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The split property for quantum field theories in flat and curved spacetimes

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Abstract

The split property expresses a strong form of independence of spacelike separated regions in algebraic quantum field theory. In Minkowski spacetime, it can be proved under hypotheses of nuclearity. An expository account is given of nuclearity and the split property, and connections are drawn to the theory of quantum energy inequalities. In addition, a recent proof of the split property for quantum field theory in curved spacetimes is outlined, emphasising the essential ideas.

Dedicated to the memory of Rudolf Haag

1 Introduction

Special relativity entails that information and influences propagate at speeds no greater than that of light, in order that causes precede effects according to all inertial clocks. Laboratories in space-like separated spacetime regions should therefore function independently. The *split property* is an expression of this independence in the algebraic formulation of quantum field theory [35] that is considerably deeper than the assumption of Einstein causality (commutation of spacelike separated observables). One aim of this paper is to present a short account of a recent extension of the split property to locally covariant quantum field theories in curved spacetimes [27]. However, it is also intended to give an expository account of the split property and its consequences, and also of the hypotheses of *nuclearity* under which the split property was first proved in a general setting [10]. This serves both to make apparent the significance of the split property, and also the physical circumstances in which it holds. Although much of the material in Sections 2 and 3 reviews other works, the discussion of links between nuclearity and Quantum Energy Inequalities
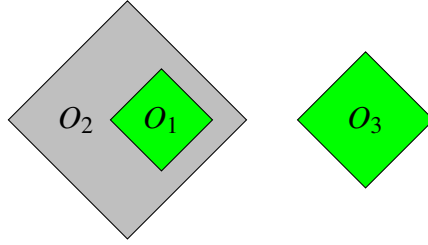
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Figure 1: Three spacetime regions in Minkowski space. Time runs up the page, and rays travelling at the speed of light have unit gradient. Thus there is no possibility of communication between regions O_2 and O_3.

is based on results proved here for the first time. The proof of our main results in curved spacetime (Section 4) is a streamlined and extended version of arguments going back to Verch [53, 54]. In fact, it provides a proof strategy for a number of other results, including the Reeh–Schlieder property [48] and modular nuclearity [41], and is therefore of independent interest.

2 The split property in Minkowski space

To begin, let us consider a configuration of three spacetime regions in Minkowski spacetime, displayed in Fig. 1. Region O_1 is contained within region O_2, which is *spacelike separated* from region O_3. All three regions are supposed to be open and relatively compact. Spacelike separation means that there are no causal curves with one endpoint in O_2 and the other in O_3; in other words, there is no possibility of communication between them, if we assume the basic precepts of special relativity, and it should be possible for experiments in O_3 to take place independently of those in region O_2.

In a theory of local quantum physics [35, 6] , each of these regions O_i (and indeed, every open relatively compact set) has a corresponding unital ∗-algebra $\mathfrak{R}(O_i)$ whose self-adjoint elements correspond to observables that can, in principle, be measured by experiments conducted within O_i. For our present discussion, the $\mathfrak{R}(O_i)$ will be von Neumann algebras, concretely represented on a Hilbert space $\mathscr{H}$, and sharing the Hilbert space identity operator as their common unit. For the moment, $\mathscr{H}$ will not be assumed separable. As any experiment conducted within O_1 is *a fortiori* conducted within O_2, we require $\mathfrak{R}(O_1) \subset \mathfrak{R}(O_2)$. This property is known as *isotony*.

How is the 'experimental independence' of regions O_2 and O_3 to be expressed mathematically? This question is much more subtle and has many more potential answers than one might first think (see [51, 52, 46]). A starting point is the requirement that measurements can be made independently in O_2 and O_3. In elementary quantum mechanics one learns that observables are independently measureable (commensurable) if and only if they commute, so one should certainly require that the algebras for regions O_2 and O_3 commute elementwise,

$$[\mathfrak{R}(O_2), \mathfrak{R}(O_3)] = 0, \tag{1}$$

which is sometimes called *Einstein causality*.[1] However, experiments not only concern measurements, but also involve a stage of preparation (corresponding to a choice of a state); what we need, then, is a way of describing the possibility for experimenters in spacelike separated regions to make both preparations and measurements independently.

This is where the split property enters. It turns out to be important to ensure that the regions in question do not touch at their boundaries, and therefore we switch attention to the independence of regions O_1, surrounded by a 'collar region' provided by O_2, and O_3. We describe the inclusion $\mathfrak{R}(O_1) \subset \mathfrak{R}(O_2)$ as *split* if there is a type I von Neumann factor $\mathfrak{N}$ such that

$$\mathfrak{R}(O_1) \subset \mathfrak{N} \subset \mathfrak{R}(O_2). \tag{2}$$

Here, $\mathfrak{N}$ is a *factor* if $\mathfrak{N} \cap \mathfrak{N}' = \mathbb{C}\mathbb{1}$, where the prime denotes the commutant, and the designation as a 'type I factor' means that $\mathfrak{N}$ is isomorphic as a von Neumann algebra to the algebra $\mathscr{B}(\mathscr{K})$ of bounded operators on a [not necessarily separable] Hilbert space $\mathscr{K}$. The *split property* requires that whenever $O_1 \Subset O_2$, i.e., that the closure of O_1 is compactly contained in O_2, then the inclusion $\mathfrak{R}(O_1) \subset \mathfrak{R}(O_2)$ is split in this way. Returning to our configuration in Fig. 1, the split property has the important consequence that $\mathfrak{R}(O_1)$ and $\mathfrak{R}(O_3)$ are *W^*-statistically independent*: for all normal states[2] φ_1 and φ_3 on these algebras there is a normal state φ on the von Neumann algebra $\mathfrak{R}(O_1) \vee \mathfrak{R}(O_3)$ they generate so that

$$\varphi(AB) = \varphi_1(A)\varphi_3(B) \qquad A \in \mathfrak{R}(O_1), B \in \mathfrak{R}(O_3). \tag{3}$$

This asserts that any pair of preparations made by the experimenters in regions O_1 and O_3 can be subsumed into a preparation of the system as a whole.

The proof of this statement is straightforward and illuminates some related issues. Because $\mathfrak{N}$ is a type I factor, there is a Hilbert space isomorphism $U : \mathscr{H} \to \mathscr{K} \otimes \mathscr{K}'$ such that $U\mathfrak{N}U^{-1} = \mathscr{B}(\mathscr{K}) \otimes \mathbb{1}_{\mathscr{K}'}$ (see, e.g., [2, III.1.5.3]). Owing to the inclusion $\mathfrak{R}(O_1) \subset \mathfrak{N}$ and the fact that $\mathfrak{R}(O_3) \subset \mathfrak{R}(O_1)'$, one finds

$$U\mathfrak{R}(O_1)U^{-1} \subset \mathscr{B}(\mathscr{K}) \otimes \mathbb{1}_{\mathscr{K}'},$$
$$U\mathfrak{N}U^{-1} = \mathscr{B}(\mathscr{K}) \otimes \mathbb{1}_{\mathscr{K}'},$$
$$U\mathfrak{R}(O_3)U^{-1} \subset \mathbb{1}_{\mathscr{K}} \otimes \mathscr{B}(\mathscr{K}'). \tag{4}$$

We now have faithful normal representations π_i of $\mathfrak{R}(O_i)$ with π_1 acting on $\mathscr{K}$ and π_3 on $\mathscr{K}'$, so that

$$\pi_1(A) \otimes \mathbb{1}_{\mathscr{K}'} = UAU^{-1}, \qquad \mathbb{1}_{\mathscr{K}} \otimes \pi_3(B) = UBU^{-1}$$

for $A \in \mathfrak{R}_1(O), B \in \mathfrak{R}(O_3)$. The unitary U clearly implements a spatial isomorphism of $\mathfrak{R}(O_1) \vee \mathfrak{R}(O_3)$ with the spatial tensor product $\pi_1(\mathfrak{R}(O_1))\bar{\otimes}\pi_3(\mathfrak{R}(O_3))$ and, in particular, the map $AB \mapsto A \otimes B$ extends to an isomorphism of von Neumann algebras

$$\mathfrak{R}(O_1) \vee \mathfrak{R}(O_3) \cong \mathfrak{R}(O_1)\bar{\otimes}\mathfrak{R}(O_3). \tag{5}$$

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[1] Fermi fields, of course, anticommute at spacelike separation, and are excluded from the algebra of *observables*, though operators constructed from products of even numbers of Fermi fields qualify as observables.

[2] A normal state on a von Neumann algebra can be defined abstractly in terms of its continuity properties; however, when the von Neumann algebra acts on a Hilbert space, the normal states are precisely those that can be represented by density matrix states on the Hilbert space [5, Thm 2.4.21].

(In the converse direction, we note that if an isomorphism of this form exists and is spatial, then the inclusion $\mathfrak{R}(O_1) \subset \mathfrak{R}(O_3)'$ is split: simply define $\mathfrak{N}$ using the second line of (4).)

Returning to the question of W^*-statistical independence and representing the normal states φ_i by density matrices ρ_i so that $\varphi_i(X) = \mathrm{Tr}\,\rho_i X$, $(X \in \mathfrak{R}(O_i))$, we obtain density matrices $\tilde{\rho}_1$ on $\mathscr{B}(\mathscr{K})$ and $\tilde{\rho}_3$ on $\mathscr{B}(\mathscr{K}')$ by the partial traces

$$\mathrm{Tr}\,\tilde{\rho}_1 K = \mathrm{Tr}\,\rho_1 U^{-1}(K \otimes \mathbb{1}_{\mathscr{K}'})U, \qquad \mathrm{Tr}\,\tilde{\rho}_3 K' = \mathrm{Tr}\,\rho_3 U^{-1}(\mathbb{1}_{\mathscr{K}} \otimes K')U \tag{6}$$

for $K \in \mathscr{B}(\mathscr{K})$, $K' \in \mathscr{B}(\mathscr{K}')$. Then the desired product state is $\varphi(C) = \mathrm{Tr}\,\rho C$ with density matrix

$$\rho = U^{-1}(\tilde{\rho}_1 \otimes \tilde{\rho}_3)U. \tag{7}$$

For, taking $A \in \mathfrak{R}(O_1)$ and $B \in \mathfrak{R}(O_3)$, and writing $UAU^{-1} = \tilde{A} \otimes \mathbb{1}_{\mathscr{K}'}$, $UBU^{-1} = \mathbb{1}_{\mathscr{K}} \otimes \tilde{B}$,

$$\begin{aligned} \mathrm{Tr}\,\rho AB = \mathrm{Tr}\,U^{-1}(\tilde{\rho}_1 \otimes \tilde{\rho}_3)UAB &= \mathrm{Tr}\,(\tilde{\rho}_1 \otimes \tilde{\rho}_3)\tilde{A} \otimes \tilde{B} = \left(\mathrm{Tr}\,\tilde{\rho}_1\tilde{A}\right)\left(\mathrm{Tr}\,\tilde{\rho}_3\tilde{B}\right) \\ &= (\mathrm{Tr}\,\rho_1 A)(\mathrm{Tr}\,\rho_3 B). \end{aligned} \tag{8}$$

The idea that suitable inclusions of local algebras should split is due to Borchers, but the first proof of the split property (already described as "an old conjecture of Borchers") was given in 1974 by Buchholz, in the particular case of the massive free scalar field [9]. There matters rested for some time – an extension to the observable algebras for the Dirac and Maxwell fields, as well as the massless scalar, was given five years later in [37], and a related result for the *field algebras* of the massive free fermion fields was given by Summers [50] in 1982.[3] There were then two breakthroughs. First, the theory of split inclusions were studied in a deep paper of Doplicher and Longo [22], showing that the split property has numerous important consequences, some of which will be discussed below; second, the development of *nuclearity* criteria [16] permitted the proof of the split property for general models with sufficiently good nuclearity properties [10]. This was a significant step, because nuclearity is closely related to questions of thermodynamic stability [16, 12, 13] and so it became apparent that – in keeping with Borchers' conjecture – the split property should indeed be a feature of suitably well-behaved quantum field theory models.

The nuclearity criterion used in [10] is defined as follows. First, let us recall that a linear map between Banach spaces $\Xi : \mathcal{X} \to \mathcal{Y}$ is said to be *nuclear* if there is a countable decomposition $\Xi(\cdot) = \sum_k \ell_k(\cdot)\psi_k$ where $\psi_k \in \mathcal{Y}$ and $\ell_k \in \mathcal{X}^*$, such that the sum $\sum_k \|\psi_k\|\|\ell_k\|$ is convergent. Under these circumstances, the *nuclearity index* $\|\Xi\|_1$ is defined as the infimum of the value of this sum over all possible decompositions.

Now let O be a nonempty open and bounded region of Minkowski spacetime, with associated von Neumann algebra $\mathfrak{R}(O)$. Denote the vacuum state vector by $\Omega \in \mathscr{H}$ and the Hamiltonian, generating time translations, by H, so that $U(\tau) = e^{iH\tau}$ satisfies

$$U(\tau)\mathfrak{R}(O)U(\tau)^{-1} = \mathfrak{R}(O_\tau), \tag{9}$$

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[3]Einstein causality must be modified for field algebras of fermionic systems. If O_2 and O_3 are spacelike separated, the corresponding field algebras $\mathfrak{F}(O_i)$ obey a graded commutation relation in place of (1); however, by introducing a suitable unitary twist map Z on $\mathscr{H}$, and defining the twisted algebra $\mathfrak{F}^t(O_3) = Z\mathfrak{F}(O_3)Z^{-1}$, Einstein causality can be reformulated as $[\mathfrak{F}(O_2), \mathfrak{F}^t(O_3)] = 0$. Then the relevant form of the split property is that the inclusion $\mathfrak{F}(O_1) \subset \mathfrak{F}^t(O_3)'$ splits.

where O_τ is the translation of O under $(t, \boldsymbol{x}) \mapsto (t + \tau, \boldsymbol{x})$. The nuclearity criterion of [10] is that for every O and $\beta > 0$, each map $\Xi_{O,\beta} : \mathfrak{R}(O) \to \mathcal{H}$ given by

$$\Xi_{O,\beta}(A) = e^{-\beta H} A \Omega \tag{10}$$

is nuclear and has nuclearity index obeying

$$\|\Xi_{O,\beta}\|_1 \le e^{(\beta_0/\beta)^n} \tag{11}$$

as $\beta \to 0+$, where the constants $n > 0$ and $\beta_0 > 0$ may depend on O (but not β). The physical understanding of these criteria is that $\|\Xi_{O,\beta}\|_1$ plays the role of a local partition function and probes the state space available on given distance and energy scales; see Section 3 for other aspects of the interpretation. Note that there is a variety of nuclearity conditions (see [14] for discussion) some of which should be used with care [30]. Given this definition, one has:

Theorem 2.1 ([10])**.** *Suppose that the net of von Neumann algebras $O \mapsto \mathfrak{R}(O)$ obeys isotony, that the Hamiltonian is nonnegative, with zero eigenspace spanned by Ω, and that the above nuclearity criterion holds. Then for any open bounded regions with $O_1 \Subset O_2$, the inclusion $\mathfrak{R}(O_1) \subset \mathfrak{R}(O_2)$ is split.*

Let us now turn to the theory of standard split inclusions [22]. Consider von Neumann algebras $\mathfrak{R}_1$ and $\mathfrak{R}_2$ acting on a Hilbert space $\mathcal{H}$. A triple $(\mathfrak{R}_1, \mathfrak{R}_2, \Omega)$, where $\Omega \in \mathcal{H}$, is said to be a *standard split inclusion* if $\mathfrak{R}_1 \subset \mathfrak{R}_2$ is split, and Ω is cyclic and separating for each of $\mathfrak{R}_1$, $\mathfrak{R}_2$ and $\mathfrak{R}_2 \wedge \mathfrak{R}_1'$ (and hence for their commutants). In the case where Ω is the vacuum vector of a quantum field theory, the latter assumptions are met as a consequence of the Reeh–Schlieder theorem [47]. A number of remarkable results are proved in [22]. For instance (setting aside a trivial case $\mathfrak{R}_2 = \mathbb{C}\mathbb{1}$) one finds that both the von Neumann algebras and the Hilbert space are substantially constrained: the $\mathfrak{R}_i$ are properly infinite and have separable preduals, while the Hilbert space $\mathcal{H}$ must be separable.

There is also a canonical choice of type I factor: because $\mathfrak{R}_1 \subset \mathfrak{R}_2$ is split, there is (cf. (5)) a von Neumann algebra isomorphism

$$\begin{aligned} \phi : \mathfrak{R}_1 \vee \mathfrak{R}_2' &\to \mathfrak{R}_1 \bar{\otimes} \mathfrak{R}_2 \\ AB' &\mapsto A \otimes B'. \end{aligned} \tag{12}$$

As Ω is cyclic and separating for $\mathfrak{R}_1 \vee \mathfrak{R}_2'$ and $\Omega \otimes \Omega$ is cyclic and separating for $\mathfrak{R}_1 \bar{\otimes} \mathfrak{R}_2$ acting on $\mathcal{H} \otimes \mathcal{H}$, there is, by Tomita–Takesaki theory, a natural choice of unitary $U : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ implementing ϕ, $UAB'U^{-1} = A \otimes B'$ for all $A \in \mathfrak{R}_1$, $B' \in \mathfrak{R}_2'$. Thus we have

$$\begin{aligned} U\mathfrak{R}_1 U^{-1} &= \mathfrak{R}_1 \otimes \mathbb{1}_{\mathcal{H}}, \\ U\mathfrak{R}_2' U^{-1} &= \mathbb{1} \otimes \mathfrak{R}_2', \\ U\mathfrak{R}_2 U^{-1} &= \mathcal{B}(\mathcal{H}) \otimes \mathfrak{R}_2, \end{aligned} \tag{13}$$

and $\mathfrak{N} = U^{-1}(\mathcal{B}(\mathcal{H}) \otimes \mathbb{1}_{\mathcal{H}})U$ is the canonical choice of an intermediate type I factor.

An important physical application arises as follows. Suppose that there is a unitary group representation $G \ni g \mapsto V(g)$ on $\mathscr{H}$, acting so that $V(g)\mathfrak{R}_i V(g)^{-1} = \mathfrak{R}_i$ for each $i = 1, 2$, $g \in G$. Then, defining $W(g) = U^{-1}(V(g) \otimes \mathbb{1})U$, we have a new unitary representation of G acting on elements $\mathfrak{R}_1 \vee \mathfrak{R}_2'$ by

$$W(g)AB'W(g)^{-1} = U^{-1}(V(g) \otimes \mathbb{1})(A \otimes B')(V(g)^{-1} \otimes \mathbb{1})U = V(g)AV(g)^{-1}B' \qquad (14)$$

for $A \in \mathfrak{R}_1$, $B' \in \mathfrak{R}_2$. Moreover, $W(g) \in U^{-1}(\mathscr{B}(\mathscr{H}) \otimes \mathbb{1})U \subset \mathfrak{R}_2$. In the context of a quantum field theory obeying the split property, this corresponds immediately to the situation in which G is a group of global gauge transformations, implemented by $V(g)$. Given any nested pair $O_1 \Subset O_2$ one may then obtain a localised representation $W(g) \in \mathfrak{R}(O_2)$ which agrees with $V(g)$ on $\mathfrak{R}(O_1)$ but acts trivially on $\mathfrak{R}(O_2)'$. This again emphasises the way in which the split property permits the physics of the inner region to be isolated from that of the causal complement of the outer region. The generators of $W(g)$ can be interpreted as suitable smearings of a conserved local current associated to the global symmetry, thus providing an abstract version of Noether's theorem (note that there is no assumption that the theory derives from a Lagrangian) [21].

There is an interesting result in the converse direction. Suppose $\mathfrak{R}_1 \subset \mathfrak{R}_2$, and that the flip automorphism $\sigma : A \otimes B \mapsto B \otimes A$ of $\mathfrak{R}_1 \otimes \mathfrak{R}_1$ is inner with respect to $\mathfrak{R}_2 \otimes \mathfrak{R}_2$ – i.e., σ agrees on $\mathfrak{R}_1 \otimes \mathfrak{R}_1$ with $\mathrm{Ad}\, U$ for some unitary $U \in \mathfrak{R}_2 \otimes \mathfrak{R}_2$. Then one has

$$U(\mathbb{1} \otimes AB')U^{-1} = U(\mathbb{1} \otimes A)U^{-1}U(\mathbb{1} \otimes B')U^{-1} = A \otimes B' \qquad (15)$$

for $A \in \mathfrak{R}_1$, $B' \in \mathfrak{R}_2'$, where we use the fact that $\mathrm{Ad}\, U$ implements the flip on $\mathfrak{R}_1 \otimes \mathfrak{R}_1$ and (as $U \in \mathfrak{R}_2 \otimes \mathfrak{R}_2$) acts trivially on $\mathbb{1} \otimes B' \in \mathfrak{R}_2' \otimes \mathfrak{R}_2'$. This establishes a von Neumann algebra isomorphism $\mathfrak{R}_1 \vee \mathfrak{R}_2' \to \mathfrak{R}_1 \bar{\otimes} \mathfrak{R}_2'$ extending $AB' \mapsto A \otimes B'$. If, in addition, there is a cyclic and separating vector for $\mathfrak{R}_1' \wedge \mathfrak{R}_2$, then one can establish a spatial isomorphism between $\mathfrak{R}_1 \vee \mathfrak{R}_2'$ and $\pi_1(\mathfrak{R}_1)\bar{\otimes}\pi_2(\mathfrak{R}_2)$, where the π_i are faithful normal representations of the $\mathfrak{R}_i$ with cyclic and separating vectors. It follows that the inclusion $\mathfrak{R}_1 \subset \mathfrak{R}_2$ is split (moreover, under these circumstances, if the inclusion is split, then the flip is inner with respect to $\mathfrak{R}_2 \otimes \mathfrak{R}_2$) [20, 22]. Clearly, the essential point here is that $\sigma(1 \otimes A) = A \otimes 1$, and the argument works for more general automorphisms than just the flip.

D'Antoni and Longo used this idea in an ingenious proof of the split property for the free scalar field. Let $O \mapsto \mathfrak{R}(O)$ be the net of von Neumann algebras of the free massive scalar field on Minkowski space in the vacuum representation. Then $O \mapsto \mathfrak{R}(O)\bar{\otimes}\mathfrak{R}(O)$ is the theory of two independent massive scalar fields, and has an internal $\mathrm{SO}(2)$ symmetry group that rotates the doublet of fields, with a rotation through $\pi/2$ precisely inducing $\mathbb{1} \otimes A \mapsto A \otimes \mathbb{1}$. The gauge symmetry is associated with a Noether current, and it is then shown that suitable local smearings of these currents in a region O_2 generate a local implementation of the gauge symmetry on a smaller region $O_1 \Subset O_2$ and acting trivially on $\mathfrak{R}(O_2)'$. The vacuum provides the cyclic and separating vector to permit the deduction that $\mathfrak{R}(O_1) \subset \mathfrak{R}(O_2)$ is split. A similar idea has been employed recently [42] to make a more explicit local implementation of the generators of gauge symmetries – actually, a family of possible implementations are constructed, with the 'canonical' implementation described above included as a special case. With appropriate modifications, local implementations of geometric symmetries can also be constructed [11, 17].

One of the most striking results to emerge from the body of work on the split property concerns the type of the local von Neumann algebras [10]. In 1985, Fredenhagen showed that, under the hypothesis of a scaling limit, local algebras are of type III_1 (for brevity, we assume here for that the local algebras are factors). Combining this with the split property and the Reeh–Schlieder theorem, and the assumption that any $\mathfrak{R}(O)$ may be generated as $\mathfrak{R}(O) = \bigvee_k \mathfrak{R}(O_k)$ for some increasing chain of nested sets $O_k \Subset O_{k+1}$, one then has $\mathfrak{R}(O) = \bigvee_k \mathfrak{N}_k$ for a sequence of nested type I factors. As the Hilbert space is separable, this leads to the conclusion that each $\mathfrak{R}(O)$ is a hyperfinite type III_1 factor, thus fixing it uniquely up to isomorphism [36]. This is a remarkable achievement, demonstrating that the distinction between different quantum field theories lies in the relationships between local algebras, rather than the content of those algebras *per se*. The timing of the results is also remarkable, with progress on the QFT side at exactly the moment that the structural results on von Neumann algebras appeared.

Finally, let us note that the split property plays a key role in the construction of certain integrable quantum field models – see [40] for a review, and that the interpretative framework of models described by funnels of type I factors has been investigated in [15].

3 Nuclearity and Quantum Energy Inequalities

The nuclearity criterion, while being physically well-motivated, is rather technical. In this section we explain more about its physical status and draw some links to the theory of quantum energy inequalities. For the purposes of illustration, we consider a class of theories, consisting of countably many independent free scalar fields with masses m_r ($r \in \mathbb{N}$) in 4 spacetime dimensions — this may also be related to a particular generalised free field [22]. For simplicity, we assume that m_r form a nondecreasing sequence and that there is a mass gap, i.e., $m_1 > 0$.

The models are constructed as follows. Each individual field has a Hilbert space $\mathscr{F}_r$ which is a copy of the symmetric Fock space over $\mathscr{H} = L^2(\mathbb{R}^3, d^3\boldsymbol{k}/(2\pi)^3)$, with vacuum vector Ω_r. The annihilation and creation operators obey

$$[\mathsf{a}_r(u), \mathsf{a}_r(v)] = 0, \qquad [\mathsf{a}_r(u), \mathsf{a}_r^*(v)] = \langle u \mid v \rangle \mathbb{1}_{\mathscr{F}_r}, \qquad (u, v \in \mathscr{H}) \tag{16}$$

and $\mathsf{a}_r(u)\Omega_r = 0$ for all $u \in \mathscr{H}$. The smeared quantum field is

$$\Phi_r(f) := \mathsf{a}_r(K_r \bar{f}) + \mathsf{a}_r(K_r f)^*, \tag{17}$$

where $K_r : C_0^\infty(\mathbb{R}^4) \to \mathscr{H}$ is defined by $(K_r f)(\boldsymbol{k}) = (2\omega_r)^{-1/2}\hat{f}(\omega_r(\boldsymbol{k}), \boldsymbol{k})$, with the Fourier transform $\hat{f}(k) = \int d^4 x\, e^{ik \cdot x} f(x)$ and $\omega_r(\boldsymbol{k}) = (\|\boldsymbol{k}\|^2 + m_r^2)^{1/2}$. Here k is a 4-vector, and the $\cdot$ denotes the Lorentz contraction in the $+ - - -$ signature. For real-valued f, the operators $\Phi_r(f)$ are essentially self-adjoint on a domain of finite particle vectors and we use $\Phi_r(f)$ also to denote the closure. The local von Neumann algebras $\mathfrak{R}(O_r)$ are then generated by the Weyl operators $\exp(i\Phi_r(f))$ as f runs over real-valued elements of $C_0^\infty(O)$. The combined theory lives on the incomplete tensor product $\mathscr{F} = \otimes_r \mathscr{F}_r$ relative to $\Omega = \otimes_r \Omega_r$ and the overall local algebras $\mathfrak{R}(O)$ are formed by taking the weak closure in $\mathscr{F}$ of the algebraic tensor product $\odot_r \mathfrak{R}_r(O)$.

For these models, we may give necessary and sufficient criteria for nuclearity.

Proposition 3.1. *Consider the theory of countably many scalar fields with mass gap.*
(a) A necessary condition for the theory to obey the nuclearity criterion is that

$$F(\beta) := \sum_r \frac{e^{-4\beta m_r}}{m_r^2} \tag{18}$$

should be finite for all $\beta > 0$, with $\log F(\beta)$ growing at most polynomially in β^{-1} as $\beta \to 0+$.
(b) A sufficient condition for the theory to obey the nuclearity criterion is that

$$G(\beta) := \sum_r e^{-\beta m_r/4} \tag{19}$$

should be finite for all $\beta > 0$ and grows only polynomially in β^{-1} as $\beta \to 0+$.

Outline of the proof. To begin, let us develop a lower bound on the nuclearity index following [14, §2]. Suppose $\Xi_{O,\beta}$ is nuclear with a decomposition of the form $\sum_k \ell_k(\cdot)\psi_k$, and note that the operator norm obeys $\|\Xi_{O,\beta}\| \leq 1$ by (10). Suppose further that one can find a sequence $A_r \in \mathfrak{R}(O)$ so that the $\Xi_{O,\beta}(A_r)$ are orthogonal in $\mathscr{F}$, assuming without loss that $\|A_r\| \leq 1$. Then

$$\sum_r \|\Xi_{O,\beta}(A_r)\|^4 = \sum_r \left| \sum_k \ell_k(A_r) \langle \Xi_{O,\beta}(A_r) \mid \psi_k \rangle \right|^2 = \left\| \sum_k B_k \psi_k \right\|^2 \leq \left(\sum_k \|\ell_k\| \|\psi_k\| \right)^2 \tag{20}$$

using orthogonality and noting that $B_k := \sum_r \ell_k(A_r)\|\Xi_{O,\beta}(A_r)\|^{-1} |\Xi_{O,\beta}(A_r)\rangle\langle\Xi_{O,\beta}(A_r) \mid$ has norm $\|B_k\| \leq \|\ell_k\|$, given our assumptions on the norms of A_r and $\Xi_{O,\beta}$. Considering all possible decompositions, this shows that

$$\sum_r \|\Xi_{O,\beta}(A_r)\|^4 \leq \|\Xi_{O,\beta}\|_1^2. \tag{21}$$

Fixing O, Buchholz and Porrmann [14] construct a sequence A_r in $\mathfrak{R}(O)$, built from Weyl operators, so that the $\Xi_{O,\beta}(A_r)$ are orthogonal in $\mathscr{F}$ and $\|\Xi_{O,\beta}(A_r)\|^4 \geq Ce^{-4\beta m_r}/m_r^2$ for sufficiently large r. This establishes part (a).

On the other hand, we note that in the full theory the nuclear maps $\Xi_{O,\beta}$ are simply tensor products of the maps for each m_r, and that the nuclearity index of the combined theory is therefore bounded above by the product of the nuclearity indices of the individual theory. An involved computation (see [16, §5] for the original version in a slightly different formulation of nuclearity, or [1, §17.3] in the present version) furnishes an upper bound

$$\|\Xi_{O,\beta}\|_1 \leq \exp\left\{ c\frac{R^3}{\beta^3} \sum_r |\log(1 - e^{-\beta m_r/2})| \right\} \tag{22}$$

if O is a diamond whose base is a ball of radius $R > m_1^{-1}$, and where $c > 0$ is a numerical constant independent of r, β, m_r. Using the fact that $\sup_{x>0} xe^{x/2}|\log(1 - e^{-x})| < \infty$, this bound can be estimated above to give

$$\|\Xi_{O,\beta}\|_1 \leq \exp\left\{ C\frac{R^3}{m_1\beta^4} \sum_r e^{-\beta m_r/4} \right\} \tag{23}$$

and part (b) is thereby proved. $\qquad\square$

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In particular, one sees that the mass spectrum

$$m_r = (2d_0)^{-1}\log(r+1) \tag{24}$$

for fixed $d_0 > 0$ is inconsistent with nuclearity, because $F(\beta)$ is infinite for $\beta < \frac{1}{2}d_0$, while a mass spectrum $m_r = rm_1$ is easily seen to satisfy the sufficient condition for nuclearity.

More insight is obtained by defining the counting function

$$N(u) = \sum_r \vartheta(u - m_r) \tag{25}$$

of fields with mass below u, for we have

$$G(\beta) = \int_0^\infty e^{-\beta u/4} dN(u) = \frac{\beta}{4}\int_0^\infty e^{-\beta u/4} N(u)\, du. \tag{26}$$

Thus by a direct computation, polynomial boundedness of $N(u)$ as $u \to \infty$ is sufficient for nuclearity. Note that the example given by (24) corresponds to an exponentially growing counting function. On the other hand, if nuclearity holds then we have $\sum_r e^{-\beta m_r/2}/m_r^2 \le e^{(\beta_0/\beta)^n}$ as $\beta \to 0+$ (we have absorbed some constants into β_0, but the value of n is as in the nuclearity criterion (11)). Thus

$$\sum_r e^{-\beta m_r} = \sum_r m_r^2 e^{-\beta m_r/2} \frac{e^{-\beta m_r/2}}{m_r^2} \le \frac{A}{\beta^2} e^{(\beta_0/\beta)^n} \tag{27}$$

as $\beta \to 0+$ for some $A > 0$ and therefore

$$N(v) \le \sum_r e^{\beta(v - m_r)} \le \frac{A}{\beta^2} e^{(\beta_0/\beta)^n + \beta v} \tag{28}$$

for any $v, \beta > 0$. We are free to optimise the right-hand side by choice of β.[4] An exact minimisation of the right-hand side over β is awkward, but we may certainly substitute $\beta = \beta_0^{n/(n+1)}(n/v)^{1/(n+1)}$ [which minimises the exponential factor] yielding

$$N(v) \le Bv^{2/(n+1)} e^{Cv^{n/(n+1)}} \tag{29}$$

for positive constants B and C. Therefore, nuclearity implies a sub-exponential growth in N, and is implied by polynomial growth.

It is important to understand that the criteria just given are correlated with physical properties. For example, Buchholz and Junglas [12] showed that convergence of the sum $\sum_n e^{-\beta m_n/2}$ is sufficient for the thermal equilibrium state ω_β at inverse temperature β to be locally normal[5] to the vacuum state; on the other hand, if ω_β is locally normal, then $\sum_n e^{-2\beta m_n}$ converges. Therefore,

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[4]This is a well-known technique in the theory of Tauberian estimates of sums and integrals, see e.g., [44].

[5]That is, its restriction to any local von Neumann algebra (in the vacuum representation) of a relatively compact region is normal.

local normality for all $\beta > 0$ is equivalent to finiteness of $F(\beta)$, which indicates a link between nuclearity criteria and the good thermodynamic behaviour. This suggestion was made precise in the general context, also by Buchholz and Junglas [13].

Theories in which $F(\beta)$ diverges for sufficiently small β have interesting behaviour relative to the split property. For example, the mass spectrum (24) produces a theory in which one has split inclusions of $\mathfrak{R}(O_1) \subset \mathfrak{R}(O_2)$ only when O_2 is sufficiently larger than O_1: the so-called *distal split property*. To be precise, consider the case where O_i ($i = 1, 2$) are Cauchy developments of concentric open balls with radii r_i lying in a common spacelike hyperplane. Then one defines a *splitting distance* $d(r_1)$ to be the infimum of $r > 0$ for which (2) holds with $r_2 = r + r_1$. In the model (24) it can be shown [18, Thm 4.3] that the splitting distance obeys $d_0 \le d(r) \le 2d_0$ for all $r > 0$. Thus the inverse splitting distance is $d(r)^{-1}$ of the same order as the maximum temperature for which locally normal equilibrium states exist.

A second (and more quantitative) illustration of the significance of nuclearity criteria is provided by quantum energy inequalities (QEIs). In classical field theory, the models typically studied have everywhere nonnegative energy densities according to inertial observers. By contrast, no quantum field theory obeying the standard assumptions can admit an energy density with nonnegative expectation values in all physical states (and vanishing in the vacuum state) [23]. However, as first suggested by Ford [33], the extent to which energy densities can remain negative turns out to be constrained by bounds – the quantum energy inequalities – in a number of models (see [25, 26] for reviews). For instance, the (Wick ordered) energy density ρ_m for a single free scalar field of mass m obeys a lower bound

$$\int \langle \Psi \mid \rho_m(t, \boldsymbol{x}) \Psi \rangle |g(t)|^2 \, dt \ge -C \int_m^\infty u^d |\hat{g}(u)|^2 \, du \tag{30}$$

for all normalised Hadamard vector states Ψ, where $g \in C_0^\infty(\mathbb{R})$ and the constant C depends on the spacetime dimension d, but not on m, g or Ψ [28, 24].[6] The convergence of the integral follows because $\hat{g}(u)$ decays faster than any inverse power. Therefore, the combined energy density ρ of the theory with countably many fields obeys

$$\int \langle \Psi \mid \rho(t, \boldsymbol{x}) \Psi \rangle |g(t)|^2 \, dt \ge -C \int_0^\infty u^d N(u) |\hat{g}(u)|^2 \, du, \tag{31}$$

where N is defined in (25) and Ψ is any state in the space $\mathfrak{S}$ of finite linear combinations of tensor product states $\bigotimes_j \psi_j$ in which all but finitely many of the ψ_j are in the vacuum state (for mass m_j) and so that the ψ_j are all Hadamard. Thus a theory in which N has polynomial growth (hence obeying nuclearity) produces a well-behaved QEI, with a finite lower bound for any $g \in C_0^\infty(\mathbb{R})$. On the other hand, if N grows exponentially (so the theory fails to obey nuclearity) then the bound is divergent for every nontrivial $g \in C_0^\infty(\mathbb{R})$.[7] Moreover, specialising to $d = 4$, the following result is proved in the Appendix.

---

[6]Much more general results are obtained in [24]. The Fourier transform is defined here by $\hat{g}(u) = \int dt \, e^{-iut} g(t)$.

[7]If the transform $\hat{g}$ decays exponentially then g extends to an analytic function in a neighbourhood of the real axis; as g is compactly supported, it would then follow that $g \equiv 0$.

Theorem 3.2. *Let $m_0 \geq 0$ be fixed. Suppose that $f \in C_0^\infty(\mathbb{R})$ is nonnegative, even, and has a Fourier transform (which is necessarily real and even) that is also nonnegative and bounded from below on $[m_0, \infty)$*

$$\hat{f}(u) \geq \varphi(|u|), \tag{32}$$

where $\varphi : [m_0, \infty) \to \mathbb{R}^+$ is monotone decreasing. Then the Klein–Gordon field of mass $m > m_0$ admits a Hadamard state, given by a normalised Fock-space vector Ψ_m, such that

$$\int \langle \Psi_m \mid \rho_m(t,\mathbf{0})\Psi_m \rangle f(t)\, dt \leq -\Gamma m^4 \varphi(2\sqrt{2}m)^2, \tag{33}$$

with a constant $\Gamma > 0$ that depends neither on m nor φ.

Returning to our model of countably many fields, we deduce that

$$\inf_{\substack{\Psi \in \mathfrak{S} \\ \|\Psi\|=1}} \int \langle \Psi \mid \rho(t,\mathbf{0})\Psi \rangle f(t)\, dt \leq -\Gamma \sum_r m_r^4 \varphi(2\sqrt{2}m_r)^2, \tag{34}$$

assuming for simplicity that $m_1 > m_0$. Noting that the rescaled function $f_\lambda(t) = \lambda^{-1}f(t/\lambda)$ obeys (32) with $\varphi(|u|)$ replaced by $\varphi(\lambda|u|)$ and m_0 by m_0/λ, the existence of a QEI for this model then implies the convergence of $\sum_r m_r^4 \varphi(2\sqrt{2}\lambda m_r)^2$ for all $\lambda > 0$.

This observation produces a link between QEIs, thermal stability and nuclearity. To do this, we first construct some test functions obeying the hypotheses of Theorem 3.2. Observe that if $\chi \in C_0^\infty(\mathbb{R})$ is even and nonnegative, then the convolution $\eta = \chi \star \chi$ is even, nonnegative and has nonnegative Fourier transform. Choose $\beta_0 > 0$ and define

$$f(t) = \frac{\beta_0 \eta(t)}{\pi(t^2 + \beta_0^2)}, \tag{35}$$

assuming without loss that $\int f(t)\, dt = 1$. Then, for any $u > 0$,

$$\hat{f}(u) = \int_{-\infty}^\infty \frac{du'}{2\pi} \hat{\eta}(u') e^{-\beta_0|u-u'|} \geq e^{-\beta_0 u} \int_{-\infty}^0 \frac{du'}{2\pi} \hat{\eta}(u') e^{\beta_0 u'} \tag{36}$$

so—as $\hat{f}$ is even—we conclude that $\hat{f}(u) \geq \kappa e^{-\beta_0|u|}$ for some positive constant κ.

Theorem 3.3. *Let $f \in C_0^\infty(\mathbb{R})$ be a fixed test function obeying the hypotheses of Theorem 3.2 with $\varphi(u) = \kappa e^{-\beta_0 u}$ (for some $\kappa, \beta_0 > 0$) and define $f_\lambda(t) = \lambda^{-1}f(t/\lambda)$. Consider the vacuum representation of the theory of countably many independent scalar fields with a mass gap and suppose it satisfies a QEI bound*

$$\inf_{\substack{\Psi \in \mathfrak{S} \\ \|\Psi\|=1}} \int \langle \Psi \mid \rho(t,\boldsymbol{x})\Psi \rangle f_\lambda(t)\, dt \geq -Q(\lambda) > -\infty \tag{37}$$

for all $\lambda > 0$. Then the thermal equilibrium states of the theory are locally normal at all temperatures. Furthermore, if Q obeys a polynomial scaling bound $Q(\lambda) \leq C\lambda^{-n}$ for some $C, n > 0$ then the theory fulfils the nuclearity criterion and hence has the split property.

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Proof. The existence of a QEI for this model implies the convergence of $\sum_r m_r^4 e^{-4\sqrt{2}\beta m_r}$, and hence $\sum_r e^{-4\sqrt{2}\beta m_r}$, for all $\beta > 0$; moreover, the scaling bound on $Q(\lambda)$ as $\lambda \to 0+$ immediately implies an inverse polynomial scaling bound on the function $G(\beta)$ as $\beta \to 0^+$. Thus, the nuclearity criterion holds and the split property follows by Theorem 2.1. $\square$

We remark that this result uses only rather minimal information on the mass spectrum derived from the existence of QEIs for test functions decaying more slowly than exponentially. One may also construct compactly supported test functions with more finely controlled decay: moderately explicit examples of smooth, even, everywhere positive functions of compact support f with transforms obeying

$$\hat{f}(u) = \kappa e^{-\gamma|u|^\alpha} + O\left(\frac{e^{-\gamma|u|^\alpha}}{|u|^{1-\alpha}}\right), \tag{38}$$

on $u \neq 0$, where $\kappa, \gamma > 0$ and $0 < \alpha < 1$ are given in [29, §II]; an older but less explicit construction of such functions appears in [38]. For the theory to obey a finite QEI for all such test functions, it is necessary that $\sum_r m_r^4 e^{-(\beta m_r)^\alpha} < \infty$ for all $\beta > 0$ and $0 < \alpha < 1$.

Connections between QEIs and thermal stability have been studied before. Indeed, arguments based on the second law of thermodynamics underlay Ford's original intuition that QFT might obey QEI-type bounds [33]. In an abstract setting [31], it was shown that the existence of QEIs entailed the existence of passive states, thereby proving that the second law of thermodynamics holds. The fact that QEIs with polynomial scaling implies nuclearity was stated without proof in [25]. By way of a converse, we give the following:

Theorem 3.4. *If the theory of countably many scalar fields with mass gap satisfies the nuclearity criterion* (11) *then one has a QEI bound of the form* (31) *for all $g \in C_0^\infty(\mathbb{R})$ such that $\hat{g}(u) = O(e^{-\gamma|u|^\alpha})$ for any $\alpha > n/(n+1)$ and $\gamma > 0$ (or with $\alpha = n/(n+1)$ and sufficiently large γ).*

Proof. We have already shown that $N(v) \leq Bv^{2/(n+1)} e^{Cv^{n/(n+1)}}$ for positive constants B and C. Substituting into (31), the QEI bound is finite for all $g \in C_0^\infty(\mathbb{R})$ meeting the hypotheses. $\square$

Note that the conditions on $\hat{g}$ become more stringent as n increases (as one would expect), but nonetheless, the result is nonempty for any $n > 0$. As the upper bound (29) on $N(u)$ arising from nuclearity is rather weak, it seems reasonable to conjecture that nuclearity for such models implies the existence of a QEI for a wide class of test functions, with scaling behaviour not much worse than polynomial.

Do all theories obeying nuclearity satisfy QEIs (or, even more, QEIs with good scaling behaviour)? One problem with this is that there are arguments suggesting that QEIs do not hold for general interacting theories in the form we have stated [45]. Allowing for a lower bound that depends on the overall energy scale of the state (but less strongly than any possible *upper* bound) then one can establish results analogous to QEIs for 'classically positive' observables in a model independent setting [4]. One drawback to those results, however, is that it has not yet been possible to show that the energy density is classically positive in general theories.

Summarising this section: we have explained how nuclearity criteria are related both to good thermodynamic behaviour and also to the QEIs. The connection with QEIs is currently restricted

to models consisting of countably many independent free fields; however the circumstantial evidence for a more general result seems strong.

4   The split property in curved spacetime

The previous sections have described the split property, and the related nuclearity criterion, in flat spacetime. Here, we turn to the question of what can be said in curved spacetimes. While the split property can be stated in much the same way in curved spacetimes as in flat, the same is not true of nuclearity because general curved spacetimes do not admit time-translation symmetries and therefore possess no Hamiltonian.

In the first instance, then, we restrict to ultrastatic spacetimes. Let (Σ, h) be a complete Riemannian metric space, assumed connected. Then the manifold $\mathbb{R} \times \Sigma$ equipped with Lorentz-signature metric

$$g = dt \otimes dt - \mathrm{pr}_\Sigma^* h \tag{39}$$

and time-orientation chosen so that t increases to the future, is by definition an ultrastatic spacetime. Moreover, the spacetime is globally hyperbolic [39, Prop. 5.2]: it admits no closed timelike curves, and every set of the form $J^+(p) \cap J^-(q)$ is compact, where $J^{+/-}(p)$ is the set of points that can be reached by future/past-directed smooth causal curves starting from p.

Let us assume that a net of local von Neumann algebras, acting on a Hilbert space $\mathscr{H}$, has been associated to this spacetime, and that the geometrical time-translation symmetry $(t, \sigma) \mapsto (t + \tau, \sigma)$ is implemented by a unitary group $U(\tau) = e^{iH\tau}$ where the generator H is a positive unbounded operator with a one-dimensional kernel spanned by a vector Ω. In this setting, one can formulate nuclearity criteria just as in Minkowski space, and derive the split property as a consequence by [1, Prop. 17.1.4] (which is an abstract version of [10]). This strategy was adopted by Verch [54] in the case of the free Klein–Gordon field on ultrastatic spacetimes, and somewhat later by D'Antoni and Hollands for Dirac fields [19].

As already mentioned, one cannot pursue this strategy in general spacetimes. Instead one proceeds indirectly by deforming spacetimes in which the split property is known to hold into others, in such a way that the split property is preserved. This approach was used by Verch in [54] for the specific case of the Klein–Gordon field, but has even older antecedents in other contexts [34].

We will present a modern, general and streamlined version of this argument, based on [27], to which the reader is referred for the full details. The general framework used is that of locally covariant quantum field theory [8] (see [32] for a recent expository review). The geometric aspects are as follows. Fixing a spacetime dimension $d \geq 2$, we define a category of spacetimes **Loc**, whose objects are globally hyperbolic spacetimes of dimension d, equipped with orientation and time-orientation and typically denoted $\boldsymbol{M}, \boldsymbol{N}$ etc. The morphisms in this category are smooth isometric embeddings, preserving orientation and time-orientation and having causally convex images – that is, every causal curve joining points in the image lies entirely within it. If the image of $\psi : \boldsymbol{M} \to \boldsymbol{N}$ contains a Cauchy surface of $\boldsymbol{N}$, ψ is called a *Cauchy morphism*.
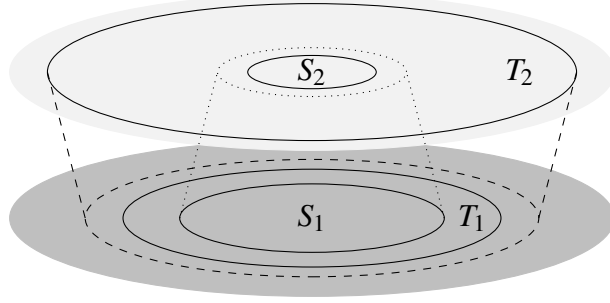
We also introduce the following terminology:

Figure 2: Regular Cauchy pairs with $(S_1, T_1) \prec_{\boldsymbol{M}} (S_2, T_2)$.

Definition 4.1. *A regular Cauchy pair (S, T) in $\boldsymbol{M} \in \mathbf{Loc}$ is an ordered pair of nonempty, open, relatively compact subsets of a common smooth spacelike Cauchy surface in which $\overline{T}$ has nonempty complement, and so that $\overline{S} \subset T$. There is a preorder on regular Cauchy pairs in $\boldsymbol{M}$ defined so that $(S_1, T_1) \prec_{\boldsymbol{M}} (S_2, T_2)$ if and only if $S_2 \subset D_{\boldsymbol{M}}(S_1)$ and $T_1 \subset D_{\boldsymbol{M}}(T_2)$.*

Here, $D_{\boldsymbol{M}}(S)$ is the *Cauchy development* of S in $\boldsymbol{M}$ – all points p in $\boldsymbol{M}$ such that every inextendible piecewise-smooth causal curve through p intersects S. The preordering is illustrated in Fig. 2.

Regular Cauchy pairs have important stability properties [27, Lem. 2.4]: if $\psi : \boldsymbol{M} \to \boldsymbol{N}$ is Cauchy, then (S, T) is a regular Cauchy pair in $\boldsymbol{M}$ if and only if $(\psi(S), \psi(T))$ is a regular Cauchy pair in $\boldsymbol{N}$ and $\overline{\psi(T)} \subset \psi(\boldsymbol{M})$. Furthermore, in all 'sufficiently nearby' Cauchy surfaces to one containing (S, T), there are regular Cauchy pairs preceding and preceded by (S, T) in the preorder. (See [27, Lem. 2.6] for the precise details and more general statements.)

Locally covariant QFT provides an axiomatic framework for quantum field theory on curved spacetimes and has led to a number of interesting developments, as described in [32]. The first assumption is:

Functoriality A theory is a covariant functor $\mathscr{A} : \mathbf{Loc} \to \mathbf{C}^*\text{-}\mathbf{Alg}$ from $\mathbf{Loc}$ to the category $\mathbf{C}^*\text{-}\mathbf{Alg}$ of unital C^*-algebras and injective unit-preserving $*$-homomorphisms.

Thus, to each spacetime $\boldsymbol{M}$ there is an object $\mathscr{A}(\boldsymbol{M})$ of $\mathbf{C}^*\text{-}\mathbf{Alg}$ and to every morphism between spacetimes $\psi : \boldsymbol{M} \to \boldsymbol{N}$ there is a $\mathbf{C}^*\text{-}\mathbf{Alg}$-morphism $\mathscr{A}(\psi) : \mathscr{A}(\boldsymbol{M}) \to \mathscr{A}(\boldsymbol{N})$ such that $\mathscr{A}(\mathrm{id}_{\boldsymbol{M}}) = \mathrm{id}_{\mathscr{A}(\boldsymbol{M})}$ and $\mathscr{A}(\phi \circ \psi) = \mathscr{A}(\phi) \circ \mathscr{A}(\psi)$. Given this, we may define a *kinematic net* in each $\boldsymbol{M} \in \mathbf{Loc}$ indexed by nonempty open causally convex subsets,

$$O \mapsto \mathscr{A}^{\mathrm{kin}}(\boldsymbol{M}; O) := \mathscr{A}(\iota_O)(\mathscr{A}(\boldsymbol{M}|_O)). \tag{40}$$

Here $\boldsymbol{M}|_O$ is the region O, equipped with the causal structures inherited from $\boldsymbol{M}$, and regarded as a spacetime (i.e., an object of $\mathbf{Loc}$) in its own right, and ι_O is the inclusion map, which becomes a morphism $\iota_O : \boldsymbol{M}|_O \to \boldsymbol{M}$ (see Fig. 3). The kinematic net is automatically isotonous, owing to the functorial nature of $\mathscr{A}$: if $O_1 \subset O_2$ then ι_{O_1} factors through ι_{O_2} and hence $\mathscr{A}^{\mathrm{kin}}(\boldsymbol{M}; O_1) \subset \mathscr{A}^{\mathrm{kin}}(\boldsymbol{M}; O_2)$. The other assumptions we make can now be stated.
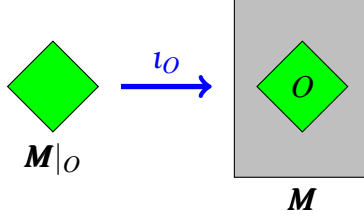
Figure 3: Illustration of the objects involved in defining the kinematic net

Einstein Causality If $O_i \subset M$ are spacelike separated then

$$[\mathscr{A}^{\mathrm{kin}}(M;O_1), \mathscr{A}^{\mathrm{kin}}(M;O_2)] = 0. \tag{41}$$

Timeslice $\mathscr{A}$ maps Cauchy morphisms to $\mathbf{C}^*$-**Alg**-isomorphisms.

The split property The algebras produced by $\mathscr{A}$ are abstract C^*-algebras. However, any state ω on $\mathscr{A}(M)$ induces a corresponding GNS representation $(\mathscr{H}, \pi, \Omega)$ such that $\omega(A) = \langle \Omega \mid \pi(A)\Omega \rangle$ for all $A \in \mathscr{A}(M)$. Passing to the representation and taking weak closures, we obtain a net of von Neumann algebras $O \mapsto \pi(\mathscr{A}^{\mathrm{kin}}(M;O))''$ associated to $\mathscr{A}$, M and ω. This is the setting in which the split property may be defined.

Definition 4.2. *Let* $\mathscr{A} : \mathbf{Loc} \to \mathbf{C}^*$-**Alg** *be a locally covariant QFT and* $M \in \mathbf{Loc}$*. A state* ω *on* $\mathscr{A}(M)$ *has the* split property *for a regular Cauchy pair* (S,T) *if, in its GNS representation* $(\mathscr{H}, \pi, \Omega)$*, there is a type-I factor* $\mathfrak{N}$ *such that*

$$\mathfrak{R}_S \subset \mathfrak{N} \subset \mathfrak{R}_T, \tag{42}$$

where $\mathfrak{R}_U = \pi(\mathscr{A}^{kin}(M;D_M(U)))''$ *for* $U = S,T$.

Apart from the use of regions based on regular Cauchy pairs, this is essentially the same formulation as used in Minkowski space. The present approach allows us to establish a key inheritance property [27, Remark 3.2]. Suppose that ω is split for (S,T). Then if $(\tilde{S}, \tilde{T})$ is some other Cauchy pair in M with $(S,T) \prec_M (\tilde{S}, \tilde{T})$, we have $\tilde{S} \subset D_M(S)$, $T \subset D_M(\tilde{T})$, so also

$$D_M(\tilde{S}) \subset D_M(S), \qquad D_M(\tilde{T}) \subset D_M(T). \tag{43}$$

Hence by isotony, $\mathfrak{R}_{\tilde{S}} \subset \mathfrak{R}_S \subset \mathfrak{N} \subset \mathfrak{R}_T \subset \mathfrak{R}_{\tilde{T}}$, so ω is split for $(\tilde{S}, \tilde{T})$. Moreover, the split property is also stable under Cauchy morphisms:

Lemma 4.3. *Suppose* $\psi : M \to N$ *is a Cauchy morphism and let* ω_M, ω_N *be a states on* $\mathscr{A}(M)$*,* $\mathscr{A}(N)$*. Then* ω_N *is split for* $(\psi(S), \psi(T))$ *if and only if* $\mathscr{A}(\psi)^* \omega_N$ *is split for* (S,T)*. Consequently, as* $\mathscr{A}(\psi)$ *is an isomorphism,* ω_M *is split for* (S,T) *if and only if* $(\mathscr{A}(\psi)^{-1})^* \omega_M$ *is split for* $(\psi(S), \psi(T))$*.*

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Proof. ω_N and $\mathscr{A}(\psi)^*\omega_N$ have unitarily equivalent GNS representations and the split inclusion is preserved because the type-I property and factor properties are stable. □

The main result is:

Theorem 4.4 (Rigidity of the split property [27, Thm 3.4]). *Suppose that $\mathscr{A}$ is a locally covariant QFT, $M, N \in \textbf{Loc}$ have oriented-diffeomorphic Cauchy surfaces, and ω_N is a state on $\mathscr{A}(N)$ that is split for all regular Cauchy pairs in N.*

Given any regular Cauchy pair (S_M, T_M) in M, there is a chain of Cauchy morphisms between M and N inducing an isomorphism $\nu : \mathscr{A}(M) \to \mathscr{A}(N)$ such that $\nu^*\omega_N$ has the split property for (S_M, T_M).

Outline proof. First, because M and N have Cauchy surfaces related by an orientation-preserving diffeomorphism, we may assume without loss that both spacetimes have common underlying manifold $\mathbb{R} \times \Sigma$, with the tangent vector to the first factor being future-directed timelike, and so that $\Sigma_M = \{\tau_M\} \times \Sigma$ for some $\tau_M \in \mathbb{R}$. The next step is to construct a globally hyperbolic metric on $\mathbb{R} \times \Sigma$ that interpolates between that of M, with which it agrees on $(-\infty, \tau_1) \times \Sigma$ and that of N, with which it agrees on $(\tau_2, \infty) \times \Sigma$, where $\tau_M < \tau_1 < \tau_2$. This also has to be arranged so that for some $\tau_* \in (\tau_1, \tau_2)$ and $\tau_N > \tau_2$ there are regular Cauchy pairs (S_*, T_*) and (S_N, T_N) lying in $\Sigma_* = \{\tau_*\} \times \Sigma$ and $\Sigma_N = \{\tau_N\} \times \Sigma$ respectively, so that

$$(S_N, T_N) \prec_I (S_*, T_*) \prec_I (S_M, T_M), \tag{44}$$

where I is the spacetime defined by the interpolating metric. Defining $P = M|_{(-\infty, \tau_1) \times \Sigma}$ and $F = N|_{(\tau_2, \infty) \times \Sigma}$, it is clear that the obvious inclusions induce a chain of Cauchy mophisms as illustrated in Fig. 4.

The inheritance and timeslice properties now allows us to pass the split property along the chain, as follows: ω_N is split for (S_N, T_N), so therefore:
- by timeslice $\omega_F = \mathscr{A}(\delta)^*\omega_N$ is split for (S_N, T_N) (regarded as a regular Cauchy pair in F)
- by timeslice $\omega_I = (\mathscr{A}(\gamma)^{-1})^*\omega_F$ is split for (S_N, T_N) in I
- by inheritance, ω_I is split for (S_*, T_*) and hence (S_M, T_M) in I
- by timeslice $\omega_P = \mathscr{A}(\beta)^*\omega_I$ is split for (S_M, T_M) in P
- by timeslice $\omega_M = (\mathscr{A}(\alpha)^{-1})^*\omega_P$ is split for (S_M, T_M) in M,

and the theorem is proved with $\nu = \mathscr{A}(\delta)\mathscr{A}(\gamma)^{-1}\mathscr{A}(\beta)\mathscr{A}(\alpha)^{-1}$. □

Now consider an arbitrary globally hyperbolic spacetime $M \in \textbf{Loc}$. Choose any smooth spacelike Cauchy surface Σ of M with an induced orientation $\mathfrak{o}$ (so that $\mathfrak{t} \wedge \mathfrak{o}$ is positively oriented for any future-pointed timelike one-form $\mathfrak{t}$), and endow Σ with a complete Riemannian metric h [43]. Then one obtains an ultrastatic spacetime N with metric (39) that is also an object of **Loc** and clearly has smooth spacelike Cauchy surfaces oriented-diffeomorphic to those of M. Together with Theorem 4.4 this shows that the problem of establishing a split property in M [in the sense that, for any regular Cauchy pair, there exists a state for which the inclusion is split] can be reduced to the ultrastatic case, for which nuclearity criteria can be employed (and have been proved in certain models [54, 19]).
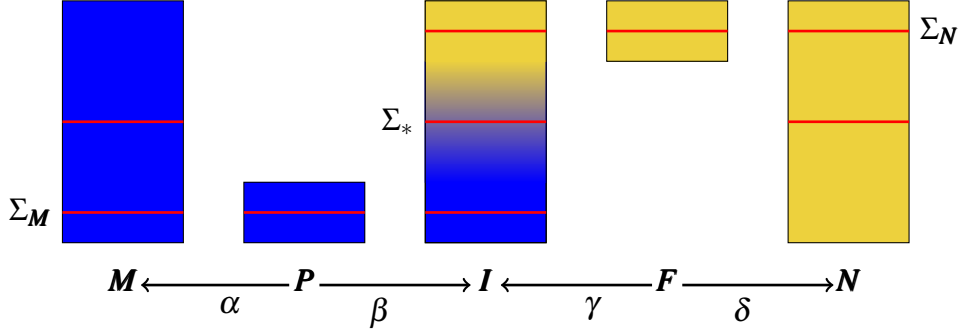
Figure 4: The chain of Cauchy morphisms in the proof of Theorem 4.4.

Theorem 4.4 can be extended in various ways. For example, one may construct states that are split for finitely many regular Cauchy pairs lying in a common Cauchy surface. Again, suppose that the theory has a state space $\mathscr{S}$ obeying the timeslice property; that is, an assignment to each $M \in \mathbf{Loc}$ of a subset $\mathscr{S}(M)$ of states on $\mathscr{A}(M)$ subject to the contravariance requirement

$$\mathscr{A}(\psi)^* \mathscr{S}(N) \subset \mathscr{S}(N) \tag{45}$$

for each morphism $\psi : M \to N$, with equality if ψ is Cauchy, and also the requirement that each $\mathscr{S}(M)$ be closed under convex combinations and operations induced by $\mathscr{A}(M)$. Suppose further that $\mathscr{S}(N)$ contains a state ω_N that is split for all regular Cauchy pairs. Then the state ω_M constructed by the theorem belongs to $\mathscr{S}(M)$ because ν is formed from Cauchy morphisms. Furthermore, if $\mathscr{S}$ obeys a condition of local quasiequivalence, one may deduce in addition that every state in $\mathscr{S}(M)$ is split for all regular Cauchy pairs in M and also for any pair of regions $O_i \in \mathcal{O}(M)$ such that

$$O_1 \subset D_M(S) \qquad D_M(T) \subset O_2 \tag{46}$$

for a regular Cauchy pair (S,T). Details appear in [27].

(Partial) Reeh–Schlieder property It should be clear that the proof of Theorem 4.4 depends on only a few ideas: the formulation in terms of regular Cauchy pairs, and the inheritance and timeslice properties. Similar arguments apply to other properties of quantum field theory (a key theme of [32] is the utility of this and related rigidity arguments). In particular, one obtains a streamlined version of Sanders' results on Reeh–Schlieder properties [48].

Definition 4.5. *Let $\mathscr{A} : \mathbf{Loc} \to \mathbf{C}^*\text{-}\mathbf{Alg}$ be a locally covariant QFT and $M \in \mathbf{Loc}$. A state ω on $\mathscr{A}(M)$ has the* Reeh–Schlieder property *for a regular Cauchy pair (S,T) if, in the GNS representation $(\mathscr{H}, \pi, \Omega)$, Ω is cyclic for $\mathfrak{R}_S$ and separating for $\mathfrak{R}_T$.*

That is, vectors of the form $A\Omega$ $(A \in \mathfrak{R}_S)$ are dense in $\mathscr{H}$, and no $B \in \mathfrak{R}_T$ can annihilate Ω. Note that the state is not assumed to be cyclic for all local algebras, so one might refer to this as a partial Reeh–Schlieder property, by comparison with the original Minkowski-space

result [47]. Our definition obeys an inheritance rule like that of the split property, but with a reversed ordering: if ω is Reeh–Schlieder for (S, T) in $\boldsymbol{M}$, then it is also Reeh–Schlieder for every $(\tilde{S}, \tilde{T})$ with

$$(\tilde{S}, \tilde{T}) \prec_{\boldsymbol{M}} (S, T) \tag{47}$$

simply because the separating property is inherited by subalgebras and cyclicity by algebras containing the given one. One can also prove that our definition is stable with respect to the timeslice condition, and this easily establishes:

Theorem 4.6 (Rigidity of Reeh–Schlieder [27, Thm 3.11], cf. [48]). *Suppose that $\mathscr{A}$ is a locally covariant QFT, $\boldsymbol{M}, \boldsymbol{N} \in \mathsf{Loc}$ have oriented-diffeomorphic Cauchy surfaces, and $\omega_{\boldsymbol{N}}$ is a state on $\mathscr{A}(\boldsymbol{N})$ that is Reeh–Schlieder for all regular Cauchy pairs.*

Given any regular Cauchy pair $(S_{\boldsymbol{M}}, T_{\boldsymbol{M}})$ in $\boldsymbol{M}$, there is a chain of Cauchy morphisms between $\boldsymbol{M}$ and $\boldsymbol{N}$ inducing an isomorphism $\nu : \mathscr{A}(\boldsymbol{M}) \to \mathscr{A}(\boldsymbol{N})$ such that $\nu^* \omega_{\boldsymbol{N}}$ has the Reeh–Schlieder property for $(S_{\boldsymbol{M}}, T_{\boldsymbol{M}})$.

Proof. Invert the argument for the split property, replacing $\prec$ by $\succ$. $\qquad\square$

As with the split property, the above Reeh–Schlieder property in general spacetimes can now be traced to the ultrastatic case. Here, one can give a *tube condition* (satisfied in general Wightman theories in Minkowski space [3] and in particular linear models in curved spacetimes [49]) that implies that ground states obey the full Reeh–Schlieder property. Thus the (partial) Reeh–Schlieder property becomes a reasonable expectation in general locally covariant theories. Our result admits various extensions, of which the most important is that the proofs of the split and Reeh–Schlieder results may be combined to yield states that are *both Reeh–Schlieder and split* for (finitely many) regular Cauchy pairs in a common Cauchy surface. Consequently, given any regular Cauchy pair (S, T) one may find a state ω with GNS vector Ω so that $(\mathfrak{R}_S, \mathfrak{R}_T, \Omega)$ is a standard split inclusion, leading to applications analogous to those described in section 2. Finally, let us mention that a slightly different definition of the split property has been studied in [7] and used to discuss the tensorial structure of locally covariant theories.

Distal split   Suppose we have a locally covariant theory $\mathscr{A}$ with state space $\mathscr{S}$. In d-dimensional Minkowski spacetime $\boldsymbol{M}$, we define the *splitting distance* of $S \subset \mathbb{R}^{d-1}$ by

$$d(S) = \inf\{\rho > 0 : (S, B(S, \rho)) \text{ is split for some state in } \mathscr{S}(\boldsymbol{M})\}, \tag{48}$$

where $B(S, \rho)$ is the open Euclidean ball of radius ρ about S. If $d(S)$ is finite but nonzero we say that the *distal split property* holds. It is of interest to understand what happens to such models in the locally covariant setting, assuming that the state space obeys local quasiequivalence. Suppressing some ε's (see [27, §3.4] for the precise statements and proof) one has the following. If $f \in \mathrm{Diff}(\mathbb{R}^{d-1})$ with uniformly bounded derivatives, and $r > d(f(S))$, then

$$d(S) \le \inf\{\rho > 0 : B(f(S), r) \subset f(B(S, \rho))\} \tag{49}$$

and hence

$$d(S) \le \kappa d(f(S)), \tag{50}$$

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where κ is the supremum of the norm of the derivative $\|D(f^{-1})\|$ over $B(f(S),r) \setminus f(S)$.

As a first example, taking $f(\boldsymbol{x}) = \boldsymbol{x}/\lambda$ implies that $d(\lambda S) \leq \lambda d(S)$ and hence the existence of a uniform splitting distance $(d(S) = d_0 \geq 0$ for all $S)$ actually implies the split property $(d_0 = 0)$. Of course, what this means is that models such as that of [18, Thm 4.3] given in (24) cannot be compatible with all the hypotheses required – one might well suspect that local quasiequivalence and/or the timeslice property become problematic.

For a second example, let S be an open ball and design $f \in \text{Diff}(\mathbb{R}^{d-1})$ with the properties that $f(S) = S$ and $B(S,d(S)) \subset f(B(S,\frac{1}{2}d(S)))$, and so that f acts trivially outside a compact set. Then (again suppressing ε's) one obtains

$$d(S) \leq \frac{1}{2}d(S), \tag{51}$$

and hence the only possible splitting distances are 0 or ∞. This proves a stronger result that the distal split property implies the split property (at least for sets diffeomorphic to balls), given our other hypotheses.

The proof of (49) takes its inspiration from cosmological inflation (see Fig. 5). Let S_1 be an open bounded subset in a constant time hypersurface and let $T_1 = B(S_1,r)$ where $r > d(S_1)$, so that some state is split for (S_2,T_2). Then we may infer that the state is split for suitable regular Cauchy pairs lying in a constant time hypersurface to the past. In the left-hand half of Fig. 5 we illustrate the situation in Minkowski space: S_2 is has a smaller diameter than S_1 while T_2 has a larger diameter than T_1, so this argument (not very helpfully) gives an upper bound on $d(S_2)$ that is larger than $d(S_1)$. In the right-half of the figure, however, we imagine that the metric undergoes a period of inflation between the regions around the two hypersurfaces and outside the Minkowski $D(S_1)$). If we arrange matters so that T_2 fits within a ball of radius $\kappa r \, \text{diam}(S_2)/\text{diam}(S_1)$ for $\kappa < 1$ then we obtain a tighter bound on $d(S_2)$ than r is on $d(S_1)$. The ε's suppressed in our discussion relate to the spreading of lightcones (at the Minkowski speed of light) which can be made arbitrarily small by considering nearby hypersurfaces.

Actually, there is some physics to go alongside what would otherwise appear to be a geometric trick. As described earlier, nonzero splitting distances are associated with the existence of a critical temperature, above which the thermal equilibrium states are no longer locally quasiequivalent to the vacuum. On the other hand, inflation cools temperature (to the future); conversely, one could expect that some states of subcritical temperature near the later hypersurface have supercritical temperatures at the earlier one. This leads to a contradiction between the assumptions of the timeslice axiom and local quasiequivalence.

5 Concluding remarks

It is hoped that this paper has explained the physical significance of the split and nuclearity properties, and also explained – in the context of the split and Reeh–Schlieder properties – a general proof strategy for establishing structural properties of locally covariant quantum field theories. As a further example, we mention that Lechner and Sanders have recently applied the machinery of regular Cauchy pairs in their proof of modular nuclearity [41]. We have sketched
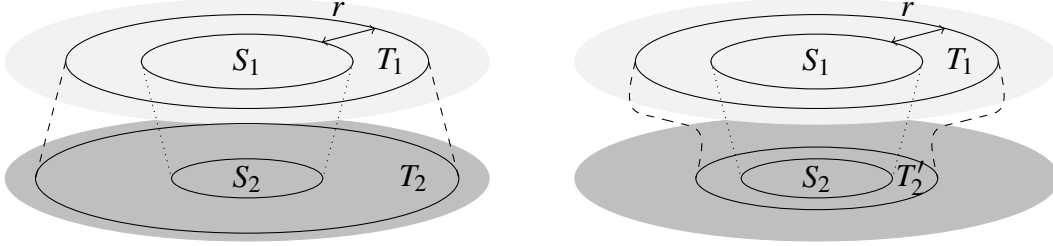
Figure 5: Illustration of the 'inflationary' proof of the distal split bound.

links with the theory of QEIs, which require further investigation. At first sight, the split and nuclearity properties might not seem to have much in common with each other, or with the QEIs. But in fact, all three can be viewed as expressions of the uncertainty principle: this is most obvious in the case of the QEIs, which provide rigorous time-energy inequalities, but it is also true of the split property (the need for a collar region to guarantee independence indicating a loss of sharp localisation) and nuclearity (understood as a constraint on the number of states per unit volume in phase space). They therefore occupy a fascinating position at the nexus of relativity and quantum theory.

A   Proof of Theorem 3.2

Recall that $m_0 \geq 0$ has been fixed and that $f \in C_0^\infty(\mathbb{R})$ is nonnegative, even, with unit integral, and has a Fourier transform that is real, even, nonnegative and bounded from below by

$$\hat{f}(u) \geq \varphi(|u|) \tag{52}$$

on $[m_0, \infty)$, where $\varphi : [m_0, \infty) \to \mathbb{R}^+$ is monotone decreasing. Note that nonnegativity of f and $\hat{f}$ implies that $\hat{f}(m_0) \leq \hat{f}(0) = 1$ and hence $\varphi(u) \leq 1$ for all $u \geq m_0$.

Consider a single Klein–Gordon field of mass $m \geq m_0$ on the symmetric Fock space $\mathscr{F}$ over $L^2(\mathbb{R}^3, d^3\boldsymbol{k}/(2\pi)^3)$, and define the usual annihilation operators $a(\boldsymbol{k})$ by

$$(a(\boldsymbol{k})\Psi)^{(n)}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n) = \sqrt{n+1}\,\Psi^{(n+1)}(\boldsymbol{k}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n), \tag{53}$$

where for $\Psi \in \mathscr{F}$, $\Psi^{(n)}$ denotes its n-particle component.[8] Writing $a^\dagger(\boldsymbol{k})$ for the adjoint of $a(\boldsymbol{k})$ as a quadratic form, the canonical commutation relations are

$$[a(\boldsymbol{k}), a(\boldsymbol{k}')] = 0, \qquad [a(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')] = (2\pi)^3 \delta(\boldsymbol{k} - \boldsymbol{k}') \mathbb{1}, \tag{54}$$

---

[8]The annihilation operators used in Section 3 are $\mathsf{a}(u) = \int d^3\boldsymbol{k}/(2\pi)^3 a(\boldsymbol{k})\overline{u(\boldsymbol{k})}$.

and the quantum field is given by

$$\Phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega}} \left(a(\mathbf{k})e^{-ik_a x^a} + a^\dagger(\mathbf{k})e^{ik_a x^a} \right), \tag{55}$$

in which $k^a = (\omega, \mathbf{k})$ with $\omega = (\|\mathbf{k}\|^2 + m^2)^{1/2}$.

The energy density (with respect to the standard time coordinate) is a sum of Wick squares

$$\rho_m(x) = \frac{1}{2}\left(:(\nabla_0\Phi(x))^2: + \sum_{i=1}^{3} :(\nabla_i\Phi(x))^2: + m^2 :\Phi(x)^2: \right), \tag{56}$$

so, again in a quadratic form sense,

$$\rho_m(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3} \frac{1}{4\sqrt{\omega\omega'}} \left\{ (\omega\omega' + \mathbf{k}\cdot\mathbf{k}' + m^2)e^{i(k-k')\cdot x}a^\dagger(\mathbf{k})a(\mathbf{k}') \right.$$
$$\left. - (\omega\omega' + \mathbf{k}\cdot\mathbf{k}' - m^2)e^{-i(k+k')\cdot x}a(\mathbf{k})a(\mathbf{k}') \right\} + \text{H.C.}, \tag{57}$$

where H.C. denotes the hermitian conjugate.

Next, choose any smooth, symmetric, nonnegative function $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, with compact support obeying

$$\operatorname{supp} B \subset \{(\mathbf{u},\mathbf{u}') : \|\mathbf{u}\|, \|\mathbf{u}'\| \in [\tfrac{1}{2}, 1]; \quad |\theta(\mathbf{u},\mathbf{u}')| < \pi/3\}, \tag{58}$$

where $\theta(\mathbf{u},\mathbf{u}')$ is the angle between the vectors $\mathbf{u}$, $\mathbf{u}'$, and normalised so that

$$\int \frac{d^3\mathbf{u}}{(2\pi)^3} \frac{d^3\mathbf{u}'}{(2\pi)^3} B(\mathbf{u},\mathbf{u}') = 1. \tag{59}$$

The function $C : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$$C(\mathbf{u},\mathbf{u}') = \int \frac{d^3\mathbf{u}''}{(2\pi)^3} B(\mathbf{u}'',\mathbf{u})B(\mathbf{u}'',\mathbf{u}') \tag{60}$$

is then pointwise nonnegative with support obeying

$$\operatorname{supp} C \subset \{(\mathbf{u},\mathbf{u}') : \|\mathbf{u}\|, \|\mathbf{u}'\| \in [\tfrac{1}{2}, 1]\}. \tag{61}$$

We now define, for $\lambda > 0$, the vacuum-plus-two-particle superposition

$$\Psi_{m,\lambda} = \mathcal{N}_{m,\lambda}\left[\Omega + \frac{\lambda}{\sqrt{2}} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3} b(\mathbf{k},\mathbf{k}')a^\dagger(\mathbf{k})a^\dagger(\mathbf{k}')\Omega \right], \tag{62}$$

where $\Omega \in \mathscr{F}$ is the Fock vacuum vector, $\mathcal{N}_{m,\tau,\lambda}$ is a normalisation constant and

$$b(\mathbf{k},\mathbf{k}') = \frac{\varphi(2\sqrt{2}m)}{m^3} B(\mathbf{k}/m, \mathbf{k}'/m). \tag{63}$$

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That is, $\Psi_{m,\lambda}^{(0)} = \mathcal{N}_{m,\lambda}$, $\Psi_{m,\lambda}^{(2)}(\boldsymbol{k},\boldsymbol{k}') = \mathcal{N}_{m,\lambda}\lambda b(\boldsymbol{k},\boldsymbol{k}')$ and all other components of $\Psi_{m,\lambda}$ vanish. As b is compactly supported, each $\Psi_{m,\lambda}$ is a Hadamard state. The normalisation constant is

$$\mathcal{N}_{m,\lambda} = \left(1 + \lambda^2 \varphi(2\sqrt{2}m)^2 \operatorname{Tr}C\right)^{-1/2} \geq \left(1 + \lambda^2 \operatorname{Tr}C\right)^{-1/2}, \tag{64}$$

where we have used $\varphi \leq 1$ and employed the short-hand notation

$$\operatorname{Tr}C = \int \frac{d^3\boldsymbol{u}}{(2\pi)^3} C(\boldsymbol{u},\boldsymbol{u}). \tag{65}$$

Using the general formulae $\langle \Omega \mid a(\boldsymbol{k})a(\boldsymbol{k}')\Psi \rangle = \sqrt{2}\Psi^{(2)}(\boldsymbol{k},\boldsymbol{k}')$ and

$$\langle \Psi \mid a^{\dagger}(\boldsymbol{k})a(\boldsymbol{k}')\Psi \rangle = 2 \int \frac{d^3\boldsymbol{k}''}{(2\pi)^3} \overline{\Psi^{(2)}(\boldsymbol{k}'',\boldsymbol{k})}\Psi^{(2)}(\boldsymbol{k}'',\boldsymbol{k}'), \tag{66}$$

for vacuum-plus-two-particle superpositions Ψ, the expected normal ordered energy density is

$$\langle \Psi_{m,\lambda} \mid \rho_m(x)\Psi_{m,\lambda} \rangle = |\mathcal{N}_{m,\lambda}|^2 \operatorname{Re} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{d^3\boldsymbol{k}'}{(2\pi)^3} \frac{1}{\sqrt{\omega\omega'}} \left(\lambda^2 c(\boldsymbol{k},\boldsymbol{k}')(\omega\omega' + \boldsymbol{k}\cdot\boldsymbol{k}' + m^2)e^{i(k-k')\cdot x}\right.$$
$$\left. - \frac{\lambda}{\sqrt{2}}b(\boldsymbol{k},\boldsymbol{k}')(\omega\omega' + \boldsymbol{k}\cdot\boldsymbol{k}' - m^2)e^{-i(k+k')\cdot x}\right), \tag{67}$$

where

$$c(\boldsymbol{k},\boldsymbol{k}') = \frac{\varphi(2\sqrt{2}m)^2}{m^3}C(\boldsymbol{k}/m,\boldsymbol{k}'/m). \tag{68}$$

As $\Psi_{m,\lambda}$ is Hadamard, the expectation value is smooth in x and we can therefore average against $f(t)$, using the fact that $\hat{f}$ is real, to find

$$\int \langle \Psi_{m,\lambda} \mid \rho_m(t,\boldsymbol{0})\Psi_{m,\lambda} \rangle f(t)\, dt = |\mathcal{N}_{m,\lambda}|^2 \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{d^3\boldsymbol{k}'}{(2\pi)^3} \frac{1}{\sqrt{\omega\omega'}}$$
$$\times \left(\lambda^2 c(\boldsymbol{k},\boldsymbol{k}')(\omega\omega' + \boldsymbol{k}\cdot\boldsymbol{k}' + m^2)\hat{f}(\omega' - \omega)\right.$$
$$\left. - \frac{\lambda}{\sqrt{2}}b(\boldsymbol{k},\boldsymbol{k}')(\omega\omega' + \boldsymbol{k}\cdot\boldsymbol{k}' - m^2)\hat{f}(\omega + \omega')\right). \tag{69}$$

We now seek an upper bound on this last quantity. First note that, for $(\boldsymbol{k},\boldsymbol{k}') \in \operatorname{supp}c$, we have $\omega, \omega' \in [\sqrt{5}m/2, \sqrt{2}m]$ and

$$\frac{1}{\sqrt{\omega\omega'}}\left(\omega\omega' + \boldsymbol{k}\cdot\boldsymbol{k}' + m^2\right) \leq \frac{4m^2}{\sqrt{5}m/2} = \frac{8m}{\sqrt{5}}, \tag{70}$$

while if $(\boldsymbol{k},\boldsymbol{k}') \in \operatorname{supp}b$ we have

$$\frac{1}{\sqrt{\omega\omega'}}\left(\omega\omega' + \boldsymbol{k}\cdot\boldsymbol{k}' - m^2\right) \geq \frac{1}{m\sqrt{2}}\left(\frac{m^2}{4} + \boldsymbol{k}\cdot\boldsymbol{k}'\right) \geq \frac{3m}{8\sqrt{2}}. \tag{71}$$

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Second, because f and $\hat{f}$ are positive, $\hat{f}(\omega' - \omega) \leq \hat{f}(0) = 1$; furthermore, for $(\boldsymbol{k}, \boldsymbol{k}') \in \operatorname{supp} b$ (and so, in particular, $m_0 < \sqrt{5} m_0 \leq \omega + \omega' \leq 2\sqrt{2} m_0$) we have

$$\hat{f}(\omega + \omega') \geq \varphi(\omega + \omega') \geq \varphi(2\sqrt{2}m). \tag{72}$$

Accordingly, as λ and the functions b and c are positive, we obtain the bound

$$\text{L.H.S. of (69)} \leq |\mathscr{N}_{m,\lambda}|^2 \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{d^3\boldsymbol{k}'}{(2\pi)^3} \left(\lambda^2 c(\boldsymbol{k},\boldsymbol{k}') \frac{8m}{\sqrt{5}} - \lambda b(\boldsymbol{k},\boldsymbol{k}') \frac{3m}{16} \varphi(2\sqrt{2}m) \right), \tag{73}$$

the right-hand side of which can be written as $-|\mathscr{N}_{m,\lambda}|^2 P(\lambda) m^4 \varphi(2\sqrt{2}m)^2$, where

$$P(\lambda) = \frac{3}{16}\lambda - \lambda^2 \frac{8}{\sqrt{5}} \int \frac{d^3\boldsymbol{u}}{(2\pi)^3} \frac{d^3\boldsymbol{u}'}{(2\pi)^3} C(\boldsymbol{u},\boldsymbol{u}'), \tag{74}$$

as follows on inserting the definitions (63), (68) of $b(\boldsymbol{k},\boldsymbol{k}')$ and $c(\boldsymbol{k},\boldsymbol{k}')$ and using the normalisation (59) of B. Now the quadratic $P(\lambda)$ has a positive maximum at some $\lambda_0 > 0$ (note that P and λ_0 are independent of m). Defining $\Psi_m = \Psi_{m,\lambda_0}$, we therefore obtain

$$\int \langle \Psi_m \mid \rho_m(t,\boldsymbol{0})\Psi_m \rangle f(t)\, dt \leq -\Gamma m^4 \varphi(2\sqrt{2}m)^2, \tag{75}$$

where

$$\Gamma = \frac{P(\lambda_0)}{1 + \lambda_0^2 \operatorname{Tr} C} \tag{76}$$

depends only on the function B (and not on m or φ). This completes the proof of Theorem 3.2.

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