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Travel-Time Models With and Without Homogeneity Over Time

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Abstract. In dynamic network loading and dynamic traffic assignment for networks, the link travel time is often taken as a function of the number of vehicles $x(t)$ on the link at time t of entry to the link, that is, $\tau(t) = f(x(t))$, which implies that the performance of the link is invariant (homogeneous) over time. Here we let this relationship vary over time, letting the travel time depend directly on the time of day, thus $\tau(t) = f(x(t), t)$. Various authors have investigated the properties of the previous (homogeneous) model, including conditions sufficient to ensure that it satisfies first-in-first-out (FIFO). Here we extend these results to the inhomogeneous model, and find that the new sufficient conditions have a natural interpretation. We find that the results derived by several previous authors continue to hold if we introduce one additional condition, namely that the rate of change of $f(x(t), t)$ with respect to the second parameter has a certain (negative) lower bound. As a prelude, we discuss the equivalence of equations for flow propagation equations and for intertemporal conservation of flows, and argue that neither these equations nor the travel-time model are physically meaningful if FIFO is not satisfied. In Section 7 we provide some examples of time-dependent travel times and some numerical illustrations of when these will or will not adhere to FIFO.

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Keywords: travel-time functions • first-in-first-out • homogeneity • dynamic network loading • dynamic traffic assignment

1. Introduction

The travel time for each link in a network, in dynamic network loading (DNL) and dynamic traffic assignment (DTA), has often been modeled as a function of the number of vehicles $x(t)$ on the link, thus $f(x(t))$, so that for a user entering a link at time t the link exit time is

$$\tau(t) = t + f(x(t)). \tag{1}$$

The variable $x(t)$ is also referred to as the link occupancy and is given by the conservation equation

$$x(t) = \int_0^t u(s)\,ds - \int_0^t v(s)\,ds, \tag{2}$$

where $u(t)$ and $v(t)$ are the inflow and outflow, respectively, at time t.

This model and its use in DNL and DTA has been investigated in many papers and is included in reviews such as Peeta and Ziliaskopoulos (2001), Szeto and Lo (2005, 2006), Friesz, Kwon, and Bernstein (2007), and Mun (2007, 2009). Some properties of the model when used in DNL or DTA are discussed and illustrated in Friesz et al. (1993), Xu et al. (1999), Zhu and Marcotte (2000), Carey and Ge (2005b), Carey (2004), Carey and McCartney (2002), Nie and Zhang (2005a), and Zhang and Nie (2005): the first four of these papers are discussed at greater length in this paper. The behavior and performance of the model is compared with other macroscopic whole-link models used in DTA in Carey and Ge (2003a), and Nie and Zhang (2005b). Computational issues for applying the model in DTA are discussed in Rubio-Ardanaz, Wu, and Florian (2003), Carey and Ge (2004, 2005a), Nie and Zhang (2005a, b), and Long, Gao, and Szeto (2011).

In the previous model it is assumed that, given the current occupancy $x(t)$ of a link, the link travel time is independent of time. However, in practice the link travel time predicted at the time of entry to a link may also vary over time during the day as a result of factors other than the number of vehicles on the link. These factors include time-varying traffic control signals, signs, speed limits, changing visibility because of the transition from day to night and vice versa, time-varying traffic mix, or changing weather conditions, such as the onset of rain, fog, snow, etc. (see Section 7). Note that changing weather conditions (rain, fog, snow, etc.) are usually of a stochastic nature, for which the deterministic models in this paper are not a suitable framework for prediction purposes. It is easy to formally extend the previous model to allow the link travel time to depend on the time of day as well as the current occupancy of the link, thus $f(x(t), t)$, so that the link exit time is

$$\tau(t) = t + f(x(t), t). \tag{3}$$

There is a long history of papers proposing and discussing various functional forms for the travel time functions $f(x(t))$ in (1), including Branston (1976), Ran and Boyce (1996), Ran et al. (1997), and Anderson and Bell (1998), so we do not repeat the discussion of functional forms here. Also, over the past 15 years many methods have been proposed and used to estimate these travel-time functions for links or routes. Many authors suggest using transit vehicles or taxi fleets as probes with sensors, or using automatic vehicle location (AVL), typically using GPS. Others recommend using automatic vehicle or number plate identification, using cell phone data, or using loop detectors. One thing that all of these methods have in common is that the time of day is automatically or easily available in the data collection process and hence in the resulting data sets. This facilitates treating the time of day as a factor in estimating and predicting travel times and thus in estimating functions of the form $f(x(t), t)$ used in (3). For example, Zhang, Wu, and Kwon (1997) note that regression based methods can easily incorporate various factors that affect travel time, one of the factors being time of day. Again, we do not repeat the discussion of functional forms or estimation methods for $f(x(t), t)$, other than in the first few paragraphs of Section 7.

The properties of the model (1)–(2) have been derived in several papers but it is not known, and is not immediately obvious, how the properties of this model are affected by extending it to allow inhomogeneity over time, as in (2)–(3). We therefore investigate this in the present paper. In particular, we investigate the conditions needed to ensure that the model still retains desirable first-in-first-out (FIFO) properties. The main mathematical properties of the model (1)–(2), including FIFO, were rigorously derived in Friesz et al. (1993), Xu et al. (1999), and Zhu and Marcotte (2000). Carey and Ge (2005b) took some of the conditions or restrictions derived in Xu et al. (1999) and Zhu and Marcotte (2000), and replaced them with conditions or restrictions that may be more easily checked or more likely to be satisfied. In this paper we take the properties derived in these four papers for model (1)–(2) and seek to extend them to the model (2)–(3).

In Section 2 we complete the models (1)–(2) and (2)–(3) by setting out flow propagation equations, which we note can be interpreted as intertemporal link conservation equations, and discuss the relationship between these and FIFO. In Sections 3–6, respectively, we take the properties of the model (1)–(2), derived in the four papers previously noted, and investigate whether and how these extend to the model (2)–(3). Section 4 assumes a linear form of $f(x(t), t)$ and Sections 4–6 assume a nonlinear form. In Section 7 we provide some examples of time-dependent travel times and some numerical illustrations of when these will or will not satisfy FIFO. Concluding remarks are in Section 8.

2. Letting Travel-Time Vary with Time: Inhomogeneous Travel-Time Functions

In the real world of road traffic, FIFO is not strictly adhered to, since many vehicles overtake and pass each other. Such overtaking or passing could potentially be modeled, but it is not explicitly modeled or included in the travel-time models (1)–(2) or (2)–(3). Nevertheless, if certain technical properties are not satisfied, these models can allow traffic cohorts to overtake or pass each other and in ways that can differ substantially from what happens in the real world and may not even be physically possible in the real world. For example, if certain technical properties are not satisfied, we find that if the inflow $u(t)$ to a link is falling rapidly over a short time interval then, based on (1)–(2), all of the traffic that enters the link in that time interval may exit before traffic that entered earlier when the inflow rate was higher, which violates FIFO. That does not reflect any real world behavior, which is why we wish to prevent such FIFO violations in the travel-time models (1)–(2) and (2)–(3).

Completing the model*: *Intertemporal flow conservation or flow propagation and FIFO. The link travel-time model is often stated as (1)–(2) together with a so-called flow propagation equation such as (4) or (5) in the next paragraph. We note that (2) is a contemporaneous conservation equation and, as we will see shortly, if FIFO holds then the flow propagation equation can also be interpreted as an intertemporal conservation equation. Thus, the travel-time model consists of (1) subject to a contemporaneous conservation Equation (2) and an intertemporal conservation equation such as (4) or (5). FIFO is not imposed as a separate or additional constraint, but must be inherent in (1) subject to these two forms of conservation equations. To show that FIFO holds for any particular form of $f(x(t))$ in (1) requires a proof: e.g., proofs are given for a linear $f(x(t))$ in Friesz et al. (1993) and for the linear and nonlinear $f(x(t))$ in Xu et al. (1999) and Zhu and Marcotte (2000). These remarks refer to the homogeneous travel-time travel function (1), but we will see that they can be extended to the inhomogeneous travel-time function (3), so that the inhomogeneous travel-time model consists of (3) subject to the same contemporaneous conservation Equation (2) and the same intertemporal conservation equation such as (4) or (5).

The flow-propagation equation is written in various forms in the literature, in particular

$$\int_{-\infty}^{t} u(s)\,ds = \int_{-\infty}^{\tau(t)} v(s)\,ds, \tag{4}$$

where $u(s)$ and $v(s)$ are the inflow and outflow from the link at time s, or alternatively

$$x(\tau(t)) = \int_{t}^{\tau(t)} u(s)\,ds, \tag{5}$$

where $x(\tau(t))$ is the number of vehicles on the link at time $\tau(t)$. The flow-propagation equation is sometimes stated with t defined as the exit time and $t-\tau^*(t)$ as the entry time, where $\tau^*(t)$ is the link travel time for traffic that exits at time t. Furthermore, the flow-propagation equation is sometimes stated as the derivative of any of these forms. All of these forms are derivable from each other hence we will not discuss them explicitly here.

If FIFO holds then it is easy to see that (4) and (5) are simply intertemporal conservation equations. More specifically, if FIFO holds for all traffic entering up to time T, then we can see that (4) holding for all time $0 \le t \le T$ is necessary and sufficient to ensure conservation of flows up to time T. Similarly for (5) if it holds for all time $0 \le t \le T$. For example, if FIFO holds then all traffic that entered up to time t must have exited by time $\tau(t)$, so that the only traffic still on the link at time $\tau(t)$ must have entered between time t and $\tau(t)$, i.e., $\int_t^{\tau(t)} u(s)\,ds$, and if flow is conserved this traffic is still on the link, so that (5) holds.

Now suppose that FIFO does not hold and consider Equations (4) or (5). If FIFO is violated then some inflows$u(s)$ that entered before time t may not exit until after time $\tau(t)$, and conversely some inflows $u(s)$ that entered after time t may exit before time $\tau(t)$. Neither of these flows is captured by (4) or (5) hence if FIFO is violated then imposing (4) or (5) would seem to have no physical justification, and is likely to produce nonsense results. If FIFO does not hold then neither (4) nor (5) nor any of the other proposed forms of flow propagation equations will ensure intertemporal conservation of flows. Also, if FIFO does not hold, it is not at all obvious that there is any form of equation that would ensure intertemporal flow conservation for the model (1)–(2) or (2)–(3).

In summary,

(a) If FIFO holds on a link, then an intertemporal flow conservation equation for the link is equivalent to the flow propagation equation used elsewhere in the literature.

(b) If FIFO does not hold then neither the flow conservation nor flow-propagation equations make physical sense and it is not appropriate to impose them.

In view of that, we will refer to intertemporal flow conservation rather than flow propagation.

Before leaving the matter in (b), namely flow conservation equations without FIFO, it is worth noting that even though this is not physically meaningful, a model that allows it may still have a mathematical solution. For example, suppose that the only traffic on a link enters between times t_1 and t_2 and these inflows all violate FIFO by exiting in the reverse of the order in which they entered and consequently $\tau(t_2) < \tau(t_1)$. Applying (5) at time t_1 and again at time $t_2 > t_1$ and subtracting the former from the latter gives an alternative form of the conservation equation, namely

$$\int_{t_1}^{t_2} u(s)\,ds = \int_{\tau(t_1)}^{\tau(t_2)} v(s)\,ds. \tag{6}$$

As a result of the FIFO violation, on the right-hand side (r.h.s.) of (6) the upper limit of integration is smaller than the lower limit, hence the r.h.s. of (6) is negative, if we treat the outflows $v(t)$ as positive. However, the left-hand side (l.h.s.) is positive thus we seem to have a contradiction. There is no mathematical difficulty in solving (6) because (6) will simply yield negative outflows $v(t)$, not because they are physically negative (they are not), but because the time span over which the outflows $v(t)$ occur is measured backwards in time.

In Sections 3–6, respectively, we take results from four different papers concerning the homogeneous case (1) and extend them to the inhomogeneous case (3). In all four cases we find that the results from the homogeneous case, including the results concerning FIFO, continue to hold for the inhomogeneous case if the following condition also holds:

$$f_t(x,t) > -u(t)/B, \tag{7}$$

where B is an upper bound on $u(t)$.

Some further implications and interpretations of (7) are discussed in Section 7. We make just two remarks concerning it here before embarking on derivations of (7) in Sections 3–6.

(i) If inflow $u(t)$ is at its upper bound B then (7) reduces to $f_t(x,t) > -1$ and if inflow is at its lower bound $u(t)=0$ then (7) reduces to $f_t(x,t) > 0$.

(ii) The FIFO sufficient condition (7) depends on the inflow profile $u(t)$ and its upper bound B, which is a disadvantage because the inflows are likely to vary over time and in a network model the inflows to each link are generally not known in advance. It would be nice to have a FIFO condition that is independent of the inflows. However, it is well known that such a sufficient condition for FIFO is not available even for the homogeneous case $f(x)$ when the travel time functions $f(x)$ are nonlinear. In that (nonlinear) homogeneous case the only available condition sufficient to ensure FIFO is as follows (see Xu et al. 1999; Zhu and Marcotte 2000) or Section 4 and 5

$$f_x(x) < 1/B, \tag{8}$$

where B is defined as in (7), i.e., it is the upper bound on the inflows $u(t)$.

Each of the four papers considered in Section 3–6 assume a travel-time function of the form $f(x(t))$ used in (1). For each of these papers we extend some key results, particularly concerning FIFO, to travel-time functions of the inhomogeneous form $f(x(t), t)$ used in (3). The theorems or propositions and their proofs in those four papers are quite lengthy so we do not wish to repeat them here. Instead we present only the changes that are needed to extend the theorems or propositions and proofs to the inhomogeneous case (3).

Each of the papers discussed in the following sections makes use of a well-known necessary and sufficient condition for FIFO for any form of travel-time model, namely that for traffic entering at time t, its exit time $\tau(t)$ should be an increasing function of t. Thus, assuming that $\tau(t)$ is differentiable, this FIFO condition is

$$\tau'(t) > 0, \tag{9}$$

where the prime (′) denotes a first derivative.

3. Extending the Results from Friesz et al. (1993) to the Inhomogeneous Case

[The notation used in Friesz et al. (1993) is the same as in this paper except that they use D to denote link travel-time functions while we use f, as in (10) and (11).]

In Section 3 of their paper Friesz et al. (1993) introduce and derive properties of a linear travel-time function or delay model

$$f(x(t)) = \alpha x(t) + \beta, \tag{10}$$

where $x(t)$ is as previously defined and $\alpha \geq 0$ and $\beta > 0$ are constants. To extend this to an inhomogeneous function, while retaining linearity, add a term $\gamma(t)$, thus

$$f(x(t), t) = \alpha x(t) + \beta + \gamma(t). \tag{11}$$

Note that β and $\gamma(t)$ can of course be combined, letting $\gamma^*(t) = \beta + \gamma(t)$. Friesz et al. (1993) derive properties of the linear model (10) in their Theorem 1 as follows.

Theorem 1 of Friesz et al. (1993). *For any linear arc delay function f, the resulting arc exit time function τ is strictly increasing and hence the inverse function τ^{-1} exists.*

Following the theorem Friesz et al. (1993) note that this (an increasing exit time function) implies that the model satisfies FIFO, as also noted in (9). We can extend their results to an inhomogeneous linear travel-time model (11) as follows.

Proposition 1. *If the travel-time function (10) is replaced by (11) then Theorem 1 from Friesz et al. (1993) continues to hold if we also let*

$$\gamma'(t) \geq -\alpha u(t), \text{ or equivalently } \gamma'(t) \geq -u(t)/B, \tag{12}$$

where $\gamma'(t)$ denotes the derivative $\mathrm{d}\gamma(t)/\mathrm{d}t$ and $1/\alpha = B$.

Remark. The parameter α in (10) is often interpreted as $1/B$ where B is the maximum flow capacity of the link in the static or steady state case. To see this, note that in the steady state case we have an identity $x = us$ where x is link occupancy, u is the flow rate, and s is the link trip time. Using the linear travel-time function $s = \alpha x + \beta$ to substitute for s in $x = us$ gives a flow-occupancy function $u = x/(\alpha x + \beta)$. The latter is everywhere increasing and is asymptotic from below to $1/\alpha$. Therefore, the linear travel-time functions (10) imply that the flow u is bounded above by $1/\alpha$. Letting $1/\alpha = B$ we can rewrite $\gamma'(t) \geq -\alpha u(t)$ from (12) as $\gamma'(t) \geq -u(t)/B$. The advantage of the latter form is that it is the linear form of (7) and hence is the linear form of the conditions assumed in the propositions in Section 4–6 to ensure FIFO for nonlinear travel-time functions.

Proof. In the proof of their Theorem 1, Friesz et al. (1993) divide the time span into intervals $[t_n, t_{n+1}]$, $n = 1, 2, 3, \ldots$, and show, in Equation (37), that "$\tau'_{n+1}(t) > \alpha u(t)$" for all time intervals where $\tau'_{n+1}(t)$ is associated with the interval $[t_n, t_{n+1}]$. Hence "$\tau'_{n+1}(t) > 0$" for all time intervals since $\alpha u(t) \geq 0$. Therefore, $\tau'(t) > 0$ for all t and, as noted in (9), $\tau'(t) > 0$ ensures FIFO.

Now replace (10) with (11), i.e., replace $f(x(t)) = \alpha x(t) + \beta$ with $f(x(t), t) = \alpha x(t) + \beta + \gamma(t)$. This adds an extra term, namely $\gamma'(t)$, to the r.h.s. of all expressions for $\tau'(t)$ since, from (3), $\tau(t) = t + f(x(t), t)$. In particular, adding $\gamma'(t)$ to the r.h.s. changes their Equation (37) from "$\tau'_{n+1}(t) > \alpha u(t)$" to "$\tau'_{n+1}(t) > \alpha u(t) + \gamma'(t)$." However, by assumption (12), $\alpha u(t) + \gamma'(t) \geq 0$ hence $\tau'_{n+1}(t) > 0$ for all time intervals $[t_n, t_{n+1}]$. Therefore, $\tau'(t) > 0$ for all t and, as noted in (9), $\tau'(t) > 0$ ensures FIFO. Hence the proof of FIFO for the travel-time function (10) is now extended to the travel-time function (11) and we are done.

The above extension of the proof of Theorem 1 of Friesz et al. (1993) is given in outline and does not list all of the specific changes that are needed in extending the proof. More specific changes in the proof are as follows.

Add $\gamma(t)$ to the r.h.s. of Equations (23), (26), (29), (32), and to the r.h.s. of the first three equations in (36). Add $\gamma'(t)$ after the equation sign in (24), (35), and (37). In (28)[1] add $\gamma'(t)$ after each of the four equation signs, and add $\gamma'[\tau_1^{-1}(t)]$ to the denominator of the quotient term in the third and fourth lines in (28). This changes the last line in (28) from "$\tau_2'(t) > \alpha u(t) \geq 0$" to "$\tau_2'(t) > \alpha u(t) + \gamma'(t)$ hence $\tau_2'(t) > 0$ since $\alpha u(t) + \gamma'(t) \geq 0$ by assumption (12)." Also change (31) from $\tau_k'(t) > \alpha u(t)$ to $\tau_k'(t) > \alpha u(t) + \gamma'(t)$, and change (34) and (37) from $\tau_{n+1}'(t) > \alpha u(t)$ to $\tau_{n+1}'(t) > \alpha u(t) + \gamma'(t)$. □

The change noted in the preceding paragraph, from "$\tau'(t) > \alpha u(t) \geq 0$" to "$\tau'(t) > \alpha u(t) + \gamma'(t) \geq 0$" is interesting though it is only an intermediate result in the proof and is not in the statement of the theorem. It means that even if the inflows $u(t)$ are "large," if $\gamma'(t)$ is negative then $\tau'(t)$ can be close to zero, so that the exit time $\tau(t)$ can increase very slowly. Note that $\tau'(t)$ "close" to zero means that the flow is "close" to violating FIFO (though it of course does not violate FIFO, as is shown by the proposition). Conversely, even if the inflows $u(t)$ are "small," if $\gamma'(t)$ is positive and large then $\tau'(t)$ will be large, so that the exit time $\tau(t)$ will increase rapidly. These outcomes are what one would expect.

4. Extending the Results from Xu et al. (1999) to the Inhomogeneous Case

[The notation used in Xu et al. (1999) is different than in this paper and in the papers discussed in Section 3, 5, and 6. Xu et al. (1999) uses $b(t)$ rather than $u(t)$ for the link inflow rate, $v(t)$ rather than $x(t)$ for the link occupancy, and $s(v(t))$ rather than $f(x(t))$ for the nonlinear travel-time function. However, for consistency, when quoting from their paper we have changed their notation to the same as in the rest of the present paper.]

Xu et al. (1999) set out two FIFO theorems, namely Theorem 3.1 that applies when the link travel-time function is nonlinear and Theorem 3.2 that applies when it is linear. We consider only Theorem 3.1 here since Theorem 3.2 is similar to the Friesz et al. (1993) Theorem 1, already considered in Section 3. As is usual, they let the link travel time be $f(x(t))$ where $x(t)$ is the number of vehicles on the link at time t so that, for a user entering the link at time t, the exit time is $\tau(t) = t + f(x(t))$.

Theorem 3.1 of Xu et al. (1999). *Assume that there exists a finite instant T' such that, for all t less than T', the entry flow rate function $u(t)$ is well defined, nonnegative, bounded from above by B, Lebesgue integrable, and that $f'(x) < 1/B$ for all x in the interval $[0, X]$ where $X = \int_0^{T'} u(t)\,dt$. Then:*

(i) *x is everywhere nonnegative and differentiable almost everywhere on $[0, \tau(T')]$;*

(ii) *τ is strictly increasing and invertible on its domain;*

(iii) *τ and τ^{-1} are differentiable almost everywhere on their respective domains, and there exists a positive constant ζ such that $\tau^{-1}(t) \geq \zeta$ for all t in $[0, \tau(T')]$;*

(iv) *v is well defined, nonnegative, Lebesque integrable, and bounded from above by B;*

(v) *the functions x, τ, τ^{-1}, and v are well defined.*

Proposition 2. *If link travel time $f(x(t))$ is replaced by $f(x(t), t)$ then Theorem 3.1 from Xu et al. (1999) continues to hold if we also let (7) hold, i.e., if we assume $f_t(x, t) > -u(t)/B$.*

Remark. If $f(x(t), t)$ is linear in this proposition, as in Equation (11) for the Friesz et al. (1993) linear model, then condition $f_t(x, t) > -u(t)/B$ in (7) and in the proposition reduces to $\gamma'(t) > -u(t)/B$, which is the same condition as in Proposition 1 for the linear model, except that in Proposition 1 we found a weak inequality ($\geq$ instead of $>$).

Proof. In proving their Theorem 3.1, Xu et al. (1999) use the derivative of the travel-time function (1), that is

$$\tau'(t) = 1 + f'(x(t))x'(t). \tag{13}$$

If we allow the travel-time function to be inhomogeneous over time as in (3), i.e., $\tau(t) = t + f(x(t), t)$, then (13) becomes

$$\tau'(t) = 1 + f_x(x(t), t)x'(t) + f_t(x(t), t), \tag{14}$$

where f_x and f_t, respectively, denote the derivatives of $f(x, t)$ with respect to the first and second argument.

As in Friesz et al. (1993) (see Section 3), they divide the time span into intervals $[t_i, t_{i+1}]$, $i = 1, \ldots, n$. Then, using (13) and assuming $f_x(x(t)) < 1/B$, they prove (in the multiline equation on page 345, column 1, lines 2 to 7) that $(\tau^2)'(t) \geq \max(\zeta, u(t)/B)$, where the "2" superscript denotes the second time interval, $[t_1, t_2]$. By their definitions $\zeta > 0$, therefore $(\tau^2)'(t) > 0$. They extend this recursively to all time intervals, hence obtain $\tau'(t) > 0$ for all t, and their FIFO result follows immediately from that, as noted in (9).

If now we replace (13) with (14), then in their multiline equation for the first time interval, referred to in the preceding paragraph

$$\begin{aligned}(\tau^2)'(t) \geq \max(\zeta, u(t)/B) \text{becomes } &(\tau^2)'(t) \\ &\geq \max(\zeta, u(t)/B) + f_t(x(t), t),\end{aligned}$$

hence $(\tau^2)'(t) \geq u(t)/B + f_t(x(t), t)$.

However, $u(t)/B + f_t(x(t), t) > 0$ by assumption (7), therefore $(\tau^2)'(t) > 0$. This can be extended recursively to all time intervals $[t_i, t_{i+1}]$, $i = 1, \ldots, n$, to give $\tau'(t) > 0$ for all t, which ensures FIFO, as noted in (9), which completes the proof. □

5. Extending the Results from Zhu and Marcotte (2000) to the Inhomogeneous Case

[The notation used in Zhu and Marcotte (2000) is the same as in this paper except that they use D to denote link travel-time functions while we use f.]

Zhu and Marcotte (2000) set out two FIFO theorems, namely Theorem 5.2 that applies when the link travel-time function is nonlinear and Theorem 5.1 that applies when it is linear. We consider only Theorem 5.2 here. Their Theorem 5.1 is similar to the Friesz et al. (1993) Theorem 1, already considered in Section 3.

Theorem 5.2 of Zhu and Marcotte (2000). *Let T' be a finite instance such that, for all t in $[0, T']$, the functions $u^p(t), p \in P$ are well defined, nonnegative, and Lebesgue integrable, and $u(t) = \sum_{p\in P} u^p(t)$ is bounded from above by B ($B \geq 1$). Let the functions f be nonnegative, nondecreasing, and differentiable with respect to x. If $f'(x) < 1/(B+\eta)$ for some positive number η, then the strong FIFO condition on the link holds with constant $\eta/(B+\eta)$.*

In the above theorem, $u(t) = \sum_{p\in P} u^p(t)$ is the sum of the inflows to the link on the paths $p \in P$ that pass through it. The condition $f'(x) < 1/(B+\eta)$ is a stronger version of the condition (8). The extra term, η, is a positive number that was introduced by Zhu and Marcotte (2000) to give a stronger form of FIFO, to ensure that the travel time function $f(x)$ is strongly monotone.

Proposition 3. *Theorem 5.2 of Zhu and Marcotte (2000) continues to hold if the homogeneous link travel-time function $f(x(t))$ is replaced with the inhomogeneous function $f(x(t), t)$ and we introduce an additional assumption, namely*

$$f_t(x, t) \geq -u(t)/(B+\eta), \tag{15}$$

and the bounded gradient condition $f'(x) < 1/(B+\eta)$ is changed to $f_x(x, t) < 1/(B+\eta)$.

Remark. The difference between (15) and (7) is the extra term η. If $\eta = 0$ then (15) reduces to (7) since B and η are nonnegative, (15) is a stronger version of the condition (7) that is used in the propositions in Sections 3–6.

Proof. The proof is the same as the proof of Theorem 5.2 in Zhu and Marcotte (2000), except for

(a) their unnumbered equation in line 13 in column 2 on page 413, and

(b) the four-line equation at the bottom of column 2 on page 413.[2]

In both of these, the result still holds but the derivation of it needs extending, as shown next.

(a) On replacing $f(x(t))$ with $f(x(t), t)$, the equation in line 13 in column 2 on page 413 becomes

$$\tau'(t) = 1 + f_x(x(t), t)u(t) + f_t(x(t), t). \tag{16}$$

By assumption, $f_t(x(t), t) \geq -u(t)/(B+\eta)$, and adding $+1$ to each side gives

$$f_t(x(t), t) + 1 \geq [(B+\eta) - u(t)]/(B+\eta).$$

Then using assumption $u(t) \leq B$ reduces this to $f_t(x(t), t) + 1 \geq \eta/(B+\eta)$. Substituting the latter in (16), and noting that the term $f_x(x(t), t)u(t)$ is always nonnegative, yields $\tau'(t) \geq \eta/(B+\eta)$, which is the same result as in Zhu and Marcotte (2000) in the sixth line after their Equation (37).

(b) In the four-line equation at the bottom of column 2 on page 413, Zhu and Marcotte (2000) show that $\tau'(t) \geq \eta/(B+\eta)$, and we wish to show that this continues to hold here.

The first line of their four-line equation is $\tau'(t) = 1 + f'(x(t))x'(t)$, which is the derivative of their exit-time equation $\tau(t) = t + f(x(t))$. When we replace the latter with the inhomogeneous form $\tau(t) = t + f(x(t), t)$, then the first line of the four-line equation becomes

$$\tau'(t) = 1 + f_x(x(t), t)x'(t) + f_t(x(t), t), \tag{17}$$

that is, it has an extra term, $f_t(x(t), t)$. Also, from the definition of $x(t)$ we have $x'(t) = u(t) - v(t)$, so (17) becomes

$$\tau'(t) = 1 + f_x(x(t), t)[u(t) - v(t)] + f_t(x(t), t). \tag{18}$$

To proceed, consider two cases, $x'(t) \geq 0$ and $x'(t) < 0$, respectively. When $x'(t) \geq 0$, the proof that $\tau'(t) \geq \eta/(B+\eta)$ is the same as in (a). When $x'(t) < 0$ then substituting the assumption $f_x(x(t), t) < 1/(B+\eta)$ into the middle term on the r.h.s. of (18) yields

$$\begin{aligned} &1 + f_x(x(t), t)[u(t) - v(t)] \\ &\quad \geq 1 + \frac{u(t) - v(t)}{B+\eta} = \frac{(B+\eta) + (u(t) - v(t))}{B+\eta} \\ &\quad \geq \frac{\eta + u(t)}{B+\eta}. \end{aligned} \tag{19}$$

The second inequality in (19) follows since Zhu and Marcotte (2000) have already shown (lines 9–11 from the end of page 413) that $v(t) \leq B$. Substituting (19) and the assumption $f_t(x, t) \geq -u(t)/(B+\eta)$ into (18) reduces (18) to $\tau'(t) \geq \eta/(B+\eta)$. □

6. Extending the Results from Carey and Ge (2005b) to the Inhomogeneous Case

[The notation used in Carey and Ge (2005b) is the same as in this paper except that they used $\tau(t)$ to denote the link travel time, thus $\tau(t) = f(x(t))$, whereas we use $\tau(t)$ to denote the link exit time, thus $\tau(t) = t + f(x(t))$ or $\tau(t) = t + f(x(t), t)$.]

As discussed in the previous two sections, Xu et al. (1999) and Zhu and Marcotte (2000) derived sufficient conditions to ensure FIFO for the model (1)–(2) and later authors have confirmed that no weaker sufficient conditions have been found for that model. They found that the condition needed to ensure FIFO for the model (1)–(2) consisted of (8), i.e., $f_x(x) < 1/B$, together with some other mild conditions on nonnegativity, differentiability, and integrability. As already noted in the paragraph preceding it, (8) is a quite severe restriction. It is not satisfied by most of the nonlinear functions $f(x)$ that have been used or proposed for this model, as is shown in several examples in Section 3 of Carey and Ge (2005b). It is a severe restriction for the following reason. The travel demand functions $f(x)$ are normally assumed to be convex or monotone increasing, with either (a) a gradient that eventually goes to +infinity for a finite value of x or as x goes to +infinity, or (b) the gradient of $f(x)$ converges to a finite value as x goes to +infinity (i.e., $f(x)$ converges to a straight line as x goes to +infinity). Thus, unless $f(x)$ eventually becomes linear as x increases, it can eventually violate (8) if the inflows $u(t)$ are sufficiently large.

One way to avoid this problem, namely violating (8), would be to restrict the inflows $u(t)$ by imposing a lower upper bound B. That would increase $1/B$, thus increase the r.h.s. of (8). It would also reduce $x(t)$, via (2), hence reduce the l.h.s. of (8) if the gradient of $f(x)$ is increasing with x. Both of these effects make it less likely that (8) would be violated. However, that raises the question, what is a rational upper bound to impose on the inflows? Carey and Ge (2005b) propose that the inflows $u(t)$ be restricted to not exceed the maximum inflows that are allowed by the flow-density function, or equivalently the flow-occupancy function, that corresponds to the given travel-time function $f(x)$. By using the identity $x = us = uf(x)$, where u is the link inflow and s is the link travel time, we obtain $u = x/f(x)$, which we can rewrite as $u = g(x)$, that is the well-known flow-occupancy function. (The flow-occupancy function is of course just the flow-density function with a change of scale on the x-axis: replacing the occupancy x with ld, where d is the density and l is the link length gives the flow-density function $u = g(ld) = g^*(d)$). The maximum of the flow-occupancy function or flow-density function is often referred to as the capacity flow rate or maximum flow rate.

Carey and Ge (2005b) denote the capacity flow rate by q^B and propose it as an upper bound on the inflows $u(t)$ in the model (1)–(2), thus $u(t) \le q^B$. They show that if this upper bound is imposed and introduced into the theorems of Xu et al. (1999) and Zhu and Marcotte (2000), set out in Sections 4 and 5, then the condition (8) will always be automatically satisfied, if the travel-time function $f(x)$ also satisfies a weak form of convexity that is satisfied by all of the travel time functions that have been proposed or used. The upper bound $u(t) \le q^B$ is like the upper bound $u(t) \le B$ that is already present in the Xu et al. (1999) and Zhu and Marcotte (2000) theorems, hence their theorems continue to hold if we drop the restrictive condition (8), so long as we redefine the upper bound B as the maximum of the corresponding flow-density/occupancy function.

It might seem that the redefinition of the upper bound on inflows has eliminated the problems associated with the restrictive condition (8) for the model (1)–(2). However, it is not as simple as that, and to see this consider the following simple example. Suppose that two or more links with high exit flow rates feed into a downstream link that has a much lower inflow capacity. The Carey and Ge (2005b) bound $u(t) \le q^B$ will not allow all of the high exit flow from the upstream links to enter the downstream link, hence the excess flow would have to be held in a queue, or queuing link, at the entrance to the downstream link. That requires extending the model (1)–(2), since the usual travel-time model (1)–(2) does not include queues. By contrast, in the usual travel-time model (1)–(2) there is no restriction on inflows to the downstream link. The bound $u(t) \le B$ in the Xu et al. (1999) and Zhu and Marcotte (2000) theorems is normally not interpreted as a restriction on inflows, but as just the upper limit of whatever inflows happen to occur. There is nothing in the travel-time model (1)–(2) to restrict the inflows $u(t)$ even if they are arbitrarily large, if they exceed some measure of physical capacity of the link, or if the link occupancy already exceeds the jam occupancy or jam density of the link.

All of the discussion in this section, and all of the discussion and results in Carey and Ge (2005b), are concerned only with the model $f(x(t))$ in (1) and not the inhomogeneous model $f(x(t), t)$ in (3). However, we can show that all of the discussion and results in this section and in Carey and Ge (2005b), continues to hold when $f(x(t))$ is replaced with $f(x(t), t)$, if we also introduce the assumption (7) and assume a weak form of convexity of $f(x, t)$ with respect to x. This can be shown by introducing these conditions into Propositions 2–4 in Carey and Ge (2005b).

7. Application and Illustrations of FIFO Adherence or Nonadherence for the Inhomogeneous Model[3]

The form of inhomogeneous link travel-time model $f(x(t), t)$ may vary throughout the day and we could assume various theoretical forms for it for various intervals within a day, for example: A separable additive form $f(x(t)) + \gamma(t)$, which implies a growth/decline rate $f_t(x(t), t) = \gamma'(t)$ or γ. A multiplicative form $f(x(t))\gamma t$, which implies a constant growth/decline rate $f_t(x(t), t) = \gamma f(x(t))$. An exponential form $f(x(t))e^{\gamma t}$ that implies a growth/decline rate $f_t(x(t), t) = \gamma e^{\gamma t} f(x(t))$. As in earlier sections, $f_t(x(t), t)$

denotes the derivative with respect to the second argument of $f(x(t), t)$. The γt or $\gamma(t)$ in these travel-time functions can be replaced by $(t - t_0)\gamma$ or $\gamma(t - t_0)$, where t_0 is the start time of the time-dependent growth or decline in travel times.

In practice however, the effect of time on the link travel time is an empirical question or may be decided by a traffic controller. An example of the latter is as follows. A traffic controller using variable message signs may relax the speed limit for some links from say s_0 to s_1 and may smooth the transition by reducing the posted speed gradually over a span of time T. We can assume that most traffic is traveling at around the speed limit and that the increased speed limit reduces the average link travel time from $r_0 = L/s_0$ to $r_1 = L/s_1$. In that case the rate of decrease of the travel time is $(r_0 - r_1)/T$.

An example of an empirically determined decrease in time-dependent travel times is as follows. In autumn and winter in more northerly or southerly latitudes the transition from daylight to darkness takes a substantial time. Drivers tend to react to the deteriorating lighting conditions, especially on unlit roads, by reducing their speeds and that results in travel times increasing over the transition period. The time taken for the transition differs depending on the latitude and the time of year. The reduction in driving speeds depend on local conditions, such as road type, whether the roads are lit, national or local driver characteristics, etc. This means that determining the rate of change of the driving speed and travel time over the transition period is an empirical issue.

A further example of empirically determined variation in time-dependent travel times is because of the effects of changing weather conditions (such as the onset of rain, snow, or fog) on driving speeds and hence on travel times. We will not discuss this further here but many articles have covered the topic: see for example Camacho, Garcia, and Belda (2010), Federal Highway Administration (2006), Lam et al. (2013) and references therein.

To illustrate the FIFO condition (23) we will tabulate the values of $\tau'(t)$ for various values of $u(t)$, $v(t)$, and γ. That is, we will illustrate how FIFO at any instant t is affected by inflows and outflows $u(t)$ and $v(t)$ at that instant.

Applying (9), the well-known necessary and sufficient condition for FIFO, namely $\tau'(t) > 0$, to the link exit time function (1), i.e., $\tau(t) = t + f(x(t))$, yields the FIFO condition

$$\tau'(t) = 1 + f_x(x(t))x'(t) > 0, \tag{20}$$

where f_x denotes the first derivative with respect to x. Applying $\tau'(t) > 0$ to the inhomogeneous exit-time function (3) gives the FIFO condition

$$\tau'(t) = 1 + f_x(x(t), t)x'(t) + f_t(x(t), t) > 0, \tag{21}$$

where f_x and f_t denote the first derivatives w.r.t. the first and second argument, respectively, of $f(x(t), t)$. If we assume that the travel-time functions are linear as in (10) and (11) then (20) reduces to $\tau'(t) = 1 + \alpha x'(t) > 0$ and (21) reduces to

$$\tau'(t) = 1 + \alpha x'(t) + \gamma'(t) > 0. \tag{22}$$

However, from (2) we have $x'(t) = u(t) - v(t)$ and, from (7), $\alpha = 1/B$ where B is an upper bound on $u(t)$, hence this necessary and sufficient condition, i.e., (22), for FIFO reduces to

$$\tau'(t) = 1 + (u(t) - v(t))/B + \gamma'(t) > 0. \tag{23}$$

We add the following two notes about conditions (20)–(23).

1. In each of the papers discussed in earlier sections it is shown that the upper bound B on the inflow rate also results in the same bound B on the outflow rate $v(t)$. Hence, in (23) the l.h.s. of the inequality is at its minimum when $v(t)$ is at its upper bound B, in which case $v(t)/B = 1$ and (23) reduces to $\gamma'(t) > -u(t)/B$ or $f_t(x, t) > -u(t)/B$. That is, the necessary and sufficient condition (23) implies the sufficient condition $f_t(x, t) > -u(t)/B$, which is also the sufficient condition (7) derived in Sections 3–6. However, in the numerical examples in Tables 1(a)–1(c) we illustrate (23) rather than just $f_t(x, t) > -u(t)/B$, because (23) is more general. The final column in those tables illustrates the case when $v(t) = B$ that yielded the condition or $f_t(x, t) > -u(t)/B$.

2. In (21)–(23) the sum of the two terms before the ">0" cannot be less than -1 if the inequality, and hence FIFO, is to be satisfied. As a result, there is a "tradeoff" between these two terms. If one of them is negative it restricts the scope for the other to be negative while still satisfying FIFO. Also, at certain times t, it is likely that one or the other of the two terms will be zero, in which case the other term should not be less than -1. For example, the first term on the r.h.s. (i.e., $f_x(x(t), t)x'(t)$) will be zero if the traffic is in free-flow conditions (i.e., $f_x(x(t), t) = 0$) or if the link inflow rate happens to equal the outflow rate, so that $x'(t) = u(t) - v(t) = 0$. The second of the two terms will be zero if there is no underlying time trend in travel times independent of link occupancy $x(t)$, and in that case (21) reduces to (20).

Illustrating how FIFO violations are affected by time-dependent travel times and by link inflow and outflow rates.

As suggested just after (3) in Section 1, we will assume that the link travel time function $f(x(t), t)$ takes the separable form $f(x(t)) + \gamma t$ so that $f_t(x(t), t) = \gamma$, where γ is a constant that is negative or positive depending on whether the link travel time decreases or increases with time. To make the results more general

Table 1(a). Values of $\tau'(t)$ for Various Values of $u(t)$ and $v(t)$, with $\gamma = -0.8$

	$v(t)=0$	$v(t)=0.2B$	$v(t)=0.4B$	$v(t)=0.6B$	$v(t)=0.8B$	$v(t)=B$
$u(t)=0$	0.2	0.0	**-0.2**	**-0.4**	**-0.6**	**-0.8**
$u(t)=0.2B$	0.4	0.2	0.0	**-0.2**	**-0.4**	**-0.6**
$u(t)=0.4B$	0.6	0.4	0.2	0.0	**-0.2**	**-0.4**
$u(t)=0.6B$	0.8	0.6	0.4	0.2	0.0	**-0.2**
$u(t)=0.8B$	1.0	0.8	0.6	0.4	0.2	0.0
$u(t)=B$	1.2	1.0	0.8	0.6	0.4	0.2

Table 1(b). Values of $\tau'(t)$ for Various Values of $u(t)$ and $v(t)$, with $\gamma = -0.5$

	$v(t)=0$	$v(t)=0.2B$	$v(t)=0.4B$	$v(t)=0.6B$	$v(t)=0.8B$	$v(t)=B$
$u(t)=0$	0.5	0.3	0.1	**-0.1**	**-0.3**	**-0.5**
$u(t)=0.2B$	0.7	0.5	0.3	0.1	**-0.1**	**-0.3**
$u(t)=0.4B$	0.9	0.7	0.5	0.3	0.1	**-0.1**
$u(t)=0.6B$	1.1	0.9	0.7	0.5	0.3	0.1
$u(t)=0.8B$	1.3	1.1	0.9	0.7	0.5	0.3
$u(t)=B$	1.5	1.3	1.1	0.9	0.7	0.5

Table 1(c). Values of $\tau'(t)$ for Various Values of $u(t)$ and $v(t)$, with $\gamma = -0.0$

	$v(t)=0$	$v(t)=0.2B$	$v(t)=0.4B$	$v(t)=0.6B$	$v(t)=0.8B$	$v(t)=B$
$u(t)=0$	1.0	0.8	0.6	0.4	0.2	**0.0**
$u(t)=0.2B$	1.2	1.0	0.8	0.6	0.4	0.2
$u(t)=0.4B$	1.4	1.2	1.0	0.8	0.6	0.4
$u(t)=0.6B$	1.6	1.4	1.2	1.0	0.8	0.6
$u(t)=0.8B$	1.8	1.6	1.4	1.2	1.0	0.8
$u(t)=B$	2.0	1.8	1.6	1.4	1.2	1.0

Note. Negative values of $\tau'(t)$ are shown in bold, to indicate that they violate FIFO.

we will measure the inflows and outflows $u(t)$ and $v(t)$ as fractions k_1 and k_2 of the link capacity B, thus $u(t) = k_1 B$ and $v(t) = k_2 B$ where $0 \le k_1 \le 1$ and $0 \le k_2 \le 1$. That reduces the necessary and sufficient condition (23) to

$$\tau'(t) = 1 + k_1 - k_2 + \gamma > 0. \tag{24}$$

Tables 1(a) to 1(c) illustrate that the more negative is γ and/or the more rapidly $x(t)$ is declining (it is declining when the outflow rate $v(t)$ exceeds the inflow rate $u(t)$), then the more likely it is that FIFO is violated. Because k_1 and k_2 are nonnegative and do not exceed 1, it follows from (24) that $\tau'(t)$ is always positive and therefore FIFO is always satisfied if $\gamma > 0$. In view of that, in Tables 1(a) to 1(c), we consider only example values of $\gamma \le 0$, in particular $\gamma = -0.8$, $\gamma = -0.5$, and $\gamma = 0$.

Tables 1(a) and 1(b) illustrate that when linear travel-time functions are inhomogeneous, as in (11), then FIFO violations can easily occur. This is in sharp contrast to the behavior of linear homogeneous travel-time functions: each of the three papers discussed in Sections 3–5 have shown that when the travel-time functions are linear and homogeneous then FIFO is always satisfied.

In the papers discussed in Sections 4 and 5, it is shown that part of the sufficient conditions for FIFO derived for the nonlinear homogeneous travel-time models is that $f_x(x) < 1/B$ where B is an upper bound on $u(t)$. In Sections 4 and 5 this is extended to nonlinear inhomogeneous travel-time functions, so that the sufficient conditions for FIFO include $f_x(x,t) < 1/B$ where B is again an upper bound on $u(t)$.

Now consider the second term before the ">0" in (21), i.e., time $f_x(x(t),t)x'(t)$. The term $f_x(x(t),t)$ is always nonnegative so we consider only the case when $x'(t)$ is negative, because that is the case that is most constraining in the condition (21). Multiplying through $f_x(x(t),t) < 1/B$ by $x'(t)$ gives $f_x(x(t),t)x'(t) > x'(t)/B$: note that the direction of the inequality is reversed as a result of multiplying through by a negative. Substituting $x'(t)/B$ for $f_x(x(t),t)x'(t)$ in (21) gives

$$\tau'(t) = 1 + x'(t)/B + f_t(x(t),t) > 0. \tag{21$'$}$$

This is still a sufficient condition because $f_x(x(t),t) \cdot x'(t) > x'(t)/B$ ensures that if (21′) is satisfied then so is (21). Recall that $x'(t) = u(t) - v(t)$ and substituting this in (21′) and also assuming that $f(x(t),t)$ takes the separable form $f(x(t)) + \gamma t$ so that $f_t(x(t),t) = \gamma$, gives

$$\tau'(t) = 1 + (u(t) - v(t))/B + \gamma > 0. \tag{21$''$}$$

This is the same form as (23) so we can again illustrate this in the same way as for (23). Recall that, to make the results illustrated in Tables 1(a)–1(c) more general, we reduced the necessary and sufficient conditions for FIFO from (23) to (24) by measuring the inflows and outflows $u(t)$ and $v(t)$ as fractions k_1 and k_2 of the link capacity B. In the same way, and for the same reason, we reduce the necessary and sufficient FIFO condition (21″) to (24), by substituting $u(t) = k_1 B$ and $v(t) = k_2 B$ into (21″). Thus Tables 1(a)–1(c) that illustrate FIFO adherence and violations for the linear inhomogeneous travel-time functions, also illustrate this for the nonlinear inhomogeneous travel-time functions.

Some simple examples of FIFO violations.

To further illustrate how a FIFO violation can occur for the travel-time models (1) and (3) it is useful to give some simple examples.

Example 1. *An intuitive example of FIFO violations for the homogeneous case* (1). We assume, as usual, that the travel-time function $f(x(t))$ is nondecreasing in x so that $f_x(x(t))$ is nonnegative hence, from (20), a FIFO violation requires that $x'(t) = u(t) - v(t)$ is negative and of sufficient magnitude to ensure that $f_x(x(t)) \cdot x'(t) \leq -1$.

Suppose that the inflow $u(t)$ and outflow $v(t)$ are positive, equal $(u(t) = v(t))$, and constant leading up to time t and suppose that the exogenous inflow rate $u(t)$ then starts decreasing rapidly. That does not affect the outflow rate $v(t)$ until the inflow has traversed the link to the exit, therefore the link occupancy $x(t)$ decreases at the same rate as $u(t)$, which causes a decreasing travel time $f(x(t))$. The latter will decrease faster if the travel-time function is sloping steeply upwards (i.e., if $f_x(x(t)) > 0$ is large) since, in that case, even a small decrease in $x(t)$ can produce a large decrease in the travel time $f(x(t))$. In that case, the travel time may decline so fast over time that the current vehicles may exit before preceding vehicles, which is a FIFO violation.

***Two simple examples of FIFO violations for the inhomogeneous case* (3).**

Example 2. Example 1 can easily be extended to allow inhomogeneity over time, by assuming (3) rather than (1), so that the FIFO condition is (21) rather than (20). From Example 1, $f_x(x(t))x'(t) \leq -1$ becomes $f_x(x(t), t)x'(t) \leq -1$ so that the first two terms on the l.h.s. of (21) become less than zero. If we let inhomogeneity over time take the form of travel time declining over time for any given x (i.e., time $f_t(x(t), t)$) then the final term on the l.h.s. of (21) is also negative. In that case, the FIFO condition (21) is violated even more easily than in Example 1.

Example 3. Up to time t or $t + \tau(t)$ let the inflow $u(t)$ equal outflow $v(t)$ so that $x'(t) = u(t) - v(t) = 0$ and let $f_t(x(t), t) = 0$, which reduces the l.h.s. of (21) to +1, hence FIFO holds. Now suppose that from time t there is inhomogeneity over time so that, for a given x, the travel time declines at a rate $f_t(x(t), t) < -1$. In that case, at time t the l.h.s. of (21) reduces to $f_t(x(t), t) < 0$ so FIFO is violated.

8. Concluding Remarks

In dynamic traffic assignment modeling, a series of papers have treated the link travel times as functions of the number of the vehicles currently on the link. That is, for traffic entering at time t the travel time is $f(x(t))$ where $x(t)$ is the number of vehicles on the link at time t. However, when this travel-time function is used to model traffic flows varying over time on a link it can violate a desirable first-in-first-out (FIFO) property. A number of papers have investigated this and other properties of the model and have derived conditions that are sufficient to ensure that these properties, including FIFO, will hold. The key sufficient condition is an upper bound on the gradient of $f(x(t))$, namely $f_x(x) < 1/B$ where B is the upper bound on the entry flow rate $u(t)$.

In this paper we note that the link travel time can also vary directly with time, independently of the number of vehicles on the link, so that the travel-time function becomes $f(x(t), t)$. We derive conditions that are sufficient to ensure that the properties, including FIFO, derived by earlier authors for the homogeneous travel-time function $f(x(t))$ will also hold for the inhomogeneous travel-time function $f(x(t), t)$. We retain the conditions needed to ensure FIFO with respect to changes in $x(t)$ and derive an additional condition that will ensure FIFO when both arguments in $f(x(t), t)$ vary, i.e., $x(t)$ and t both vary. We derive this additional condition first for linear travel-time functions, in Section 3, by extending results from Friesz et al. (1993), and derive it for nonlinear travel-time functions, in Sections 4–6, by extending results from Xu et al. (1999), Zhu and Marcotte (2000); and Carey and Ge (2005b), respectively.

For nonlinear travel-time functions the additional condition, that is sufficient to ensure FIFO, is a lower bound on the gradient $f_t(x, t)$, namely $f_t(x, t) > -u(t)/B$. For linear travel-time functions this additional condition reduces to $f_t(x, t) = \gamma'(t) \geq -u(t)/B$. Both of these bounds show some similarity to the bound $f_x(x) < 1/B$ already derived, in the earlier papers referred to in Sections 2–6, for the case of travel-time functions $f(x(t))$. All of these conditions show that the link inflows, and especially their upper bound B, play a major role in determining whether FIFO is ensured or not.

In Section 7 we give some examples of link travel-time functions varying over time. We also give some numerical examples to illustrate when travel-time functions, especially the inhomogeneous functions, will or will not adhere to FIFO. From these examples, and from the results in the earlier sections, it is clear that letting link travel times vary with time of day, in addition to varying with link occupancy, significantly increases the chances of FIFO violations.

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Endnotes

[1] There seems to be a minor typing error on the l.h.s. of Equation (28) in Friesz et al. (1993): the $\pi_t'(t)$ should presumably be $\tau_2'(t)$.

[2] There is a minor typing error on the last line of page 413 of Zhu and Marcotte (2000): the ≤ should be ≥. Also, in the seventh line after Equation (37), $\sum_{p\in P} v^p(t) = 0$ is accidentally typed as $\sum_{p\in P} u^p(t) = 0$.

[3] The FIFO conditions (20)–(23) are derived much more simply than the FIFO conditions derived in earlier sections. However, (a) they can be shown to be consistent with the latter, and (b) they do not replace the latter since they are not "operational," that can be explained as follows. The FIFO condition (8) that is used in Sections 3–6 depends on the gradient $\mathrm{d}f(x)/\mathrm{d}x - f_x(x)$ of the given travel-time function $f(x)$. This has the advantage that the range of possible values of $f_x(x)$ can be determined in advance from the given function $f(x)$. By contrast, the FIFO conditions (20) and (21) depend on $\mathrm{d}f(x(t))/\mathrm{d}t = f_x(x(t))x'(t)$ and $\mathrm{d}f(x(t),t)/\mathrm{d}t = f_x(x(t),t)x'(t)$ that in turn depend on the current values of $x(t)$, $u(t)$, and $v(t)$ that cannot be evaluated in advance. Nevertheless, they provide interesting insights into when FIFO will or will not be satisfied and enable us to construct simple examples of FIFO violations.

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