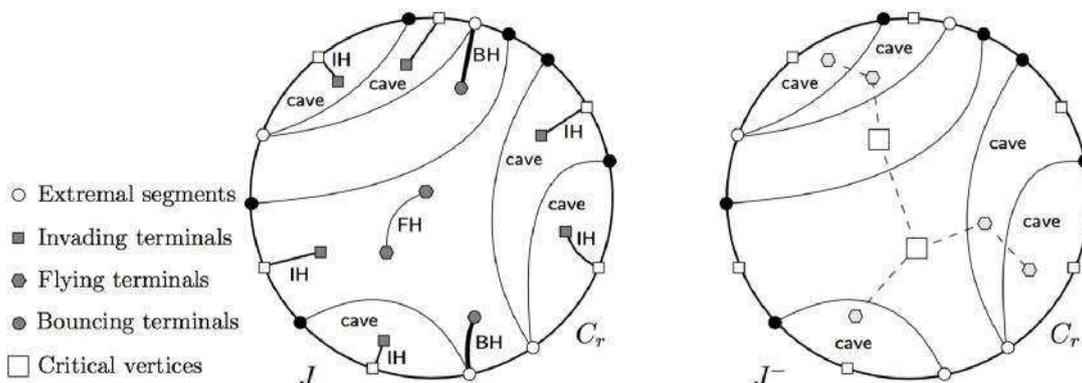
Figure 8: Examples of the graphs J and J− in the proof of Lemma 4 (the outer face in the picture corresponds to the interior of $D_r$). The faces that are caves contain the word cave. FH: flying hair, BH: bouncing hair, IH: invading hair. The forest $\Psi = J^-\backslash E(C_r)$ has 6 edges and 4 connected components. The weak dual T of J− is depicted with dashed lines. The large white square vertices are the rich vertices of T.

To complete the proof, it is enough to show that

$$|S| \le |T_1| - 2 \qquad\qquad (2)$$

Indeed the truth of (2) along with (1), would imply that p is at most $|T_2| + |S| \le |T_2| + |T_1| - 2 \le |T| - 2 = 2k - 2$.

We now return to the proof of (2). For this, we need two more claims.

Claim 2: $|F_0| \le \kappa(\Psi) - 1$.
Proof. We use induction on $\kappa(\Psi)$. Let $K_1, \dots, K_{\kappa(\Psi)}$ be the connected components of $\Psi$. If $\kappa(\Psi) = 1$ then all faces in F are caves, therefore $|F_0| = 0$ and we are done. Assume now that $\Psi$ contains at least two connected components.

We assert that there exists at least one connected component Kh of $\Psi$ with the

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property that only one non-cave face of J− contains edges of Kh in its boundary. To see this, consider the weak dual T of J−. Recall that, as J− is biconnected, T is a tree. Let $K_{*i}$ be the subtree of T containing the duals of the edges in $E(K_i)$, $i \in \{1, \dots, \kappa(\Psi)\}$, and observe that $E(K_{*1}), \dots, E(K_{*\kappa(\Psi)})$ is a partition of E(T) into $\kappa(\Psi)$ cyclically connected sets. We say that a vertex of T is rich if it is incident with edges in more than one members of $\{K_{*1}, \dots, K_{*\kappa(\Psi)}\}$, otherwise it is called poor (see Figure 8). Notice that a vertex of T is rich if and only if its dual face in J− is a non-cave. We call a subtree $K_{*i}$ peripheral if $V(K_{*i})$ contains at most one rich vertex of T. Notice that the claimed property for a component in $\{K_1, \dots, K_{\kappa(\Psi)}\}$ is equivalent to the existence of a peripheral subtree in $\{K_{*1}, \dots, K_{*\kappa(\Psi)}\}$. To prove that such a peripheral subtree exists, consider a path P in T intersecting the vertex sets of a maximum number of members of

$W_{*1}, \ldots, _{K_{*K(4)1}}$. Let $e_*$ be the first edge of P and let $K_{*h}$ be the unique subtree whose edge set contains $e_*$. Because of the maximality of the choice of P, $V(K_{*h})$ contains exactly one rich vertex $v_h$, therefore $K_{*h}$ is peripheral and the assertion follows. We denote by $f_h$ the non-cave face of $J_-$ that is the dual of $v_h$.

Let $II_-$ be the outerplanar graph obtained from $J_-$ after removing the edges of $K_h$. Notice that this removal results in the unification of all faces that are incident to the edges of $K_h$, including $f_h$, to a single face $f_+$. By the inductive hypothesis the number of non-cave faces of $II_-$ is at most $\kappa(\Psi)-2$. Adding back the edges of $K_h$ in $J_-$ restores $f_h$ as a distinct non-cave face of $J_-$. If $f_+$ was a non-cave of $II_-$ then $|F_0|$ is equal to the number of non-cave faces of $II_-$, else $|F_0|$ is one more than this number. In any case, $|F_0| < \kappa(\Psi) - 1$, and the claim follows.

Claim 3: $|V(\Psi)| < |T_1| + 2 - \kappa(\Psi) - 2$.
Proof. Let T be the weak dual of $J^-$. Observe that $|F_0| + |F_1| = |F| = |V(T)| = |E(T)| + 1 = |E(\Psi)| + 1 = |V(\Psi)| - \kappa(\Psi) + 1$. Therefore $|V(\Psi)| = |F_0| + |F_1| + \kappa(\Psi) - 1$. Recall that, by Claim 1, $|F_1| < |T_1|$ and, taking into account Claim 2, we conclude that $|V(\Psi)| < |T_1| + 2 \bullet \kappa(\Psi) - 2$. Claim 3 follows.

Notice now that a vertex of $J^-$ has degree 4 iff it is an internal vertex of some path in $\Psi$. Therefore, as all connected components of $\Psi$ are non-trivial paths, it holds that $|V(\Psi)| = |S| + |L(\Psi)| = |S| + 2 \bullet \kappa(\Psi)$, where $L(\Psi)$ is the set of leaves of $\Psi$. By Claim 3,

$$|S| + 2 - \kappa(\Psi) = |V(\Psi)| < |T_1| + 2 - \kappa(\Psi) - 2 = |S| < |T_1| - 2.$$

Therefore, (2) holds and this completes the proof of the lemma. $\qquad\square$

## 3.4 Bounding the number and size of segment types

In this section we introduce the notion of segment type that partitions the segments into classes of mutually "parallel" segments. We next prove that, in the light of the results of the previous section, the number of these classes is bounded by a linear function of the number k of linkage paths. In Subsections 3.5 and 3.6 we show that if one of these equivalence classes has size more than $2^k$, then an equivalent cheaper linkage can be found. All these facts will be employed in the culminating Subsection 3.7 in order to prove that a cheap linkage cannot go very "deep" into the cycles of a cheap CL-configuration. That way we will be able to quantify the depth at which an irrelevant vertex is guaranteed to exist.

Types of segments. Let G be a plane graph and let $Q = (C, L)$ be a convex CL-configuration of G. Let $S_1, S_2$ be two segments of Q and let P and $P^0$ be the two paths on $C_r$ connecting an endpoint of $S_1$ with an endpoint of $S_2$ and passing through no other endpoint of $S_1$ or $S_2$. We say that $S_1$ and $S_2$ are parallel, and we write $S_1 \, II \, S_2$, if

(1) no segment of Q has both endpoints on P.

(2) no segment of Q has both endpoints on $P^0$.

(3) the closed-interior of the cycle $P \cup S_1 \cup P^0 \cup S_2$ does not contain the disk $D_0$.

A type of segment is an equivalence class of segments of Q under the relation k .

Given a linkage L of G and a closed disk D of $R^2$ whose boundary is a cycle of G, we define $inD(L)$ to be the graph obtained from $(L \cup bnd(D))$ fl D after dissolving all vertices of degree 2.

Notice that $inD_T(L)$ is the biconnected outerplanar graph formed if we dissolve all vertices of degree 2 in the graph that is formed by the union of $C_r$ and the segments of Q. As Q is convex, one of the faces of $inD_T(L)$ contains the interior of $D_0$ and we call this face central face. We define the segment tree of Q, denoted by T(Q), as follows.

- Let $T^-$ be the weak dual of $inD_T(L)$ rooted at the vertex that is the dual of the central face.

- Let Q be the set of leaves of $T^-$. For each vertex l E Q do the following: Notice first that l is the dual of a face $l_*$ of $inD_T(L)$. Let $W_1,...$ , $_{W_{ρl}}$ be the extremal segments in the boundary of $l_*$ (notice that, by the convexity of Q, for every l, $ρl \geq$ 1). Then, for each i E {1,. . . , ρl}, create a new leaf wi corresponding to the extremal segment Wi and make it adjacent to l.

The height of T(Q) is the maximum distance from its root to its leaves. The real height of T(Q) is the maximum number of internal vertices of degree at least 3 in a path from its root to its leaves plus one. The dilation of T(Q) is the maximum length of a path all whose internal vertices have degree 2 and are different from the root.

**Observation 3.** Let G be a plane graph and let Q = (C, L) be a convex CL-configuration of G. Then the dilation of T(Q) is equal to the maximum cardinality of an equivalence class of ||.

**Observation 4.** Let G be a plane graph and let Q = (C, L) be a convex CL-configuration of G. Then the height of T(Q) is upper bounded by the dilation of T(Q) multiplied by the real height of T(Q).

The following lemma is an immediate consequence of Lemma 4 and the definition of a segment tree. The condition that L fl $C_r$ =6 0 simply requires that the CL-configuration that we consider is non-trivial in the sense that the linkage L enters the closed disk $D_r$.

**Lemma 5.** Let G be a plane graph and Q = (C, L) be a touch-free CL-configuration of G where C is tight in G, L is C-cheap, and L fl $C_r$ =6 0. Then Q is convex and the real height of the segment tree T(Q) is at most 2 · |P(L)| − 3.
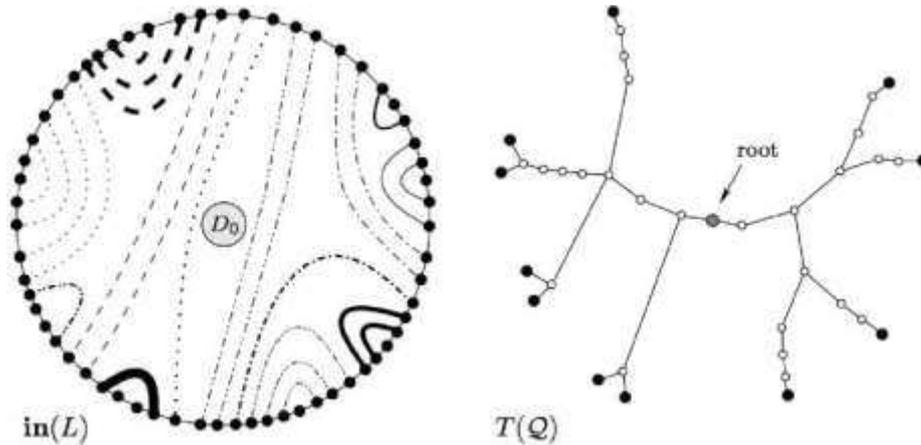
Figure 9: The graph $\mathrm{in}_{D_r}(L)$ for some convex CL-configuration $Q = (C, L)$ and the tree $T(Q)$. Internal edges in $\mathrm{in}_{D_r}(L)$ of the same type are drawn as lines of the same type. $Q$ has 11 extremal segments, as many as the leaves of $T(Q)$. The relation $k$ has 19 equivalent classes. The dilation of $T(Q)$ is 4, its height is 8 and its real height is 4.

Proof. Certainly, the convexity of $Q$ follows directly from Lemma 3. We examine the non-trivial case where $T(Q)$ contains at least one edge. We first claim that $|P(L)| \geq 2$. Assume to the contrary that $L$ consists of a single path $P$. As $Q$ is convex and $L \cap C_r = 6$ $0$, $Q$ has at least one extremal segment. Suppose now that $Q$ has more than one extremal segment all of which are connected components of $C_r \cap P$. Let $P_1$ and $P_2$ be the closures of the connected components of $L \setminus D_r$ that contain the terminals of $P$. Let $p_i \in V(C_r)$ be the endpoint of $P_i$ that is not a terminal, $i \in \{1, 2\}$. Let also $P'$ be any path in $C_r$ between $p_1$ and $p_2$. Notice now that $P_1 \cup P' \cup P_2$ is a cheaper linkage with the same pattern as $L$, a contradiction to the fact that $L$ is C-cheap. Therefore we conclude that $Q$ has exactly one extremal segment, which contradicts the fact that $Q$ is touch-free. This completes the proof that $|P(L)| \geq 2$.

Recall that, by the construction of $T(Q)$ there is a 1–1 correspondence between the leaves of $T(Q)$ and the extremal segments of $Q$. From Lemma 4, $T(Q)$ has at most $2 \cdot |P(L)| - 2$ leaves. Also $T(Q)$ has at least 2 leaves, because $Q$ is touch-free. It is known that the number of internal vertices of degree $\geq 3$ in a tree with $r \geq 2$ leaves is at most $r - 2$. Therefore, $T(Q)$ has at most $2 \cdot |P(L)| - 4$ internal vertices of degree $\geq 3$. Therefore the real height of $T(Q)$ is at most $2 \cdot |P(L)| - 3$. $\qquad\square$

## 3.5 Tidy grids in convex configurations

In this subsection we prove that the existence of many "parallel" segments implies the existence of a big enough grid-like structure.

**Topological minors.** We say that a graph H is a topological minor of a graph C if there exists an injective function $\sigma_0 : V(H) \to V(C)$ and a function $\sigma_1$ mapping the edges of H to paths of C such that

- for every edge $\{x,y\} \in E(H)$, $\sigma_1(\{x,y\})$ is a path between $\sigma_0(x)$ and $\sigma_0(y)$.

- if two paths in $\sigma_1(E(H))$ have a common vertex, then this vertex should be an endpoint of both paths.

Given the pair $(\sigma_0, \sigma_1)$, we say that H is a topological minor of C via $(\sigma_0, \sigma_1)$.

**Tilted grids and L-tidy grids.** Let C be a graph. A tilted grid of C is a pair $U = (X, Z)$ where $X = \{X_1,\dots, X_r\}$ and $Z = \{Z_1,\dots, Z_r\}$ are both collections of r vertex-disjoint paths of C such that

- for each $i,j \in \{1,\dots,r\}$ $I_{i,j} = X_i \cap Z_j$ is a (possibly edgeless) path of C,

- for $i \in \{1,\dots,r\}$ the subpaths $I_{i,1}, I_{i,2}, \dots, I_{i,r}$ appear in this order in $X_i$.

- for $j \in \{1,\dots, r\}$ the subpaths $I_{1,j}, I_{2,j},\dots, I_{r,j}$ appear in this order in $Z_j$.

- $E(I_{1,1}) = E(I_{1,r}) = E(I_{r,r}) = E(I_{r,1}) = 0$,

- Let
$$U_{C_U} = \left(\bigcup_{i\in\{1,\dots,r\}} X_i\right) \cup \left(\bigcup_{i\in\{1,\dots,r\}} Z_i\right)$$

and let $C_* U$ be the graph taken from the graph after contracting all edges in $\bigcup_{(i,j)\in\{1,\dots,r\}^2} I_{i,j}$. Then $C_* U$ contains the (r x r)-grid F as a topological minor via a pair $(\chi_0, \chi_1)$ such that

A. the upper left (resp. upper right, down right, down left) corner of F is mapped via $\chi_0$ to the (single) endpoint of $I_{1,1}$ (resp. $I_{1,r}$, $I_{r,r}$, and $I_{r,1}$).

B. $\bigcup_{e\in E(\Gamma)} \chi_1(e) = C_* U$ (this makes $C_* U$ to be a subdivision of F).

We call the subgraph $C_U$ of C realization of the tilted grid U and the graph $C_* U$ representation of U. We treat both $C_U$ and $C_* U$ as plane graphs. We also refer to the cardinality r of X (or Z) as the capacity of U. The perimeter of $C_U$ is the cycle $X_1 \cup Z_1 \cup X_r \cup Z_r$. Given a graph C and a linkage L of C we say that a tilted grid $U = (X, Z)$ of C is an L-tidy tilted grid of C if $D_U \cap L = \cup Z$ where $D_U$ is the closed-interior of the perimeter of $C_U$.

**Lemma 6.** Let C be a plane graph and let $Q = (C, L)$ be a convex CL-configuration of C. Let also S be an equivalence class of the relation . Then C contains a tilted grid $U = (X, Z)$ of capacity $\lceil |S|/2 \rceil$ that is an L-tidy tilted grid of C.
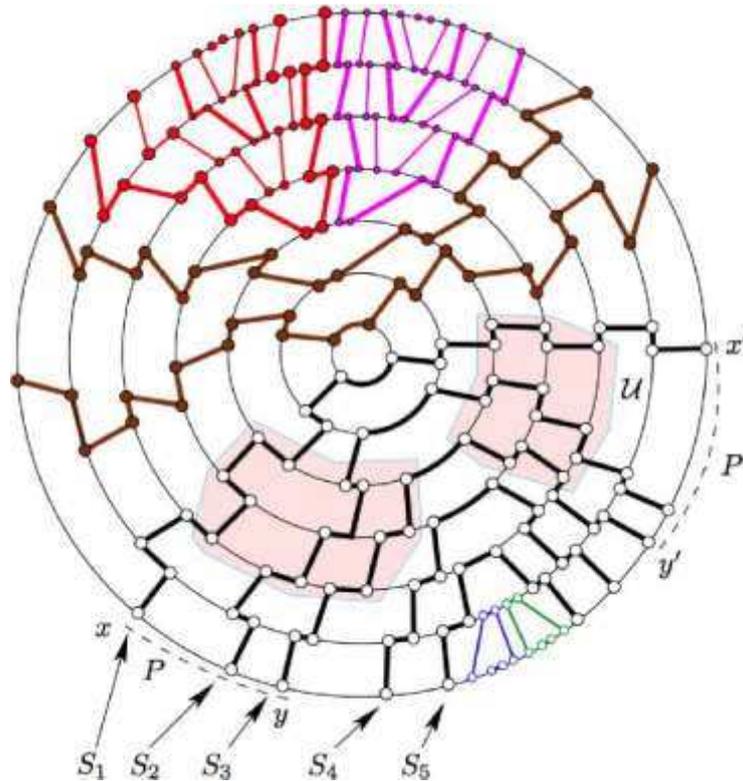
Figure 10: A visualisation of the proof of Lemma 6. It holds that $a_1 = 0$ and $a_5 = 4$, $m = 5$, $m' = 3$. The two shadowed regions indicate the two connected components of $D_S \cap A_C$.

Proof. Let $C = \{C_0,\dots, C_r\}$ and let $S = \{S_1,\dots, S_m\}$. For each $i \in \{1,\dots, m\}$, let $a_i$ be the eccentricity of $S_i$ and let $a^{max} = \max\{a_i \mid i \in \{1,\dots, m\}\}$ and $a^{min} = \min\{a_i \mid i \in \{1,\dots, m\}\}$. Convexity allows us to assume that $S_1, \dots, S_m$ are ordered in a way that

- $a_1 = a^{min}$,

- $a_m = a^{max}$, and

- for all $i \in \{1,\dots,m-1\}$, $a_{i+1} = a_i + 1$.

- for all $i \in \{1,\dots, m\}$, $I_{i,\sigma_i} = S_i \cap C_{\sigma_i}$ is a subpath of $C_{\sigma_i}$.

Let $m' = \lceil \frac{m}{2} \rceil$ and let $x, x'$ (resp. $y, y'$) be the endpoints of the path $S_1$ (resp. $S_{m'}$) such that the one of the two $(x, y)$-paths (resp. $(x', y')$-paths) in $C_r$ contains both $x', y'$ $(x, y)$ and the other, say $P$ (resp. $P'$), contains none of them. Let $D_S$ be the closed-interior of the cycle $S_1 \cup P' \cup S_{m'} \cup P$. Let also $A_C$ be the closed annulus defined by the cycles

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$C_{\sigma_{max}-(m'-1)}$ and $C_{\sigma_{max}}$. Let $\Delta$ be any of the two connected components of $D_S \cap A_C$. We now consider the graph

$$(L \cup uC) \cap \Delta.$$

It is now easy to verify that the above graph is the realization $G_U$ of a tilted grid $U = (X, Z)$ of capacity $m^0$, where the paths in $X$ are the portions of the cycles $C_{\sigma_{max}-(m'-1)}, \ldots, C_{\sigma_{max}}$ cropped by $\Delta$, while the paths in $Z$ are the portions of the paths in $\{S_1, \ldots, S_{m'}\}$ cropped by

$\Delta$ (see Figure 10). As S is an $\qquad\qquad\qquad\square$

equivalence class of k, it follows that U is L-tidy, as required. 3.6 Replacing linkages by cheaper ones

In this section we prove that a linkage L of k paths can be rerouted to a cheaper one, given the existence of an L-tidy tilted grid of capacity greater than $2^k$. Given that L is a cheap linkage, this will imply an exponential upper bound on the capacity of an L-tidy tilted grid.

Let G be a plane graph and let L be a linkage in G. Let also D be a closed disk in the surface where G is embedded. We say that L crosses vertically D if the outerplanar graph defined by the boundary of D and $L \cap D$ has exactly two simplicial faces. This naturally partitions the vertices of $\mathrm{bnd}(D) \cap L$ into the up and down ones. The following proposition is implicit in the proof of Theorem 2 in [3] (see the derivation of the unique claim in the proof of the former theorem). See also [8] for related results.

**Proposition 3.** Let G be a plane graph and let D be a closed disk and a linkage L of G of order k that crosses D vertically. Let also $L \cap D$ consist of $r > 2^k$ lines. Then there is a collection $A_r$ of strictly less than r mutually non-crossing lines in D each connecting two points of $\mathrm{bnd}(D) \cap L$, such that there exists some linkage R that is a subgraph of $L \setminus \mathrm{int}(D)$ such that $R \cup \cup A_r$ is a linkage of the graph $(G \setminus D) \cup \cup A_r$ that is equivalent to L.

**Lemma 7.** Let $k, k^0, \rho$ be integers such that $0 < \rho < k^0 < k$. Let $\Gamma$ be a $(k \times k^0)$-grid and let $\{p^{u_1 p}, \ldots, p_\rho{}^{up}\}$ (resp. $\{p_{down}{}_1, \ldots, {}_\rho{}^{down}\}$) be vertices of the higher (resp. lower) horizontal line arranged as they appear in it from left to right. Then the grid $\Gamma$ contains $\rho$ pairwise disjoint paths $P_1, \ldots, P_\rho$ such that, for every $h \in [\rho]$, the endpoints of $P_h$ are $p_h{}^{up}$ and $^{p_h{}^{down}}$.

Proof. We use induction on $\rho$. Clearly the lemma is obvious when $\rho = 0$. Let $(i, j) \in [k]^2$ such that $p_\rho{}^{up}$ (resp. $p_\rho{}^{down}$) is the i-th (resp. j-th) vertex of the higher (lower) horizontal line counting from left to right. We examine first the case where $i > j$. Let $P_\rho$ be the path created by starting from $p_\rho{}^{up}$, moving $k^0 - 1$ edges down, and then $i - j$ edges to the left. For $h \in [\rho - 1]$ let $P_h{}^{(down)0}$ be the path created by starting from $p_{down\ h}$ and ${}_h$ moving one edge up (clearly, $P_h{}^{(down)0}$ consists of a single edge). We also denote by

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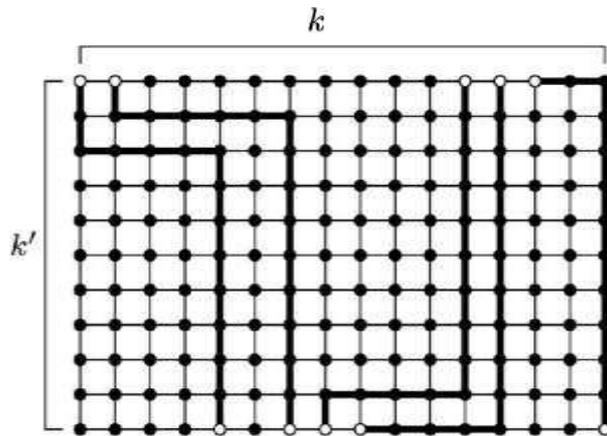
Figure 11: An example of the proof of Lemma 7, where $k = 16$, $k^0 = 11$, and $\rho = 5$. The white vertices of the higher (resp. lower) horizontal line are the vertices in $\{p^{up}_1, \ldots, p^{up}_5\}$ (resp. $\{p^{down}_1, \ldots, p^{down}_5\}$).

$p^{(down)0}$ the other endpoint of $P^{(down)0}$. We now define $\Gamma_i$ as the subgrid of $\Gamma$ that occurs from $\Gamma$ after removing its lower horizontal line and, for every $h \in [i,k]$, its $h$-th vertical line. By construction, none of the edges or vertices of $P_\rho$ belongs in $\Gamma^0$. Notice also that the higher (resp. lower) horizontal line of $\Gamma^0$ contains all vertices in $\{p^{up}_1, \ldots, p^{up}_{\rho-1}\}$ (resp. $\{p^{(down)0}_1, \ldots, p^{(down)0}\}$). From the induction hypothesis, $\Gamma^0$ contains $\rho-1$ pairwise disjoint paths $P^0_1, \ldots, P^0_{\rho-1}$ such that for every $h \in [\rho - 1]$, the endpoints of $P_h$ are $p^u_h{}^p$ and $P^{(down)0}$. It is now easy to verify that $P^0_1 \cup P^{(down)0}_h, \ldots, P^0_{\rho-1} \cup P^{(down)0}_1$, $P_\rho$ is the required collection of pairwise disjoint paths. For the case where $i < j$, just reverse the same grid upside down and the proof is identical (see Figure 11). $\square$

Lemma 8. Let $\Gamma$ be a $(k \times k)$-grid embedded in the plane and assume that the vertices of its outer cycle, arranged in clockwise order, are:

$$\{v^{up}_1, \ldots, v^{up}_k, v^{right}_2, \ldots, v^{right}_{k-1}, v^{down}_k, \ldots, v^{down}_1, v^{left}_{k-1}, \ldots, v^{left}_2, v_1\}.$$

Let also H be a graph whose vertices have degree 0 or 1 and they can be cyclically arranged in clockwise order as

$$\{x^{up}_1, \ldots, x^{up}_k, x^{down}_k, \ldots, x^{down}_1, x^{up}_1\}$$

such that if we add to H the edges formed by pairs of consecutive vertices in this cyclic ordering, the resulting graph $H^+$ is outerplanar. Let $V^1$ be the vertices of H that have degree 1 and let $H^1 = H[V^1]$. Then $H^1$ is a topological minor of $\Gamma$ via some pair $(\varphi_0, \varphi_1)$, satisfying the following properties:

1. $\varphi_0(x^{up}$

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$_i$ ) $= v^{up}_i$, $i \in \{1, \ldots, k\} \cap V_1$

2. $0_0(x_i^{down}) = v_{i,i}^{down} \in \{1, \ldots, k\} \cap V_1$.

Proof. Let $U = \{x_1^{up}, \ldots, x_k^{up}\} \cap V^1$ and $D = \qquad \cup V^1$. We define $0_0$ as in the statement of the lemma. In the rest of the proof we provide the definition of $0_1$. We partition the edges of $H^1$ into three sets: the upper edges EU that connect vertices in U, the down edges EL that connect vertices in D, and the crosssing edges EC that have one endpoint in U and one in D. As $|V(H^1)| \leq 2k$ we obtain that $|E(H^1)| \leq k$ and therefore $|EU| + |ED| + |EC| = |E(H^1)| \leq k$. We set $\rho = |EC|$.

We recursively define the depth of an edge $\{x_i^{up}, x_j^{up}\}$ in EU as follows: it is 0 if there is no edge of EU with an endpoint in $\{x_{i+1}^{up}, \ldots, x_{j-1}^{up}\}$ and is $i > 0$ if the maximum depth of an edge with an endpoint in $\{x_{i+1}^{up}, \ldots, x_{j-1}^{up}\}$ is $i - 1$. The depth of an edge $\{x_i^{down}, x_j^{down}\}$ is defined analogously. It directly follows, by the definition of depth that:

$$q_{up} = \max\{depth(e) \mid e \in EU\} + 1 \leq |EU| \qquad (3)$$
$$q_{down} = \max\{depth(e) \mid e \in ED\} + 1 \leq |ED| \qquad (4)$$

We now continue with the definition of $0_1$ as follows:
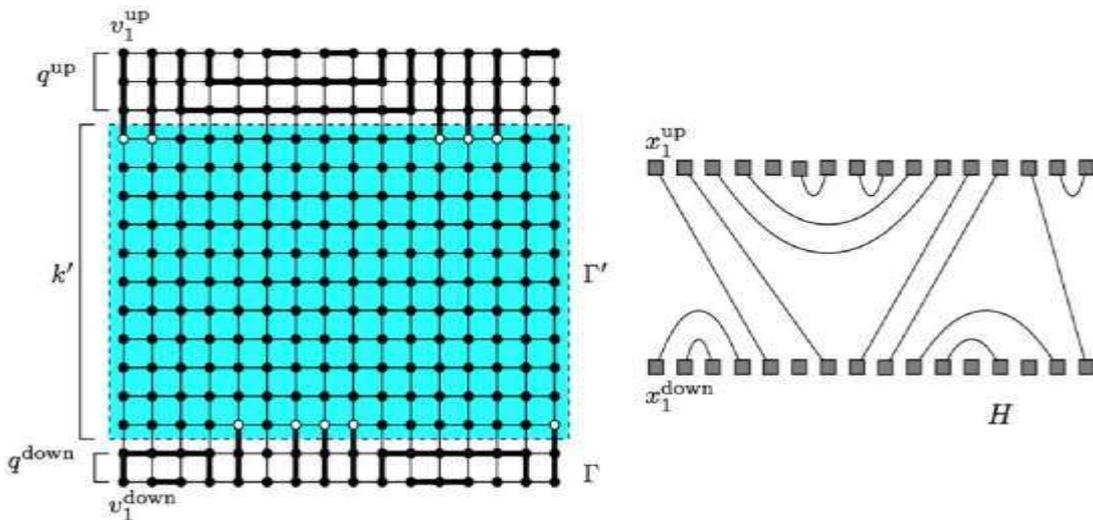


Figure 12: An example of the proof of Lemma 8. On the left, the $(16 \times 16)$-grid G is depicted along with the way the graph H (depicted on the right) is (partially) routed in it. In the figure $q^{up} = 3$, $q_{down} = 2$, $k' = 11$, and $\rho = 5$.

- for every edge $e = \{x_i^{up}, x_j^{up}\}$ in EU, of depth $l$ and such that $i < j$, let $0_1(e)$ be the path defined if we start in the grid G from $v_i^{up}$, move $l$ steps down, then $j - i$ steps to the right, and finally move $l$ steps up to the vertex $v_j^{up}$ (by "number of steps" we mean number of edges traversed).

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- for every edge $e = \{x^{down}_i, x^{down}_j\}$ in $E_D$, of depth $l$ and such that $i < j$, let $0_1(e)$ be the path defined if we start in the grid $G$ from $v^{down}_i$, move $l$ steps up, then $j - i$ steps to the right, and finally move $l$ steps down to the vertex $v^{down}_j$.

Notice that the above two steps define the values of $0_1$ for all the upper and down edges. The construction guarantees that all paths in $0_1(E_U \cup E_D)$ are mutually non-crossing. Also, the distance between $0_0(U)$ and some horizontal line of $\Gamma$ that contains edges of the images of the upper edges is $\max\{depth(e) \mid e \in E_U\}$ that, from (3), is equal to $q^{up} - 1$. Symmetrically, using (4) instead of (3), the distance between $0_0(D)$ and the horizontal lines of $\Gamma$ that contain edges of the images of the down edges is equal to $q_{down} - 1$. As a consequence, the graph

$$\Gamma_0 = \Gamma \setminus \{x \in V(\Gamma) \mid dist_\Gamma(x, 0_0(U)) < q^{up} \vee dist_\Gamma(x, 0_0(D)) < q_{down}\}$$

is a $(k \times k^0)$-grid $\Gamma^0$, where $k^0 = k - (q^{up} + q^{down})$, whose vertices do not appear in any of the paths in $0_1(E_U \cup E_D)$. Given a crossing edge $e = \{x^{up}_i, x^{down}_j\} \in E_C$, we define the path $P^{up}_e$ as the subpath of $\Gamma$ created if we start from $x^{up}_i$ and then go $q^{up}$ steps down. Similarly, we define $P_e^{down}$ as the subpath of $\Gamma$ created if we start from $x^{down}_j$ and then go $q_{down}$ steps up. Notice that each of the paths $P_e^{up}$ (resp. $P_e^{down}$) share only one vertex, say $p_e^{up}$ (resp. $p_e^{down}$), with $\Gamma^0$ that is one of their endpoints (these endpoints are depicted as white vertices in the example of Figure 12). We use the notation $\{p^u_1{}^p, \ldots, p_\rho{}^{up}\}$ (resp. $\{p_1^{down}, \ldots, p_\rho{}^{down}\}$) for the vertices of the set $\{p_e^{up} \mid e \in E_C\}$ (resp. $\{p_e^{down} \mid e \in E_C\}$) such that, for every $h \in [\rho]$, there exists an $e \in E_C$ such that $p^u_h{}^p$ is an endpoint of $P_e^{up}$ and $p_h^{down}$ is an endpoint of $P_e^{down}$. We also agree that the vertices in $\{p^u_1{}^p, \ldots, p_\rho{}^{up}\}$ (resp. $\ldots, p_\rho{}^{down}\}$) are ordered as they appear from left to right in the upper (lower) horizontal line of $\Gamma^0$ (this is possible because of the outeplanarity of $H^+$).

Notice that $\rho = |E(H^1)| - (|E_U| + |E_D|) < k - (|E_U| + |E_D|)$ which by (3) and (4) implies that $\rho < k^0$.

As $\rho < k^0 < k$, we can now apply Lemma 7 on $\Gamma^0$, $\{p^u_1{}^p, \ldots, p_\rho{}^{up}\}$ and $\{p_1^{down}, \ldots, p_\rho{}^{down}\}$ and obtain a collection $\{P_e \mid e \in E_C\}$ of $\rho$ pairwise disjoint paths in $\Gamma^0$ between the vertices of $\{p_e^{up} \mid e \in E_C\}$ and the vertices of $\{p_e^{down} \mid e \in E_C\}$. It is now easy to verify that $\{P^{up}_e \cup P_e \cup P_e^{down} \mid e \in E_C\}$ is a collection of $\rho$ vertex disjoint paths between $U$ and $D$. We can now complete the definition of $0_1$ for the crossing edges of $H$ by setting, for each $e \in E_C$, $0(e) = P^{up}_e \cup P_e \cup P_e^{down}$. By the above construction it is clear that $(0_1, 0_2)$ provides the claimed topological isomorphism. □ **Lemma 9.** Let $G$ be a graph with a linkage $L$ consisting of $k$ paths. Let also $U = (X, Z)$ be an $L$-tidy tilted grid of $G$ with capacity $m$. Let also $\Delta$ be the closed-interior of the perimeter of $G_U$. If $m > 2^k$, then $G$ contains a linkage $L^0$ such that

1. $L$ and $L^0$ are equivalent,

2. $L^0 \setminus \Delta \subset L \setminus \Delta$, and

3. $|E(UZ \cap L^0)| < |E(UZ \cap L)|$.

Proof. We use the notation $X = \{X_1, \ldots, X_m\}$ and $Z = \{Z_1, \ldots, Z_m\}$. Let GU be the realization of U in G and let $G^*$ (resp. $L^*$) be the graph (resp. linkage) obtained from G (resp. L) if we contract all edges in the paths of $1_{j(i,j)\in\{1,\ldots,r\}^2}$ $I_{i,j}$, $^{where}$ $I_{i,j} = X_i \cap Z_j$, $i, j \in \{1, \ldots, m\}$. We also define $X^*$ and $Z^*$ by applying the same contractions to their paths. Notice that $U^* = (X^*, Z^*)$ is an $L^*$-tidy tilted grid of $G^*$ with capacity m and that the lemma follows if we find a linkage $L^{0*}$ such that the above three conditions are true for $A^*$, $L^*$, $L^{0*}$, and $Z^*$, where $A^*$ is the closed-interior of the perimeter of $G^*_U$ (recall that $G^*_U$ is the representation of U that is isomorphic to GU).

Let $G^{*-} = (G^* \setminus A^*) \cup UZ$ and apply Proposition 3 on $G^{*-}$, $A^*$, and $L^*$. Let Ar be a collection of strictly less than m mutually non-crossing lines in D each connecting two points of $bnd(A^*) \cap L^*$ and a linkage $R \subseteq L^* \setminus int(A^*)$ such that $L_0 = R \cup UA_r$ is a linkage of the graph $(G^* \setminus A^*) \cup UA_r$ that is equivalent to $L^*$. Let $H = (L_0 \cap A^*) \cup (L^* \cap bnd(A^*))$. Notice that in H, the set $V(L_0 \cap A^*)$ contains the vertices of H of degree 1 while the rest of the vertices of H have degree 0 and all edges of H have their endpoints in $V(L_0 \cap A^*)$. Recall that the $(m \times m)$-grid $\Gamma$ is a topological minor of $G^*_U$ via some pair $(\chi_0, \chi_1)$ satisfying the conditions A and B in the definition of tilted grid.

We are now in position to apply Lemma 8 for the $(m \times m)$-grid $\Gamma$ and H. We obtain that $H^1 = L_0 \cap A^*$ is a topological minor of $\Gamma$ via some pair $(0_0, 0_1)$. We now define the graph

$$\left[ L = E(0_1(e)). \atop e \in E(H^1) \right.$$

Notice that L is a subgraph of $\Gamma$. We also define the graph

$$Q = \cup \chi_1(e) \atop e \in E(L)$$

which, in turn, is a subgraph of $G^*_U$. Observe that $L^{0*} = R \cup Q$ is a linkage of $G^*$ that is equivalent to $L^*$. This proves Condition 1. Condition 2 follows from the fact that $R \subseteq L^* \setminus int(A^*)$. Notice now that, as $|Ar| < m$, $E(UZ^* \cap Q)$ is a proper subset of $E(UZ^*)$. By construction of $L^{0*}$, it holds that $E(UZ \cap L^{0*}) = E(UZ \cap Q)$. Moreover, as $U^* = (X^*, Z^*)$ is an $L^*$-tidy ⊔ tilted grid of $G^*$, it follows that $E(UZ^*) = E(UZ^* \cap L^*)$. Therefore, Condition 3 follows.

## 3.7 Existence of an irrelevant vertex

We now bring together all results from the previous subsections in order to prove Theorem 1.

Lemma 10. There exists an algorithm that, given an instance $(G, P = \{(s_i, t_i) \in V(G)^2, i \in \{1, \ldots, k\}\})$ of PDPP, either outputs a tree-decomposition of G of width at most $9 \cdot (k \cdot 2^{k+2} + 1) \cdot \sqrt{2k} + 11$ or outputs an irrelevant vertex $x \in V(G)$ for $(G, P)$. This algorithm runs in $2^{2^{O(k)}} \cdot n$ steps.

Proof. Let $T = \{s_1,\ldots,s_k,t_1,\ldots,t_k\}$. By applying the algorithm of Lemma 1, for $r = k.2^{k+2}$ either we output a tree-decomposition of G of width at most $9(r+1).\digamma\sqrt{2k}\_+11)$ or we find an internally chordless cycle C of G such that G contains a tight sequence of cycles $\mathcal{C} = \{C_0, \ldots, C_r\}$ in G where $C_0 = C$ and all vertices of T are in the open exterior of $C_r$. From Lemma 1, this can be done in $2^{(r\cdot\sqrt{|T|})^{O(1)}} \cdot n = 2^{2^{O(k)}} \cdot n$ steps.

Assume that G has a linkage whose pattern is P and, among all such linkages, let L be a C-cheap one. Our aim is to prove that $V(L \cap C_0) = 0$, i.e., we may pick x to be any of the vertices in $D_0$.

First, we can assume that $k \geq 2$. Otherwise, if $k = 1$, the fact that L is C-cheap, implies that $L \cap D_{r-1} = 0 \Rightarrow L \cap D_0 = 0$ and we are done.

For every $i \in \{0, \ldots, r\}$, we define $Q^{(i)} = (C^{(i)}, L^{(i)})$ where $C^{(i)} = \{C_0, \ldots, C_i\}$ and $L_{(i)}$ is the subgraph of L consisting of the union of the connected components of L that have common points with $D_i$. As $r + 1 > k$, at least one of $Q^{(i)}$, $i \in \{0, \ldots, r\}$ is touch-free. Let $Q^0 = (C^0, L^0)$ be the touch-free CL-configuration in $\{Q^{(1)}, \ldots, Q^{(r)}\}$ of the highest index, say h. In other words, $C^0 = C^{(h)}$ and $L^0 = L^{(h)}$. Moreover, $C^0$ is tight in G and $L^0$ is $C^0$-cheap. Let $k^0$ be the number of connected components of $L^0$. We set $d = r - h$ and observe that $k^0 \leq k - d$, while $C^0$ has $r^0 = r + 1 - d > 0$ concentric cycles. Again, we assume that $k^0 \geq 2$ as, otherwise, the fact that $L^0$ is $C^0$-cheap implies that $L_0 \cap D_{r-1} = 0 \Rightarrow L^0 \cap D_0 = 0$ and we are done. Therefore $0 \leq d \leq k - 2$.

As $C^0$ is tight in G and $L^0$ is $C^0$-cheap, by Lemma 3, $Q^0$ is convex. To prove that $V(L \cap C_0) = 0$ it is enough to show that all segments of Q have positive eccentricity and for this it is sufficient to prove that all segments of $Q^0$ have positive eccentricity. Assume to the contrary that some segment $P_0$ of $Q^0$ has eccentricity 0. Then, from the third condition in the definition of convexity we can derive the existence of a sequence $P_0, \ldots, P_{r'-1}$ of segments such that for each $i \in \{0, \ldots, r^0 - 1\}$, $P_{i+1}$ is inside the zone of $P_i$. This implies the existence in the segment tree $T(Q^0)$ of a path of length $r_0$ from its root to one of its leaves, therefore $T(Q^0)$ has height $r^0$. By Lemma 5, the real height of $T(Q^0)$ is at most $2k^0 - 3$. By Observation 4, the dilation of $T(Q^0)$ is at least $\frac{2k'}{3} \geq \frac{k \cdot 2k}{2} 2^{k-+2-2d} \quad \frac{2d}{2} \frac{k}{2}^{k+k}2 = 2^{k+1}$. By Observation 3 and Lemma 6, G contains an $L^0$-tidy tilted grid $U = (X, Z)$ of capacity $> 2^k$. From Lemma 9, G contains another linkage $L^{00}$ with the same pattern as $L^0$ and such that $c(L^{00}) < c(L^0)$, a contradiction to the fact that $L^0$ is $C^0$-cheap.

Since $V(L \cap C_0) = 0$, any vertex of G in the closed-interior of $C_0$ is irrelevant. $\quad\square$

Proof of Theorem 1. The proof follows from Lemma 10, taking into account that, for every $k \geq 1$,

$\square$

$$82 \cdot k^{3/2} \cdot 2^k > 9 \cdot (k \cdot 2^{k+2} + 1) \cdot F\sqrt{2k} + 11)$$

# 4 An algorithm for PDPP

In this section we prove Theorem 2. In particular, we briefly describe an algorithm that, given an instance $(C, 2)$ of DPP where $C$ is planar, provides a solution to PDPP, if one exists, in $2^{2^{O(k)}} \cdot n^{O(1)}$ steps.

Our algorithm is based on the following proposition.

**Proposition 4** ( [25]). There exists an algorithm that, given an instance $(C, 2)$ of PDPP and a tree decomposition of $C$ of width at most $w$, either reports that $(C, 2)$ is a NO-instance or outputs a solution of PDPP for $(C, 2)$ in $2^{O(w \log w)} \cdot n$ steps.

**Proof of Theorem 2.** By applying the algorithm of Lemma 10, we either find an irrelevant vertex v for $(C, 2)$ or we obtain a tree-decomposition of C of width $2^{O()}$. In the first case, we again look for an irrelevant vertex in the equivalent instance $(C, 2) +\!\!-\!\!- (C\backslash v, 2)$. This loop breaks if the second case appears, namely when a tree decomposition of C of width $2^{O()}$ is found. Then we apply the algorithm of Proposition 4, that solves the problem in $2^{2^{O(k)}} \cdot n$ steps. As, unavoidably, the loop will break in less than n steps, the claimed running time follows. □

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