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Existence of stable hairy black holes in $\mathfrak{su}(2)$ Einstein-Yang-Mills theory with a negative cosmological constant

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Abstract

We consider black holes in EYM theory with a negative cosmological constant. The solutions obtained are somewhat different from those for which the cosmological constant is either positive or zero. Firstly, regular black hole solutions exist for continuous intervals of the parameter space, rather than discrete points. Secondly, there are non-trivial solutions in which the gauge field has no nodes. We show that these solutions are linearly stable.

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1 Introduction

Non-Abelian Einstein-Yang-Mills (EYM) theories have been the subject of detailed study since the discovery of particle-like [1] and black hole [2] solutions when the gauge group is $\mathfrak{su}(2)$. Since then, there has been a great deal of numerical and analytic work on various aspects of $\mathfrak{su}(N)$ EYM black holes and solitons, in both asymptotically flat and asymptotically de Sitter geometries (see [3] for a wide-ranging, recent review of work to date, and [4] for some analytic results). The behaviour of the cosmological solutions is similar in many respects to that for asymptotically flat geometries, for sufficiently small, positive, cosmological constant [5, 6] and in particular, the configurations are unstable [7]. Our main result is that there are black hole solutions of $\mathfrak{su}(2)$ EYM theory with a negative cosmological constant which are *stable*.

In this paper we are concerned with EYM black holes in spacetimes which are asymptotically anti-de Sitter (AdS). AdS space has recently enjoyed a revival due to the work of Witten and others in connection with conformal field theories and large N gauge theories (see, for example, [8]). Initial interest in black holes that are asymptotically AdS was due to the result that (sufficiently large) Schwarzschild-anti-de Sitter black holes are thermodynamically stable [9], which has been the inspiration for much of the recent work. Another trigger was the discovery of three-dimensional black holes constructed from AdS space [10], which have interesting thermodynamical properties. This suggests that, although geometries with a negative cosmological constant have some undesirable properties (such as closed timelike curves, although these are readily removed by considering the covering space) it has many advantages for studying quantum field theory and may prove to be a useful probe of quantum gravity effects. Black holes with hair in such theories may therefore be useful for probing not only quantum gravity, but also may be interesting tests of the AdS-CFT conjecture [11], particularly since we can find such objects which are classically stable.

Here we restrict attention to $\mathfrak{su}(2)$ EYM theory for simplicity, and investigate black hole solutions which asymptotically approach AdS space. Numerical calculations will be backed up with analytic work. We find some surprising results, which are rather different from the corresponding ones for asymptotically flat and de Sitter black holes, of which stability of some of our solutions is the most important.

The outline of the paper is as follows. Firstly, in section 2 we outline the field equations and ansätze for the field variables we shall be using, before

discussing the numerical solution of these field equations, integrating out from the event horizon. The results found are rather different from those observed in the case of a positive cosmological constant [5, 6]. For example, here the solutions are distributed continuously in the space of initial parameters, rather than discretely, and there are solutions in which the gauge field has no zeros. These numerical results are backed up with analysis of the defining differential equations in section 3. This section emphasizes the differences in the behaviour of the field equations when the cosmological constant is positive, negative, or zero. Those solutions for which the gauge field has no zeros are particularly interesting since they are shown to be stable, by a mixture of analytic and numerical techniques, in section 4. Finally, our conclusions are presented in section 5.

The metric signature is $(-+++)$ throughout, and we use units in which $G = 1 = c$.

2 Solving the field equations

2.1 Preliminaries

The Einstein equations with a cosmological constant Λ take the form

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}. \quad (1)$$

For simplicity we restrict attention to spherically symmetric solutions of these equations, in the case that $\Lambda < 0$. We require the geometry to approach (the covering space of) AdS spacetime at infinity, and therefore a suitable ansatz for the metric is

$$ds^2 = - \left(1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3} \right) S^2(r) dt^2 + \left(1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2)$$

where $m(r)$ will be the ‘quasi-local mass’ for the geometry relative to anti-de Sitter spacetime itself. The most general static, spherically symmetric ansatz for the $\mathfrak{su}(2)$ Yang-Mills gauge field, with a purely magnetic field and an appropriate choice of gauge, is

$$A = (1 + \omega) [-\tau_\phi d\theta + \tau_\theta \sin \theta d\phi] \quad (3)$$

where ω is a function of r alone, and the $\tau_{r,\theta,\phi}$ are given in terms of the usual Pauli matrices $\tau_{1,2,3}$ by $\tau_{r,\theta,\phi} = \boldsymbol{\tau} \cdot \mathbf{e}_{r,\theta,\phi}$.

The theory has, in addition to Newton's constant G (which we have set equal to unity by an appropriate choice of units), a gauge coupling constant g and cosmological constant Λ , all of which are parameters. We shall set the gauge coupling constant g also equal to unity, leaving Λ as the only parameter. This is the approach taken by the authors of ref. [5]. In [6] a different combination of coupling constants was used as the single free parameter.

The field equations then take the following form, with $'$ denoting d/dr :

$$m' = \left(1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}\right) \omega'^2 + \frac{(\omega^2 - 1)^2}{2r^2} \quad (4)$$

$$\frac{S'}{S} = \frac{2\omega'^2}{r} \quad (5)$$

$$0 = r^2 \left(1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}\right) \omega'' + \left(2m - \frac{2\Lambda r^3}{3} - \frac{(\omega^2 - 1)^2}{r}\right) \omega' + (1 - \omega^2) \omega. \quad (6)$$

These equations are invariant under the transformation $\omega \rightarrow -\omega$, although the ansatz (3) is not. The gauge potentials (3) for $\pm\omega$ are related by a gauge transformation of the form [3]

$$A \rightarrow UAU^{-1} + UdU^{-1}, \quad U = \exp[\pi\tau_r]. \quad (7)$$

Since we are interested in black hole solutions, there are two length scales to consider, firstly, the event horizon radius r_h , and secondly the length scale $l^{-2} = -\frac{\Lambda}{3}$. For simplicity, in our numerical work in section 2.2, we shall set $r_h = 1$ and vary Λ in order to change the relative magnitude of the two length scales. However, the analytic results that follow will hold for all r_h .

For black hole solutions having a regular event horizon at $r = r_h$,

$$m(r_h) = \frac{r_h}{2} - \frac{\Lambda r_h^3}{6} > 0 \quad \forall r_h > 0 \quad (8)$$

since $\Lambda < 0$. In order to have a regular event horizon at $r = r_h$, it must be the case that

$$\left. \frac{d}{dr} \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) \right|_{r_h} > 0. \quad (9)$$

This places a bound on $m'(r_h)$:

$$2m'(r_h) = \frac{[\omega(r_h)^2 - 1]^2}{r_h^2} < 1 - \Lambda r_h^2. \quad (10)$$

From the field equations, for fixed r_h , there is just one initial parameter, $\omega_h = \omega(r_h)$, since then $\omega'(r_h)$ is given by

$$\omega'(r_h) = \frac{[\omega_h^2 - 1]\omega_h}{r_h - \Lambda r_h^3 - \frac{[\omega_h^2 - 1]^2}{r_h}}. \quad (11)$$

Since the cosmological constant is negative, we do not expect there to be a cosmological event horizon. This places a bound on $m(r)$, namely,

$$m(r) < \frac{r}{2} - \frac{\Lambda r^3}{6}. \quad (12)$$

This is quite a weak restriction, whereas when Λ is positive, there will always be a cosmological horizon, and fairly strict conditions have to be placed on the field variables at the cosmological horizon in order that it is regular. When $\Lambda = 0$, the restriction (12) is stronger than for $\Lambda < 0$, and will be violated for generic initial data [12]. As $r \rightarrow \infty$, the solutions are expected to be charged as in the $\Lambda > 0$ case [5], and the field variables will have the following asymptotic forms:

$$\begin{aligned} m(r) &= M + \frac{M_1}{r} + O\left(\frac{1}{r^2}\right) \\ \omega(r) &= \omega_\infty + \frac{a}{r} + \frac{b}{r^2} + O\left(\frac{1}{r^3}\right) \\ S(r) &= 1 + O\left(\frac{1}{r}\right), \end{aligned} \quad (13)$$

where ω_∞ , M and a are parameters and

$$\begin{aligned} M_1 &= \frac{\Lambda a^2}{3} - \frac{1}{2}(\omega_\infty^2 - 1)^2 \\ b &= \frac{3\omega_\infty(1 - \omega_\infty)}{2\Lambda}. \end{aligned} \quad (14)$$

2.2 Numerical Results

The field equations (4–6) were integrated numerically, fixing $r_h = 1$, so that Λ was the only parameter in the problem, and l (where $l^{-2} = -\frac{\Lambda}{3}$) is the only varying length scale. Using the initial conditions on the event horizon (8,11), the equations were integrated for increasing r , for various values of $|\Lambda|$ in the range 10^{-4} – 10^6 , and a range of values of ω_h . Since the equations (4-6) are invariant under the transformation $\omega \rightarrow -\omega$, only values of $\omega_h > 0$ were

considered. The equation for S decouples from the rest, and the requirement that $S \rightarrow 1$ as $r \rightarrow \infty$ can be relaxed during the numerical integration, S subsequently being multiplied by an appropriate constant factor so that the correct asymptotic behaviour is recovered. Some of the results found were rather surprising, and are listed below.

1. For $|\Lambda|$ sufficiently large (i.e. ≥ 0.1) there exist regular solutions for which the gauge field has no zeros (nodes). Examples of this type of solution are illustrated in figures 1 to 5, for the value $\Lambda = -100$. For this large value of $|\Lambda|$, the field variables do not vary much as r increases. This is in accordance with proposition 6 in the next section. This is the first example of behaviour unique to the negative cosmological constant theory. For $\Lambda \geq 0$, the gauge field must have at least one zero.
2. For every value of $\Lambda < 0$, there was a critical value, $\omega_h^c > 0$, of ω_h , such that, for all $0 < \omega_h < \omega_h^c$ integrating the field equations gave a regular black hole solution which was asymptotically AdS. The value of ω_h^c increases as $|\Lambda|$ increases. For $|\Lambda|$ sufficiently large (i.e. ≥ 0.1), $\omega_h^c > 1$. Figure 3 illustrates the solution when $\Lambda = -100$ and $\omega_h = 1.1$. When $\Lambda \geq 0$, the regular solutions occur only for discrete values of the initial parameter, although these values do have an accumulation point at zero. For $\Lambda = 0$, it has been shown [12] that it is not possible to have regular black holes for which $|\omega(r)| > 1$ for any r , and all evidence to date shows that we may expect this to hold also for positive cosmological constant [5, 6].
3. Unlike the situation when $\Lambda \geq 0$, the regular black hole solutions did not occur for discrete values of ω_h , but, even for $\omega_h > \omega_h^c$, over continuous intervals of ω_h . For smaller values of $|\Lambda|$ ($\sim 10^{-4}$), these intervals were separated by intervals on which there were no regular solutions. For each $\Lambda < 0$, there is a second, maximal, value of ω_h , $\omega_h^m > \omega_h^c$, such that for all $\omega_h > \omega_h^m$ there are no regular black hole solutions. This behaviour is also noticed for $\Lambda \geq 0$, and follows from the condition (10) for the existence of a regular event horizon.
4. There are regular solutions for which $|\omega_\infty| > 1$, although $|\omega_h| < 1$. Examples of this type of solution can be seen in figures 6 to 8, when $\Lambda = -0.001$. Again, when $\Lambda \geq 0$, this type of solution is forbidden since $|\omega(r)| < 1$ for all r inside the cosmological event horizon.
5. As $|\Lambda|$ increases, the values of ω_h for which the regular solutions have one or more nodes decrease rapidly.

Otherwise, many of the properties of the numerical solutions were as found in the asymptotically flat [2] or asymptotically de Sitter [5, 6] models:

1. For each regular black hole solution extending to $r \rightarrow \infty$, the gauge field ω has a finite number of zeros.
2. The number of zeros of ω increases as ω_h decreases.

3 Analytic explanation of the numerical results

In this section we prove some analytic results which explain the behaviour observed numerically in the previous section. We shall stress, where appropriate, the similarities and differences between our results and the corresponding ones for asymptotically flat and asymptotically de Sitter spacetimes. In the results of this section, we shall pay little attention to the behaviour of the metric function $S(r)$. The regularity of this function can easily be established in each case by the appropriate behaviour of the other field variables, and integrating its defining equation (5), with the inclusion of a constant (see discussion in the previous section).

3.1 Elementary results

We begin with some simple lemmas, the proofs of which are identical to those in the $\Lambda = 0$ case [12], and which are valid for all values of Λ .

Lemma 1 *For each fixed value of Λ , there exists a family of local solutions of the field equations (4-6) satisfying the initial conditions (8,11), defined for $r_h > 0$, and ω_h such that (10) is satisfied, and analytic in r_h , ω_h , r and Λ .*

Lemma 2 *For each value of Λ , there exists a family of local solutions of the field equations (4-6) satisfying the boundary conditions (13,14), and analytic in $1/r$, ω_∞ , a , M , and Λ .*

Lemma 3 *As long as*

$$1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3} > 0, \tag{15}$$

the solutions are regular functions of r .

Note that lemma 3 only guarantees regularity of the solutions having a positive cosmological constant up to the cosmological event horizon. Conditions (similar to those for a black hole event horizon) have to be imposed on the field variables if the cosmological horizon is to be regular. It is the necessity of these conditions which makes the behaviour of the solutions of the Einstein-Yang-Mills equations with positive cosmological constant so different from those we discuss here with negative cosmological constant.

The argument in the next subsection will make use of the asymptotic solutions of the field equations as $r \rightarrow \infty$. Therefore we consider the Yang-Mills equation in the regime $r \gg 1$, $|\Lambda|r^2 \gg 1$, which takes the form:

$$-\frac{\Lambda r^4}{3}\omega'' - \frac{2\Lambda r^2}{3}\omega' + (1 - \omega^2)\omega = 0. \quad (16)$$

This equation is made autonomous by the use of the change of variable

$$\tau = \frac{l}{r}, \quad (17)$$

where $l^{-2} = -\frac{\Lambda}{3}$. Then the equation is,

$$\frac{d^2\omega}{d\tau^2} = (\omega^2 - 1)\omega. \quad (18)$$

The phase portrait of this equation is shown in figure 9. For $\Lambda > 0$, the right-hand side of equation (18) has the opposite sign, which means that the phase plane is rather different. However, this is less crucial than the inevitability of a cosmological event horizon in this case. The phase plane for the asymptotically flat case in which $\Lambda = 0$ is discussed in [12]. In order to make the equation corresponding to (16) in that case autonomous, the variable has to be changed to $\tilde{\tau} = \log r$, rather than (17). This means that as $r \rightarrow \infty$, the new variable $\tilde{\tau} \rightarrow \infty$ also, whereas in our case $\tau \in [0, \tau_1]$ for very large r , as $r \rightarrow \infty$, in other words, τ remains in a finite interval. This is responsible for the surprising results of propositions 4-6 below, since phase paths that are close to the saddle points in figure 9 do not necessarily have to travel off to infinity as $r \rightarrow \infty$, since τ remains finite. However, for asymptotically flat black holes, the phase portrait also has saddle points at $\omega = \pm 1$, and $\omega' = 0$ (together with a spiral point at $\omega = 0 = \omega'$ rather than a centre), but paths passing close to the saddle points must zoom off to infinity because $\tilde{\tau}$ cannot remain bounded as r increases.

3.2 Existence of a continuum of asymptotically AdS black holes

The propositions in this section make use of the asymptotic equation (18) and the reader may ask whether they still apply when $\Lambda > 0$. The answer is no, because we can no longer guarantee that if a solution has a regular cosmological event horizon, then integrating the field equations with sufficiently close initial parameters will also yield a solution with a regular cosmological horizon. Indeed, one may suspect that in general this will not be the case. In asymptotically flat spacetime [12], all solutions close to the regular black holes are in fact singular because phase paths close to the saddle points must reach infinity, as discussed in the previous subsection.

Proposition 4 *For fixed r_h and $\Lambda < 0$, and for every ω_h sufficiently small, there is a regular, asymptotically AdS black hole solution.*

Proof

If $\omega_h = 0$, then $\omega'(r_h) = 0$ and we have the Reissner-Nordstrom AdS (RNAdS) solution. From the local existence theorem (Lemma 1), the solutions depend continuously on ω_h and therefore, for sufficiently small ω_h , all field variables remain close to their values for the RNAdS solution with the same r_h until $r \gg 1$ and $2m(r)/r \ll 1$. In this situation the asymptotic form of the Yang-Mills equations (18) is valid. The phase plane of this equation possesses a centre at $\omega = 0 = \frac{d\omega}{d\tau}$, and therefore ω and $\omega' = -\frac{1}{r^{2l}} \frac{d\omega}{d\tau}$ will remain very small for all values of increasing r . Therefore $m(r)/r$ will also continue to be extremely small for all r and we have a regular black hole solution. \square

When $\Lambda = 0$, the proof breaks down because ω and ω' no longer remain small as r increases [12], and the solution will in general become singular.

Proposition 5 *For fixed r_h and $\Lambda < 0$, if $\bar{\omega}_h$ leads to a regular black hole solution with the gauge field having n nodes, then all ω_h sufficiently close to $\bar{\omega}_h$ will also lead to a regular black hole solution with n nodes.*

Proof

Since the solutions are continuous in ω_h from Lemma 1, we can choose $r_1 \gg 1$ such that $|\Lambda|r_1^2 \gg 1$ and for all ω_h sufficiently close to $\bar{\omega}_h$, the gauge field function ω has n nodes in the interval $r_h < r < r_1$ and $m(r_1)/r_1 \ll 1$. Therefore, as in the proof of the previous proposition, the asymptotic equation (18) is applicable, where $\tau = \tau_1 \ll 1$. If we have chosen r_1 sufficiently large,

the field variables ω and $\frac{d\omega}{d\tau}$ will not move very far on the appropriate phase plane trajectory (figure 9) as τ decreases from τ_1 to zero. In particular, ω will not cross zero again, provided r_1 has been chosen to be sufficiently large. In this case $m(r)/r$ will continue to be very small and this will not affect the Yang-Mills equation. Therefore we have a regular black hole solution with the gauge field having n nodes. \square

Here again, when $\Lambda = 0$, since ω and ω' will grow indefinitely as r increases, we can no longer guarantee that a regular solution will be formed for ω_h sufficiently close to $\bar{\omega}_h$. In general, a singular solution will be formed [12].

3.3 Solutions for $|\Lambda| \gg 1$

We close this section by proving the existence of regular black hole solutions for which the gauge field has no nodes. These solutions will be particularly important as we shall show in the next section that they are (linearly) *stable*. Note that for $\Lambda = 0$, it has been proved [12] that the gauge field must have a least one node. This also appears to be the case for $\Lambda > 0$ [5, 6], although there is more limited analytic work in this area [13].

Proposition 6 *For fixed r_h , given any value of ω_h , for $|\Lambda|$ sufficiently large, there exists a regular black hole solution in which ω has no nodes.*

Proof

First note that for all $|\Lambda|$ sufficiently large, the condition (10) for a regular black hole event horizon at $r = r_h$ is satisfied. The condition (8) for a regular event horizon suggests that $m \sim |\Lambda|$ as $|\Lambda| \rightarrow \infty$. In the light of this, define a new variable p by $p(r) = m(r)/\Lambda$ which will then remain finite as $|\Lambda| \rightarrow \infty$. In terms of p the field equations (4–6) are, where $\xi = \frac{1}{\Lambda}$,

$$\begin{aligned} p' &= \left(\xi - \frac{2p}{r} - \frac{r^2}{3} \right) \omega'^2 + \frac{\xi (\omega^2 - 1)^2}{2r^2} \\ \frac{S'}{S} &= \frac{2\omega'^2}{r} \\ 0 &= r^2 \left(\xi - \frac{2p}{r} - \frac{r^2}{3} \right) \omega'' + \left(2p - \frac{2r^3}{3} - \frac{\xi (\omega^2 - 1)^2}{r} \right) \omega' \\ &\quad + \xi (1 - \omega^2) \omega. \end{aligned} \tag{19}$$

When $\xi = 0$, we have the solution $p(r) = -\frac{r^3}{6}$, $\omega(r) = \omega_h$, $S(r) = 1$ for any value of ω_h . By a local existence theorem (similar to Lemma 1), the solutions

are continuous in the initial values and ξ . Therefore, given a value of ω_h , for ξ sufficiently small, the solution will remain close to its values for $\xi = 0$, until $r \gg 1$ and $p(r)/r \ll 1/|\Lambda|$. In particular, ω will have no nodes in this regime. Since $m(r)/r \ll 1$, we may consider the asymptotic equation (18). The argument used at the end of the proof of Proposition 5 then shows that we have a solution for all r and ω will have no zeros. \square

Once again, this proposition does not carry over for $\Lambda > 0$. Quite the opposite may be inferred from [3, 5, 6], namely that non-trivial solutions exist only for sufficiently small values of $|\Lambda|$. This result is the crux of our proof that there exist stable black holes in EYM theory with a negative cosmological constant. Black holes for which $|\Lambda| \gg 1$ are such that the length scale set by the cosmological constant l (given by $l^{-2} = -\Lambda/3$) is much smaller than the radius of the event horizon. This means that we have in effect a gauge field perturbation of anti-de Sitter space. For illustrative purposes, we have used the value $\Lambda = -100$ in figures 1-5, which gives $l < 1 = r_h$. However, it should be noted that the same effects occur for comparatively small values of $|\Lambda|$, for example, there are black holes for which ω has no zeros for $\Lambda = -0.1$ which satisfy the criteria for stability. Therefore, the analytic results suggest that the stable configurations are merely perturbations of anti-de Sitter space, in fact the results hold for black holes in which $l > r_h$, so that the geometry is quite different from AdS. The thermodynamics of Schwarzschild-AdS [9] suggests that relatively large black holes (having $r_h > l$) may in fact be the most interesting since they (in the non-hairy case) are thermodynamically stable.

4 The existence of stable black holes

4.1 Linearized perturbation equations

Consider spherically symmetric perturbations of the black holes described in the previous section. The metric is now time-dependent but remains spherically symmetric:

$$ds^2 = -N(r, t)S^2(r, t) dt^2 + N^{-1}(r, t) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (20)$$

and we consider the following, more general, ansatz for the gauge field:

$$A = a_0 \tau_r dt + b \tau_r dr + [\nu \tau_\theta - (1 + \omega) \tau_\phi] d\theta + [(1 + \omega) \tau_\theta + \nu \tau_\phi] \sin \theta d\phi, \quad (21)$$

where a_0 , b , ω and ν depend on t and r . All the field variables will be written as follows, for example,

$$\omega(r, t) = \omega(r) + \delta\omega(r, t) \quad (22)$$

where $\omega(r)$ is the static equilibrium solution whose stability we are investigating, and $\delta\omega(r, t)$ is the perturbation. We shall work to first order only in small perturbations. In the case where $a_0 \equiv 0$, the linearized perturbation equations decouple into two sectors, as in the $\Lambda = 0$ case [14]. Firstly, the equations for the perturbations δb and $\delta\nu$ decouple from the remaining equations to form the *sphaleronic sector* (with $\dot{}$ denoting d/dt):

$$0 = \ddot{\delta b} + \frac{2NS^2}{r^2} [\omega (\delta\nu' + \omega\delta b) - \omega'\delta\nu] \quad (23)$$

$$0 = \ddot{\delta\nu} - NS(NS)' [\delta\nu' + \omega\delta b] - N^2S^2 [\delta\nu'' + \omega\delta b' + 2\omega'\delta b] + \frac{NS^2}{r^2} \delta\nu (\omega^2 - 1). \quad (24)$$

The variables δb and $\delta\nu$ must also satisfy the *Gauss constraint* [14]:

$$\left(\frac{2}{rS^2} - \frac{S'}{S^3} \right) \dot{\delta b} + \frac{1}{S^2} \dot{\delta b}' - \frac{2}{r^2NS^2} \omega \dot{\delta\nu} = 0. \quad (25)$$

The remaining perturbations, $\delta\omega$, δm and δS form the *gravitational sector*.

4.2 Sphaleronic sector

Volkov et al [15] used a powerful method to show that the n -th asymptotically flat coloured black hole has exactly n unstable modes in this sector. The same method will now be used to show that the solutions described in section 2 in which the gauge field ω has no nodes have no instabilities in this sector.

Firstly define a new variable α by

$$\alpha = \frac{r^2\delta b}{2S}, \quad (26)$$

in terms of which the Gauss constraint (25) reads

$$\dot{\delta\nu} = -\frac{NS}{\omega} \dot{\alpha}', \quad (27)$$

so that $\delta\nu = NS\alpha'/\omega$. As usual, next define a “tortoise” co-ordinate r^* by

$$\frac{dr^*}{dr} = \frac{1}{NS}. \quad (28)$$

The perturbation equation (23) takes a simple form when we introduce the quantity $\beta = \alpha/\omega$ and consider periodic perturbations $\beta(r^*, t) = e^{i\sigma t}\beta(r^*)$

$$\sigma^2\beta = -\frac{d^2\beta}{dr^{*2}} + \beta \left\{ \frac{NS^2}{r^2} [1 + \omega^2] + \frac{2}{\omega^2} \left(\frac{d\omega}{dr^*} \right)^2 \right\}, \quad (29)$$

where we have used the static field equation (6) for ω . This has the form of a standard Schrödinger equation. The potential is positive and regular everywhere in the case where ω has no nodes. As $r \rightarrow r_h$, $r^* \rightarrow -\infty$, the potential tends to zero, and in the other asymptotic regime, $r, r^* \rightarrow \infty$, the potential approaches the constant value

$$-\frac{\Lambda}{3}(1 + \omega_\infty^2) > 0. \quad (30)$$

Then a standard theorem of quantum mechanics [16] tells us that there are no bound states for this system, since the potential is everywhere greater than the lower of its two asymptotic values. In other words, there are no negative eigenvalues for σ^2 and no unstable modes.

If the gauge function ω has n nodes, then we would expect that the method of Volkov et al [15] could be extended to this case to show that there are exactly n unstable sphaleronic modes. This is the situation for asymptotically flat and asymptotically de Sitter black holes.

4.3 Gravitational sector

The perturbations δm and δS can be eliminated from the equations for this sector to leave a single equation for $\delta\omega(r, t) = e^{i\sigma t}\delta\omega(r^*)$:

$$\sigma^2\delta\omega = -\frac{d^2\delta\omega}{dr^{*2}} + U(r^*)\delta\omega \quad (31)$$

where the potential U is given by

$$U = \frac{NS^2}{r^2} \left\{ 3\omega^2 - 1 - 4r\omega'^2 \left[\frac{1}{r} - \Lambda r - \frac{(1 - \omega^2)^2}{r^3} \right] + \frac{8}{r}\omega\omega'(\omega^2 - 1) \right\}, \quad (32)$$

with $'$ denoting d/dr , as previously. Near the event horizon, as $r \rightarrow r_h$ and $r^* \rightarrow -\infty$, the potential $U \rightarrow 0$. At infinity, $r \rightarrow \infty$, and U approaches its asymptotic value:

$$U \rightarrow -\frac{\Lambda}{3}(3\omega_\infty^2 - 1). \quad (33)$$

As for the sphaleronic sector, a standard theorem [16] tells us that the equation (31) will have no bound states if the potential U is everywhere greater than the lower of its two asymptotic values. These conditions will be satisfied if

$$\omega_\infty \geq 1/\sqrt{3} \quad \text{and} \quad U \geq 0 \text{ everywhere.} \quad (34)$$

We are particularly interested in those solutions in which the gauge field has no zeros, since these are stable in the sphaleronic sector. Figure 10 illustrates numerically that there are such black holes which also satisfy our two requirements for stability in the gravitational sector. These solutions are therefore linearly stable in both sectors of the theory.

We shall now proceed to show analytically that there exist black holes in which the gauge field has no zeros and the gravitational potential satisfies the conditions (34). Firstly, we restrict attention to those solutions for which $\omega_h > 1/\sqrt{3}$. Then, from proposition 6, for $|\Lambda|$ sufficiently large, ω will remain sufficiently close to its initial value, so that $\omega(r) > 1/\sqrt{3}$ for all r , and in particular, $\omega_\infty > 1/\sqrt{3}$. In this case, ω' will be very small for all r , and the dominant behaviour of the potential U will be:

$$U = \frac{NS^2}{r^2} [3\omega^2 - 1 + 4\Lambda r^2 \omega'^2]. \quad (35)$$

The third term in the brackets is negative, whereas the other two terms together are positive. If we can show that this third term can be made as small as we like by taking $|\Lambda|$ sufficiently large, then $U > 0$ for all r^* and the black hole has no unstable modes in the gravitational sector. Therefore, we need to show that $\sqrt{|\Lambda|}\omega' \rightarrow 0$ as $|\Lambda| \rightarrow \infty$.

Firstly, consider the expression (11) for $\omega'(r_h)$:

$$\omega'_h = \frac{(\omega_h^2 - 1)\omega_h}{r_h - \Lambda r_h^3 - \frac{(\omega_h^2 - 1)^2}{r_h}} \sim -\frac{1}{\Lambda r_h^3} (\omega_h^2 - 1) \omega_h \quad (36)$$

as $|\Lambda| \rightarrow \infty$. From (36), it is clear that $\sqrt{|\Lambda|}\omega'_h \rightarrow 0$ as $|\Lambda| \rightarrow \infty$. In order to show that this is true for other values of r also, we turn to the equations (19) for the field variables, where $\xi = 1/\Lambda$ and $p(r) = m(r)/\Lambda$. Introduce a new variable $\eta(r)$ by $\eta(r) = \omega'(r)/\sqrt{|\xi|}$, in terms of which the equations (19) read:

$$\begin{aligned} p' &= \left(\xi - \frac{2p}{r} - \frac{r^2}{3} \right) |\xi| \eta^2 + \frac{\xi}{2r^2} (\omega^2 - 1)^2 \\ \frac{S'}{S} &= \frac{2|\xi| \eta^2}{r} \\ 0 &= r^2 \left(\xi - \frac{2p}{r} - \frac{r^2}{3} \right) \eta' + \left(2p - \frac{\xi}{r} (\omega^2 - 1)^2 - \frac{2r^3}{3} \right) \eta \\ &\quad + \sqrt{|\xi|} (1 - \omega^2) \omega. \end{aligned} \quad (37)$$

When $\xi = 0$, we have the solution $p \equiv -r_h^3/6$, $\omega \equiv \omega_h$, $S \equiv 1$ and the equation for η decouples from the rest to give

$$r(r_h^3 - r^3)\eta' - (r_h^3 + 2r^3)\eta = 0. \quad (38)$$

This is a linear first order differential equation with solution

$$\eta(r) = \frac{\mathcal{A}r}{r^3 - r_h^3}, \quad (39)$$

where \mathcal{A} is a constant of integration. However, we know that η must vanish at the event horizon from (36), and therefore $\mathcal{A} = 0$, which means that η vanishes identically. Therefore $\sqrt{|\Lambda|}\omega' \rightarrow 0$ as $|\Lambda| \rightarrow \infty$, and the potential U is positive for sufficiently large $|\Lambda|$. This proves that the black holes for which $\omega_h > 1/\sqrt{3}$ have no unstable modes in the gravitational sector for sufficiently large $|\Lambda|$. These black holes will have a gauge field which has no zeros and will therefore be stable in both the gravitational and sphaleronic sectors.

5 Conclusions

In this paper we have studied the $\mathfrak{su}(2)$ Einstein-Yang-Mills equations and looked for spherically symmetric black hole solutions which approach anti-de Sitter space at infinity. The numerical integration of the field equations yielded behaviour rather different from that observed for black holes in asymptotically flat or asymptotically de Sitter spacetime. For example, there are regular black hole solutions for continuous intervals of the initial parameter space, rather than discrete points; there are solutions for which the gauge field has no zeros; solutions exist for all values of $\Lambda < 0$; and the gauge field ω does not necessarily satisfy $|\omega| < 1$. Analysis of the field equations has confirmed this observed behaviour. In particular, we showed that the solutions in which the gauge field has no zeros are (linearly) *stable*.

The results of this paper may also have consequences for the “no-hair” conjecture. It is already suspected [17] that the asymptotic nature of the spacetime can affect the existence of hair. In [17] it is shown that the “no-scalar-hair” theorem of Bekenstein [18] is no longer valid for asymptotically de Sitter space, and specific solutions exhibited. However, although these solutions evade the “letter” of the no-hair conjecture, they do not alter its “spirit” since they are unstable. The fact that we have found stable solutions in EYM theory when the cosmological constant is negative suggests that it

may be possible to find primary scalar hair in asymptotically anti-de Sitter geometries. We hope to return to this question in the near future.

The behaviour of a quantum field on a non-trivial black hole background geometry with a negative cosmological constant remains an interesting open question, since it incorporates both hair (which itself has radical effects on quantum field theory [19, 20]) and anti-de Sitter space (which renders Schwarzschild black holes thermodynamically stable [9]). The stability of the solutions we have found means that a study of the quantum field theory would have more wide-ranging implications, particularly since interesting effects have been observed in theories with a negative cosmological constant (see, for example [21]). It may also be relevant to some of the deep issues being tackled presently in the light of Maldacena's [11] conjecture that thermodynamics of quantum gravity with a negative cosmological constant is equivalent to the large N thermodynamics of standard quantum field theory. We hope to return to this question in a subsequent publication.

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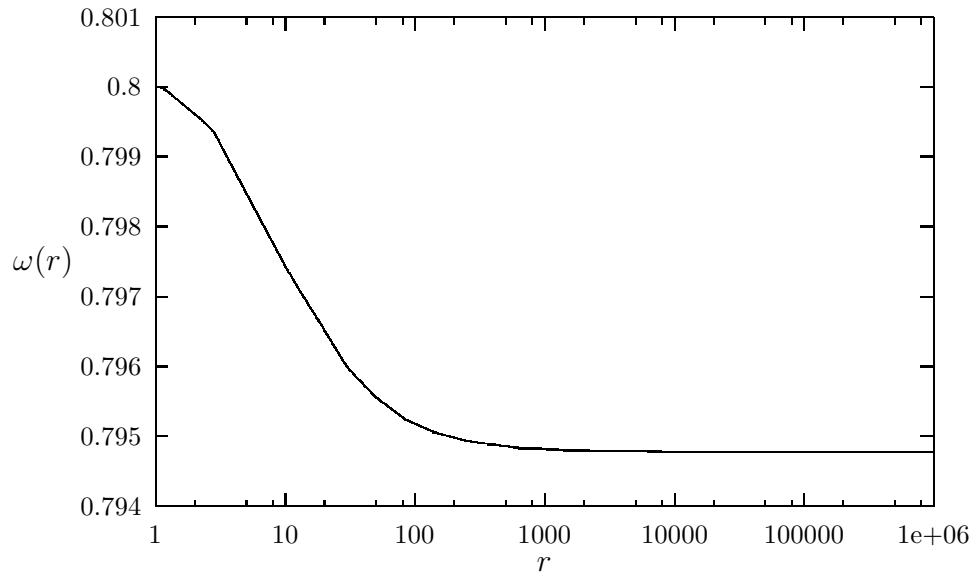


Figure 1: Black hole solution for which ω has no nodes, with $\Lambda = -100$ and $\omega_h = 0.8$.

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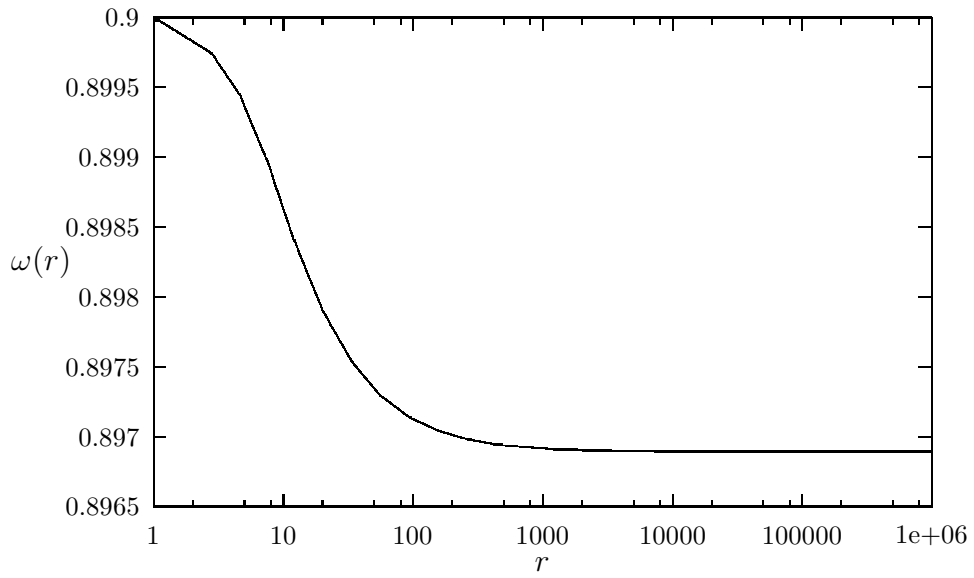


Figure 2: As figure 1, but with $\omega_h = 0.9$.

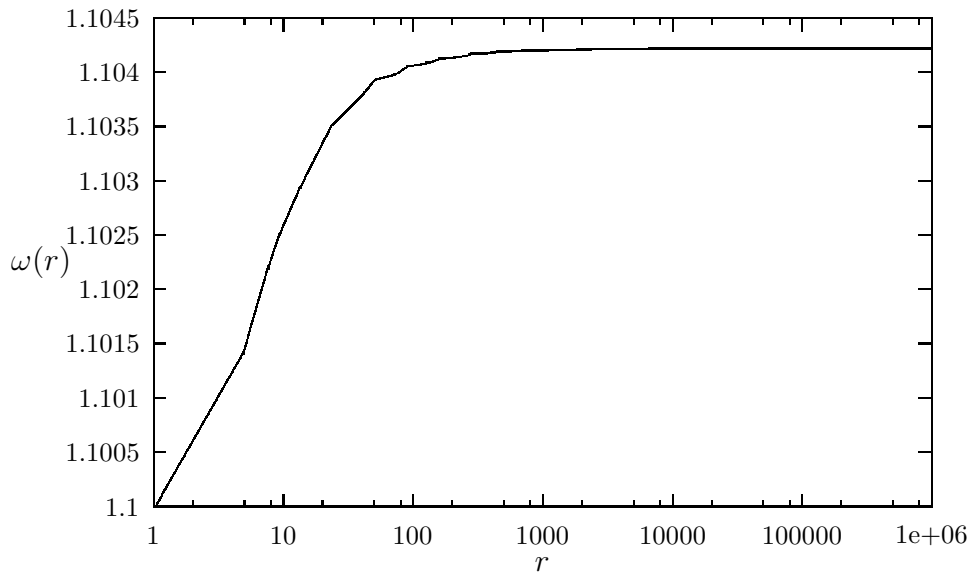


Figure 3: Black hole solution in which $|\omega| > 1$ for all r . Here $\Lambda = -100$, and $\omega_h = 1.1$.

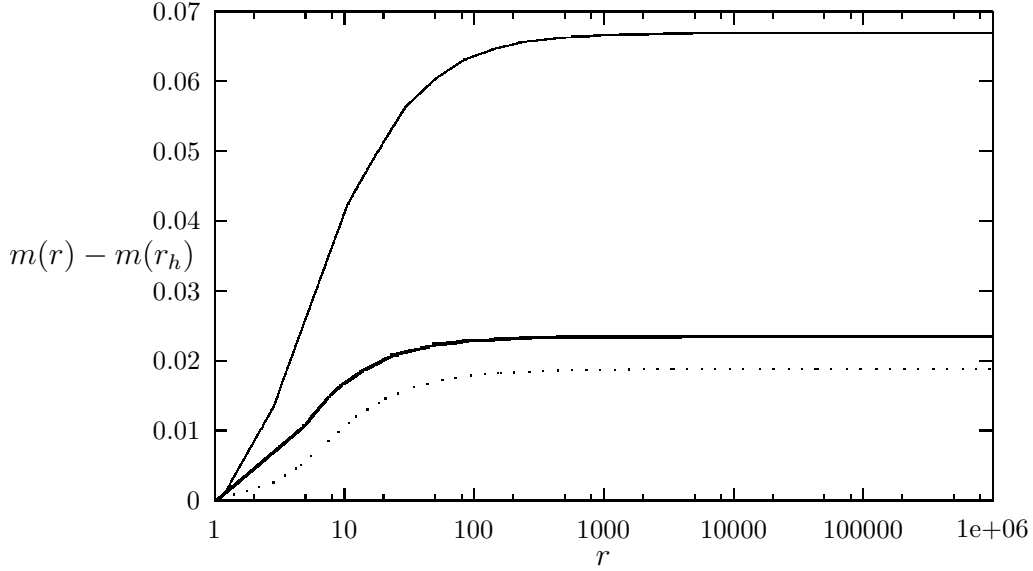


Figure 4: Black hole solutions in which the gauge field ω has no nodes, for $\Lambda = -100$. Here we plot $m(r) - m(r_h)$, with $m(r_h) = 103/6$ for this value of Λ . In this figure, and also figures 5 and 10, the solid line represents the solution for $\omega_h = 0.8$, the dotted line $\omega_h = 0.9$ and the bold solid line $\omega_h = 1.1$.

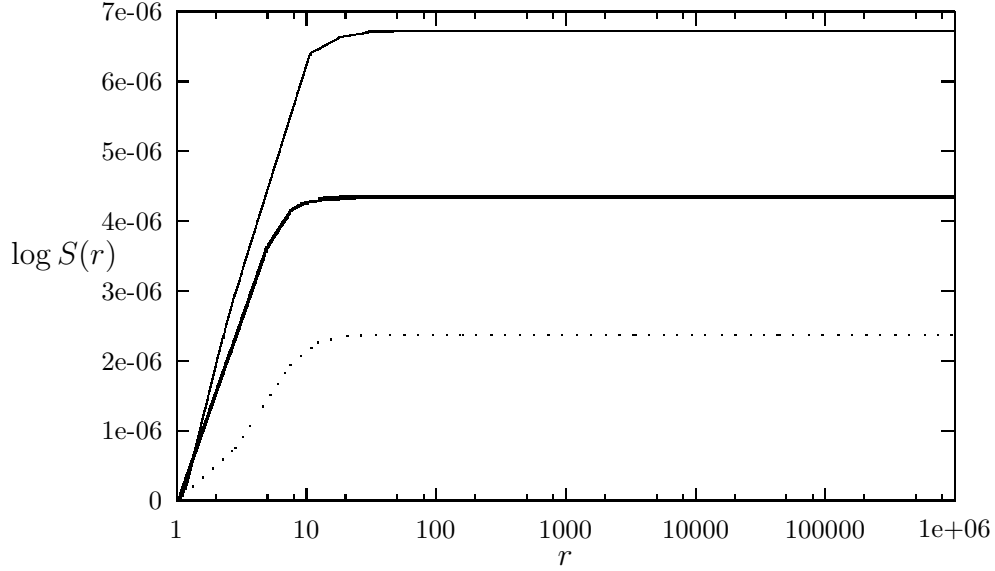


Figure 5: The same solutions as figure 4, but here $\log S(r)$ is plotted.

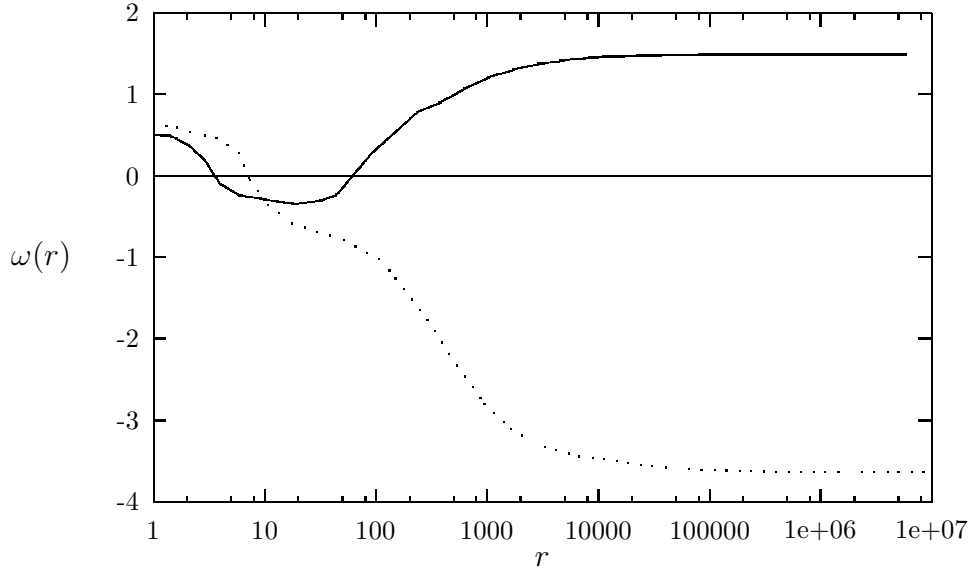


Figure 6: Black hole solutions in which $|\omega(r)| > 1$ for r sufficiently large, although $|\omega(r_h)| < 1$. Here, and in figures 7 and 8, we use cosmological constant $\Lambda = -0.001$. The solid line represents the solution with $\omega_h = 0.5$ (the gauge field having 2 nodes) and the dotted line corresponds to $\omega_h = 0.64$ (1 node).

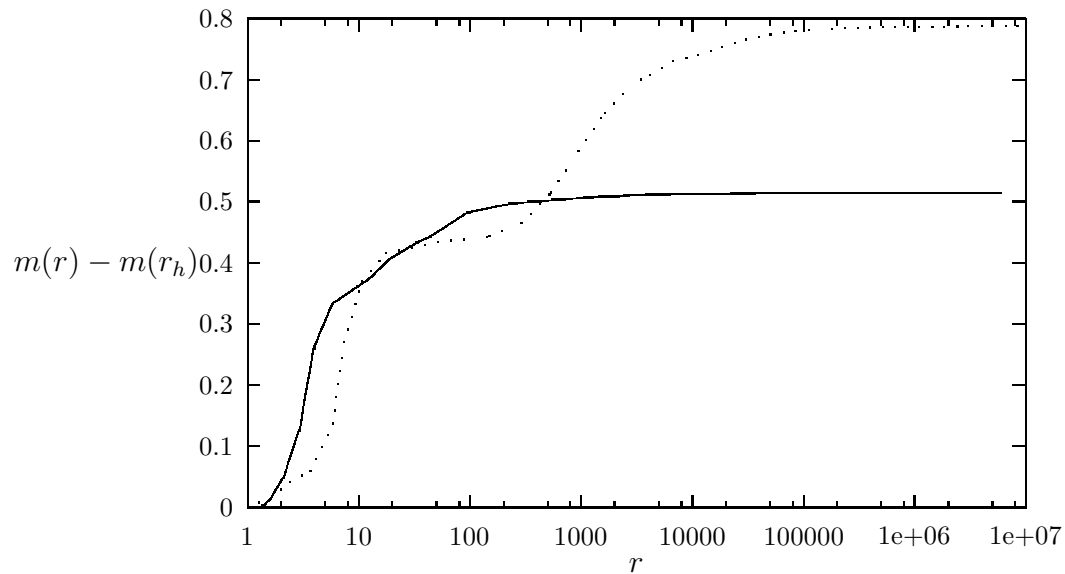


Figure 7: The metric function $m(r)$ plotted for the solutions of figure 6. For this value of Λ , note that $m(r_h) \simeq 0.5$.

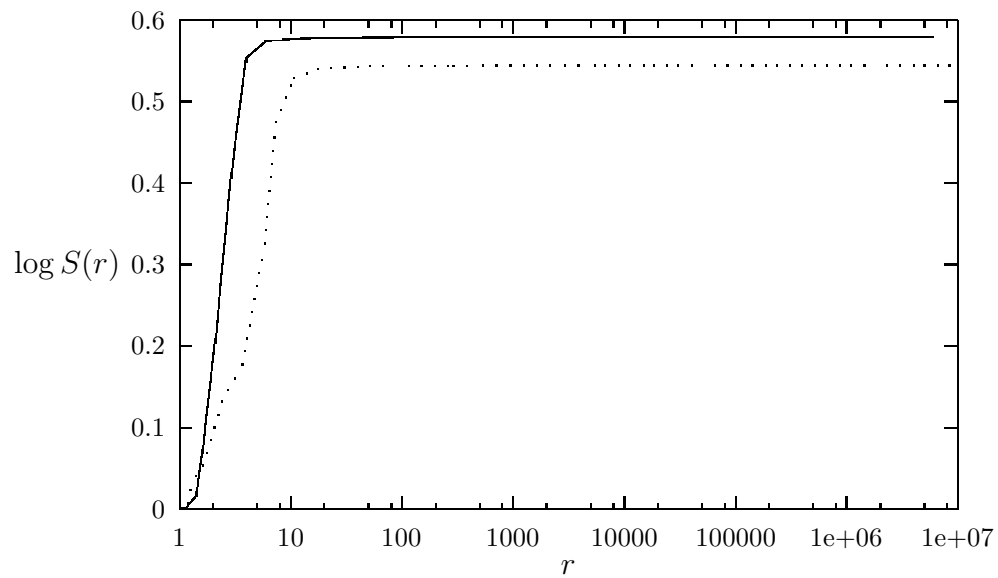


Figure 8: The same solutions as figures 6 and 7, but with the metric function $\log S(r)$ plotted.

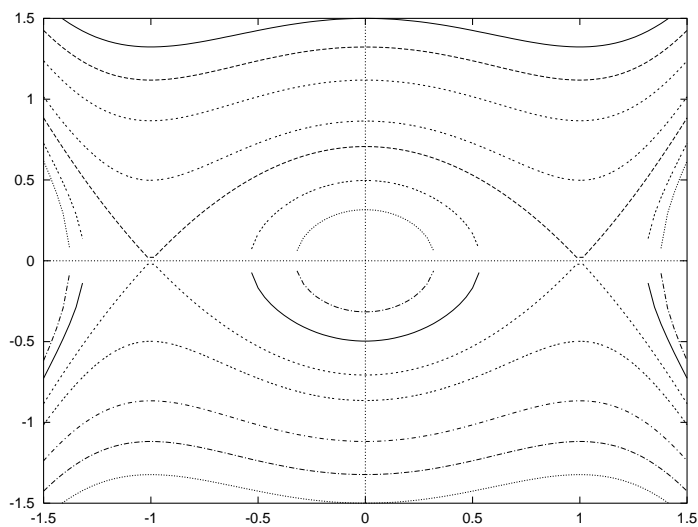


Figure 9: The phase plane for the equation $\frac{d^2\omega}{dt^2} = (\omega^2 - 1)\omega$.

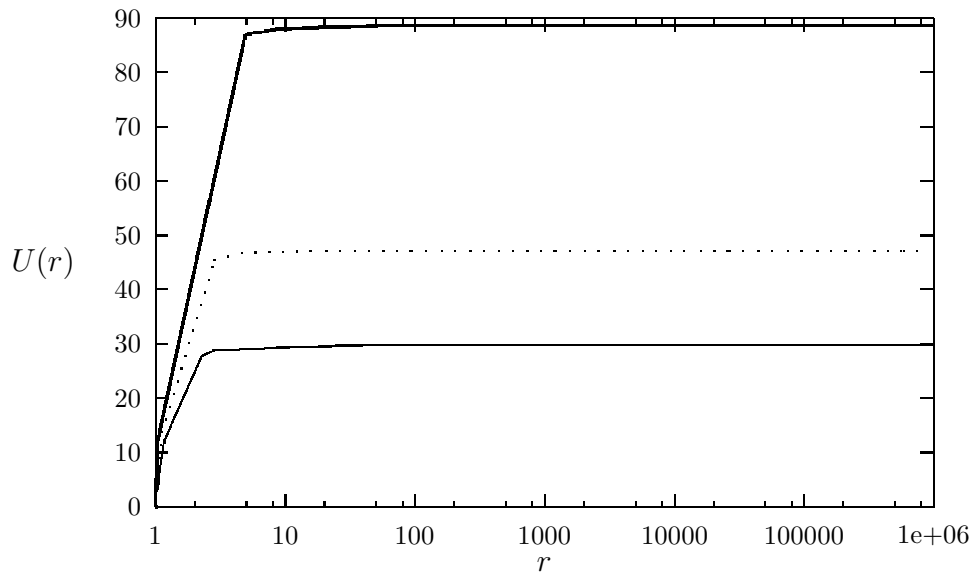


Figure 10: The potential $U(r)$ arising in the gravitational perturbation equations (31) and (32), plotted for the black holes of figures 1, 2 and 3. The key fact here is that U is everywhere positive, so that these solutions are stable in the gravitational sector.