Article:

Hussein, MS, Kinash, N, Lesnic, D et al. (1 more author) (2017) Retrieving the time-dependent thermal conductivity of an orthotropic rectangular conductor. Applicable Analysis, 96 (15). pp. 2604-2618. ISSN 0003-6811

https://doi.org/10.1080/00036811.2016.1232401

Retrieving the time-dependent thermal conductivity of an orthotropic rectangular conductor

M.S. Hussein[1,2], N. Kinash[3], D. Lesnic[2] and M. Ivanchov[3]
[1]*Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq*
[2]*Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK*
[3]*Faculty of Mechanics and Mathematics, Department of Differential Equations, Ivan Franko National University of Lviv, 1, Universytetska str., Lviv, 79000, Ukraine*
E-mails: mmmsh@leeds.ac.uk (M.S. Hussein), kinashnataliia@gmail.com (N. Kinash), amt5ld@maths.leeds.ac.uk (D. Lesnic), ivanchov@franko.lviv.ua (M.I. Ivanchov).

Abstract
The aim of this paper is to determine the thermal properties of an orthotropic planar structure characterised by the thermal conductivity tensor in the coordinate system of the main directions (Oxy) being diagonal. In particular, we consider retrieving the time-dependent thermal conductivity components of the an orthotropic rectangular conductor from nonlocal overspecified heat flux conditions. Since only boundary measurements are considered, this inverse formulation belongs to the desirable approach of non-destructive testing of materials. The unique solvability of this inverse coefficient problem is proved based on the Schauder fixed point theorem and the theory of Volterra integral equations of the second kind. Furthermore, the numerical reconstruction based on a nonlinear least-squares minimization is performed using the MATLAB optimization toolbox routine *lsqnonlin*. Numerical results are presented and discussed in order to illustrate the performance of the inversion for orthotropic parameter identification.

Keywords: Orthotropic heat conductor; heat equation; inverse problem; thermal conductivity; regularization.

2010 Mathematics Subject Classification: 65M30, 65M32, 80A23

1 Introduction

Factors such as manufacturing and curing process may affect the material properties of a structure often introducing additional variations such as anisotropy, [4], which are difficult to measure directly. Such a coefficient identification problem is challenging because it is inverse, nonlinear and, in general, ill-posed.

At steady-state, research on the determination of the diffusivity/conductivity of a layered and orthotropic medium has been initiated in [1, 3] and the general case of identification of anisotropic spacewise dependent conductivity in the Laplace-Beltrami elliptic equation has been much investigated in the last two decades, [15].

In the time-dependent case the situation is much less investigated and here we only mention the nonlinear identification of a temperature-dependent orthotropic material, [14], the space-dependent anisotropic case considered in [11], and the recovery of the leading coefficients of a heterogeneous orthotropic medium, [1, 5].

In this study, for the first time to the authors' knowledge, the identification of the time-dependent thermal conductivity of an orthotropic rectangular material from nonlocal heat flux overdetermination boundary conditions is investigated both theoretically and numerically. The inverse problem that we formulate in Section 2 and propose to study combines the features of multi-dimensions, multiple coefficient identification, [6], as well as non-local overdetermination data, [2,7,8,10]. All these papers are unified by the approach utilized to prove the existence of solution: the inverse problem is reformulated as a fixed point problem for a certain nonlinear compact operator, so that the Schauder theorem can be applied to it. Afterwards, uniqueness of solution follows from the theory of Volterra integral equations of the second kind. In this paper, we also follow this approach and prove the existence and uniqueness of the solution in Section 2. The numerical solutions of the direct and inverse problems are described in Section 3 and 4, respectively, and numerical results are presented and discussed in Section 5. Finally, conclusions are given in Section 6.

2 Statement of the problem

We consider an inverse problem of identifying the thermal conductivity coefficients $a_1(t)$ and $a_2(t)$ in the two-dimensional orthotropic heat equation

$$u_t = a_1(t)u_{xx} + a_2(t)u_{yy} + f(x,y,t),$$
$$(x,y,t) \in Q_T := \{(x,y,t) : 0 < x < h, 0 < y < l, 0 < t < T\}, \tag{1}$$

with initial condition

$$u(x,y,0) = \varphi(x,y), \quad (x,y) \in [0,h] \times [0,l], \tag{2}$$

Dirichlet boundary data

$$u(0,y,t) = \mu_{11}(y,t), \quad u(h,y,t) = \mu_{12}(y,t), \quad (y,t) \in [0,l] \times [0,T], \tag{3}$$
$$u(x,0,t) = \mu_{21}(x,t), \quad u(x,l,t) = \mu_{22}(x,t), \quad (x,t) \in [0,h] \times [0,T] \tag{4}$$

and nonlocal overspecification conditions

$$a_1(t)(\nu_{11}(t)u_x(0,y_0,t) + \nu_{12}(t)u_x(h,y_0,t)) = \varkappa_1(t), \quad t \in [0,T], \tag{5}$$

$$a_2(t)(\nu_{21}(t)u_y(x_0,0,t) + \nu_{22}(t)u_y(x_0,l,t)) = \varkappa_2(t), \quad t \in [0,T], \tag{6}$$

where x_0, y_0 are fixed values from the intervals $(0,h)$ and $(0,l)$ respectively.

Let $G_k(x,t,\xi,\tau)$ be the Green function for the equation $u_t = a_1(t)u_{xx}$ with Dirichlet boundary conditions for $k=1$, and with Neumann boundary conditions for $k=2$. These functions are defined by, [9],

$$G_k(x,t,\xi,\tau) = \frac{1}{2\sqrt{\pi(\theta_1(t)-\theta_1(\tau))}} \sum_{n=-\infty}^{+\infty} \left(\exp\left(-\frac{(x-\xi+2nh)^2}{4(\theta_1(t)-\theta_1(\tau))} \right) \right.$$
$$\left. +(-1)^k \exp\left(-\frac{(x+\xi+2nh)^2}{4(\theta_1(t)-\theta_1(\tau))} \right) \right), \quad k=1,2, \quad \theta_1(t) = \int_0^t a_1(\tau)d\tau, t \in [0,T]. \tag{7}$$

At the same time we define the function $G_m(y,t,\eta,\tau)$ for the equation $u_t = a_2(t)u_{yy}$ analogously to $G_k(x,t,\xi,\tau)$.

Then the Green function of the problem (1)-(4) is defined by

$$G_{km}(x,y,t,\xi,\eta,\tau) = G_k(x,t,\xi,\tau)G_m(y,t,\eta,\tau), \text{ when } k=m=1. \tag{8}$$

Theorem 1. *Let the following assumptions be satisfied:*

(A1) $\varphi \in C^2([0,h]\times[0,l])$, μ_{11}, $\mu_{12} \in C^{2,1}([0,l]\times[0,T])$, μ_{21}, $\mu_{22} \in C^{2,1}([0,h]\times[0,T])$,$f \in C^{1,0}(\overline{Q_T})$, $\varkappa_1, \varkappa_2, \nu_{11}, \nu_{12} \in C([0,T])$;

(A2) $\varkappa_1(t) > 0$, $\nu_{11}(t) + \nu_{12}(t) > 0$, $t \in [0,T]$, $\varphi_x(x,y) > 0$, $(x,y) \in [0,h] \times [0,l]$, $\mu_{21x}(x,t) > 0$, $\mu_{22x}(x,t) > 0$, $(x,t) \in [0,h] \times [0,T]$, $\mu_{11t}(y,t) - f(0,y,t) \leqslant 0$, $\mu_{11yy}(y,t) \geqslant 0$, $\mu_{12t}(y,t) - f(h,y,t) \geqslant 0$, $\mu_{12yy}(y,t) \leqslant 0$, $(y,t) \in [0,l] \times [0,T]$, $f_x(x,y,t) \geqslant 0$, $(x,y,t) \in Q_T$;

(A3) $\varkappa_2(t) > 0$, $\nu_{21}(t) + \nu_{22}(t) > 0$, $t \in [0,T]$, $\varphi_y(x,y) > 0$, $(x,y) \in [0,h] \times [0,l]$, $\mu_{11y}(y,t) > 0$, $\mu_{12y}(y,t) > 0$, $(y,t) \in [0,l] \times [0,T]$, $\mu_{21t}(x,t) - f(x,0,t) \leqslant 0$, $\mu_{21xx}(x,t) \geqslant 0$, $\mu_{22t}(x,t) - f(x,l,t) \geqslant 0$, $\mu_{22xx}(x,t) \leqslant 0$, $(x,t) \in [0,h] \times [0,T]$, $f_y(x,y,t) \geqslant 0$, $(x,y,t) \in Q_T$;

(A4) $\varphi(0,y) = \mu_{11}(y,0)$, $\varphi(h,y) = \mu_{12}(y,0)$, $y \in [0,l]$, $\varphi(x,0) = \mu_{21}(x,0)$, $\varphi(x,l) = \mu_{22}(x,0)$, $x \in [0,h]$, $\mu_{11}(0,t) = \mu_{21}(0,t)$, $\mu_{11}(l,t) = \mu_{22}(0,t)$, $\mu_{12}(0,t) = \mu_{21}(h,t)$, $\mu_{12}(l,t) = \mu_{22}(h,t)$, $t \in [0,T]$.

Then the problem (1)-(6) *has at least one solution* $(a_1, a_2, u) \in (C([0,T]))^2 \times C^{2,1}(\overline{Q}_T)$.

Proof. To prove the existence of a solution to (1)-(6) we are first going to reduce it to an equivalent in a certain sense operator equation with respect to (a_1, a_2) and afterwards apply the Schauder fixed point theorem.

If $a_1(t) > 0, a_2(t) > 0$ are known functions, the solution to the direct problem (1)-(4) can be represented as

$$
\begin{aligned}
u(x,y,t) = &\int_0^h\int_0^l G_{11}(x,y,t,\xi,\eta,0)\varphi(\xi,\eta)d\xi d\eta + \int_0^l\int_0^t G_{11\xi}(x,y,t,0,\eta,\tau)a_1(\tau)\mu_{11}(\eta,\tau)d\eta d\tau \\
&- \int_0^l\int_0^t G_{11\xi}(x,y,t,h,\eta,\tau)a_1(\tau)\mu_{12}(\eta,\tau)d\eta d\tau + \int_0^h\int_0^t G_{11\eta}(x,y,t,\xi,0,\tau)a_2(\tau)\mu_{21}(\xi,\tau)d\xi d\tau \\
&- \int_0^h\int_0^t G_{11\eta}(x,y,t,\xi,l,\tau)a_2(\tau)\mu_{22}(\xi,\tau)d\xi d\tau + \int_0^h\int_0^l\int_0^t G_{11}(x,y,t,\xi,\eta,\tau)f(\xi,\eta,\tau)d\xi d\eta d\tau.
\end{aligned} \tag{9}
$$

Denote $w_1 := u_x(x,y,t)$ and differentiate (9) with respect to x to obtain

$$
\begin{aligned}
w_1(x,y,t) = &\int_0^h\int_0^l G_{21}(x,y,t,\xi,\eta,0)\varphi_\xi(\xi,\eta)d\xi d\eta - \int_0^l\int_0^t G_{21}(x,y,t,0,\eta,\tau)(\mu_{11\tau}(\eta,\tau) \\
&- f(0,\eta,\tau) - a_2(\tau)\mu_{11\eta\eta}(\eta,\tau))d\eta d\tau + \int_0^l\int_0^t G_{21}(x,y,t,h,\eta,\tau)(\mu_{12\tau}(\eta,\tau) - f(h,\eta,\tau) \\
&- a_2(\tau)\mu_{12\eta\eta}(\eta,\tau))d\eta d\tau + \int_0^h\int_0^t G_{21\eta}(x,y,t,\xi,0,\tau)a_2(\tau)\mu_{21\xi}(\xi,\tau)d\xi d\tau
\end{aligned}
$$

$$-\int_0^h\int_0^t G_{21\eta}(x,y,t,\xi,l,\tau)a_2(\tau)\mu_{22\xi}(\xi,\tau)d\xi d\tau+\int_0^h\int_0^l\int_0^t G_{21}(x,y,t,\xi,\eta,\tau)f_\xi(\xi,\eta,\tau)d\xi d\eta d\tau. \tag{10}$$

An operator equation with respect to a_1 is obtained from (5) as

$$a_1=P_1(a_1,a_2), \tag{11}$$

where

$$P_1(a_1,a_2)(t)=\frac{\varkappa_1(t)}{\nu_{11}(t)w_1(0,y_0,t)+\nu_{12}(t)w_1(h,y_0,t)},\quad t\in[0,T]$$

and $w_1$is defined by (10).

Analogously, in order to get an operator equation with respect to $a_2(t)$, we differentiate (9) with respect to y and use the notation $w_2:=u_y(x,y,t)$ to obtain

$$\begin{aligned}
w_2(x,y,t)=&\int_0^h\int_0^l G_{12}(x,y,t,\xi,\eta,0)\varphi_\eta(\xi,\eta)d\xi d\eta\\
&+\int_0^l\int_0^t G_{12\xi}(x,y,t,0,\eta,\tau)a_1(\tau)\mu_{11\eta}(\eta,\tau)d\eta d\tau\\
&-\int_0^l\int_0^t G_{12\xi}(x,y,t,h,\eta,\tau)a_1(\tau)\mu_{12\eta}(\eta,\tau)d\eta d\tau\\
&-\int_0^h\int_0^t G_{12}(x,y,t,\xi,0,\tau)(\mu_{21\tau}(\xi,\tau)-f(\xi,0,\tau)-a_1(\tau)\mu_{21\xi\xi}(\xi,\tau))d\xi d\tau\\
&+\int_0^h\int_0^t G_{12}(x,y,t,\xi,l,\tau)(\mu_{22\tau}(\xi,\tau)-f(\xi,l,\tau)-a_1(\tau)\mu_{22\xi\xi}(\xi,\tau))d\xi d\tau\\
&+\int_0^h\int_0^l\int_0^t G_{12}(x,y,t,\xi,\eta,\tau)f_\eta(\xi,\eta,\tau)d\xi d\eta d\tau.
\end{aligned} \tag{12}$$

Therefore, an operator equation with respect to a_2 is obtained from (6)

$$a_2=P_2(a_1,a_2), \tag{13}$$

where

$$P_2(a_1,a_2)(t)=\frac{\varkappa_2(t)}{\nu_{21}(t)w_2(x_0,0,t)+\nu_{22}(t)w_2(x_0,l,t)},\quad t\in[0,T]$$

and w_2 is defined by (12).

Denote:

$\mathcal{N}:=\{a_1,a_2\in C([0,T]):\ \alpha_1\leqslant a_1(t)\leqslant A_1,\ \alpha_2\leqslant a_2(t)\leqslant A_2\}$, where the constants $\alpha_1,\alpha_2,A_1,A_2\in\mathbb{R}_+$ are to be determined below;

$P : \mathcal{N} \to \mathcal{N}$ such that $P(a_1, a_2) = \begin{pmatrix} P_1(a_1, a_2) \\ P_2(a_1, a_2) \end{pmatrix}$.

Thus, the problem (1)-(6) is reduced to the operator equation

$$(a_1, a_2) = P(a_1, a_2), \quad (a_1, a_2) \in \mathcal{N}. \tag{14}$$

The problem (1)-(6) is equivalent to the equation (14) in the following sense: if (a_1, a_2, u) is a solution to (1)-(6), then (a_1, a_2) is a solution to (14), and, conversely, if $(a_1, a_2) \in \mathcal{N}$ is a solution to (14), then (a_1, a_2, u) is a solution to (1)-(6), where u is defined by formula (9). This follows from the way the equation (14) has been obtained.

To make sure that the operator P maps $\mathcal{N}$ into itself let us estimate the constants $\alpha_1, \alpha_2, A_1, A_2 \in \mathbb{R}_+$. From the uniqueness of the solution to the problem

$$\begin{cases} u_t = a_1(t) u_{xx}, \quad (x,t) \in (0,h) \times (0,T), \\ u(x,0) = 1, \quad x \in [0,h], \\ u_x(0,t) = 0, \quad u_x(h,t) = 0, \quad t \in [0,T], \end{cases}$$

the following identity is obtained

$$\int\limits_0^h G_2(x,t,\xi,0) d\xi = 1. \tag{15}$$

According to the properties of the Green function (namely, that $G_{1\eta}(y,t,0,\eta) \geqslant 0$, $G_{1\eta}(y,t,l,\eta) \leqslant 0$ and (15) holds) it follows from **(A2)** applied to (10) that

$$w_1(x,y,t) \geqslant \min_{[0,h]\times[0,l]} \varphi_x(x,y) \int\limits_0^l G_1(y,t,\eta,0) d\eta + \min_{[0,h]\times[0,T]} \mu_{21x}(x,t) \int\limits_0^t G_{1\eta}(y,t,0,\tau) a_2(\tau) d\tau$$

$$- \min_{[0,h]\times[0,T]} \mu_{22x}(x,t) \int\limits_0^t G_{1\eta}(y,t,l,\tau) a_2(\tau) d\tau.$$

Similarly to (15), from the uniqueness of the solution to the problem

$$\begin{cases} u_t = a_2(t) u_{yy}, \quad (y,t) \in (0,l) \times (0,T), \\ u(y,0) = 1, \quad y \in [0,l], \\ u(0,t) = 1, \quad u(l,t) = 1, \quad t \in [0,T]. \end{cases}$$

we obtain the formula

$$\int\limits_0^l G_1(y,t,\eta,0) d\xi + \int\limits_0^t G_{1\eta}(y,t,0,\tau) a_2(\tau) d\tau - \int\limits_0^t G_{1\eta}(y,t,l,\tau) a_2(\tau) d\tau = 1. \tag{16}$$

The formula (16) implies the estimate

$$w_1(x,y,t) \geqslant \min\{ \min_{[0,h]\times[0,l]} \varphi_x(x,y), \min_{[0,h]\times[0,T]} \mu_{21x}(x,t), \min_{[0,h]\times[0,T]} \mu_{22x}(x,t) \} =: W_1.$$

Thus, the following estimate for the operator P_1 holds:

$$P_1(a_1, a_2)(t) \leqslant \frac{\max\limits_{[0,T]} \varkappa_1(t)}{\min\limits_{[0,T]}(\nu_{11}(t) + \nu_{12}(t))W_1} =: A_1, \quad t \in [0, T].$$

Similarly,

$$P_2(a_1, a_2)(t) \leqslant A_2, \quad t \in [0, T],$$

where

$$A_2 := \frac{\max\limits_{[0,T]} \varkappa_2(t)}{\min\limits_{[0,T]}(\nu_{21}(t) + \nu_{22}(t))W_2},$$
$$W_2 := \min\{\min_{[0,h]\times[0,l]} \varphi_y(x, y), \min_{[0,l]\times[0,T]} \mu_{11y}(y, t), \min_{[0,l]\times[0,T]} \mu_{12y}(y, t)\}.$$

The next step is to obtain the upper bound estimate of $w_1(0, y_0, t)$. It implies from (15) and (16) that

$$\int_0^h \int_0^l G_{21}(x, y, t, \xi, \eta, 0)\varphi_\xi(\xi, \eta)d\xi d\eta + \int_0^h \int_0^t G_{21\eta}(x, y, t, \xi, 0, \tau)a_2(\tau)\mu_{21\xi}(\xi, \tau)d\xi d\tau$$
$$- \int_0^h \int_0^t G_{21\eta}(x, y, t, \xi, l, \tau)a_2(\tau)\mu_{22\xi}(\xi, \tau)d\xi d\tau$$
$$\leqslant \max\{\max_{[0,h]\times[0,l]} \varphi_x(x, y), \max_{[0,h]\times[0,T]} \mu_{21x}(x, t), \max_{[0,h]\times[0,T]} \mu_{22x}(x, t)\}.$$

It is shown in [9] that $G_2(0, t, 0, \tau) \leqslant \left(\frac{1}{h} + \frac{1}{\sqrt{\pi\alpha_1(t-\tau)}}\right)$ and $G_2(0, t, h, \tau) \leqslant \frac{1}{h}$. Thus,

$$- \int_0^l \int_0^t G_{21}(0, y_0, t, 0, \eta, \tau)(\mu_{11\tau}(\eta, \tau) - f(0, \eta, \tau) - a_2(\tau)\mu_{11\eta\eta}(\eta, \tau))d\eta d\tau$$
$$\leqslant \left(\frac{T}{h} + \frac{2\sqrt{T}}{\sqrt{\pi\alpha_1}}\right)\big(\max_{[0,l]\times[0,T]}(-\mu_{11t}(y, t) + f(0, y, t)) + A_2 \max_{[0,l]\times[0,T]} \mu_{11yy}(y, t)\big);$$
$$\int_0^l \int_0^t G_{21}(0, y_0, t, h, \eta, \tau)(\mu_{12\tau}(\eta, \tau) - f(h, \eta, \tau) - a_2(\tau)\mu_{12\eta\eta}(\eta, \tau))d\eta d\tau$$
$$\leqslant \frac{1}{h}\big(\max_{[0,l]\times[0,T]}(\mu_{12t}(y, t) - f(h, y, t)) + A_2 \max_{[0,l]\times[0,T]}(-\mu_{12yy}(y, t))\big).$$

Finally, the last term of (10) is evaluated by

$$\int_0^h \int_0^l \int_0^t G_{12}(x, y, t, \xi, \eta, \tau)f_\xi(\xi, \eta, \tau)d\xi d\eta d\tau \leqslant T \max_{\overline{Q}_T} f_x(x, y, t).$$

After the same procedure is applied to $w_1(h, y_0, t)$, the following inequality is obtained:

$$\nu_{11}(t)w_1(0, y_0, t) + \nu_{12}(t)w_1(h, y_0, t) \leqslant C_1 + \frac{C_2}{\sqrt{\alpha_1}}.$$

Therefore, for P_1 we obtain the lower bound estimate

$$\frac{C_3}{C_1 + \dfrac{C_2}{\sqrt{\alpha_1}}} \leqslant P_1(a_1, a_2)(t), \quad t \in [0, T], \quad \text{where } C_3 := \min_{[0,T]} \varkappa_1(t).$$

To ensure that P maps $\mathcal{N}$ into itself α_1 must satisfy the equation

$$\alpha_1 = \frac{C_3}{C_1 + \dfrac{C_2}{\sqrt{\alpha_1}}}.$$

This equation has a unique positive solution

$$\alpha_1 := \left(\frac{-C_2 + \sqrt{C_2^2 + 4C_1C_3}}{2C_1} \right)^2.$$

Similarly,

$$\nu_{21}(t)w_2(x_0, 0, t) + \nu_{22}(t)w_2(x_0, l, t) \leqslant C_4 + \frac{C_5}{\sqrt{\alpha_2}},$$

which yields the lower bound estimate for P_2

$$\frac{C_6}{C_4 + \frac{C_5}{\sqrt{\alpha_2}}} \leqslant P_2(a_1, a_2)(t), \quad t \in [0, T], \quad \text{where } C_6 := \min_{[0,T]} \varkappa_2(t).$$

Then, α_2 satisfies the equation

$$\alpha_2 = \frac{C_6}{C_4 + \frac{C_5}{\sqrt{\alpha_2}}},$$

which has only one positive solution

$$\alpha_2 := \left(\frac{-C_5 + \sqrt{C_5^2 + 4C_4C_6}}{2C_4} \right)^2.$$

Taken such values for $\alpha_1, \alpha_2, A_1, A_2 \in \mathbb{R}_+$, the operator P maps the set $\mathcal{N}$ into itself. Compactness of the operator P follows from [9]. According to the Schauder theorem there exists a solution to (14), and therefore to the problem (1)-(6).

Theorem 2. *Provided that $\varkappa_1(t) \neq 0$, $\varkappa_2(t) \neq 0$ for $t \in [0, T]$, the solution (a_1, a_2, u) to the problem (1)-(6) is unique in $C([0, T])^2 \times C^{2,1}(\overline{Q}_T)$.*

Proof. Suppose that there are two solutions $(a_1(t), a_2(t), u(x, y, t))$ and $(a_1^*(t), a_2^*(t), u^*(x, y, t))$ to the problem (1)-(6). Denote $\widehat{a}_1(t) = a_1(t) - a_1^*(t)$, $\widehat{a}_2(t) = a_2(t) - a_2^*(t)$ $\widehat{u}(x, y, t) = u(x, y, t) - u^*(x, y, t)$. Then $(\widehat{a}_1(t), \widehat{a}_2(t), \widehat{u}(x, y, t))$ is solution to the problem

$$\widehat{u}_t = a_1(t)\widehat{u}_{xx} + a_2(t)\widehat{u}_{yy} + \widehat{a}_1(t)u^*_{xx}(x, y, t) + \widehat{a}_2(t)u^*_{yy}(x, y, t), \quad (x, y, t) \in Q_T, \tag{17}$$

$$\widehat{u}(x, y, 0) = 0, \quad (x, y) \in [0, h] \times [0, l], \tag{18}$$

$$\widehat{u}(0, y, t) = 0, \quad \widehat{u}(h, y, t) = 0, \quad (y, t) \in [0, l] \times [0, T], \tag{19}$$

$$\widehat{u}(x, 0, t) = 0, \quad \widehat{u}(x, l, t) = 0, \quad (x, t) \in [0, h] \times [0, T], \tag{20}$$

$$\begin{aligned}&\widehat{a}_1(t)(\nu_{11}(t)u^*_x(0, y_0, t) + \nu_{12}(t)u^*_x(h, y_0, t)) + a_1(t)(\nu_{11}(t)\widehat{u}_x(0, y_0, t)\\ &+ \nu_{12}(t)\widehat{u}_x(h, y_0, t)) = 0, \quad t \in [0, T],\end{aligned} \tag{21}$$

$$\begin{aligned}&\widehat{a}_2(t)(\nu_{21}(t)u^*_y(x_0, 0, t) + \nu_{22}(t)u^*_y(x_0, l, t)) + a_2(t)(\nu_{21}(t)\widehat{u}_y(x_0, 0, t)\\ &+ \nu_{22}(t)\widehat{u}_y(x_0, l, t)) = 0, \quad t \in [0, T].\end{aligned} \tag{22}$$

The solution to the problem (17)-(20) can be calculated by formula (9) to read

$$\widehat{u}(x, y, t) = \int_0^t \int_0^l \int_0^h G_{11}(x, y, t, \xi, \eta, \tau)(\widehat{a}_1(\tau)u^*_{xx}(\xi, \eta, \tau) + \widehat{a}_2(\tau)u^*_{yy}(\xi, \eta, \tau))d\xi d\eta d\tau,$$
$$(x, y, t) \in \overline{Q}_T. \tag{23}$$

Substituting (23) into (21), (22) we obtain the system

$$\begin{aligned}&\widehat{a}_1(t)(\nu_{11}(t)u^*_x(0, y_0, t) + \nu_{12}(t)u^*_x(h, y_0, t)) = -a_1(t)\int_0^t \int_0^l \int_0^h (\nu_{11}(t)G_{11x}(0, y_0, t, \xi, \eta, \tau)\\ &+ \nu_{12}(t)G_{11x}(h, y_0, t, \xi, \eta, \tau))(\widehat{a}_1(\tau)u^*_{xx}(\xi, \eta, \tau) + \widehat{a}_2(\tau)u^*_{yy}(\xi, \eta, \tau))d\xi d\eta d\tau, \ t \in [0, T],\end{aligned} \tag{24}$$

$$\begin{aligned}&\widehat{a}_2(t)(\nu_{21}(t)u^*_y(x_0, 0, t) + \nu_{22}(t)u^*_y(x_0, l, t)) = -a_2(t)\int_0^t \int_0^l \int_0^h (\nu_{21}(t)G_{11y}(x_0, 0, t, \xi, \eta, \tau)\\ &+ \nu_{22}(t)G_{11y}(x_0, l, t, \xi, \eta, \tau))(\widehat{a}_1(\tau)u^*_{xx}(\xi, \eta, \tau) + \widehat{a}_2(\tau)u^*_{yy}(\xi, \eta, \tau))d\xi d\eta d\tau, \ t \in [0, T].\end{aligned} \tag{25}$$

Thus, (24) and (25) form a system of homogeneous Volterra integral equations of the second kind. Since (a_1^*, a_2^*, u^*) is solution to the problem (1)–(6), it implies from the conditions (5), (6) and assumptions of the theorem that

$$\nu_{11}(t)u^*_x(0, y_0, t) + \nu_{12}(t)u^*_x(h, y_0, t) \neq 0, \ \nu_{21}(t)u^*_y(x_0, 0, t) + \nu_{22}(t)u^*_y(x_0, l, t) \neq 0, \ t \in [0, T].$$

Therefore, the system of equations (24) and (25) has a unique trivial solution. The uniqueness is proved.

3 Solution of direct problem

In this section, we consider the direct initial boundary value problem (1)–(4), where $a_1(t)$, $a_2(t)$, $f(x, y, t)$, $\phi(x, y)$, and μ_{ij}, $i, j = 1, 2$, are known and the solution $u(x, y, t)$

is to be determined. To achieve this, we use the Forward-Time-Central-Space (FTCS) finite-difference scheme which is conditionally stable.

We subdivide the solution domain Q_T into M_x, M_y and N subintervals of equal step lengths Δx and Δy, and uniform time step Δt, where $\Delta x = h/M_x$, $\Delta y = \ell/M_y$ and $\Delta t = T/N$, for space and time, respectively. At the node (i, j, k) we denote $u_{i,j}^k := u(X_i, Y_j, t_k)$, where $X_i = i\Delta x$, $Y_j = j\Delta y$, $t_k = k\Delta t$, $a_1^k := a(t_k)$, $a_2^k := a_2(t_k)$ and $f_{i,j}^k := f(X_i, Y_j, t_k)$ for $i = \overline{0, M_x}$, $j = \overline{0, M_y}$ and $k = \overline{0, N}$.

The simplest explicit difference scheme for equation (1) is given by

$$\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\Delta t} = a_1^k \frac{u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k}{(\Delta x)^2} + a_2^k \frac{u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k}{(\Delta y)^2} + f_{i,j}^k \tag{26}$$

for $i = \overline{1, M_x - 1}$, $j = \overline{1, M_y - 1}$ and $k = \overline{0, N}$. The initial and boundary conditions (2)–(4) give

$$u_{i,j}^0 = \phi_{i,j}, \quad i = \overline{0, M_x}, \quad j = \overline{0, M_y}, \tag{27}$$

$$u_{0,j}^k = \mu_{11}(Y_j, t_k), \quad u_{M_x,j}^k = \mu_{12}(Y_j, t_k), \quad j = \overline{0, M_y}, \quad k = \overline{1, N}, \tag{28}$$

$$u_{i,0}^k = \mu_{21}(X_i, t_k), \quad u_{i,M_y}^k = \mu_{22}(X_i, t_k), \quad i = \overline{0, M_x}, \quad k = \overline{1, N}. \tag{29}$$

Let $\tilde{a}_1$ and $\tilde{a}_2$ be the maximum values of $a_1(t)$ and $a_2(t)$, respectively, then, the stability condition for the explicit FDM scheme (26) will be [13],

$$\frac{\tilde{a}_1 \Delta t}{(\Delta x)^2} + \frac{\tilde{a}_2 \Delta t}{(\Delta y)^2} \leq \frac{1}{2}. \tag{30}$$

The fluxes (5) and (6) can be calculated using the second-order FDM approximations:

$$\varkappa_1(t_k) = a_1^k \Big(\nu_{11}(t_k) u_x(0, y_0, t_k) + \nu_{12}(t_k) u_x(h, y_0, t_k) \Big), \quad k = \overline{1, N}, \tag{31}$$

$$\varkappa_2(t_k) = a_2^k \Big(\nu_{21}(t_k) u_y(x_0, 0, t_k) + \nu_{22}(t_k) u_y(x_0, \ell, t_k) \Big), \quad k = \overline{1, N}, \tag{32}$$

where

$$u_x(0, y_0, t_k) = \frac{4u(X_1, y_0, t_k) - u(X_2, y_0, t_k) - 3\mu_{11}(y_0, t_k)}{2\Delta x}, \quad k = \overline{1, N}, \tag{33}$$

$$u_x(h, y_0, t_k) = \frac{4u(X_{M_x-1}, y_0, t_k) - u(X_{M_x-2}, y_0, t_k) - 3\mu_{12}(y_0, t_k)}{-2\Delta x}, \quad k = \overline{1, N}, \tag{34}$$

$$u_y(x_0, 0, t_k) = \frac{4u(x_0, Y_1, t_k) - u(x_0, Y_2, t_k) - 3\mu_{21}(x_0, t_k)}{2\Delta y}, \quad k = \overline{1, N}, \tag{35}$$

$$u_y(x_0, \ell, t_k) = \frac{4u(x_0, Y_{M_y-1}, t_k) - u(x_0, Y_{M_y-2}, t_k) - 3\mu_{22}(x_0, t_k)}{-2\Delta y}, \quad k = \overline{1, N}. \tag{36}$$

4 Solution of inverse problem

In this section, we aim to obtain stable reconstructions for the principal direction components $a_1(t) > 0$ and $a_2(t) > 0$ of the two-dimensional orthotropic rectangular medium together with the temperature $u(x, y, t)$ satisfying the equations (1)–(6). One can remark that at initial time $t = 0$ the values $a_1(0)$ and $a_2(0)$ can be obtained from the overdetermination conditions (5) and (6) as

$$a_1(0) = \frac{\varkappa_1(0)}{\nu_{11}(0)\phi_x(0, y_0) + \nu_{12}(0)\phi_x(h, y_0)}, \tag{37}$$

$$a_2(0) = \frac{\varkappa_2(0)}{\nu_{21}(0)\phi_y(x_0, 0) + \nu_{22}(0)\phi_y(x_0, \ell)}. \tag{38}$$

The inverse problem is solved based on the nonlinear minimization of the least-squares objective function

$$\begin{aligned} F(a_1, a_2) := & \left\| a_1(t)\Big(\nu_{11}(t)u_x(0, y_0, t) + \nu_{12}(t)u_x(h, y_0, t)\Big) - \varkappa_1(t) \right\|^2 \\ & + \left\| a_2(t)\Big(\nu_{21}(t)u_y(x_0, 0, t) + \nu_{22}(t)u_y(x_0, l, t)\Big) - \varkappa_2(t) \right\|^2, \end{aligned} \tag{39}$$

or, in discretised form

$$\begin{aligned} F(\underline{a}_1, \underline{a}_2) = & \sum_{k=1}^{N} \left[a_1^k \Big(\nu_{11}(t_k)u_x(0, y_0, t_k) + \nu_{12}(t_k)u_x(h, y_0, t_k) \Big) - \varkappa_1(t_k) \right]^2 \\ & + \sum_{k=1}^{N} \left[a_2^k \Big(\nu_{21}(t_k)u_y(x_0, 0, t_k) + \nu_{22}(t_k)u_y(x_0, l, t_k) \Big) - \varkappa_2(t_k) \right]^2. \end{aligned} \tag{40}$$

The minimization of the objective functional (40), subject to the physical simple bound constraints $\underline{a}_1 > \underline{0}$ and $\underline{a}_2 > \underline{0}$ is accomplished using the MATLAB optimization toolbox routine *lsqnonlin*, which does not require supplying (by the user) the gradient of the objective function, [12]. Furthermore, within *lsqnonlin* we use the Trust-Region algorithm which is based on the interior-reflective Newton method. Each iteration involves a large linear system of equations whose solution, based on a preconditioned conjugate gradient method, allows a regular and sufficiently smooth decrease of the objective functional (40). Upper and lower bounds on the thermal conductivities a_1 and a_2 can be specified according to *a priori* information on these physical parameters.

In the numerical computation, we take the parameters of the routine *lsqnonlin*, as follows:

- Maximum number of iterations = $10^5\times$ (number of variables).
- Maximum number of objective function evaluations = $10^6\times$ (number of variables).
- Solution and objective function tolerances = 10^{-10}.

The inverse problem (1)–(6) is solved subject to both exact and noisy measurements (5) and (6). The noisy data is numerically simulated as

$$\varkappa_1^{\epsilon 1}(t_k) = \varkappa_1(t_k) + \epsilon 1_k, \quad \varkappa_2^{\epsilon 2}(t_k) = \varkappa_2(t_k) + \epsilon 2_k, \quad k = \overline{1, N}, \tag{41}$$

where $\epsilon 1_k$ and $\epsilon 2_k$ are random variables generated from a Gaussian normal distribution with mean zero and standard deviations $\sigma 1$ and $\sigma 2$ given by

$$\sigma 1 = p \times \max_{t \in [0,T]} |\varkappa_1(t_k)|, \quad \sigma 2 = p \times \max_{t \in [0,T]} |\varkappa_2(t_k)|, \tag{42}$$

where p represents the percentage of noise. We use the MATLAB function *normrnd* to generate the random variables $\underline{\epsilon 1} = (\epsilon 1_k)_{k=\overline{1,N}}$ and $\underline{\epsilon 2} = (\epsilon 2_k)_{k=\overline{1,N}}$, as follows:

$$\underline{\epsilon 1} = normrnd(0, \sigma 1, N), \quad \underline{\epsilon 2} = normrnd(0, \sigma 2, N). \tag{43}$$

In the case of noisy data (41), we replace $\varkappa_1(t_k)$ and $\varkappa_2(t_k)$ by $\varkappa_1^{\epsilon 1}(t_k)$ and $\varkappa_2^{\epsilon 1}(t_k)$, respectively, in (40).

5 Numerical results and discussion

In this section, we present numerical results for the reconstruction of the orthotropic thermal conductivity components $a_1(t)$, $a_2(t)$ and the temperature $u(x, y, t)$, in the case of exact and noisy data (41). To assess the accuracy of the numerical solution we employ the root mean square errors ($rmse$) defined by:

$$rmse(a_1) = \left[\frac{1}{N} \sum_{k=1}^{N} \left(a_1^{numerical}(t_k) - a_1^{exact}(t_k) \right)^2 \right]^{1/2}, \tag{44}$$

$$rmse(a_2) = \left[\frac{1}{N} \sum_{k=1}^{N} \left(a_2^{numerical}(t_k) - a_2^{exact}(t_k) \right)^2 \right]^{1/2}. \tag{45}$$

For simplicity, we take $h = \ell = T = 1$. The bounds on the physical variables a_1 and a_2 are 10^{-9} (lower bounds) and 10^2 (upper bounds). Although the initial guess for $a_1(t)$ and $a_2(t)$ could be taken as $a_1(0)$ and $a_2(0)$ which are known form (37) and (38), in order to investigate the robustness of the numerical inversion we take them (arbitrary), say equal to unity.

5.1 Example 1

Consider the inverse problem (1)–(6) with unknown coefficients $a_1(t)$ and $a_2(t)$, with the input data φ, μ_{ij}, ν_{ij} and $\varkappa_i$, $i, j = \overline{1, 2}$, as follows:

$$\begin{aligned}
&\varphi(x, y) = u(x, y, 0) = -(-2 + x)^2 - (-2 + y)^2, \quad f(x, y, t) = \frac{101.5 + 3t + x + y}{50}, \\
&\mu_{11}(y, t) = u(0, y, t) = -4 + 2t - (-2 + y)^2, \quad \mu_{12}(y, t) = u(1, y, t) = -1 + 2t - (-2 + y)^2, \\
&\mu_{21}(x, t) = u(x, 0, t) = -4 + 2t - (-2 + x)^2, \quad \mu_{22}(x, t) = u(x, 1, t) = -1 + 2t - (-2 + x)^2, \\
&\nu_{11}(t) = 1, \ \nu_{12}(t) = 1, \ \nu_{22}(t) = 1, \ \nu_{21}(t) = 1, \ \varkappa_1(t) = \frac{3(t + 1)}{100}, \ \varkappa_2(t) = \frac{3(2t + 0.5)}{50}, \\
&x_0 = 0.5, \quad y_0 = 0.5.
\end{aligned}$$

One can observe that conditions of Theorem 2 are satisfied and therefore, the uniqueness of the solution is guaranteed. In fact, it can easily be checked by direct substitution that the analytical solution is given by

$$a_1(t) = \frac{t + 1}{100}, \quad a_2(t) = \frac{2t + 0.5}{100}, \quad t \in [0, 1], \tag{46}$$

$$u(x, y, t) = -(x - 2)^2 - (y - 2)^2 + 2t, \quad (x, y, t) \in \overline{Q}_T. \tag{47}$$

We take $M_x = M_y = N = 10$ which together with the upper bound 10^2 for the unknown coefficients a_1 and a_2 ensure that the stability condition (30) is always satisfied at each iteration of the minimization process.

We start the investigation for simultaneously determining the time-dependent unknowns a_1 and a_2 for exact and noisy input data, i.e., for the cases $p \in \{0, 1, 5, 10\}\%$ of noise. Figure 1 presents the objective function (40), as a function of the number of iterations. From this figure one can notice that a rapid convergence is achieved in 42 iterations. The objective function (40) converges to a very small minimum value of $O(10^{-28})$.

Figure 2 shows the unknown reconstructions for $a_1(t)$ and $a_2(t)$ for various noise levels. As expected, the numerically obtained results become more stable and accurate as the percentage of noise p decrease from 10% to 5% and then to 1%, see $rmse(a_1)$ and $rmse(a_2)$ in Table 1.

5.2 Example 2

The previous example has recovered the smooth time-dependent orthotropic conductivity components $a_1(t)$ and $a_2(t)$ given by (46). In this example we asses the performance of the numerical method for reconstructing a non-smooth test case given by

$$a_1(t) = \frac{1}{10}\left|t - \frac{1}{2}\right| + \frac{1}{20}, \quad a_2(t) = \frac{1}{10}\left|t^2 - \frac{1}{2}\right| + \frac{1}{20}, \quad t \in [0, 1], \tag{48}$$

and u given by (47). The input data φ, $\mu_{i,j}$, $\nu_{i,j}$, x_0 and y_0 are the same as in Example 1 but

$$f(x, y, t) = \frac{1}{5}\left|t - \frac{1}{2}\right| + \frac{1}{5}\left|t^2 - \frac{1}{2}\right| + \frac{21}{10}, \tag{49}$$

$$\varkappa_1(t) = \frac{3}{5}\left|t - \frac{1}{2}\right| + \frac{3}{10}, \quad \varkappa_2(t) = \frac{3}{5}\left|t^2 - \frac{1}{2}\right| + \frac{3}{10}. \tag{50}$$

We take $M_x = M_y = 10$, $N = 40$ which, as in Example 1, together with the upper bound of 10^2 imposed ensure that the stability condition (30) is always satisfied during the iterative procedure. As we did in Example 1, Figures 3, 4 and Table 1 present the plots of objective function (40) as a function of the number of iterations, the numerically obtained reconstructions for the non-smooth coefficients and the *rmse* values (44) and (45) for Example 2, respectively. The same conclusions can be drawn about the stable reconstructions for the unknown coefficients.

6 Conclusions

The inverse problem concerning the simultaneous identification of the orthotropic thermal conductivity components $a_1(t)$ and $a_2(t)$ in a rectangular domain has been theoretically and numerically investigated. The unique solvability of the inverse problem has been established using Schauder's fixed point theorem and the theory of Volterra integral equations of the second kind.

The orthotropic heat equation has been discretised using an explicit FDM. Further, the inverse problem has been solved as a constrained minimization problem using the MATLAB optimization toolbox routine *lsqnonlin*. Numerical results presented and discussed for both exact and noisy data show that accurate and stable solutions have been obtained. No regularization has been found necessary indicating that the inverse problem is in fact well-posed.

Acknowledgments

M.S. Hussein would like to thank the Ministry of Higher Education and Scientific Research for their financial support in this research. He would also like to thank the hospitality of Leeds University.

References

[1] Barans'ka, I.E. The inverse problem in a domain with free bound for an anisotropic equation of parabolic type, *Naykovy Visnyk Chernivetskogo Universytetu Matematyka*, 2008;374:13–28.

[2] Bereznytska, I.B. Inverse problem for a parabolic equation with non-local overdetermination condition. *Math. Methods Phys.-Mech. Fields*, 2001;44(1):54–62.

[3] Cannon, J.R. and Jones Jr., B.F. Determination of the diffusivity of an anisotropic medium, *Int. J. Eng. Sci.*, 1963;1:457–460.

[4] Huang, L., Sun, X., Liu, Y. and Cen, Z. Parameter identification for two-dimensional orthotropic material bodies by the boundary element method, *Eng. Anal. Boundary Elements*, 2004;28:109–121.

[5] Hussein, M.S. and Lesnic, D. and Ivanchov, M.I. Identification of a heterogeneous orthotropic conductivity in a rectangular domain, *Int. J. Novel Ideas: Mathematics*, (accepted).

[6] Hussein, M.S., Lesnic, D., Ivanchov, M.I. and Snitko, H.A., Multiple time-dependent coefficient identification thermal problems with a free boundary, *Appl. Num. Math.*, 2016; 99:24–50.

[7] Ivanchov, N.I. Some inverse problems for the heat equation with nonlocal boundary condition. *Ukr. Math. J.*, 1993; 45(8):1066–1071.

[8] Ivanchov, M.I. On an inverse problem of heat conduction with nonlocal overdetermination condition. *Visnyk L'vivskogo Universytetu, Ser. Mech.-Mat.*, 1994;40:12–15.

[9] Ivanchov, M.I. *Inverse Problems for Equations of Parabolic Type*, VNTL Publishers, Liviv, Ukraine, 2003.

[10] Kinash, N. Inverse problem for the parabolic equation with nonlocal overdetermination condition. *Bukovynsky Math. J.*, 2015; 3(1):64–73.

[11] Knowles, I. and Yan, A. The recovery of an anisotropic conductivity in groundwater modelling, *Appl. Anal.*, 2002; 81:1347–1365.

[12] Mathworks R2012 Documentation Optimization Toolbox-Least Squares (Model Fitting) Algorithms, available from www.mathworks.com/help/toolbox/optim/ug /brnoybu.html.

[13] Morton, K.W. and Mayers, D.F. *Numerical Solution of Partial Differential Equations: An Introduction*, Cambridge University Press, Cambridge, 2005.

[14] Sawaf, B., Ozisik, M.N. and Jarny, Y. An inverse analysis to estimate linearly temperature dependent thermal conductivity components and heat capacity of an orthotropic medium, *Int. J. Heat Mass Transfer*, 1995;38:3005–3010.

[15] Uhlmann, G. Developments in inverse problems since Calderón's foundational paper, In: *Harmonic Analysis and Partial Differential Equations*, University of Chicago Press, Chicago, IL, USA, pp. 295–345, 1999.

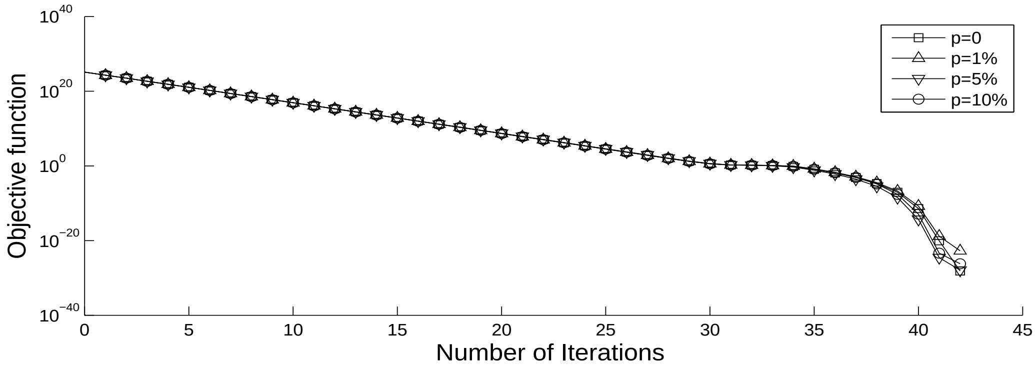
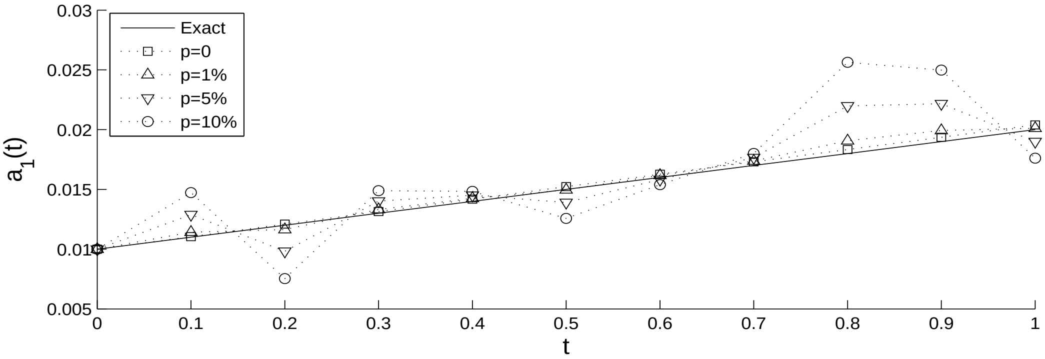
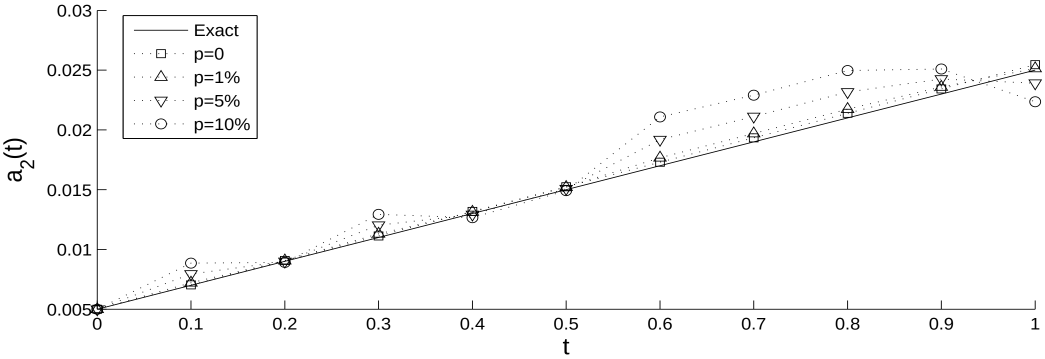


Figure 1: The objective function (40), for various noise levels $p \in \{0, 1, 5, 10\}\%$, for Example 1.



(a)



(b)

Figure 2: The exact solution (—) and numerical solutions for various noise levels $p \in \{0, 1, 5, 10\}\%$ for (a) $a_1(t)$ and (b) $a_2(t)$, for Example 1.

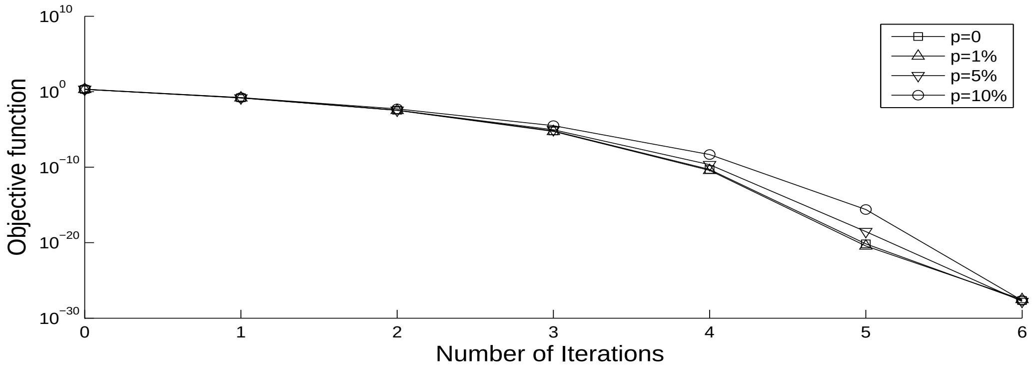
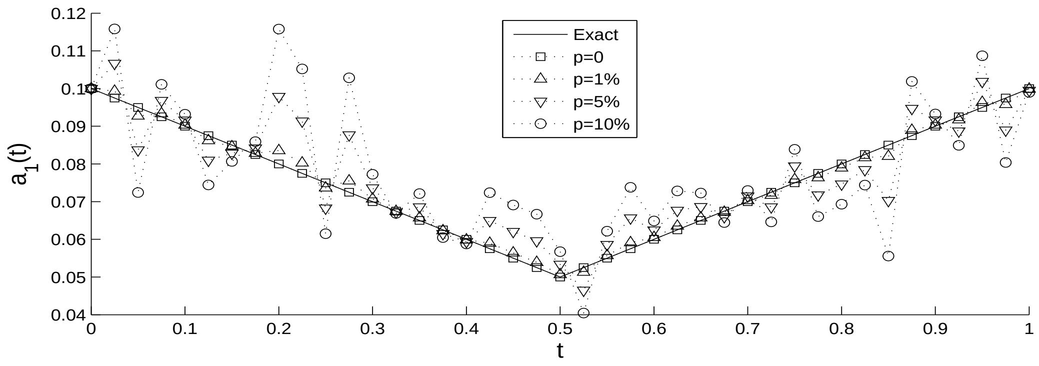
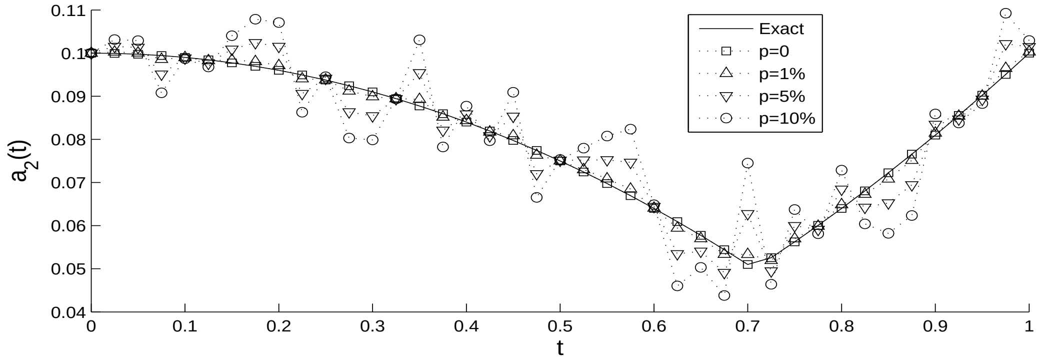


Figure 3: The objective function (40), for various noise levels $p \in \{0, 1, 5, 10\}\%$, for Example 2.



(a)



(b)

Figure 4: The exact solution (—) and numerical solutions for various noise levels $p \in \{0, 1, 5, 10\}\%$ for (a) $a_1(t)$ and (b) $a_2(t)$, for Example 2.

Table 1: The $rmse$ values (44) and (45) for various noise levels $p \in \{0, 1, 5, 10\}\%$.

Example 1	$p = 0$	$p = 1\%$	$p = 5\%$	$p = 10\%$
$rmse(a_1)$	2.6E-4	5.1E-4	0.0020	0.0038
$rmse(a_2)$	2.9E-4	4.5E-4	0.0014	0.0026
Example 2	$p = 0$	$p = 1\%$	$p = 5\%$	$p = 10\%$
$rmse(a_1)$	2.9E-16	0.0014	0.0070	0.0140
$rmse(a_2)$	2.9E-4	9.3E-4	0.0047	0.0093