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Dynamics of modulationally unstable ion-acoustic wavepackets in plasmas with negative ions

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Abstract. In this paper we study the propagation of nonlinear ion-acoustic waves in plasmas with negative ions. The Gardner equation governing these waves in plasmas with the negative ion concentration close to critical is derived. The weakly nonlinear theory of modulational instability based on the use of the nonlinear Schrödinger equation is discussed. The investigation of the nonlinear dynamics of modulationally unstable quasi-harmonic wavepackets is carried out by the numerical solution of the Gardner equation. The results are compared with the predictions of the weakly nonlinear theory.

1. Introduction

Nonlinear ion-acoustic waves in plasmas have been studied for more than four decades. Washimi and Taniuti (1966), Su and Gardner (1969), Tappert (1972) and Tappert (1973) derived the Korteweg–de Vries (KdV) equation for ion-acoustic waves propagating in a plasma that consists of electrons and one type of positive ions. The KdV solitons in electron–ion plasmas were experimentally studied by Ikezi (1973), Watanabe (1975), Tran (1979), Nakamura (1982) and Lonngren (1983). Das and Tagare (1975), Das (1977, 1979), Tagare (1986), Kalita and Devi (1993), Kalita and Barman (1995) and Kalita and Das (2002) generalized the derivation of the KdV equation for multicomponent plasmas, where there are both positive and negative ions. In particular, it was shown that the solitons in a plasma with negative ions can be either compressional or rarefactional depending on the negative ion concentration. In plasmas without negative ions solitons are always compressional. The theoretical results were experimentally confirmed by Watanabe (1978), Ludwig et al. (1984) and Nakamura et al. (1985).

When the concentration of negative ions is equal to a critical value, the coefficient at the nonlinear terms in the KdV equation is equal to zero, so that the cubic nonlinearity has to be taken into account. As a result, the nonlinear ion-acoustic waves are described by the modified Korteweg–de Vries (mKdV) equation. The mKdV equation for ion-acoustic waves in a plasma with the critical concentration of negative ions was derived by Watanabe (1984), Nakamura and Tsukabayashi (1985), Tagare (1986), Verheest (1988), Kalita and Kalita (1990) and Kalita and Das (2002). The mKdV solitons in such a plasma were experimentally observed by Nakamura et al. (1985). Weakly two-dimensional nonlinear ion-acoustic waves in a plasma with the critical concentration of negative ions are described by the modified Kadomtsev–Petviashvili equations. Recently Tsuji and Oikawa (2004) used this equation to study the oblique interaction of solitary waves.

If the concentration of negative ions is not exactly equal to the critical value, but close to it, then both the quadratic and cubic nonlinearity should be taken into account. Watanabe (1984) has shown that, in this case, the nonlinear ion-acoustic waves are described by the Gardner equation.

In all studies on nonlinear ion-acoustic wave propagation most of the attention was paid to solitons, in particular, to the dependences of their parameters (the amplitude, width and propagation speed) on plasma parameters (e.g. negative ion concentration). Another interesting aspect of the nonlinear theory of ion-acoustic waves is the nonlinear development of modulationally unstable wavepackets. Recently Grimshaw et al. (2005) carried out a general study of this problem for the mKdV equation. The aim of this paper is to extend this study to the Gardner equation with a particular emphasis on its application to the ion-acoustic waves. The paper is organized as follows. In the next section we give a brief derivation of the Gardner equation for ion-acoustic waves in plasmas with negative ions under slightly more general assumptions than those adopted by Watanabe (1984). In Sec. 3 we discuss the analytical results on modulational instability obtained on the basis of the nonlinear Schrödinger equation, and then present the results of numerical modelling of modulational instability of the initial quasi-harmonic wavepackets with small and moderate amplitudes. Section 4 contains a summary of the results and our conclusions.

2. Derivation of Gardner equation

In this section we give a brief derivation of the Gardner equation for ion-acoustic waves in a plasma with negative ions. Similar to Watanabe (1984) we assume that the ions are cold, the electrons are isothermal and neglect the electron inertia. However, in contrast to Watanabe (1984), who assumed that both positive and negative ions bear only one elementary charge, we allow arbitrary ion charges. Then the system of equations governing the one-dimensional plasma motion can be written as

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial (n_\alpha u_\alpha)}{\partial x} = 0, \tag{2.1a}$$

$$\frac{\partial u_\alpha}{\partial t} + u_\alpha \frac{\partial u_\alpha}{\partial x} = -\frac{\chi_\alpha q_\alpha}{m_\alpha}\frac{\partial \phi}{\partial x}, \tag{2.1b}$$

$$\varepsilon_0 \frac{\partial^2 \phi}{\partial x^2} = e n_{\mathrm{e}} - q_+ n_+ + q_- n_-, \tag{2.1c}$$

$$n_{\mathrm{e}} = n_0 e^{e\phi/\kappa\Theta}. \tag{2.1d}$$

Here n_α, u_α, m_α and q_α are the number density, the velocity, the mass and the charge of the positive ions when $\alpha = +$ and the negative ions when $\alpha = -$; n_{e}

and n_0 are the electron number density and the unperturbed electron number density respectively; Θ is the electron temperature (assumed constant), ϕ is the electric potential, e is the elementary charge, κ is the Boltzmann constant, ε_0 is the permittivity of empty space and $\chi_\pm = \pm 1$.

We assume that the unperturbed plasma is neutral. Introducing the notation $Z_\alpha = q_\alpha / e$, we write this condition as

$$Z_+ n_{0+} = Z_- n_{0-} + n_0, \tag{2.2}$$

where the subscript '0' indicates an unperturbed quantity.

The standard derivation of the KdV equation (see, e.g., Das and Tagare 1975) gives the coefficient at the nonlinear term proportional to the quantity

$$W = \frac{1}{3n_0}\left(\frac{n_{0+}Z_+^2}{m_+} + \frac{n_{0-}Z_-^2}{m_-}\right)^2 - \frac{n_{0+}Z_+^3}{m_+^2} + \frac{n_{0-}Z_-^3}{m_-^2} \tag{2.3}$$

with the proportionality coefficient that is always different from zero. To derive the mKdV equation Watanabe (1984) assumed that $W = 0$. It follows from (2.2) and the condition $W = 0$ that

$$n_{0+} = n_{0+}^{(0)} \equiv \frac{n_0[3\eta - 1 + \sqrt{3(3\eta^2 - 2\eta + 3)}]}{2(\eta + 1)Z_+}, \tag{2.4a}$$

$$n_{0-} = n_{0-}^{(0)} \equiv \frac{n_0[\eta - 3 + \sqrt{3(3\eta^2 - 2\eta + 3)}]}{2(\eta + 1)Z_-}, \tag{2.4b}$$

where $\eta = Z_+ m_- / Z_- m_+$. It is easy to show that $n_{0+}Z_+/n_0$ increases monotonically from 1 to 3 and $n_{0-}Z_-/n_0$ increases monotonically from 0 to 2 when η varies from 0 to ∞.

Since we want to derive the Gardner equation, where both the quadratic and cubic nonlinear terms are present, we take

$$n_{0\alpha} = n_{0\alpha}^{(0)} + \epsilon n_{0\alpha}^{(1)}, \quad n_{0\alpha}^{(1)} = \gamma n_0 / Z_\alpha, \tag{2.5}$$

where $\epsilon \ll 1$ is the dimensionless wave amplitude and γ is a free dimensionless parameter of order unity. With such a choice of n_{0+} and n_{0-}, (2.3) is still satisfied, however W is not equal to zero any longer; instead $W = \mathcal{O}(\epsilon)$.

To derive the Gardner equation we use the reductive perturbation method (e.g. Engelbrecht et al. 1988; Kakutani et al. 1968; Taniuti and Wei 1968). In accordance with this method we introduce the same stretching variables as are used to derive the mKdV equation (e.g. Watanabe 1984), $\xi = \epsilon(x - Vt)$ and $\tau = \epsilon^3 t$. In the new variables (2.1a)–(2.1c) are rewritten as

$$\epsilon^2 \frac{\partial n_\alpha}{\partial \tau} - V\frac{\partial n_\alpha}{\partial \xi} + \frac{\partial(n_\alpha u_\alpha)}{\partial \xi} = 0, \tag{2.6a}$$

$$\epsilon^2 \frac{\partial u_\alpha}{\partial t} - V\frac{\partial u_\alpha}{\partial \xi} + u_\alpha \frac{\partial u_\alpha}{\partial \xi} = -\frac{e\chi_\alpha Z_\alpha}{m_\alpha}\frac{\partial \phi}{\partial \xi}, \tag{2.6b}$$

$$\epsilon^2 \varepsilon_0 \frac{\partial^2 \phi}{\partial \xi^2} = e(n_e - Z_+ n_+ + Z_- n_-). \tag{2.6c}$$

Now we are looking for the solution of the system of equations (2.1d) and (2.6) in the form of expansions

$$
\begin{aligned}
n &= n_0 + \epsilon n^{(1)} + \epsilon^2 n^{(2)} + \epsilon^3 n^{(3)} + \cdots, \\
n_\alpha &= n_{0\alpha}^{(0)} + \epsilon\big(n_{0\alpha}^{(1)} + n_\alpha^{(1)}\big) + \epsilon^2 n_\alpha^{(2)} + \epsilon^3 n_\alpha^{(3)} + \cdots, \\
u_\alpha &= \epsilon u_\alpha^{(1)} + \epsilon^2 u_\alpha^{(2)} + \epsilon^3 u_\alpha^{(3)} + \cdots, \\
\phi &= \epsilon \phi^{(1)} + \epsilon^2 \phi^{(2)} + \epsilon^3 \phi^{(3)} + \cdots.
\end{aligned}
\tag{2.7}
$$

We also write $V = V_0 + \epsilon V_1$. We do not include terms of higher order in the expansion for V because they can be incorporated in the dependence on τ. Substituting these expansions in (2.1d) and (2.6) we obtain in the first-order approximation

$$V_0 \frac{\partial n_\alpha^{(1)}}{\partial \xi} - n_{0\alpha}^{(0)} \frac{\partial u_\alpha^{(1)}}{\partial \xi} = 0, \tag{2.8a}$$

$$V_0 \frac{\partial u_\alpha^{(1)}}{\partial \xi} - \frac{e\chi_\alpha Z_\alpha}{m_\alpha} \frac{\partial \phi^{(1)}}{\partial \xi} = 0, \tag{2.8b}$$

$$n_{\mathrm{e}}^{(1)} - Z_+ n_+^{(1)} + Z_- n_-^{(1)} = 0, \tag{2.8c}$$

$$n_{\mathrm{e}}^{(1)} - \frac{e n_0}{\kappa\Theta} \phi^{(1)} = 0. \tag{2.8d}$$

It is straightforward to show that the system of equations (2.8) has a non-trivial solution if and only if V_0^2 is given by

$$V_0^2 = \frac{\kappa\Theta}{n_0}\left(\frac{n_{0+}^{(0)} Z_+^2}{m_+} + \frac{n_{0-}^{(0)} Z_-^2}{m_-}\right) = \frac{\kappa\Theta Z_+}{2m_+\eta}\big[3\eta - 3 + \sqrt{3(3\eta^2 - 2\eta + 3)}\big]. \tag{2.9}$$

It is easy to show that $V_0^2(\kappa\Theta Z_+/m_+)^{-1}$ monotonically increases from 1 to 3 when η varies from 0 to ∞.

In what follows we consider only waves propagating in the positive x-direction and take $V_0 > 0$. Using (2.8) we can express all of the quantities of the first-order approximation in terms of ϕ_1:

$$u_\alpha^{(1)} = \frac{e\chi_\alpha Z_\alpha}{m_\alpha V_0}\phi_1, \quad n_\alpha^{(1)} = \frac{e\chi_\alpha n_{0\alpha}^{(0)} Z_\alpha}{m_\alpha V_0^2}\phi_1, \quad n_{\mathrm{e}}^{(1)} = \frac{e n_0}{\kappa\Theta}\phi_1. \tag{2.10}$$

In the second-order approximation we obtain, with the aid of (2.5),

$$V_0 \frac{\partial n_\alpha^{(2)}}{\partial \xi} - n_{0\alpha}^{(0)} \frac{\partial u_\alpha^{(2)}}{\partial \xi} = \frac{\partial}{\partial \xi}\big[u_\alpha^{(1)}\big(n_{0\alpha}^{(1)} + n_\alpha^{(1)}\big)\big] - V_1 \frac{\partial n_\alpha^{(1)}}{\partial \xi}, \tag{2.11a}$$

$$V_0 \frac{\partial u_\alpha^{(2)}}{\partial \xi} - \frac{e\chi_\alpha Z_\alpha}{m_\alpha}\frac{\partial \phi^{(2)}}{\partial \xi} = u_\alpha^{(1)} \frac{\partial u_\alpha^{(1)}}{\partial \xi} - V_1 \frac{\partial u_\alpha^{(1)}}{\partial \xi}, \tag{2.11b}$$

$$n_{\mathrm{e}}^{(2)} - Z_+ n_+^{(2)} + Z_- n_-^{(2)} = 0, \tag{2.11c}$$

$$n_{\mathrm{e}}^{(2)} - \frac{e n_0}{\kappa\Theta}\phi^{(2)} = \frac{e^2 n_0}{2\kappa^2\Theta^2}{\phi^{(1)}}^2. \tag{2.11d}$$

Using (2.11a), (2.11b) and (2.11d) we express all quantities of the second-order approximation in terms of $\phi^{(1)}$ and $\phi^{(2)}$:

$$u_\alpha^{(2)} = \frac{e\chi_\alpha Z_\alpha}{m_\alpha V_0}\phi^{(2)} + \frac{e^2 Z_\alpha^2}{2m_\alpha^2 V_0^3}\phi^{(1)\,2} - \frac{e\chi_\alpha Z_\alpha V_1}{m_\alpha V_0^2}\phi^{(1)}, \tag{2.12a}$$

$$n_\alpha^{(2)} = \frac{e\chi_\alpha n_{0\alpha}^{(0)} Z_\alpha}{m_\alpha V_0^2}\phi^{(2)} + \frac{3e^2 n_{0\alpha}^{(0)} Z_\alpha^2}{2m_\alpha^2 V_0^4}\phi^{(1)\,2} + \frac{e\chi_\alpha Z_\alpha\left(V_0 n_{0\alpha}^{(1)} - 2V_1 n_{0\alpha}^{(0)}\right)}{m_\alpha V_0^3}\phi^{(1)}, \tag{2.12b}$$

$$n_{\mathrm{e}}^{(2)} = \frac{en_0}{\kappa\Theta}\phi^{(2)} + \frac{e^2 n_0}{2\kappa^2\Theta^2}\phi^{(1)\,2}. \tag{2.12c}$$

Substituting (2.12b) and (2.12c) in (2.11c) and using the fact that $W^{(0)} = 0$, where $W^{(0)}$ is given by (2.3) with $n_{0\alpha}^{(0)}$ substituted for $n_{0\alpha}$, we obtain

$$2V_1\left(\frac{Z_+^2 n_{0+}^{(0)}}{m_+} + \frac{Z_-^2 n_{0-}^{(0)}}{m_-}\right) = V_0\left(\frac{Z_+^2 n_{0+}^{(1)}}{m_+} + \frac{Z_-^2 n_{0-}^{(1)}}{m_-}\right). \tag{2.13}$$

With the aid of (2.4) and (2.5) this equation reduces to

$$V_1 = \frac{\gamma(\eta+1)V_0}{3\eta - 3 + \sqrt{3(3\eta^2 - 2\eta + 3)}}. \tag{2.14}$$

It is easy to show that $V_1/\gamma V_0$ monotonically decreases from ∞ to 1/6 when η varies from 0 to ∞.

In the third-order approximation we obtain, from (2.1d) and (2.6),

$$V_0\frac{\partial n_\alpha^{(3)}}{\partial\xi} - n_{0\alpha}^{(0)}\frac{\partial u_\alpha^{(3)}}{\partial\xi} = \frac{\partial n_\alpha^{(1)}}{\partial\tau} + \frac{\partial}{\partial\xi}\left[u_\alpha^{(2)}\left(n_{0\alpha}^{(1)} + n_\alpha^{(1)}\right) + u_\alpha^{(1)} n_\alpha^{(2)}\right] - V_1\frac{\partial n_\alpha^{(2)}}{\partial\xi}, \tag{2.15a}$$

$$V_0\frac{\partial u_\alpha^{(3)}}{\partial\xi} - \frac{e\chi_\alpha Z_\alpha}{m_\alpha}\frac{\partial\phi^{(3)}}{\partial\xi} = \frac{\partial u_\alpha^{(1)}}{\partial\tau} + \frac{\partial}{\partial\xi}\left(u_\alpha^{(1)} u_\alpha^{(2)}\right) - V_1\frac{\partial u_\alpha^{(2)}}{\partial\xi}, \tag{2.15b}$$

$$n_{\mathrm{e}}^{(3)} - Z_+ n_+^{(3)} + Z_- n_-^{(3)} = \frac{\varepsilon_0}{e}\frac{\partial^2\phi^{(1)}}{\partial\xi^2}, \tag{2.15c}$$

$$n_{\mathrm{e}}^{(3)} - \frac{en_0}{\kappa T}\phi^{(3)} = \frac{e^2 n_0}{\kappa^2\Theta^2}\phi^{(1)}\phi^{(2)} + \frac{e^3 n_0}{6\kappa^3\Theta^3}\phi^{(1)\,3}. \tag{2.15d}$$

Now we use (2.15a), (2.15b) and (2.15d) to express $n_+^{(3)}$, $n_-^{(3)}$ and $n_{\mathrm{e}}^{(3)}$ in terms of $\phi^{(3)}$ and the quantities of the first- and second-order approximation. Substituting the obtained expressions in (2.15c) we find that the terms proportional to $\phi^{(3)}$ cancel each other, so that we arrive at the equation relating the quantities of the first- and second-order approximation. Using (2.12) we obtain that the quantities of the second-order approximation also cancel each other, so that the derived equation contains only the quantities of the first-order approximation. Using (2.10)we write this equation in the form

$$\frac{2}{V_0^3}\left(\frac{n_{0+}^{(0)}Z_+^2}{m_+}+\frac{n_{0-}^{(0)}Z_-^2}{m_-}\right)\frac{\partial\phi^{(1)}}{\partial\tau}-\frac{2V_1^2}{V_0^4}\left(\frac{n_{0+}^{(0)}Z_+^2}{m_+}+\frac{n_{0-}^{(0)}Z_-^2}{m_-}\right)\frac{\partial\phi^{(1)}}{\partial\xi}$$
$$+\frac{e}{V_0^4}\left[\frac{3n_{0+}^{(1)}Z_+^3}{m_+^2}-\frac{3n_{0-}^{(1)}Z_-^3}{m_-^2}-\frac{12V_1}{V_0}\left(\frac{n_{0+}^{(0)}Z_+^3}{m_+^2}-\frac{n_{0-}^{(0)}Z_-^3}{m_-^2}\right)\right]\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi}$$
$$-\left(\frac{e^2n_0}{2\kappa^3\Theta^3}-\frac{15e^2n_{0+}^{(0)}Z_+^4}{2m_+^3V_0^6}-\frac{15e^2n_{0-}^{(0)}Z_-^4}{2m_-^3V_0^6}\right){\phi^{(1)}}^2\frac{\partial\phi^{(1)}}{\partial\xi}+\frac{\varepsilon_0}{e^2}\frac{\partial^3\phi^{(1)}}{\partial\xi^3}=0. \quad (2.16)$$

Introducing the dimensionless quantities

$$\psi=\frac{e\phi^{(1)}}{\kappa\Theta},\quad \tau'=\tau e\left(\frac{n_0Z_+}{\varepsilon_0m_+}\right)^{1/2},\quad \xi'=e\left(\frac{n_0}{\varepsilon_0\kappa\Theta}\right)^{1/2}\left(\xi+\frac{V_1^2}{V_0}\tau\right), \quad (2.17)$$

using (2.4) and (2.9), and dropping the prime at τ and ξ, we rewrite (2.16) as

$$\frac{\partial\psi}{\partial\tau}-a\psi\frac{\partial\psi}{\partial\xi}+b\psi^2\frac{\partial\psi}{\partial\xi}+\beta\frac{\partial^3\psi}{\partial\xi^3}=0, \quad (2.18)$$

where

$$a=\frac{\gamma(\eta+1)\sqrt{6(3\eta^2-2\eta+3)}}{\eta^{1/2}[3\eta-3+\sqrt{3(3\eta^2-2\eta+3)}]^{3/2}}, \quad (2.19a)$$

$$b=\frac{\sqrt{6}[3(5\eta^2-6\eta+5)-5(\eta-1)\sqrt{3(3\eta^2-2\eta+3)}]}{3\eta[\sqrt{3(3\eta^2-2\eta+3)}-3\eta+3]^{1/2}}, \quad (2.19b)$$

$$\beta=\left(\frac{3\eta-3+\sqrt{3(3\eta^2-2\eta+3)}}{8\eta}\right)^{1/2}. \quad (2.19c)$$

The quantity β is real because

$$\sqrt{3(3\eta^2-2\eta+3)}>\sqrt{9\eta^2-18\eta+9}=|3\eta-3|.$$

It can be verified that equation (30) of Watanabe (1984) rewritten in the variables used in this paper coincides with (2.18) in the case where $Z_+=Z_-=1$.

The dependences of a/γ, b and β on η are shown in Fig. 1. The first two quantities are monotonically decreasing functions of η, while β is a monotonically increasing function of η, and

$$a\approx\frac{3\gamma}{2\eta^2}\quad\text{for }\eta\ll 1,\quad a\to\frac{\gamma\sqrt{3}}{6}\quad\text{as }\eta\to\infty, \quad (2.20a)$$

$$b\approx\frac{10}{\eta}\quad\text{for }\eta\ll 1,\quad b\to\frac{2\sqrt{3}}{3}\quad\text{as }\eta\to\infty, \quad (2.20b)$$

$$\beta\to\frac{1}{2}\quad\text{as }\eta\to 0,\quad \beta\to\frac{\sqrt{3}}{2}\quad\text{as }\eta\to\infty. \quad (2.20c)$$

Equation (2.18) is the Gardner equation (also called the extended KdV equation). It is used in the next section to study the nonlinear evolution of modulationally unstable wavepackets.

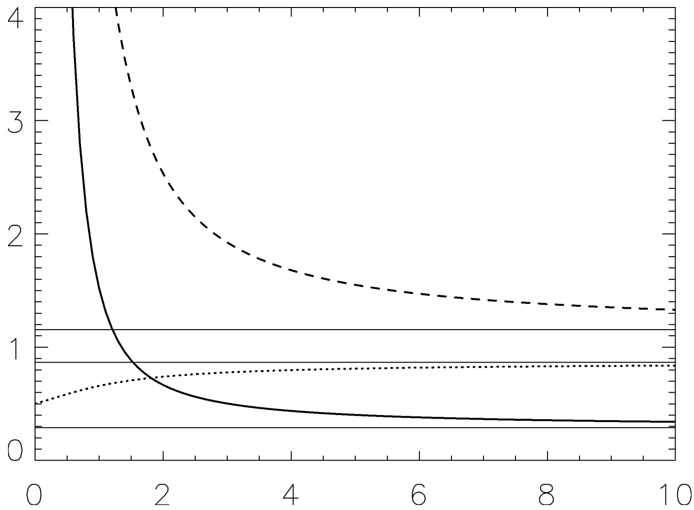

Figure 1. The solid, dashed and dotted curves show the dependences on η of a/γ, b and β, respectively. The horizontal lines show the asymptotic values of a/γ, b and β as $\eta \to \infty$.

3. Modulational instability of nonlinear wavepackets

In this section we use the Gardner equation (2.18) to study the dynamics of modulationally unstable wavepackets. When the ion densities are equal to their critical values, i.e. $\gamma = 0$, we have $a = 0$ and (2.18) reduces to the mKdV equation. The dynamics of modulationally unstable wavepackets described by the mKdV equation has been already extensively studied (see, e.g., Grimshaw et al. 2005). Here we aim to study the role of the quadratic nonlinearity described by the second term in (2.18).

First of all we recall the results for the weakly nonlinear limit of the modulational instability. In this case the dynamics of wavepackets is described by the nonlinear Schrödinger equation for the complex amplitude of the wavepacket (Grimshaw et al. 2001; Parkes 1987)

$$i\frac{\partial \Psi}{\partial T} = 3\beta k \frac{\partial^2 \Psi}{\partial X^2} + \delta k |\Psi|^2 \Psi. \tag{3.1}$$

This equation governs the dynamics of quasi-monochromatic wave solutions of (2.18), i.e. solutions that have the form

$$\psi(\xi, \tau) = \varepsilon \Psi(X, T) \exp(i\Theta) + \text{c.c.}, \tag{3.2}$$

where $X = \varepsilon(\xi + 3\beta k^2 \tau)$, $T = \varepsilon^2 \tau$, $\Theta = k\xi - \omega\tau$, k is the carrier wavenumber, $\omega = -\beta k^3$, ε is an arbitrary small parameter and c.c. denotes the complex conjugate. The coefficient δ in (3.1) is given by (Grimshaw et al. 2001)

$$\delta = b - \frac{a^2}{6\beta k^2}. \tag{3.3}$$

Here it is worth making one comment. The nonlinear Schrödinger equation describing the nonlinear evolution of modulations of a harmonic carrier wave can be derived directly from the system of equations (2.1). A similar derivation has been performed for ion-electron plasmas by, e.g., Chan and Seshadri (1975), for ion-electron plasmas with two populations of electron with different temperatures by Kourakis and Shukla (2003a), for plasmas with negative ions by Saito et al. (1984),

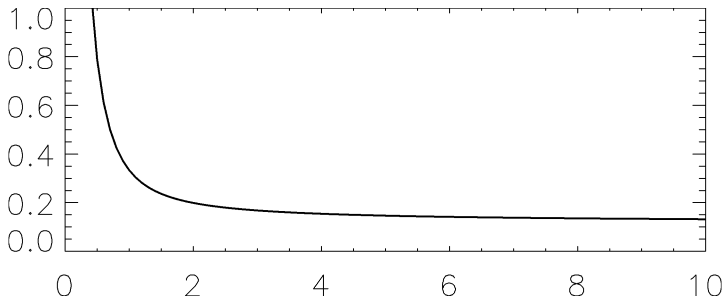

Figure 2. The dependences of $k_c/|\gamma|$ on η.

and for multicomponent plasma by, e.g., Mishra et al. (1994) (see also the derivation of the nonlinear Schrödinger equation for dusty plasma by Kourakis and Shukla (2003b, 2005)). In all of these papers the derivation of the nonlinear Schrödinger equation has been carried out for arbitrary frequency of the carrier wave. If we use a similar approach to derive the nonlinear Schrödinger equation form (2.1), then we obtain an equation similar to (3.1), but with much more complicated expressions for the coefficients β and δ. This equation should, in principle, reduce to (3.1) in the limit of low wave frequency, i.e. when the frequency of the carrier wave is much smaller than the ion plasma frequency ω_{pi}.

Equation (3.1) has the solution in the form of monochromatic wave given by

$$\Psi_0 = A_0 \exp[i(KX - \Omega T)], \quad \Omega = k(\delta A_0^2 - 3\beta K^2). \tag{3.4}$$

To study the stability of this solution we write $\Psi = Ae^{i\theta}$. The substitution of this expression in (3.1) results in a system of two equations for the real variables A and θ. Then we take $A = A_0 + A'$ and $\theta = KX - \Omega T + \theta'$, and linearize the system of equations for A and θ with respect to A' and θ'. Next we look for the solution of the obtained linear system of equations in the form $A', \theta' \sim \exp[i(\kappa X - \sigma T)]$. As a result we arrive at the dispersion equation

$$(\sigma + 6k\beta K\kappa)^2 = 3\beta k^2\kappa^2(3\beta\kappa^2 - 2\delta A_0^2). \tag{3.5}$$

It follows from this equation that the solution (3.4) is stable when $\beta\delta < 0$, and unstable when $\beta\delta > 0$. This is the well-known criterion for the modulational or Benjamin–Feir instability (e.g. Benjamin and Feir 1967; Newell 1985). Since, in accordance with (2.19c), $\beta > 0$, the stability of monochromatic ion-acoustic waves is completely determined by the sign of δ. It follows from (3.3) that the condition for the onset of the modulational instability is

$$k > k_c = \frac{|a|}{\sqrt{6b\beta}}. \tag{3.6}$$

In a particular case when the negative ion density is equal to its critical value, i.e. $\gamma = 0$, we have $a = 0$ and the monochromatic wave is unstable for any value of k. In the general case this wave is unstable if and only if the carrier wavenumber k is sufficiently large. The dependence of $k_c/|\gamma|$ on η is shown in Fig. 2. It is a monotonically decreasing function, and its behaviour for small and large values of η is given by

$$\frac{k_c}{|\gamma|} \approx \frac{\sqrt{30}}{20\eta^{3/2}} \quad \text{for } \eta \ll 1, \qquad \frac{k_c}{|\gamma|} \to \frac{\sqrt{2}}{12} \quad \text{as } \eta \to \infty. \tag{3.7}$$

It is also follows from (3.5) that the solution (3.4) is only unstable with respect to relatively long-wavelength modulations with the wavenumber κ satisfying

$$\kappa < \kappa_{\rm lim} = A_0\sqrt{\frac{2\delta}{3\beta}}. \qquad (3.8)$$

The instability increment takes its maximum value, δA_0^2, at $\kappa = \kappa_{\rm lim}/\sqrt{2}$.

To study the nonlinear dynamics of modulationally unstable wavepackets we solved the initial value problem for the discrete nonlinear Schrödinger equation numerically. To do this it is convenient to rewrite (2.18) in such new variables that it contains only one coefficient characterizing the effect of quadratic nonlinearity. These new variables are given by

$$\Upsilon = -(b/6)^{1/2}\beta^{-1/6}\psi\,{\rm sgn}(a), \quad \zeta = \beta^{-1/3}\xi, \qquad (3.9)$$

where we take ${\rm sgn}(a) = 1$ when $a = 0$. In these new variables (2.18) takes the form

$$\frac{\partial\Upsilon}{\partial\tau} + 6\alpha\Upsilon\frac{\partial\Upsilon}{\partial\zeta} + 6\Upsilon^2\frac{\partial\Upsilon}{\partial\zeta} + \frac{\partial^3\Upsilon}{\partial\zeta^3} = 0, \qquad (3.10)$$

where $\alpha = |a|(6b)^{-1/2}\beta^{-1/6}$. When $\alpha = 0$, (3.8) becomes the mKdV equation. Note that (3.10) cannot be reduced to the KdV equation because the coefficient at the term describing the cubic nonlinearity is fixed. This form of equation is convenient for studying the effect of quadratic nonlinearity.

In the new variables the criterion for the modulational instability (3.5) and the inequality (3.8) are rewritten as

$$k > \alpha, \quad \kappa < 2A_0\sqrt{1 - \frac{\alpha^2}{k^2}}. \qquad (3.11)$$

The wavenumber of the fastest growing perturbation is given by

$$\kappa = A_0\sqrt{2 - \frac{2\alpha^2}{k^2}}. \qquad (3.12)$$

At this wavenumber the instability increment takes its maximum value $\tilde{\delta}A_0^2$, where

$$\tilde{\delta} = 6\left(1 - \frac{\alpha^2}{k^2}\right). \qquad (3.13)$$

For the numerical solution of (3.10) we used the finite-difference scheme described by Berezin (1987). The number of mesh points in the calculation domain was 8000. The size of the calculational domain was 400, and the periodic boundary conditions were imposed at the domain boundaries. The time step was taken in accordance with the Courant criterion. The initial condition was chosen in the form of a modulated harmonic wave,

$$\Upsilon = A\{1 + m\cos(\kappa\zeta - \pi)\}\sin(k\zeta). \qquad (3.14)$$

The main purpose of our numerical modelling was to study the role of the quadratic nonlinearity. To do this we carried out calculations with $\alpha = 1$ and $\alpha = 0$ (corresponding to the mKdV equation), and then compared the results. In all calculations we took $m = 0.05$, $k = 1.256$ and $\kappa = 0.0157$, so that the first inequality in (3.11) was satisfied for both values of α. Figure 3 shows the shape of the initial wavepacket with $A = 0.05$. In accordance with (3.12) the increment of the modulational instability

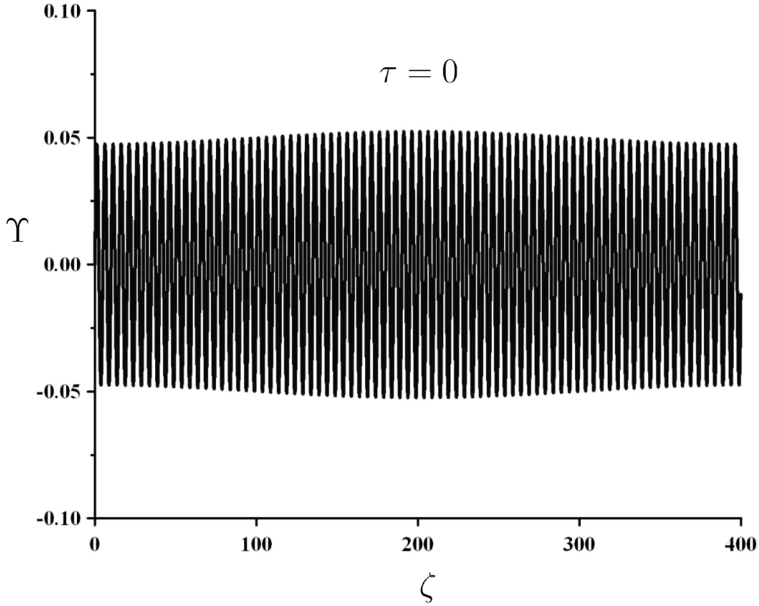

Figure 3. Initial modulated wave group.

takes its maximum value at $A = 2A_0 = 0.0367$ for $\alpha = 1$, and at $A = 2A_0 = 0.0222$ for $\alpha = 0$.

We performed calculations for two values of the wave amplitude, $A = 0.05$ (weakly nonlinear wave) and $A = 0.23$ (moderately nonlinear wave). The first value is close to the value corresponding to the maximum growth rate for $\alpha = 1$, and about twice as large as the value corresponding to the maximum growth rate for $\alpha = 0$. Since this value is small, we expected that the wave dynamics would be close to that described by the nonlinear Schrödinger equation. It is important to note that $\tilde{\delta} = 6$ for the mKdV equation, while $\tilde{\delta} \approx 2.2$ for the Gardner equation with $\alpha = 1$. Since the instability increment is proportional to $\tilde{\delta}$, we can expect that the modulational instability will develop more slowly for the Gardner equation than for the mKdV equation. This preliminary conclusion is confirmed by the numerical solution of (3.10) and illustrated in Fig. 4, where the formation of the first wave group with the maximum amplitude is presented. The corresponding time is about 2.3 times as large for the Gardner equation than for the mKdV equation. The spatial scales are also different with the width of the large-amplitude wave group about twice as large for the Gardner equation than for the mKdV equation. In this figure we can also observe the recurrence phenomenon. In the upper right panel corresponding to $\tau = 2400$ for the mKdV equation the waveform is almost the same as it was at $\tau = 0$.

Figure 5 displays the time evolution of the maximum amplitude (crest and trough amplitudes) of the wavepacket. This figure once again clearly demonstrates the difference in the time scales with the wave amplitude changing more rapidly for the mKdV equation than for the Gardner equation. The variation of the wave amplitude is more regular for the Gardner equation. There is no visible difference in the crest and trough amplitudes in complete agreement with the results obtained by using the nonlinear Schrödinger equation. The maximum amplification of the

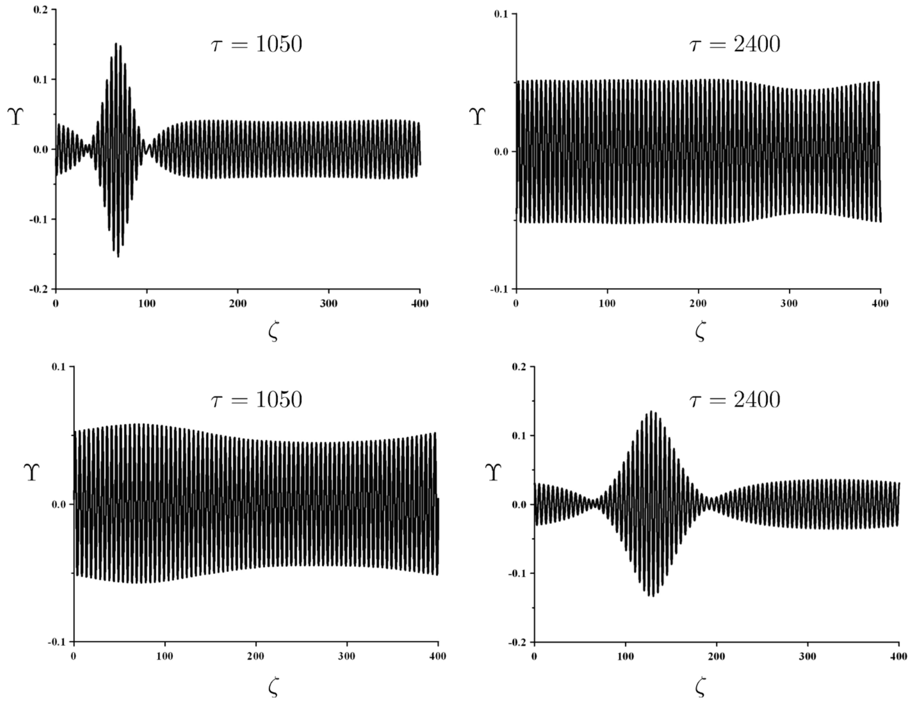

Figure 4. Formation of the first intense wave groups owing to modulational instability of the wave with the initial amplitude $A = 0.05$. The upper panels correspond to $\alpha = 0$ (the mKdV equation) and the lower to $\alpha = 1$ (the Gardner equation).

wave amplitude owing to the modulational instability is approximately 3 for the mKdV equation and (2.7) for the Gardner equation. In both cases the maximum amplification does not exceed the maximum possible value, 3, predicted by the nonlinear Schrödinger equation.

The long-time wave dynamics is illustrated in Fig. 6. In the case of the Gardner equation the intense wave groups appear and disappear periodically. The wave dynamics described by the mKdV equation is richer with the various wave groups appearing and disappearing mode or less randomly.

The second run was done for a moderately nonlinear quasi-monochromatic wave with $A = 0.23$. In this case the characteristic time of the development of the modulational instability is much shorter than it was with $A = 0.05$. This time is approximately 140 for the mKdV equation, and 270 for the Gardner equation. Maximum amplification exceeds the theoretical value, 3, predicted by the weakly nonlinear theory. It is 3.15 for the mKdV equation, and 3.25 for the Gardner equation. Owing to the effect of quadratic nonlinearity the wave profile described by the Gardner equation becomes asymmetric, with the crest amplitude exceeding the trough amplitude. This effect is absent in the case of the mKdV equation because this equation is invariant under the substitution $-\Upsilon \to \Upsilon$. The time evolution of the crest and trough amplitudes are shown in Fig. 7.

In the case of moderate amplitude the generated intense wave groups are shorter than those in the case of small amplitude (see Fig. 8). Usually a few wave groups are generated at approximately the same time, and the dynamics of these groups cannot

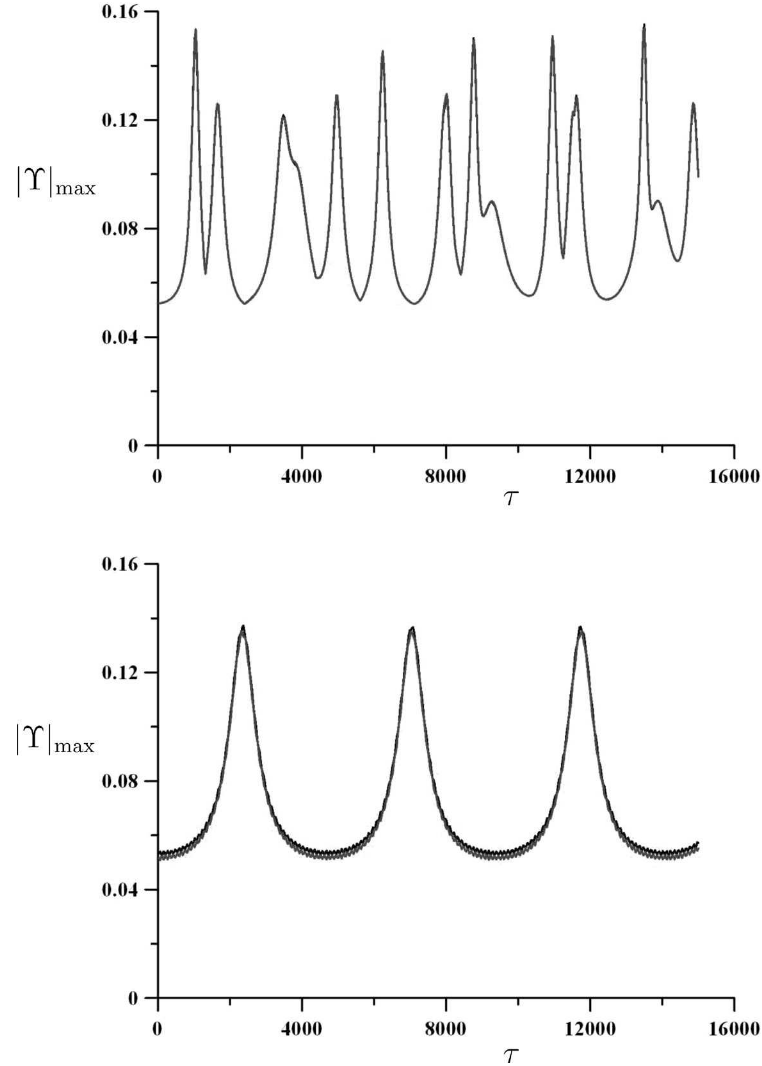

Figure 5. Maximum crest (black) and trough (grey) amplitudes versus time for the wave with the initial amplitude $A = 0.05$. The upper panel corresponds to $\alpha = 0$ (the mKdV equation) and the lower to $\alpha = 1$ (the Gardner equation). In the upper panel the two curves are indistinguishable.

be described by the weakly nonlinear theory based on the nonlinear Schrödinger equation. In average, the number of short-lived intense wave groups increases with time, the number of wave groups being larger in the case of the mKdV equation than in the case of the Gardner equation (see Fig. 9).

4. Summary and conclusions

In this paper we have studied the dynamics of the modulational instability of ion-acoustic waves in plasmas with negative ions. In general, ion-acoustic wave in plasmas are described by the KdV equation. However, when the negative ion

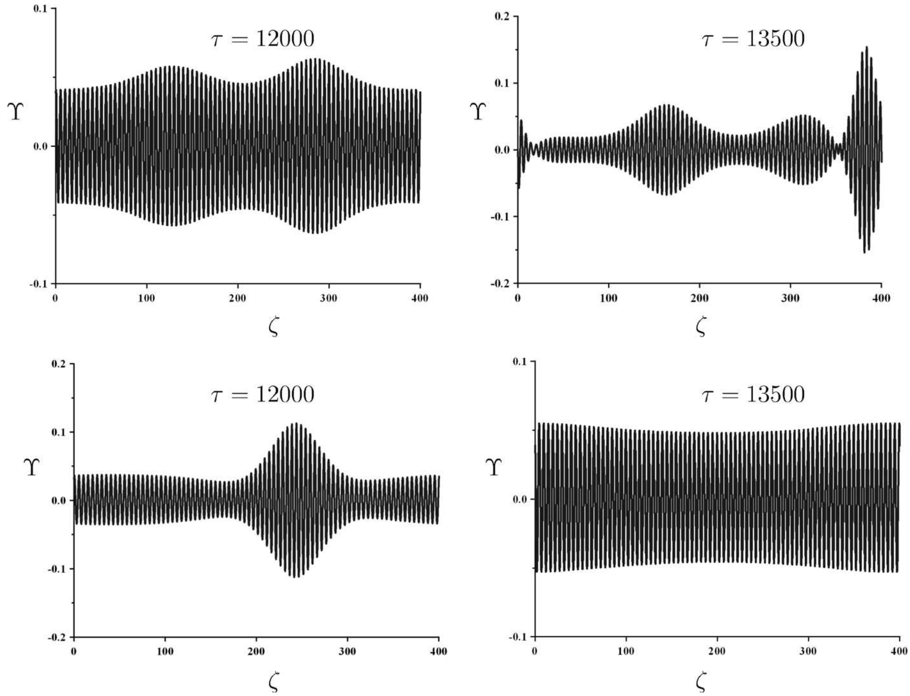

Figure 6. Long-time evolution of the wave with the initial amplitude $A = 0.05$. The upper panels correspond to $\alpha = 0$ (the mKdV equation) and the lower to $\alpha = 1$ (the Gardner equation).

concentration is equal to its critical value, the coefficient at the nonlinear term in the KdV equation vanishes, and the cubic nonlinearity has to be taken into account. As a result, the wave dynamics is described by the mKdV equation. When the negative ion concentration is not exactly equal to its critical value, but close to it, both quadratic and cubic nonlinearity contribute in the wave dynamics, which is now described by the Gardner equation (also called the extended KdV equation).

The Gardner equation for the ion-acoustic waves in plasmas with negative ions was first derived by Watanabe (1984). We repeated this derivation under slightly more general conditions, and using slightly different method. We presented the Gardner equation in dimensionless variables, the units for measuring the time, length and electric potential being the inverse ion plasma frequency, $\omega_{\rm pi}^{-1}$, the thermal speed of electrons times the inverse electron plasma frequency, and the thermal energy of the electrons (or ions) divided by the elementary charge (see (2.17)). A very important property of the Gardner equation for the ion-acoustic waves is that the coefficient at the term describing the cubic nonlinearity is positive.

We then used the Gardner equation to study the dynamics of modulationally unstable wavepackets. First we briefly recalled the results of weakly nonlinear theory based on the use of the nonlinear Schrödinger equation. When the negative ion concentration is equal to its critical value, the Gardner equation reduces to the mKdV equation. Since the coefficient at the nonlinear term in this equation is positive, the weakly nonlinear theory predicts that quasi-monochromatic wave-packets are always modulationally unstable. On the other hand, when the negative

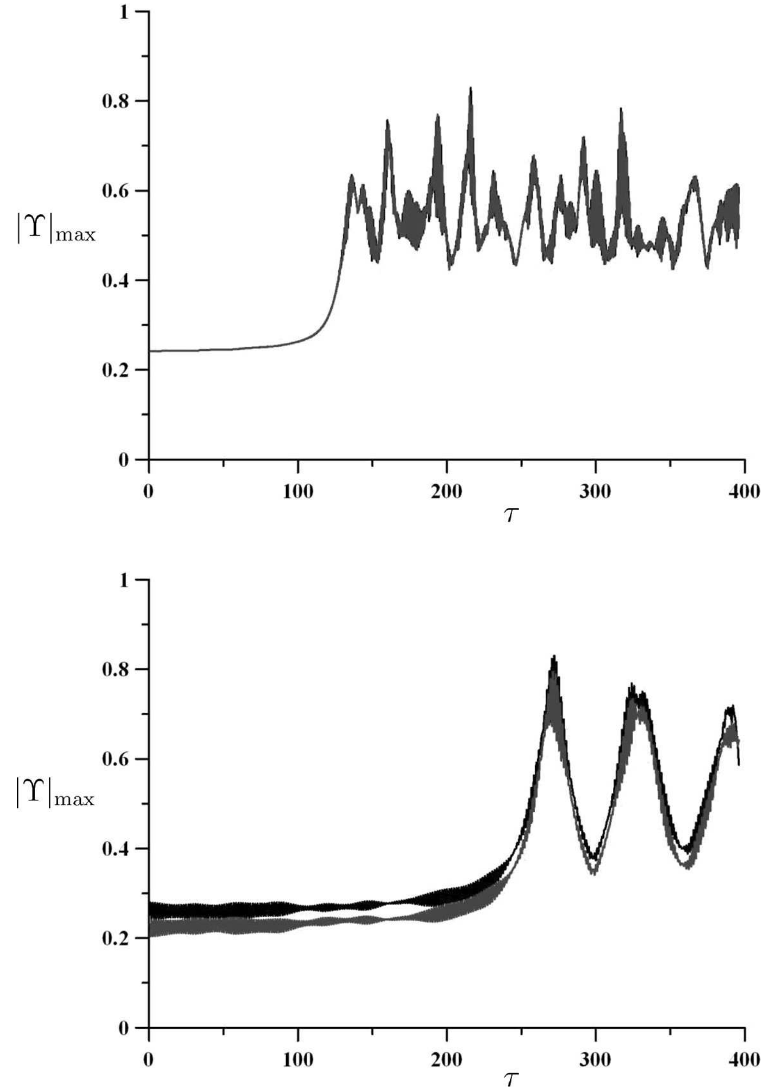

Figure 7. Maximum crest (black) and trough (grey) amplitudes versus time for the wave with the initial amplitude $A = 0.23$. The upper panel corresponds to $\alpha = 0$ (the mKdV equation) and the lower to $\alpha = 1$ (the Gardner equation). In the upper panel the two curves are indistinguishable.

ion concentration deviates from the critical value, so that the coefficient at the term describing the quadratic nonlinearity in the Gardner equation is non-zero, a quasi-monochromatic wavepacket is unstable only when the carrier wavenumber is larger than the critical value (see (3.8)).

To study the nonlinear development of the modulational instability and compare it with the prediction of the weakly nonlinear theory we solved the Gardner equation numerically. To do this we introduced new dimensionless variables in such a way that the transformed equation contains only one coefficient α at the term describing the quadratic nonlinearity. In these new dimensionless variables the time

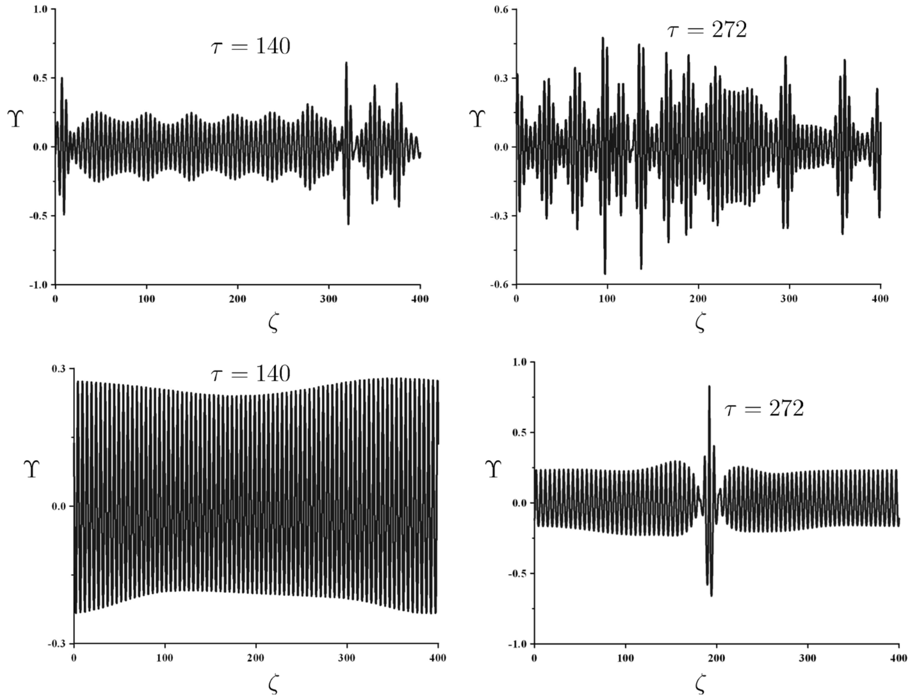

Figure 8. Formation of the first intense wave groups owing to modulational instability of the wave with the initial amplitude $A = 0.23$. The upper panels correspond to $\alpha = 0$ (the mKdV equation) and the lower to $\alpha = 1$ (the Gardner equation).

remains the same, while the length and potential are multiplied by quantities of the order of unity. We carried out the calculations for two different values of α: $\alpha = 0$, which corresponds to the mKdV equation, and $\alpha = 1$, which corresponds to the Gardner equation. In all calculations the initial condition was chosen in the form of harmonic modulated wave with the carrier wavenumber satisfying the condition of the modulational instability given by the weakly nonlinear theory.

Fist we took the amplitude of the initial wave equal to 0.05, which corresponds to the weak nonlinearity. In this case the numerical results are in complete agreement with the predictions of the weakly nonlinear theory. In particular, the modulational instability was developing slower in the case when $\alpha = 1$ than in the case when $\alpha = 0$. The wave remains symmetric with equal amplitudes of crests and troughs.

We then studied the evolution of the wave with the amplitude 0.23, which corresponds to the moderate nonlinearity. In this case the modulational instability was developing much faster than in the case of wave with the amplitude 0.05, although, once again, it was developing slower in the case when $\alpha = 1$ than in the case when $\alpha = 0$. The wave dynamics strongly deviated from what was predicted using the weakly nonlinear theory. In particular, the wave become asymmetric with the amplitudes of crests exceeding the amplitudes of troughs. The long-time dynamics of the modulational instability also reveals the freak wave phenomenon, when large-amplitude perturbations appear for a short period of time, and then disappear quickly.

It is also worth noting that the modulational instability can be considered as a relatively slow process. Even for moderate initial amplitude the characteristic time

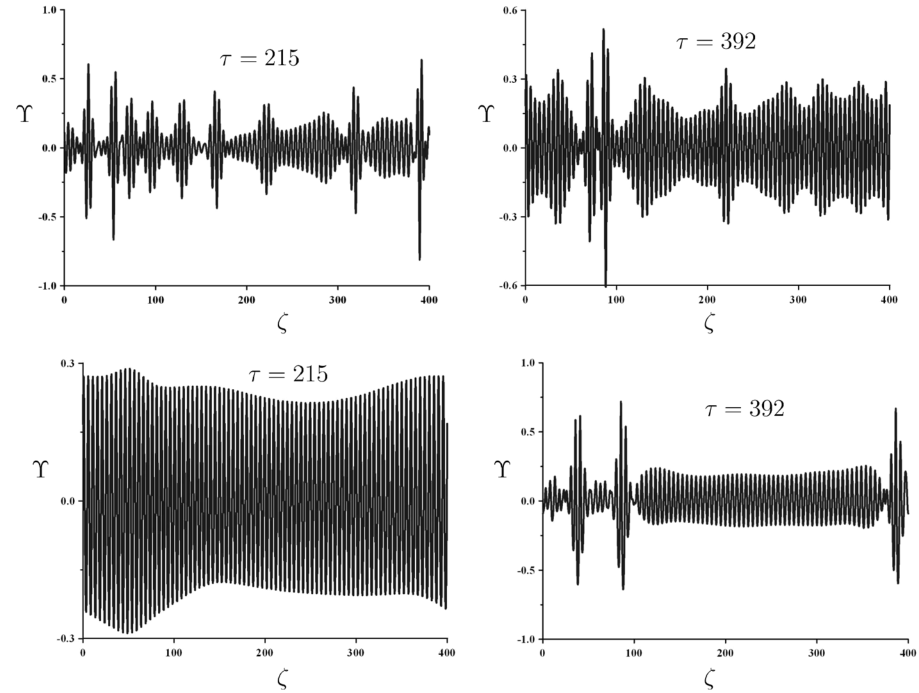

Figure 9. Wave profiles for large moments of time for the wave with the initial amplitude $A = 0.23$. The upper panels correspond to $\alpha = 0$ (the mKdV equation) and the lower to $\alpha = 1$ (the Gardner equation).

of its development is a hundred of $\omega_{\mathrm{pi}}^{-1}$, and this time increases to a thousand of $\omega_{\mathrm{pi}}^{-1}$ for a small-amplitude initial wave.

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