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REVERSIBLE SKEW LAURENT POLYNOMIAL RINGS AND DEFORMATIONS OF POISSON AUTOMORPHISMS

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Abstract. A skew Laurent polynomial ring $S = R[x^{\pm 1}; \alpha]$ is reversible if it has a reversing automorphism, that is, an automorphism θ of period 2 that transposes x and x^{-1} and restricts to an automorphism γ of R with $\gamma = \gamma^{-1}$. We study invariants for reversing automorphisms and apply our methods to determine the rings of invariants of reversing automorphisms of the two most familiar examples of simple skew Laurent polynomial rings, namely a localization of the enveloping algebra of the two-dimensional non-abelian solvable Lie algebra and the coordinate ring of the quantum torus, both of which are deformations of Poisson algebras over the base field $\mathbb{F}$. Their reversing automorphisms are deformations of Poisson automorphisms of those Poisson algebras. In each case, the ring of invariants of the Poisson automorphism is the coordinate ring B of a surface in $\mathbb{F}^3$ and the ring of invariants S^θ of the reversing automorphism is a deformation of B and is a factor of a deformation of $\mathbb{F}[x_1, x_2, x_3]$ for a Poisson bracket determined by the appropriate surface.

1. Introduction

Notation 1. Throughout $\mathbb{F}$ denotes a field, $\mathrm{Aut}(R)$ denotes the group of automorphisms of a ring R and if R is an $\mathbb{F}$-algebra then $\mathrm{Aut}_{\mathbb{F}}(R)$ is the group of $\mathbb{F}$-automorphisms of R. Whenever we discuss Poisson $\mathbb{F}$-algebras, we shall assume that $\mathrm{char}\,\mathbb{F} = 0$. We denote by $\mathbb{N}_0$ the set of non-negative integers.

If R is any ring then there is an automorphism θ of the Laurent polynomial ring $R[x^{\pm 1}]$ such that $\theta(x) = x^{-1}$ and $\theta(r) = r$ for all $r \in R$. A skew Laurent polynomial ring $S = R[x^{\pm 1}; \alpha]$, where α is an automorphism of R, has no such automorphism unless $\alpha^2 = \mathrm{id}_R$. However there may exist $\theta \in \mathrm{Aut}(S)$, of order 2, such that $\theta(x) = x^{-1}$ and the restriction $\theta|_R$ is an automorphism γ of R, necessarily such that $\gamma^2 = \mathrm{id}_R$. We shall see, in Proposition 2.2, that such an automorphism θ exists if and only if $\gamma\alpha\gamma^{-1} = \alpha^{-1}$, in which case we say that α is γ-reversible, that θ is a *reversing* automorphism and that S is a *reversible* skew Laurent polynomial ring. The concept of reversibility arises in dynamical systems and the theory of flows, for example see [4, 5, 13, 14, 23].

One of the two best known examples of simple skew Laurent polynomial rings is the localization $V(\mathfrak{g}) = \mathbb{F}[x^{\pm 1}, y : xy - yx = x]$ at the powers of the normal element x of the enveloping algebra $U(\mathfrak{g}) = \mathbb{F}[x, y : xy - yx = x]$ of the two-dimensional non-abelian solvable Lie algebra $\mathfrak{g}$. This is $\mathbb{F}[y][x^{\pm 1}; \alpha]$, where,$\alpha(y) = y + 1$ and it is simple provided $\mathrm{char}\,\mathbb{F} = 0$. The second is the coordinate ring $W_q = \mathbb{F}[x^{\pm 1}, y^{\pm 1} : xy = qyx] = \mathbb{F}[y^{\pm 1}][x^{\pm 1}; \alpha]$ of the quantum torus, where $q \in \mathbb{F}\backslash\{0\}$ and $\alpha(y) = qy$, and is simple provided q is not a root of unity. Both these examples are reversible,

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the appropriate automorphisms γ being such that $\gamma(y) = -y$, for $V(\mathfrak{g})$, and $\gamma(y) = y^{-1}$, for W_q. A common approach will be used to compute the invariants for the reversing automorphisms of $V(\mathfrak{g})$ and W_q, together, in each case, with those for a reversing automorphism of an associated reversible skew Laurent polynomial ring of which it is a factor. For $V(\mathfrak{g})$, this is the localized homogenized enveloping algebra $V_t(\mathfrak{g}) = \mathbb{F}[x^{\pm1}, y, t : xy - yx = x, xt = tx, yt = ty]$ of $\mathfrak{g}$ and for W_q it is $W_Q = \mathbb{F}[x^{\pm1}, y^{\pm1}, Q^{\pm1} : xy = Qyx, xQ = Qx, yQ = Qy]$, the coordinate ring of the generic quantum torus.

Suppose now that $\operatorname{char}\mathbb{F} = 0$. If T is an $\mathbb{F}$-algebra with a central non-unit non-zero-divisor t such that $B := T/tT$ is commutative then there is a Poisson bracket $\{-,-\}$ on B such that $\{\overline{u}, \overline{v}\} = \overline{t^{-1}[u, v]}$ for all $\overline{u}, \overline{v} \in B$. In this situation, we shall follow [3, Chapter III.5] in referring to T as a quantization of the Poisson algebra B and we shall refer to an $\mathbb{F}$-algebra of the form $T_\lambda = T/(t - \lambda)T$, where $\lambda \in \mathbb{F}$ is such that the central element $t - \lambda$ is a non-unit in T, as a *deformation* of B. In this sense, $V_t(\mathfrak{g})$ and $V(\mathfrak{g})$ are, respectively, a quantization and deformation of $\mathbb{F}[x^{\pm1}, y]$, with $\{x, y\} = x$, while, taking $t = Q - 1$ and $\lambda = q - 1$, W_Q and W_q are, respectively, a quantization and deformation of $\mathbb{F}[x^{\pm1}, y^{\pm1}]$, with $\{x, y\} = xy$.

With T, t, B and λ as above, let $\theta \in \operatorname{Aut}_{\mathbb{F}}(T)$ be such that $\theta(t) = t$. Such an automorphism induces, in obvious ways, a Poisson $\mathbb{F}$-automorphism π of B and an automorphism θ_λ of T_λ. We shall refer to θ and θ_λ, respectively, as a quantization and a deformation of π. The reversing automorphisms θ of $V_t(\mathfrak{g})$ and θ_1 of $V(\mathfrak{g})$ are, respectively, a quantization and deformation of the Poisson automorphism π of $\mathbb{F}[x^{\pm1}, y]$ such that $\pi(x) = x^{-1}$ and $\pi(y) = -y$. The ring of invariants of π is a Poisson subalgebra of $\mathbb{F}[x^{\pm1}, y]$ and is readily seen to be isomorphic to the coordinate ring of the surface $x_1(x_2^2 - 4) - x_3^2 = 0$ in $\mathbb{F}^3$.

The reversing automorphism θ of W_Q is a quantization of the Poisson automorphism π of the coordinate ring $\mathbb{F}[x^{\pm1}, y^{\pm1}]$ of the torus such that $\pi(x) = x^{-1}$ and $\pi(y) = y^{-1}$. We shall write θ_q, rather than θ_{q-1}, for the corresponding automorphism of W_q which deforms π. The ring of invariants of π is a Poisson subalgebra of $\mathbb{F}[x^{\pm1}, y^{\pm1}]$ and is known to be isomorphic to the coordinate ring of the surface $x_1x_2x_3 = x_1^2 + x_2^2 + x_3^2 - 4$ in $\mathbb{F}^3$. For example, see [20, Example 3.5], although there the base ring is $\mathbb{Z}$ rather than a field.

For each of our main examples, the situation is represented in Figure 1 where the top row consists of Poisson algebras and Poisson homomorphisms. Here $A := \mathbb{F}[x_1, x_2, x_3]$, $f \in A$ is irreducible and $B = \mathbb{F}[x^{\pm1}, y]$ or $\mathbb{F}[x^{\pm1}, y^{\pm1}]$. The second and third rows are a quantization and a deformation of the first. Here g and p are central elements of the quantization T and deformation T_λ of A. In the last two columns, S and S_λ are reversible skew Laurent polynomial rings, either $V(\mathfrak{g})$ and $U(\mathfrak{g})$ or W_Q and W_q, and θ and θ_λ are reversing automorphisms. Each j_i is inclusion and each p_i, q_i or d_i is a natural surjection. The maps p_2 and p_3 may be regarded as a quantization and deformation of the embedding of the appropriate surface in $\mathbb{F}^3$.

In the case of the localized enveloping algebra, the deformation T_1 is an iterated skew polynomial ring in three indeterminates but, for the quantum torus, no such structure is apparent for the deformation which has been of interest elsewhere in the literature. It arises as the cyclically q-deformed enveloping algebra $U_q'(so_3)$ [11, 12] and as the algebra determined by a special case of the Askey-Wilson relations [28,

$$
\begin{array}{ccccccccc}
A & \xrightarrow{p_1} & A/fA & \xrightarrow{\simeq} & B^{\pi} & \xrightarrow{j_1} & B & \xrightarrow{\pi} & B \\
{\scriptstyle q_1}\big\uparrow & & {\scriptstyle q_2}\big\uparrow & & {\scriptstyle q_3}\big\uparrow & & {\scriptstyle q_4}\big\uparrow & & {\scriptstyle q_4}\big\uparrow \\
T & \xrightarrow{p_2} & T/gT & \xrightarrow{\simeq} & S^{\theta} & \xrightarrow{j_2} & S & \xrightarrow{\theta} & S \\
{\scriptstyle d_1}\big\downarrow & & {\scriptstyle d_2}\big\downarrow & & {\scriptstyle d_3}\big\downarrow & & {\scriptstyle d_4}\big\downarrow & & {\scriptstyle d_4}\big\downarrow \\
T_\lambda & \xrightarrow{p_3} & T_\lambda/pT_\lambda & \xrightarrow{\simeq} & S_\lambda^{\theta_\lambda} & \xrightarrow{j_3} & S_\lambda & \xrightarrow{\theta_\lambda} & S_\lambda
\end{array}
$$

FIGURE 1. Quantization, deformation and invariants

29]. Both deformations are examples of algebras determined by noncommutative potentials, as described in [9].

The invariants for the reversing automorphisms of the localized enveloping algebra and its homogenization are computed in Section 5 and those for the quantum torus and generic quantum torus are computed in Section 6. Section 2 contains the definitions and the main examples of reversing automorphisms together with some general results on generators and relations for their rings of invariants. Basic material on Poisson structures, quantization and deformation appears in Section 3 while Section 4 presents some technical material, on filtrations and the Diamond Lemma, that is applied in Sections 5 and 6.

Some of the results of the paper appeared in the PhD thesis of the second author [26]. The study will be continued in two papers by the first author. In [18] the notion of reversing automorphism, and the methods introduced in this paper, will be extended to other algebras, including $U(sl_2)$ and $U_q(sl_2)$. A connection between the reversing automorphisms of $U_q(sl_2)$ and $U'_q(so_3)$ will be exploited to determine the prime spectrum $\mathrm{Spec}(U'_q(so_3))$. In [17], the Poisson spectrum for certain Poisson brackets on A, including the two main examples of this paper, will be determined. For each of those examples, there is a homeomorphism between the Poisson spectrum and the completely prime subspace of the spectrum of the corresponding deformation S_λ.

2. Reversing automorphisms and invariants

Definition 2.1. Let R be a ring and let $\alpha, \gamma \in \mathrm{Aut}(R)$ be such that $\gamma^2 = \mathrm{id}_R$. We say that α is *γ-reversible* if $\gamma\alpha\gamma^{-1} = \alpha^{-1}$. In other words α and γ provide a representation of the infinite dihedral group in $\mathrm{Aut}(R)$.

It is easy to check that γ-reversibility of α is equivalent to each of the following four statements: (a) $(\alpha\gamma)^2 = \mathrm{id}_R$; (b) $(\gamma\alpha)^2 = \mathrm{id}_R$; (c) $\alpha = \gamma\tau$ for some $\tau \in \mathrm{Aut}(R)$ such that $\tau^2 = \mathrm{id}_R$; (d) $\alpha = \tau'\gamma$ for some $\tau' \in \mathrm{Aut}(R)$ such that $\tau'^2 = \mathrm{id}_R$.

Proposition 2.2. *Let R be a ring and let $\alpha, \gamma \in \mathrm{Aut}(R)$ be such that $\gamma^2 = \mathrm{id}_R$. Let $S = R[x^{\pm 1}; \alpha]$. There exists $\theta \in \mathrm{Aut}(S)$ such that $\theta|_R = \gamma$ and $\theta(x) = x^{-1}$ if and only if α is γ-reversible.*

Proof. Suppose that such an automorphism θ exists. For each $r \in R$, $xr = \alpha(r)x$ and $x^{-1}r = \alpha^{-1}(r)x^{-1}$. Applying θ to the first of these, $x^{-1}\gamma(r) = \gamma\alpha(r)x^{-1}$, whence $\alpha^{-1}\gamma = \gamma\alpha$ and $\alpha^{-1} = \gamma\alpha\gamma^{-1}$. Thus α is γ-reversible.

Conversely, suppose that α is γ-reversible and let $\eta = \iota\gamma : R \to S$, where ι is the embedding of R in S. The unit x^{-1} in S is such that $x^{-1}\eta(r) = \eta\alpha(r)x^{-1}$. By the universal mapping property for skew Laurent polynomial rings, as specified in [10, Exercise 1N], there is a (unique) ring endomorphism θ of S such that $\theta|_R = \gamma$ and $\theta(x) = x^{-1}$. Being self-inverse, $\theta \in \mathrm{Aut}(R)$. □

Definition 2.3. When α is *γ-reversible* for some $\gamma \in \mathrm{Aut}(R)$ such that $\gamma^2 = \mathrm{id}_R$, we shall say that S is a *reversible* skew Laurent polynomial ring and that the automorphism θ of S such that $\theta(x) = x^{-1}$ and $\theta|_R = \gamma$ is the *reversing* automorphism of S determined by γ.

The main examples are the two pairs of related skew Laurent polynomial rings discussed in the Introduction.

Example 2.4. (i) Let $R = \mathbb{F}[y]$, let $\alpha, \gamma \in \mathrm{Aut}_{\mathbb{F}}(R)$ be such that $\alpha(y) = y + 1$ and $\gamma(y) = -y$ and let $S_1 = R[x^{\pm 1}; \alpha]$. Then α is γ-reversible, $R[x; \alpha] = \mathbb{F}[y, x : xy - yx = x]$ is the enveloping algebra $U(\mathfrak{g})$ of the two-dimensional non-abelian solvable Lie algebra $\mathfrak{g}$ and S_1 is its localization, which we denote $V(\mathfrak{g})$, at the powers of the normal element x. The reversing automorphism θ_1 of $V(\mathfrak{g})$ determined by γ is such that $\theta_1(y) = -y$ and $\theta_1(x) = x^{-1}$. It is well-known that $V(\mathfrak{g})$ is simple if $\mathrm{char}\,\mathbb{F} = 0$, for example see [10, Exercise 1V].

(ii) Let $R = \mathbb{F}[y, t]$, let $\alpha, \gamma \in \mathrm{Aut}_{\mathbb{F}}(R)$ be such that $\alpha(y) = y + 1$, $\gamma(y) = -y$ and $\alpha(t) = t = \gamma(t)$. Let $S = R[x^{\pm 1}; \alpha]$. Then α is γ-reversible and $R[x; \alpha]$ is the $\mathbb{F}$-algebra generated by t, x and y subject to the relations

$$xy - yx = tx, \quad tx = xt, \quad ty = yt.$$

This is the *homogenized enveloping algebra* $U_t(\mathfrak{g})$, with $\mathfrak{g}$ as in (i), and S is its localization, which we denote $V_t(\mathfrak{g})$, at the powers of x. The reversing automorphism θ of $V_t(\mathfrak{g})$ determined by γ is such that $\theta(x) = x^{-1}$, $\theta(y) = -y$ and $\theta(t) = t$.

Example 2.5. (i) Let $q \in \mathbb{F}\backslash\{0\}$, let $R = \mathbb{F}[y^{\pm 1}]$ and let $\alpha, \gamma \in \mathrm{Aut}_{\mathbb{F}}(R)$ be such that $\alpha(y) = qy$ and $\gamma(y) = y^{-1}$. Then $\gamma^2 = \mathrm{id}$ and α is γ-reversible. Here the skew Laurent polynomial ring $S = R[x^{\pm 1}; \alpha]$ is the quantized coordinate ring $W_q = \mathcal{O}_q((\mathbb{F}^*)^2)$ or, more informally, the quantum torus, see [10, p.16]. It is well-known that W_q is simple if q is not a root of unity, for example see [10, Corollary 1.18]. The reversing automorphism θ_q of W_q determined by γ is such that $\theta_q(y) = y^{-1}$ and $\theta_q(x) = x^{-1}$.

(ii) Let $R = \mathbb{F}[y^{\pm 1}, Q^{\pm 1}]$ and let $\alpha, \gamma \in \mathrm{Aut}_{\mathbb{F}}(R)$ be such that $\alpha(y) = Qy$, $\alpha(Q) = Q = \gamma(Q)$ and $\gamma(y) = y^{-1}$. The *generic quantum torus* is the skew Laurent polynomial ring $W_Q = R[x^{\pm 1}; \alpha]$. Then $xy = Qyx$, α is γ-reversible and the reversing automorphism θ of W_Q determined by γ is such that $\theta(x) = x^{-1}$, $\theta(y) = y^{-1}$ and $\theta(Q) = Q$.

For the remainder of this section, let R be a ring and let $\alpha, \gamma \in \mathrm{Aut}(R)$ be such that $\gamma^2 = \mathrm{id}_R$ and α is γ-reversible. Let $S = R[x^{\pm 1}; \alpha]$ and let $\theta \in \mathrm{Aut}(S)$ be the reversing automorphism determined by γ. We now identify some elements of S^θ and some relations that hold between them.

Lemma 2.6. *For $r \in R$ and $n \geq 0$, let $s_n(r) := rx^n + \gamma(r)x^{-n}$. In particular $s_0(r) = r + \gamma(r)$. Then $s_n(r) \in S^\theta$. If $r, r' \in R$ and $rr' = r'r$ then*

$$s_0(r)s_1(r') - s_1(r')s_0(\alpha^{-1}(r)) = s_1((\gamma(r) - \alpha^2(\gamma(r)))r'). \tag{2.1}$$

In particular, with $r' = 1$,

$$s_0(r)s_1(1) - s_1(1)s_0(\alpha^{-1}(r)) = s_1(\gamma(r) - \alpha^2\gamma(r)). \tag{2.2}$$

Also,

$$s_1(r)s_1(1) - s_1(1)s_1(\alpha^{-1}(r)) = s_0(r - \alpha^{-2}(r)). \tag{2.3}$$

Proof. It is immediate from the definition of θ that $s_n(r) \in S^\theta$. The relations (2.1), (2.2) and (2.3) are routinely checked using the equations $\alpha^{-1}\gamma\alpha^{-1} = \gamma$ and $\alpha\gamma\alpha^{-1} = \gamma\alpha^{-2} = \alpha^2\gamma$. □

Proposition 2.7. *The fixed ring S^θ is generated by the fixed ring R^γ and the set $\{s_1(r) : r \in R\}$.*

Proof. Let S_1 be the subring of S generated by R^γ and $\{s_1(r) : r \in R\}$. It is clear that $S_1 \subseteq S^\theta$. Let $s = \sum_m^n r_i x^i \in S^\theta$, where each $r_i \in R$. Then $s = \theta(s) = \sum_m^n \gamma(r_i)x^{-i}$ from which it follows that $m = -n$, $r_0 = \gamma(r_0)$ and, for $1 \leq i \leq n$, $r_{-i} = \gamma(r_i)$. Thus $s = r_0 + \sum_1^n s_i(r_i)$. As $r_0 \in R^\gamma \subset S_1$, it now suffices to show that, for all $r \in R$ and all $i \geq 1$, $s_i(r) \in S_1$. This is certainly true when $i = 1$ and it follows inductively using the formula

$$s_{i+1}(r) = s_i(r)s_1(1) - s_{i-1}(r).$$

□

The following result, whose hypothesis is satisfied if R is left Noetherian, by [24, Corollary 26.13], will be applicable to give finite sets of generators for our examples.

Proposition 2.8. *Suppose that R is finitely generated, as a left R^γ-module, by $r_1 = 1, r_2, \ldots, r_n$. Then S^θ is generated by R^γ and $\{s_1(r_i) : 1 \leq i \leq n\}$.*

Proof. If $r = c_1r_1 + c_2r_2 + \ldots + c_nr_n$, where $c_1, c_2, \ldots, c_n \in R^\gamma$, then $s_1(r) = c_1s_1(r_1) + c_2s_1(r_2) + \ldots + c_ns_1(r_n)$. The result follows from Proposition 2.7. □

Corollary 2.9. *Let $S = V_t(\mathfrak{g})$ and θ be as in Example 2.4(ii) and suppose that* $\operatorname{char}\mathbb{F} \neq 2$. *Then S^θ is generated by t, y^2, $x + x^{-1}$ and $yx - yx^{-1}$.*

Proof. Here R^γ is generated by t and y^2 and $R = R^\gamma + R^\gamma y$ so the result follows from Proposition 2.8. □

Corollary 2.10. *Let $S_1 = V(\mathfrak{g})$ and θ_1 be as in Example 2.4(i) and suppose that* $\operatorname{char}\mathbb{F} \neq 2$. *Then S^{θ_1} is generated by y^2 and $x + x^{-1}$.*

Proof. Let $r = y^2/2$, $a_1 = y^2 = s_0(r)$, $a_2 = yx - yx^{-1} = s_1(y)$ and $a_3 = x + x^{-1} = s_1(1)$. Then R^γ is generated by a_1 and $R = R^\gamma + R^\gamma y$ so, by Proposition 2.8, $S_1^{\theta_1}$ is generated by a_1, a_2 and a_3. Also $\alpha^{-1}(r) = 1$ and $\alpha^2\gamma(r) = r + 2y + 2$ so, by (2.2), $a_1a_3 - a_3(a_1 + 1) = s_1(-2y - 2) = -2a_2 - 2a_3$, whence, as $\operatorname{char}\mathbb{F} \neq 2$, a_2 is in the $\mathbb{F}$-subalgebra generated by a_1 and a_3. □

Corollary 2.11. *Let $S = W_Q$ and θ be as in Example 2.5(ii). Then S^θ is generated by $y + y^{-1}$, Q, Q^{-1}, $x + x^{-1}$ and $yx + y^{-1}x^{-1}$.*

Proof. Here R^γ is generated by Q, Q^{-1} and $y + y^{-1}$. For $n \geq 0$, let V_n be the $\mathbb{F}[Q^{\pm 1}]$-submodule of $\mathbb{F}[y^{\pm 1}, Q^{\pm 1}]$ generated by $\{y^m : n \geq m \geq -n\}$. For $r, s \in \mathbb{F}[Q^{\pm 1}]$, $ry + sy^{-1} = s(y + y^{-1}) + (r - s)y$, whence $V_1 \subseteq R^\gamma + R^\gamma y$. For $n \geq 2$,

$$ry^n + sy^{-n} = sy^n + sy^{-n} + ((r - s)y^{n-1} + (r - s)y^{1-n})y + (s - r)y^{2-n},$$

so $V_n \subseteq R^\gamma + R^\gamma y + V_{n-1}$. It follows, inductively, that $V_n \subseteq R^\gamma + R^\gamma y$ for all n and hence that $R = R^\gamma + R^\gamma y$. The result follows from Proposition 2.8. □

Corollary 2.12. *Let $S_q = W_q$ and θ_q be as in Example 2.5(i). Then $S_q^{\theta_q}$ is generated by $y + y^{-1}$, $x + x^{-1}$ and $yx + y^{-1}x^{-1}$. If $q^2 \neq 1$ then S^{θ_q} is generated by $y + y^{-1}$ and $x + x^{-1}$.*

Proof. Here R^γ is generated by $y + y^{-1}$ and, as in the proof of Corollary 2.11, $R = R^\gamma + R^\gamma y$. The first conclusion follows from Proposition 2.8. Let $a_1 = y + y^{-1} = s_0(y)$, $a_2 = x + x^{-1} = s_1(1)$ and $a_3 = yx + y^{-1}x^{-1} = s_1(y)$. Note that $s_0(\alpha^{-1}(y^{-1})) = qa_1$ and $\gamma(y^{-1}) - \alpha^2\gamma(y^{-1}) = (1 - q^2)y$. By (2.2) with $r = y^{-1}$, $a_1a_2 - qa_2a_1 = (1 - q^2)a_3$ so as $q^2 \neq 1$, a_3 is in the subalgebra generated by a_1 and a_2. □

Remark 2.13. If $\operatorname{char} \mathbb{F} = 0$ in Example 2.4(i) then $S_1^{\theta_1}$ is simple as an easy consequence of [24, Theorem 28.3(ii)]. If $\operatorname{char} \mathbb{F} \neq 2$ and q is not a root of unity in Example 2.5(i) then $S_q^{\theta_q}$ is simple for the same reason.

Remark 2.14. Let α be a γ-reversible automorphism of a ring R and let $S = R[x^{\pm 1}; \alpha]$. Then α is also $\alpha\gamma$-reversible and $\gamma\alpha$-reversible and there are reversing automorphisms θ' and θ'' of S determined by $\alpha\gamma$ and $\gamma\alpha$ respectively. Suppose that there is a γ-reversible automorphism β of R such that $\beta^2 = \alpha^{-1}$. If $\operatorname{char} \mathbb{F} \neq 2$ this is the case in Example 2.4(i), with $\beta(y) = y - \frac{1}{2}$, and if q is a square in $\mathbb{F}$ it is the case in Example 2.5(ii), with $\beta(y) = q^{-1/2}y$. Then β extends to an automorphism of S with $\beta(x) = x$ and $\beta\theta'\beta^{-1} = \theta = \beta^{-1}\theta''\beta$. Thus the reversing automorphisms θ, θ' and θ'' are conjugate in $\operatorname{Aut}(S)$ and hence their rings of invariants are isomorphic.

3. Poisson algebras, quantization and deformation

In this section $\operatorname{char} \mathbb{F} = 0$. By a *Poisson algebra* we mean a commutative $\mathbb{F}$-algebra A with a bilinear product $\{-, -\} : A \times A \to A$ such that A is a Lie algebra under $\{-, -\}$ and, for all $a \in A$, $\{a, -\}$ is an $\mathbb{F}$-derivation of A. Such a product is a *Poisson bracket* on A. For $a \in A$, the derivation $\{a, -\}$ is a *hamiltonian* derivation (or hamiltonian vector field) and is written $\operatorname{ham} a$.

A subalgebra B of A is a *Poisson subalgebra* of A if $\{b, c\} \in B$ for all $b, c \in B$ and an ideal I of A is a *Poisson ideal* if $\{i, a\} \in I$ for all $i \in I$ and all $a \in A$. If I is a Poisson ideal of A then A/I is a Poisson algebra in the obvious way: $\{a + I, b + I\} = \{a, b\} + I$. The *Poisson centre* of a Poisson algebra A is $\operatorname{PZ}(A) := \{a \in A : \{a, b\} = 0 \text{ for all } b \in A\}$.

An $\mathbb{F}$-automorphism π of a Poisson algebra A is a *Poisson automorphism* of A if $\{\pi(a), \pi(b)\} = \pi(\{a, b\})$ for all $a, b \in A$, in which case π^{-1} is also a Poisson automorphism. If π is a Poisson automorphism of A then the ring of invariants A^π is a Poisson subalgebra of A.

Definitions 3.1. Let T be an $\mathbb{F}$-algebra with a central non-unit non-zero-divisor t such that $B := T/tT$ is commutative. Then $[-, -]$ in T induces a well-defined Poisson bracket $\{-, -\}$ on B by the rule

$$\{\overline{u}, \overline{v}\} = \overline{t^{-1}[u, v]} \text{ for all } \overline{u} = u + tT, \overline{v} = v + tT \in B.$$

For more detail, see [3, Chapter III.5]. Following [3], we say that T is a *quantization* of the Poisson algebra B. Let $\lambda \in \mathbb{F}$ be such that the central element $t - \lambda$ is a non-unit in T and let $T_\lambda = T/(t - \lambda)T$. We shall refer to T_λ as a *deformation* of B.

Now suppose that there exists $\theta \in \mathrm{Aut}_{\mathbb{F}}(T)$ such that $\theta(t) = t$. Then there is a well-defined $\mathbb{F}$-automorphism π of B such that $\pi(\overline{u}) = \overline{\theta(u)}$ for all $\overline{u} \in B$. Let $a = u + tT, b = v + tT \in B$. Then

$$\begin{aligned}\{\pi(a), \pi(b)\} &= \{\theta(u) + tT, \theta(v) + tT\} = t^{-1}[\theta(u), \theta(v)] + tT\\ &= \theta(t^{-1}[u, v]) + tT = \pi(\{u + tT, v + tT\}) = \pi(\{a, b\}).\end{aligned}$$

Thus π is a Poisson automorphism of B. There is also an automorphism θ_λ of the deformation $T/(t-\lambda)T$ with $\theta_\lambda(\overline{u}) = \overline{\theta(u)}$ for all $\overline{u} \in T/(t-\lambda)T$. We shall refer to θ as a quantization of π and to θ_λ as a *deformation* of π.

In this situation, $t \in T^\theta$, where it must be a regular central non-unit, so the $\mathbb{F}$-algebra $C := T^\theta/tT^\theta$ becomes a Poisson algebra. On the other hand the ring of invariants B^π is a Poisson subalgebra of B. There is an injective $\mathbb{F}$-algebra homomorphism $\psi : C \to B^\pi$ given by $\psi(u + tT^\theta) = u + tT$. If θ has the property that $\{u \in T : u - \theta(u) \in tT\} = tT + T^\theta$ then ψ is an isomorphism and we may identify the Poisson algebras C and B^π.

Example 3.2. In Example 2.4, $V_t(\mathfrak{g})$ is a quantization of the Poisson algebra $B := \mathbb{F}[x^{\pm 1}, y]$, with the Poisson bracket such that $\{x, y\} = x$, and $V(\mathfrak{g})$ is a deformation of B. By the quotient rule for the derivation $\mathrm{ham}\, y$, $\{x^{-1}, y\} = -x^{-1}$. The reversing automorphisms θ and θ_1 of $V_t(\mathfrak{g})$ and $V(\mathfrak{g})$ are, respectively, a quantization and a deformation of the Poisson automorphism π of B such that $\pi(x) = x^{-1}$ and $\pi(y) = -y$. There are three obvious invariants under π, namely $a_1 := y^2$, $a_2 := y(x - x^{-1})$ and $a_3 := x + x^{-1}$. It is a routine matter to check that these generate B^π and that $a_2^2 = a_1(a_3^2 - 4)$. Thus, if $A = \mathbb{F}[x_1, x_2, x_3]$ and $f = x_1(4 - x_3^2) + x_2^2$, then there is a surjection $\phi : A \to B^\pi$ with $f \in \ker \phi$. As B^π and A/fA are domains of Krull dimension 2, they are isomorphic.

Example 3.3. In Example 2.5, let $t = Q - 1$, which is a regular central non-unit in W_Q, and let $B = W_Q/tW_Q$ which we identify with $\mathbb{F}[x^{\pm 1}, y^{\pm 1}]$. Then W_Q is a quantization of the Poisson algebra B, where the Poisson bracket is such that $\{x, y\} = xy$, and, taking $\lambda = q - 1$, W_q is a deformation. In B, $\mathrm{ham}\, x = xy\partial/\partial y$ and $\mathrm{ham}\, y = -xy\partial/\partial x$. Consequently $\{x, y^{\pm 1}\} = -xy^{\pm 1}$, $\{x^{\pm 1}, y\} = -x^{\pm 1}y$ and $\{x^{\pm 1}, y^{\pm 1}\} = x^{\pm 1}y^{\pm 1}$. There is a Poisson automorphism π of B such that $\pi(x) = x^{-1}$ and $\pi(y) = y^{-1}$ and the reversing automorphisms θ, of W_Q, and θ_q, of W_q are, respectively, a quantization and deformation of π.

The ring of invariants of the commutative Laurent polynomial $\mathbb{Z}[y^{\pm 1}, x^{\pm 1}]$ for the automorphism $x \mapsto x^{-1}$, $y \mapsto y^{-1}$ is discussed in [20, Example 3.5]. The ring of invariants is generated by $a_1 := y + y^{-1}, a_2 := x + x^{-1}$ and $a_3 := xy + x^{-1}y^{-1}$ and is isomorphic to $\mathbb{Z}[x_1, x_2, x_3]/f\mathbb{Z}[x_1, x_2, x_3]$ where $f = x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2 + 4$. The same calculations are valid over $\mathbb{F}$, where $B^\pi \simeq \mathbb{F}[x_1, x_2, x_3]/f\mathbb{F}[x_1, x_2, x_3]$.

Remark 3.4. Let $A = \mathbb{F}[x_1, x_2, x_3]$. In each of the Examples 3.2 and 3.3, we have identified B^π as the coordinate ring A/fA of a surface determined by an irreducible polynomial f. In each case there is a Poisson bracket on A, determined by f as in the following definition, such that B^π is a factor of A as a Poisson algebra.

Definition 3.5. Let $A = \mathbb{F}[x_1, x_2, x_3]$ and let $f \in A$. There is a Poisson bracket $\{-,-\}_f$ on A given by $\{x_1, x_2\}_f = \partial f/\partial x_3$, $\{x_2, x_3\}_f = \partial f/\partial x_1$ and $\{x_3, x_1\}_f = \partial f/\partial x_2$. Such brackets are considered, for example, in [25], [22, p.1312 (1) with $n = 3$ and $\lambda = 1$] and [7, p. 252].

For $g, h \in A$,

$$\{g,h\}_f = \begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{vmatrix}$$

from which it is clear that $f \in \mathrm{PZ}(A)$ and hence that fA is a Poisson ideal of A. Writing $f_i = \partial f/\partial x_i$ for $i = 1, 2, 3$, the hamiltonian derivations of A for the three generators are

$$\begin{aligned} \operatorname{ham} x_1 &= f_3\partial/\partial x_2 - f_2\partial/\partial x_3; \\ \operatorname{ham} x_2 &= f_1\partial/\partial x_3 - f_3\partial/\partial x_1; \\ \operatorname{ham} x_3 &= f_2\partial/\partial x_1 - f_1\partial/\partial x_2. \end{aligned}$$

We shall call a Poisson bracket on A *exact* (determined by f) if it has the form $\{-,-\}_f$ for some $f \in A$.

Examples 3.6. In 3.2, it is a straightforward exercise to check that

$$\{a_1, a_2\} = -2a_1a_3,\ \{a_2, a_3\} = 4 - a_3^2 \text{ and } \{a_3, a_1\} = 2a_2.$$

Thus, as a Poisson algebra, B^π is the factor A/fA of A under the exact Poisson bracket $\{-,-\}_f$, where $f = x_1(4 - x_3^2) + x_2^2$ and each $a_i = x_i + fA$.

Similarly, in Example 3.3, it can be checked that

$$\{a_1, a_2\} = a_1a_2 - 2a_3,\ \{a_2, a_3\} = a_2a_3 - 2a_1 \text{ and } \{a_3, a_1\} = a_1a_3 - 2a_2$$

so that the Poisson algebra B^π is again a Poisson factor A/fA of A, with the exact Poisson bracket $\{-,-\}_f$, where, on this occasion, $f = x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2 + 4$.

Remark 3.7. In Example 2.4(ii) the automorphism α, used in constructing $S = V_t(\mathfrak{g})$ as a skew Laurent polynomial ring, extends to a $\mathbb{F}$-automorphism of S such that $\alpha(x) = x$, $\alpha(t) = t$ and $\alpha(y) = y + t \equiv y \bmod tS$. Thus α quantizes the identity automorphism on $B = \mathbb{F}[y, x^{\pm 1}]$. The same is true in Example 2.5(ii) where the automorphism α of W_Q such that $\alpha(x) = x, \alpha(y) = Qy$ and $\alpha(Q) = Q$ quantizes the identity automorphism on $B = \mathbb{F}[y^{\pm 1}, x^{\pm 1}]$.

Remark 3.8. In Examples 3.2 and 3.3, the Poisson algebra B is simple. In each case B is a localization of $\mathbb{F}[x, y]$ and the hamiltonian derivations $\{x, -\}$ and $\{y, -\}$ have the forms $u\partial/\partial y$ and $v\partial/\partial x$ respectively, where u and v are units in B. Consequently any non-zero Poisson ideal of B must intersect $\mathbb{F}[x, y]$ in a non-zero ideal invariant under the derivations $\partial/\partial x$ and $\partial/\partial y$ and therefore cannot be proper.

There is an example in [1] of a Poisson automorphism ($x \mapsto -x, y \mapsto -y$) of a simple Poisson algebra ($\mathbb{F}[x, y]$ with $\{x, y\} = 1$) such that the ring of invariants is not a simple Poisson algebra, having a maximal ideal (generated by x^2, xy, y^2) that is Poisson. The situation for our two examples is similar in that, although B is simple, there are finitely many maximal ideals of B^π that are Poisson. In Example 3.2, these are the two maximal ideals $(a_1, a_2, a_3 \pm 2)$ while, in Example 3.3, they are $(a_1 - 2, a_2 - 2, a_3 - 2)$, $(a_1 + 2, a_2 + 2, a_3 - 2)$, $(a_1 + 2, a_2 - 2, a_3 + 2)$ and $(a_1 - 2, a_2 + 2, a_3 + 2)$. Thus B^π is not simple in either example.

4. Filtrations

We have identified finite sets of generators for the rings of invariants for the four reversing automorphisms in Examples 2.4 and 2.5. Identifying finite sets of relations is much more technical. A model is given by the argument, using Krull dimension and domain recognition, sketched in Examples 3.6 for the invariants of the automorphism π of $\mathbb{F}[x^{\pm 1}, y]$ with $\pi(x) = x^{-1}$ and $\pi(y) = -y$. Essentially, we need to impose sufficiently many relations, from those identified in Lemma 2.6, to obtain a domain of the correct dimension. The appropriate dimension is Gelfand-Kirillov dimension, for which we refer to [19] and [21]. Other key methods involve Bergman's Diamond Lemma, for which we refer to [3, Appendix I.11], and filtrations with their associated graded rings, for which references are [21, §1.6] and [3, Appendix I.12]. The methods will occasionally be sensitive to the choice of filtration. The filtrations considered are slightly more general than those described in [3, I.12.2(c)].

The orderings on monomials that we use in applying the Diamond Lemma are modifications of the length-lexicographic ordering. Let $n \geq 1$, let M_n be the free monoid on $\{z_1, z_2, \ldots, z_n\}$ and let F_n be the free algebra $\mathbb{F}\langle z_1, z_2, \ldots, z_n\rangle$. By a *degree function* on M_n, we mean a monoid homomorphism $d : M_n \to (\mathbb{N}_0, +)$. Such a function is determined by its values on $z_1, z_2, \ldots, z_n$.

Definitions 4.1. Given a degree function $d : M_n \to (\mathbb{N}_0, +)$ we modify the length-lexicographic ordering $\preceq_{\text{lex}}$ by ordering words first by the degree function d and then lexicographically with $z_1 \succ z_2 \succ \ldots \succ z_n$. More formally, we define $m \leq m'$ if and only if either $d(m) < d(m')$ or $d(m) = d(m')$ and $m \preceq_{\text{lex}} m'$. We do not require that $d(z_i) \geq d(z_j)$ whenever $i \leq j$ so we may have $z_2 > z_1$ although $z_1 \succ z_2$. We shall refer to this as the *d-length-lexicographic ordering* . It is clearly a semigroup ordering and if $d(z_i) > 0$ for each i it has DCC.

If $d(z_i) = 0$ for some i then a semigroup ordering with DCC can be defined using a complementary degree function e, such that $e(z_i) = 0$ if $d(z_i) > 0$ and $e(z_i) = 1$ if $d(z_i) = 0$. The ordering is then given by the rules: $m \leq m'$ if and only if either $d(m) < d(m')$ or $(d(m) = d(m')$ and $e(m) < e(m'))$ or $(d(m) = d(m')$ and $e(m) = e(m')$ and $m \preceq_{\text{lex}} m')$. We shall refer to this as the *augmented d-length-lexicographic ordering.* It has DCC because for each $m \in M_n$ there are only finitely many monomials $< m$.

Let A be an algebra with a presentation of the form F_n/I, where I is an ideal of F_n, and let $x_i = z_i + I \in A$, $1 \leq i \leq n$. Let $d : M_n \to (\mathbb{N}_0, +)$ be a degree function. Set $A_0 = \mathbb{F}$, and, for $i \geq 1$, let A_i be the $\mathbb{F}$-subspace of A spanned by the images $m + I$ in A of words $m \in M_n$ with $d(m) \leq i$. Then $A_0 \subseteq A_1 \subseteq A_2 \ldots$ is a filtration of A. We shall call this the *d-standard filtration* of A.

Suppose now that I is the ideal generated by the elements $w_\sigma - f_\sigma$, for some reduction system $S = \{(w_\sigma, f_\sigma)\}$. Let $\leq$ be a semigroup ordering on M_n that has DCC and is compatible with S. We say that a degree function $d : M_n \to \mathbb{N}_0$ is *compatible* with S if, for each $(w_\sigma, f_\sigma) \in S$, f_σ is a linear combination of words m with $d(m) \leq d(w_\sigma)$. If $\leq$ is the augmented d-length-lexicographic ordering then compatibility of d with S is a consequence of compatibility of $\leq$ with S.

The following Proposition will be applicable to identify associated graded rings for the filtrations that we use.

Proposition 4.2. *Let A, M_n, F_n and S be as above. Let $\leq$ be a semigroup ordering on M_n, with DCC, that is compatible with S and let d be a degree function that is compatible with S. Denote by $\operatorname{Irr}(M_n)$ the set of images $m+I$ in A of those words m in M_n that are irreducible with respect to S. Suppose that all ambiguities in S are resolvable. Then, for the d-standard filtration, $\operatorname{gr}(A)$ has basis $\{\overline{m+I} : m \in \operatorname{Irr}(M_n)\}$ and there is a vector space isomorphism $\psi : A \to \operatorname{gr}(A)$ given by $\psi(m+I) = \overline{m+I}$ for all $m+I \in \operatorname{Irr}(M_n)$.*

Proof. By the Diamond Lemma, $\operatorname{Irr}(M_n)$ is a basis for A. Set $B_0 = \mathbb{F}$, and, for $i \geq 1$, let B_i be the $\mathbb{F}$-subspace spanned by the images $m+I$ in A of the irreducible words m with $d(m) \leq i$. Thus $B_i \subseteq A_i$ and B_i has basis $B_i \cap \operatorname{Irr}(M_n)$. We claim that $B_i = A_i$ for each i. Suppose not. By DCC, there exists a word $m \in M_n$ that is minimal, under $\leq$, with the property that $m+I \in A_{d(m)} \backslash B_{d(m)}$. Then m cannot be irreducible so there exists $(w_\sigma, f_\sigma) \in S$ such that $m = aw_\sigma b$ for some $a, b \in M_n$. Then $m = af_\sigma b$ is a linear combination of words w with $w < m$ and $d(w) \leq d(m)$, whence $m+I \in B_{d(m)}$, contradicting the minimality of m. Therefore, for all i, $A_i = B_i$, A_i has basis $A_i \cap \operatorname{Irr}(M_n)$ and, in $\operatorname{gr}(A)$, each summand A_i/A_{i-1} has a basis consisting of the elements $\overline{m+I}$ where $m+I \in (A_i \cap \operatorname{Irr}(M_n)) \backslash A_{i-1}$. Therefore $\{\overline{m+I} : m \in \operatorname{Irr}(M_n)\}$ is a basis of $\operatorname{gr}(A)$ and there is a vector space isomorphism $\psi : A \to \operatorname{gr}(A)$ given by $\psi(m+I) = \overline{m+I}$ for all $m+I \in \operatorname{Irr}(M_n)$. □

5. Invariants for the localized enveloping algebra and its homogenization

Having identified, in Section 2, generators for the rings of invariants of our principal examples of reversing automorphisms, we now aim to identify full sets of defining relations, beginning in this section with those specified in Example 2.4. We shall assume that $\operatorname{char} \mathbb{F} \neq 2$. The methods for $V(\mathfrak{g})$ and $V_t(\mathfrak{g})$ are similar and we begin with the latter. We know from Corollary 2.9 that $V_t(\mathfrak{g})^\theta$ is generated by t, y^2, $x + x^{-1}$ and $yx - yx^{-1}$.

Proposition 5.1. *Let $S = V_t(\mathfrak{g})$ and the reversing automorphism θ be as in Example 2.4(ii). Let T be the $\mathbb{F}$-algebra generated by t, x_1, x_3 and x_2 subject to the relations*

$$x_i t = t x_i \text{ for } i = 1,2,3, \quad (5.1)$$

$$x_1 x_2 = x_2 x_1 - 2t x_3 x_1 + 3t^2 x_2 + 2t^3 x_3, \quad (5.2)$$

$$x_2 x_3 = x_3 x_2 - t x_3^2 + 4t, \quad (5.3)$$

$$x_1 x_3 = x_3 x_1 - t^2 x_3 - 2t x_2. \quad (5.4)$$

(i) T is an iterated skew polynomial ring $\mathbb{F}[t, x_3][x_2; \delta][x_1; \sigma, \delta_1]$, where δ is a derivation of $\mathbb{F}[t, x_3]$, σ is an automorphism of $\mathbb{F}[t, x_3][x_2; \delta]$ and δ_1 is a σ-derivation of $\mathbb{F}[t, x_3][x_2; \delta]$.

(ii) Let $g = (4 - x_3^2)x_1 + x_2^2 + 3t x_3 x_2 + t^2 x_3^2 + 4t^2$. Then g is a central element of T and T/gT is a domain.

(iii) S^θ is isomorphic to T/gT.

Proof. (i) Let R_1 be the commutative polynomial ring $\mathbb{F}[t, x_3]$ and let δ be the $\mathbb{F}$-derivation $t(4 - x_3^2)\partial/\partial x_3$ of R_1. Let $R_2 = R_1[x_2; \delta]$ so that (5.3) is satisfied. Let F be the free algebra $\mathbb{F}\langle u, z_2, z_3\rangle$ and I be the ideal of F generated by $g :=$

$uz_2 - z_2u, h := uz_3 - z_3u$ and $f := z_2z_3 - z_3z_2 + uz_3^2 - 4u$. Then, by [6, Proposition 1], there is an isomorphism $\phi : R_2 \to F/I$ with $\phi(t) = \overline{u}, \phi(x_2) = \overline{z}_2$ and $\phi(x_3) = \overline{z}_3$.

There exists $\tau \in \mathrm{Aut}_{\mathbb{F}}(F)$ such that $\tau(u) = u$, $\tau(z_3) = z_3$ and $\tau(z_2) = z_2 - 2uz_3$, its inverse being such that $\tau^{-1}(u) = u$, $\tau^{-1}(z_3) = z_3$ and $\tau^{-1}(z_2) = z_2 + 2uz_3$. It is readily checked that $\tau(g) = g - 2uh$, $\tau(h) = h$ and $\tau(f) = f - 2hz_3$, whence $\tau(I) = I$ and there is an induced $\mathbb{F}$-automorphism σ of R_2 such that $\sigma(t) = t$, $\sigma(x_3) = x_3$ and $\sigma(x_2) = x_2 - 2tx_3$.

A left τ-derivation ∂ of F is determined by specifying $\partial(u)$, $\partial(z_3)$ and $\partial(z_2)$ and using the definition of a left τ-derivation [10, p. 33] to extend to arbitrary elements of F. Here we set $\partial(u) = 0$, $\partial(z_3) = -u^2z_3 - 2uz_2$ and $\partial(z_2) = 2u^3z_3 + 3u^2z_2$. Modulo I, u commutes with z_3 and z_2 so we find that

$$\begin{aligned}\partial(z_2z_3) &\equiv 2u^2z_2z_3 + 4u^2z_3z_2 + 4u^3z_3^2 - 2uz_2^2 \bmod I,\\ \partial(z_3z_2) &\equiv 2u^2z_3z_2 + 2u^3z_3^2 - 2uz_2^2 \bmod I \text{ and}\\ \partial(uz_3^2) &\equiv -2u^2z_2z_3 - 2u^2z_3z_2 - 2u^3z_3^2 \bmod I,\end{aligned}$$

whence $\partial(f) \in I$. Also $\partial(g) \in I$ and $\partial(h) \in I$. Thus $\partial(I) \subseteq I$ so there is an induced σ-derivation δ_1 of R_2 such that $\delta_1(t) = 0$, $\delta_1(x_3) = -t^2x_3 - 2tx_2$ and $\delta_1(x_2) = 2t^3x_3 + 3t^2x_2$. By [6, Proposition 1], $R_2[x_1; \sigma, \delta_1]$ is the $\mathbb{F}$-algebra generated by t, x_1, x_2 and x_3 subject to the relations (5.1), (5.2), (5.3) and (5.4).

(ii) Let ad_1 (*resp* ad_3) be the inner derivation of T given by $a \mapsto x_1a - ax_1$ (*resp* $a \mapsto x_3a - ax_3$). Then, using (5.4), (5.2) and (5.3),

$$\begin{aligned}\mathrm{ad}_1(g) &= \mathrm{ad}_1(x_2^2) + \mathrm{ad}_1(x_3^2)(t^2 - x_1) + 3t\,\mathrm{ad}_1(x_3x_2)\\ &= -4tx_3x_2x_1 + 6t^2x_3^2x_1 - 8t^2x_1 + 6t^2x_2^2 - 2t^3x_3x_2 - 6t^4x_3^2 + 8t^4\\ &\quad - 4t^3x_3x_2 - 8t^4 + 4tx_3x_2x_1 + 8t^2x_1 - 6t^2x_3^2x_1 + 6t^3x_3x_2 + 6t^4x_3^2 - 6t^2x_2^2\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\mathrm{ad}_3(g) &= \mathrm{ad}_3(x_2^2) + (4 - x_3^2)\,\mathrm{ad}_3(x_1) + 3x_3\,\mathrm{ad}_3(x_2)\\ &= 2tx_3^2x_2 - 2t^2x_3^3 - 8tx_2 + 8t^2x_3 + 4t^2x_3 + 8tx_2\\ &\quad - t^2x_3^3 - 2tx_3^2x_2 + 3t^2x_3^3 - 12t^2x_3\\ &= 0.\end{aligned}$$

Thus $gx_1 = x_1g$ and $gx_3 = x_3g$. By (5.4) and (5.1), g commutes with tx_2. As t is central and T is a domain, g commutes with x_2. Therefore g is central.

Let $d = 4 - x_3^2$ and $e = x_2^2 + t^2x_3^2 + 3tx_3x_2 + 4t^2$. Applying [16, Proposition 1] to the central element $g = dx_1 + e$ in the ring $T = R_2[x_1; \sigma, \delta_1]$, we see that dR_2 is an ideal of R_2 and that, provided e is regular modulo dR_2, T/gT is a domain. Now R_2/dR_2 is commutative and may be identified with $C/(4 - x_3^2)C$ where C is the commutative polynomial ring $\mathbb{F}[t, x_2, x_3]$. Thus R_2/dR_2 has two minimal primes P_1 and P_2, generated by the images of $2 - x_3$ and $2 + x_3$, and intersecting in 0. The set of zero-divisors in R_2/dR_2 is $P_1 \cup P_2$ and $e + dR_2 \notin P_1 \cup P_2$. Hence T/gT is a domain and gT is a (completely) prime ideal of T.

(iii) We have seen in Corollary 2.9 that, in the notation of Lemma 2.6, S^θ is generated by t, $a_1 = y^2 = s_0(\frac{1}{2}y^2)$, $a_2 = yx - yx^{-1} = s_1(y)$ and $a_3 = x + x^{-1} = s_1(1)$. We now check that (5.1), (5.2), (5.3) and (5.4) hold when x_1, x_2 and x_3 are replaced by a_1, a_2 and a_3. This is certainly true for (5.1), t being central in S.

Note that $s_0(y^i) = 0$ if i is odd, that $s_0(1) = 2$ and that $s_0(\alpha^{-1}(\frac{1}{2}y^2)) = a_1 + t^2$. By (2.2),

$$a_1a_3 - a_3(a_1 + t^2) = s_1(-2ty - 2t^2) = -2ta_2 - 2t^2a_3,$$

whence a_1, a_2 and a_3 satisfy (5.4), and, by (2.3),

$$a_2a_3 = s_1(1)s_1(y - t) + s_0(2t) = a_3a_2 - ta_3^2 + 4t,$$

whence a_1, a_2 and a_3 satisfy (5.3). For (5.2), note that, by (2.1),

$$a_1a_2 = a_2(a_1 + t^2) + s_1(-2ty^2 - 2t^2y) = a_2a_1 + t^2a_2 + s_1(-2ty^2) - 2t^2a_2. \quad (5.5)$$

Let $r = -\frac{1}{3}y^3 - ty^2 - \frac{2}{3}t^2y$. Then $s_0(r) = s_0(-ty^2) = -2ta_1$, $s_0(\alpha^{-1}(r)) = s_0(-\frac{1}{3}y^3 + \frac{1}{3}t^2y) = 0$ and $\gamma(r) - \alpha^2\gamma(r) = -2t^2y$. By (2.2) and (5.4) for a_1, a_2, a_3,

$$s_1(-2ty^2) = -2ta_1a_3 = -2ta_3a_1 + 2t^3a_3 + 4t^2a_2.$$

Combining this with (5.5) shows that a_1, a_2 and a_3 satisfy (5.2).

Thus (5.1), (5.2), (5.3) and (5.4) hold when x_1, x_2 and x_3 are replaced by a_1, a_2 and a_3 so there is a surjective ring homomorphism $\eta : T \to S^\theta$ such that $\eta(t) = t, \eta(x_1) = a_1, \eta(x_2) = a_2$ and $\eta(x_3) = a_3$. It remains to show that $\ker\eta = gT$.

A simple calculation, using (5.4), (5.2) and (5.3), shows that

$$a_1a_3^2 = a_3^2a_1 - 4ta_3a_2 - 8t^2. \quad (5.6)$$

In S^θ,

$$\begin{aligned} a_2^2 &= (yx - yx^{-1})^2 \\ &= y^2(x^2 - 2 + x^{-2}) + yt(x^2 - x^{-2}) \\ &= a_1(a_3^2 - 4) + ta_2a_3 \\ &= (a_3^2 - 4)a_1 - 4ta_3a_2 - 8t^2 + ta_3a_2 - t^2a_3^2 + 4t^2 \text{ (by (5.6) and (5.3))} \\ &= (a_3^2 - 4)a_1 - 3ta_3a_2 - 4t^2 - t^2a_3^2. \end{aligned}$$

Therefore $g \in \ker\eta$.

To show that $gT = \ker\eta$, we shall use Gelfand-Kirillov dimension and a filtration of T. Let F be the free algebra F_4 and M be the free monoid M_4. It will be convenient to write z_4 as u. Let $\psi : F \to T$ be the surjective homomorphism such that $\psi(z_i) = x_i$, $1 \le i \le 3$, and $\psi(u) = t$. Thus we may identify T and $F/\ker\psi$. Let d be the degree function such that

$$d(z_1) = 6,\ d(z_2) = 4,\ d(z_3) = 2 \text{ and } d(u) = 1. \quad (5.7)$$

Consider the d-standard filtration on T and note that, as T has been presented as an iterated skew polynomial ring $\mathbb{F}[t, x_3][x_2; \delta][x_1; \sigma, \delta_1]$, with σ an automorphism, it has a basis $\{t^i x_3^j x_2^k x_1^\ell\}$. It follows from this that, in presentation of T given in (i), and with the d-length lexicographic ordering, all ambiguities are resolvable. Alternatively, this may be checked directly. Computing degrees of the monomials appearing in (5.2), (5.3), and (5.4) and applying Proposition 4.2, we see that $\mathrm{gr}(T)$ is a commutative polynomial ring in four variables so, by [19, Proposition 6.6], $\mathrm{GKdim}(T) = 4$. On the other hand, it follows from [19, Proposition 3.5] that $\mathrm{GKdim}(S) = \mathrm{GKdim}(\mathbb{F}[x^{\pm 1}, t][y; -tx\partial/\partial x]) = 3$. By [24, Corollary 26.13(ii)], S is finitely generated as a right module over S^θ, so, by [21, Proposition 8.2.9], $\mathrm{GKdim}(S^\theta) = 3$.

As S is a domain, so too is its subalgebra S^θ. Therefore $\ker\eta$ is a prime ideal P, say, of T, such that $T/P \simeq S^\theta$. By [19, Corollary 3.16],

$$4 = \mathrm{GKdim}(T) \geq \mathrm{GKdim}(T/P) + \mathrm{ht}(P) = \mathrm{GKdim}(S^\theta) + \mathrm{ht}(P) = 3 + \mathrm{ht}(P).$$

Hence $\mathrm{ht}(P) \leq 1$. As T and T/gT are domains, by (i) and (ii), and as $0 \neq g \in P$, it must be the case that $gT = P = \ker\eta$. Therefore $S^\theta \simeq T/gT$. □

Remark 5.2. Having identified invariants for the quantization θ of π, we proceed to consider the deformations. All the deformations $S/(t-\lambda)S$, $\lambda \in \mathbb{F}\backslash\{0\}$, are isomorphic and the same is true of $T/(t-\lambda)T$. So we shall only consider the case $\lambda = 1$.

Proposition 5.3. *Let $S_1 = V(\mathfrak{g})$ and let the reversing automorphism θ_1 be as in Example 2.4(i). Suppose that* $\mathrm{char}\,\mathbb{F} \neq 2$. *Let T_1 be the $\mathbb{F}$-algebra generated by x_1, x_2 and x_3 subject to the relations*

$$x_1x_3 = x_3x_1 - x_3 - 2x_2, \tag{5.8}$$

$$x_2x_3 = x_3x_2 - x_3^2 + 4, \tag{5.9}$$

$$x_1x_2 = x_2x_1 - 2x_3x_1 + 3x_2 + 2x_3. \tag{5.10}$$

(i) T_1 is an iterated skew polynomial ring $\mathbb{F}[x_3][x_2;\delta][x_1;\sigma,\delta_1]$, where δ is a derivation of $\mathbb{F}[x_3]$, σ is an automorphism of $\mathbb{F}[x_3][x_2;\delta]$ and δ_1 is a σ-derivation of $\mathbb{F}[x_3][x_2;\delta]$.

(ii) Let $p = (4-x_3^2)x_1 + x_2^2 + 3x_3x_2 + x_3^2 + 4$. Then p is a central element of T_1 and T_1/pT_1 is a domain.

(iii) $S_1^{\theta_1}$ is isomorphic to T_1/pT_1.

Proof. This can be proved by the same methods as Proposition 5.1. The details are somewhat simpler, with the indeterminates t and u being replaced by 1. Alternatively, it may be deduced from Proposition 5.1 using standard skew polynomial ring results and Corollary 2.10. □

Remark 5.4. It is a routine matter to check that, with A as in 3.4 and $f = x_1(4-x_3^2) + x_2^2$ as in Example 3.2, T and T/gT are quantizations of A and A/fA respectively, that T_1 and T_1/pT_1 are deformations of A and A/fA respectively and that the situation is as illustrated in Figure 1.

Remark 5.5. In the proof of Proposition 5.1, use was made of a filtration of T for which $\mathrm{gr}(T)$ is commutative. There are other filtrations for which $\mathrm{gr}(T)$ is a noncommutative iterated skew polynomial ring and a quantization of another exact Poisson bracket. If we take the degree function d on M_4 such that

$$d(z_1) = 2,\ d(z_2) = 3,\ d(z_3) = 2 \text{ and } d(u) = 1,$$

then (5.1), (5.2) and (5.4) become homogeneous while, in (5.3), only the term $4t$ has degree less than 5. For the d-standard filtration of T, $\mathrm{gr}(T)$ has, like T, the form $C = \mathbb{F}[t, x_3][x_2;\delta][x_1;\sigma,\delta_1]$ but with $\delta = -tx_3^2\partial/\partial x_3$ and with the central element $g = -x_3^2x_1 + x_2^2 + 3tx_3x_2 + t^2x_3^2$. Methods similar to those used in the study of T confirm the existence of such a skew polynomial ring C and that $\mathrm{gr}(T) \simeq C$. Setting $f = x_2^2 - x_3^2x_1 \in A = \mathbb{F}[x_1, x_2, x_3]$, the algebras $\mathrm{gr}(T)$ and $\mathrm{gr}(T/gT)$ are respectively quantizations of A, for the exact Poisson bracket $\{-,-\}_f$ and the coordinate ring A/fA of the Whitney umbrella. Alternatively, if we take $d(z_1) = 3, d(z_2) = 4, d(z_3) = 2$ and $d(u) = 1$ and set $h = x_2^2$ then $\mathrm{gr}(T)$ is a quantization of A with the exact Poisson bracket $\{-,-\}_h$ and is isomorphic to the enveloping algebra of the three-dimensional Heisenberg Lie algebra.

Remark 5.6. The $\mathbb{F}$-algebra T in Proposition 5.3 is an example of an algebra determined by a noncommutative potential. In the notation of [9], it is $\mathfrak{U}(F,\Phi)$ where $F = \mathbb{F}\langle x_1, x_2, x_3\rangle$ and

$$\Phi = x_1x_2x_3 - x_3x_2x_1 + x_1x_3^2 - x_2x_3 - x_2^2 - \tfrac{1}{2}x_3^2 - 4x_1,$$

so that

$$\frac{\partial\Phi}{\partial x_2} = x_3x_1 - x_1x_3 - x_3 - 2x_2,$$
$$\frac{\partial\Phi}{\partial x_1} = x_2x_3 - x_3x_2 + x_3^2 - 4 \text{ and}$$
$$\frac{\partial\Phi}{\partial x_3} = x_1x_2 - x_2x_1 + x_3x_1 + x_1x_3 - x_2 - x_3.$$

6. Invariants for the quantum torus

The aim of this section is to complete Figure 1 for the quantum torus $S_q = W_q$ and the generic quantum torus $S = W_Q$ and their reversing automorphisms, θ_q and θ, specified in Example 2.5. We shall assume that q is not a root of unity and, in order to apply standard results on fixed rings, that $\operatorname{char}\mathbb{F} \neq 2 = |\langle\theta\rangle|$.

The situation is more complex than in the previous section due to the invertibility of Q and the lack of any apparent iterated skew polynomial ring structure for the quantization T or the deformations, which in this case are parametrized by $q \in \mathbb{F}\backslash\{0\}$ and will be written T_q rather than T_{q-1}. The ring T_q has been of interest in mathematical physics [8, 11, 12] and in the work of Terwilliger [28] and others on Leonard pairs and Askey-Wilson relations.

Proposition 6.1. *Let T be the $\mathbb{F}$-algebra generated by Q, Q^{-1}, x_1, x_2 and x_3 subject to the relations*

$$x_iQ = Qx_i,\ x_iQ^{-1} = Q^{-1}x_i \text{ for } i = 1,2,3,\ QQ^{-1} = Q^{-1}Q = 1, \tag{6.1}$$
$$x_1x_2 = Qx_2x_1 + (1-Q^2)x_3, \tag{6.2}$$
$$x_2x_3 = Qx_3x_2 + (Q^{-1}-Q)x_1, \tag{6.3}$$
$$x_1x_3 = Q^{-1}x_3x_1 + (1-Q^{-2})x_2. \tag{6.4}$$

The algebra T has a partially localized PBW basis $\{Q^ix_3^jx_2^kx_1^\ell : i \in \mathbb{Z}, j,k,\ell \in \mathbb{N}_0\}$.

Proof. (i) Let F be the free algebra F_5 and M be the free monoid M_5. It will be convenient to write u for z_4 and v for z_5. Let $\psi : F \to T$ be the surjective homomorphism such that $\psi(z_i) = x_i$, $1 \le i \le 3$, $\psi(u) = Q$ and $\psi(v) = Q^{-1}$. Let $d_1 : M_5 \to \mathbb{N}_0$ be the degree function such that

$$d_1(u) = d_1(v) = d_1(z_1) = 0 \text{ and } d_1(z_2) = d_1(z_3) = 1.$$

We shall apply the Diamond Lemma with the augmented d_1-length-lexicographic ordering, as defined in Definitions 4.1. In the d_1-standard filtration of T, the largest monomials appearing in (6.2), (6.3) and (6.4) are x_1x_2, x_2x_3 and x_1x_3 respectively. Overlap ambiguities involving the relations in (6.1) are easily resolved. The only other ambiguity is the overlap ambiguity $(x_1x_2)x_3 = x_1(x_2x_3)$. Reducing $(x_1x_2)x_3$

by applying (6.2) followed by (6.4) and (6.3), together with several applications of (6.1), one obtains

$$Qx_3x_2x_1 + (Q^{-1} - Q)x_1^2 + (Q - Q^{-1})x_2^2 + (1 - Q^2)x_3^2.$$

The same result is obtained by reducing $x_1(x_2x_3)$ using (6.3), (6.1), (6.4) and (6.2). It follows, by the Diamond Lemma, that T has the stated basis. □

Although the degree function d_1 used above will be helpful in showing that T/gT is a domain, for a central element g to be specified in Proposition 6.4, we shall also make use of the degree function $d_2 : M_5 \to \mathbb{N}_0$ for which

$$d_2(u) = d_2(v) = 0 \text{ and } d_2(z_1) = d_2(z_2) = d_2(z_3) = 1.$$

This has the advantage that, after passage, via localization at $\mathbb{F}[Q^{\pm 1}]\backslash\{0\}$, to a filtered algebra over $\mathbb{F}(Q)$, the d_2-standard-filtration becomes finite. The following Lemma identifies the associated graded rings for the d_i-standard-filtrations, $i = 1, 2$.

Lemma 6.2. *(i) There exist* $\sigma \in \mathrm{Aut}_{\mathbb{F}}(\mathbb{F}[Q^{\pm 1}, y_3])$, $\sigma_1 \in \mathrm{Aut}_{\mathbb{F}}(\mathbb{F}[Q^{\pm 1}, y_3][y_2; \sigma])$ *and a* σ_1*-derivation* δ *of* $\mathbb{F}[Q^{\pm 1}, y_3][y_2; \sigma]$ *such that, for the* d_1*-standard filtration of* T, $\mathrm{gr}(T)$ *is an iterated skew polynomial ring* $\mathbb{F}[Q^{\pm 1}, y_3][y_2; \sigma][y_1; \sigma_1, \delta]$. *The algebra* $\mathrm{gr}(T)$ *is generated by* $Q^{\pm 1}, y_1, y_2$ *and* y_3 *subject to the relations:*

$$y_1Q = Qy_1, \quad y_2Q = Qy_2, \quad y_3Q = Qy_3, \quad QQ^{-1} = 1 = Q^{-1}Q,$$
$$y_1y_2 = Qy_2y_1 + (1 - Q^2)y_3, \quad y_2y_3 = Qy_3y_2, \quad y_1y_3 = Q^{-1}y_3y_1 + (1 - Q^{-2})y_2.$$

(ii) For the d_2*-standard filtration of* T, $\mathrm{gr}(T)$ *is an iterated skew polynomial ring* $\mathbb{F}[Q^{\pm 1}, y_3][y_2; \sigma][y_1; \sigma_1]$, *where* σ *and* σ_1 *are as in (i). The algebra* $\mathrm{gr}(T)$ *is generated by* $Q^{\pm 1}, y_1, y_2$ *and* y_3 *subject to the relations:*

$$y_1Q = Qy_1, \quad y_2Q = Qy_2, \quad y_3Q = Qy_3, \quad QQ^{-1} = 1 = Q^{-1}Q,$$
$$y_1y_2 = Qy_2y_1, \quad y_2y_3 = Qy_3y_2, \quad y_1y_3 = Q^{-1}y_3y_1.$$

Proof. (i) Let $\sigma \in \mathrm{Aut}_{\mathbb{F}}(\mathbb{F}[Q^{\pm 1}, y_3])$ be such that $\sigma(y_3) = Qy_3$ and $\sigma(Q) = Q$. A method similar to that used in the proof of Proposition 5.1 shows that there exist $\sigma_1 \in \mathrm{Aut}_{\mathbb{F}}(\mathbb{F}[Q^{\pm 1}, y_3][y_2; \sigma])$ and a σ_1-derivation δ of $\mathbb{F}[Q^{\pm 1}, y_3][y_2; \sigma]$ such that $\sigma_1(Q) = Q$, $\sigma_1(y_2) = Qy_2$, $\sigma_1(y_3) = Q^{-1}y_3$, $\delta(Q) = 0$, $\delta(y_2) = (1 - Q^2)y_3$ and $\delta(y_3) = (1 - Q^{-2})y_2$. Let $B = \mathbb{F}[Q^{\pm 1}, y_3][y_2; \sigma][y_1, \sigma_1, \delta]$. Then B is generated by $Q^{\pm 1}, y_1, y_2$ and y_3 subject to the stated relations and has basis $\{Q^i y_3^j y_2^k y_1^\ell : i \in \mathbb{Z}, j, k, \ell \in \mathbb{N}_0\}$.

The degree function d_1 is compatible with the reduction scheme represented by the defining relations for T, as shown in Proposition 6.1. The defining relations for B are satisfied in $\mathrm{gr}(T)$, with each y_i replaced by $\overline{x_i}$. It follows from [6, Proposition 1] that there is a surjection $\phi : B \to \mathrm{gr}(T)$ such that $\phi(y_i) = \overline{x_i}$, $i = 1, 2, 3$, and $\phi(Q) = Q$. (As $Q \in T_0$, we write Q and Q^{-1} rather than $\overline{Q}$ and $\overline{Q^{-1}}$ in $\mathrm{gr}(T)$.) By Proposition 6.1 and Proposition 4.2, with the augmented d_1-length-lexicographic ordering, $\mathrm{gr}(T)$ has basis $\{Q^i \overline{x_3}^j \overline{x_2}^k \overline{x_1}^\ell : i \in \mathbb{Z}, j, k, \ell \in \mathbb{N}_0\}$ so ϕ is an isomorphism.

(ii) The proof is similar to that of (i), but simpler, with the σ_1-derivation δ replaced by 0. □

Corollary 6.3. *The algebra* T *is a domain.*

Proof. This follows from [21, Proposition 1.6.6(i)] and either part of Lemma 6.2. □

Proposition 6.4. *In T, let $g = x_3x_2x_1 - Qx_3^2 - Q^{-2}x_2^2 - x_1^2 + 2(1+Q^{-2})$. Then g is a central element of T and $T/(g-\kappa)T$ is a domain for all $\kappa \in \mathbb{F}$.*

Proof. Using the defining relations, it can be checked that

$$x_1x_2^2 = Q^2x_2^2x_1 + (1-Q^4)x_3x_2 + (Q^2-1)^2x_1, \quad (6.5)$$

$$x_1x_3^2 = Q^{-2}x_3^2x_1 + (Q-Q^{-3})x_3x_2 - Q^{-1}(Q-Q^{-1})^2x_1, \quad (6.6)$$

$$x_2^2x_3 = Q^2x_3x_2^2 + (Q^{-1}-Q^3)x_2x_1 + (Q^2-1)^2x_3, \quad (6.7)$$

$$x_1^2x_3 = Q^{-2}x_3x_1^2 + (Q-Q^{-3})x_2x_1 - (Q-Q^{-1})^2x_3, \quad (6.8)$$

$$x_2x_1x_3 = x_3x_2x_1 + (Q^{-2}-1)x_1^2 + (1-Q^{-2})x_2^2 \text{ and} \quad (6.9)$$

$$x_1x_3x_2 = x_3x_2x_1 + (Q^{-1}-Q)x_3^2 + (1-Q^{-2})x_2^2. \quad (6.10)$$

It follows routinely that $x_1g = gx_1$ and $x_3g = gx_3$. By (6.4), g commutes with $(1-Q^{-2})x_2$ so, as Q is central and T is a domain, g is central.

Let $h = g - \kappa$. We filter T/hT using the d_1-standard filtration, where d_1 is as in the proof of Proposition 6.1, and, for application of the Diamond Lemma, we use the augmented d_1-length-lexicographic ordering. We write u_i for $x_i + hT$, $1 \le i \le 3$, and, with a mild abuse of notation, Q for $Q + hT$. Thus the defining relations for T/hT, each written with the largest term isolated on the left hand side, are:

$$u_iQ = Qu_i,\ u_iQ^{-1} = Q^{-1}u_i,\ i = 1,2,3,\ QQ^{-1} = 1 = Q^{-1}Q \quad (6.11)$$

$$u_1u_2 = Qu_2u_1 + (1-Q^2)u_3, \quad (6.12)$$

$$u_2u_3 = Qu_3u_2 + (Q^{-1}-Q)u_1, \quad (6.13)$$

$$u_1u_3 = Q^{-1}u_3u_1 + (1-Q^{-2})u_2, \quad (6.14)$$

$$u_2^2 = Q^2u_3u_2u_1 - Q^3u_3^2 - Q^2u_1^2 + 2(Q^2+1) - \kappa Q^2. \quad (6.15)$$

There are no inclusion ambiguities and the only overlap ambiguities, apart from those that involve (6.11), are

$$(u_1u_2)u_3 = u_1(u_2u_3),\ (u_1u_2)u_2 = u_1(u_2^2) \text{and } u_2(u_2u_3) = (u_2^2)u_3.$$

The first is resolved as in the proof of Proposition 6.1 but two formulae obtained during this calculation, namely

$$u_2u_1u_3 = Q^2u_3u_2u_1 + (Q^{-2}-Q^2)u_1^2 + (Q-Q^3)u_3^2 + 2(Q^2-Q^{-2}) + \kappa(1-Q^2) \quad (6.16)$$

and

$$u_1u_3u_2 = Q^2u_3u_2u_1 + (Q^{-1}-Q^3)u_3^2 + (1-Q^2)u_1^2 + 2(Q^2-Q^{-2}) + \kappa(1-Q^2) \quad (6.17)$$

are used in resolving the other two ambiguities. We reduce $(u_1u_2)u_2$ and $u_1(u_2^2)$, beginning the former by applying (6.12) and the latter by applying (6.15), and making use of (6.12), (6.13), (6.14), (6.16) and (6.17), to obtain the result

$$Q^4u_3u_2u_1^2 - Q^5u_3^2u_1 - Q^4u_1^3 + (1-Q^4)u_3u_2 + ((3-\kappa)Q^4+1)u_1$$

in both cases. Similarly, $u_2(u_2u_3)$ and $(u_2^2)u_3$ both reduce to

$$Q^4u_3^2u_2u_1 - Q^4u_3u_1^2 - Q^5u_3^3 + (Q^{-1}-Q^3)u_2u_1 + ((3-\kappa)Q^4+1)u_3.$$

By the Diamond Lemma, $\{Q^iu_3^ju_2^ku_1^\ell : i \in \mathbb{Z}, j,k,\ell \ge 0, k < 2\}$ is a basis for T/hT.

We claim that $\mathrm{gr}(T/hT) \simeq \mathrm{gr}(T)/\overline{h}\,\mathrm{gr}(T)$. For $i = 1, 2, 3$, we write $v_i = \overline{u_i} \in \mathrm{gr}(T/hT)$ (and we write Q for $\overline{Q}$). By Proposition 4.2, $\{Q^i v_3^j v_2^k v_1^\ell : i \in \mathbb{Z}, j, k, \ell \geq 0, k < 2\}$ is a basis for $\mathrm{gr}(T/hT)$ whose generators $Q^{\pm 1}, v_1, v_2$ and v_3 satisfy the following relations:

$$\begin{aligned} v_iQ &= Qv_i,\ v_iQ^{-1} = Q^{-1}v_i,\ i = 1, 2, 3,\ QQ^{-1} = 1 = Q^{-1}Q, \\ v_1v_2 &= Qv_2v_1 + (1 - Q^2)v_3, \\ v_2v_3 &= Qv_3v_2, \\ v_1v_3 &= Q^{-1}v_3v_1 + (1 - Q^{-2})v_2, \\ v_2^2 &= Q^2v_3v_2v_1 - Q^3v_3^2. \end{aligned}$$

The same relations, with each v_i replaced by $w_i := y_i + \overline{h}\,\mathrm{gr}(T) \in \mathrm{gr}(T)/\overline{h}\,\mathrm{gr}(T)$, are defining relations for $\mathrm{gr}(T)/\overline{h}\,\mathrm{gr}(T)$ so there is a surjection $\phi : \mathrm{gr}(T)/\overline{h}\,\mathrm{gr}(T) \to \mathrm{gr}(T/hT)$ such that $\phi(Q) = Q$ and $\phi(w_i) = v_i$, $i = 1, 2, 3$. Effectively the same Diamond Lemma calculations as for T/hT, but with some low degree terms deleted, show that $\mathrm{gr}(T)/\overline{h}\,\mathrm{gr}(T)$ has basis $\{Q^i w_3^j w_2^k w_1^\ell : i \in \mathbb{Z}, j, k, \ell \geq 0, k < 2\}$. It follows that ϕ is an isomorphism.

By [21, Proposition 1.6.6(i)], it now suffices to show that $\mathrm{gr}(T)/\overline{h}\,\mathrm{gr}(T)$ is a domain. For this, recall, from Lemma 6.2(i), that $\mathrm{gr}(T) = \mathbb{F}[Q^{\pm 1}, y_3][y_2; \sigma][y_1; \sigma_1, \delta]$. We apply [16, Proposition 1] to the central element $\overline{h} = y_3y_2y_1 - Qy_3^2 - Q^{-2}y_2^2$ of degree 1 in y_1. Let $D := \mathbb{F}[Q^{\pm 1}, y_3][y_2; \sigma]$, $d = y_3y_2$ and $e = -Qy_3^2 - Q^{-2}y_2^2$. Here $\sigma(y_3) = Qy_3$ and $\sigma(Q) = Q$. To conclude that $\mathrm{gr}(T)/\overline{h}\,\mathrm{gr}(T)$ is a domain, we need to check that e is regular modulo the ideal Dd. In D, Dd is the intersection of two height one primes Dy_3 and Dy_2 which are completely prime. Hence all zero-divisors modulo Dd are in $Dy_3 \cup Dy_2$ and so e is regular modulo Dd. This completes the proof that T/hT is a domain. □

Proposition 6.5. *Let $S = W_Q$ and the reversing automorphism θ be as in Example 2.5(ii). Then S^θ is isomorphic to T/gT.*

Proof. In the notation of Lemma 2.6, let $a_1 = s_0(y) = s_0(y^{-1}) = y + y^{-1}$, let $a_2 = s_1(1) = x + x^{-1}$ and let $a_3 = s_1(y) = yx + y^{-1}x^{-1}$. Note that $s_0(\alpha^{-1}(y^{-1})) = s_0(Qy^{-1}) = Qa_1$ and $\gamma(y^{-1}) - \alpha^2\gamma(y^{-1}) = (1 - Q^2)y$. By (2.2) with $r = y^{-1}$,

$$a_1a_2 - Qa_2a_1 = (1 - Q^2)a_3.$$

By (2.3), $a_3a_2 - Q^{-1}a_2a_3 = (1 - Q^{-2})a_1$ so

$$a_2a_3 - Qa_3a_2 = (Q^{-1} - Q)a_1.$$

Applying (2.1) with $r = r' = y$, we obtain $a_1a_3 - Q^{-1}a_3a_1 = s_1((y^{-1} - Q^{-2}y^{-1})y) = (1 - Q^{-2})a_2$, whence

$$a_3a_1 - Qa_1a_3 = (Q^{-1} - Q)a_2.$$

By Corollary 2.12, a_1, a_2, a_3, Q and Q^{-1} generate S^θ so there is a surjective $\mathbb{F}$-homomorphism $\eta : T \to S^\theta$, such that $\eta(x_i) = a_i$, $i = 1, 2, 3$, and $\eta(Q) = Q$. Therefore $S^\theta \simeq T/\ker\eta$.

Note that $a_1^2 = y^2 + 2 + y^{-2}$, $a_2^2 = x^2 + 2 + x^{-2}$, $a_3^2 = Qy^2x^2 + 2Q^{-1} + Qy^{-2}x^{-2}$ and

$$\begin{aligned} a_3a_2a_1 &= Q^2y^2x^2 + Q^2y^{-2}x^{-2} + Q^{-2}x^2 + Q^{-2}x^{-2} + y^2 + y^{-2} + 2 \\ &= Qa_3^2 + Q^{-2}a_2^2 + a_1^2 - 2(1 + Q^{-2}). \end{aligned}$$

Thus $g \in \ker \eta$.

As in the proof of Proposition 5.1, we use GK-dimension to show that $\ker \eta = gT$. However it will be convenient to work over the rational function field $\mathbb{K} := \mathbb{F}(Q)$ rather than over $\mathbb{F}$. To this end, let $\mathcal{C}$ denote the central multiplicatively closed set $\mathbb{F}[Q^{\pm 1}]\backslash\{0\}$. It follows from Proposition 6.1 that the localization $\hat{T}$ of T at $\mathcal{C}$ is a $\mathbb{K}$-algebra with a PBW basis $\{x_3^i x_2^j x_1^k\}$. As $d_2(Q) = 0$, the filtration in Lemma 6.2(ii) extends to a filtration of $\hat{T}$ as a $\mathbb{K}$-algebra and, as each $d_2(x_i) > 0$, this filtration is finite. The associated graded $\mathbb{K}$-algebra $\mathrm{gr}(\hat{T})$ is the quantum coordinate ring of $\mathbb{K}^3$ generated by y_1, y_2, y_3 subject to the relations

$$y_1 y_2 = Q y_2 y_1, \quad y_2 y_3 = Q y_3 y_2, \quad y_3 y_1 = Q y_1 y_3.$$

If V is the generating subspace $\mathbb{K}y_1 + \mathbb{K}y_2 + \mathbb{K}y_3$ then $\dim_{\mathbb{K}} V^n$ is the same as for the commutative polynomial ring $\mathbb{K}[y_1, y_2, y_3]$ so $\mathrm{GKdim}_{\mathbb{K}}\, \mathrm{gr}(\hat{T}) = 3$ and, by [19, Proposition 6.6] or [21, Proposition 8.6.5], $\mathrm{GKdim}_{\mathbb{K}}(\hat{T}) = 3$. Let $\hat{S}$ denote the localization of S at $\mathcal{C}$, so that, as $\theta(\mathcal{C}) = \mathcal{C}$, θ extends to $\hat{S}$ in the obvious way and the surjective homomorphism $\eta : T \to S^\theta$ extends to a surjective homomorphism $\hat{\eta} : \hat{T} \to \hat{S}^\theta$.

Now $\mathrm{GKdim}_{\mathbb{K}}(\hat{S}^\theta) = \mathrm{GKdim}(\hat{S}) = 2$, by [24, Corollary 26.13(ii)] and [21, Proposition 8.2.9]. As $\hat{S}^\theta$ is a domain, $\ker \hat{\eta}$ is a prime ideal P, say, of $\hat{T}$ and, by [19, Corollary 3.16],

$$3 = \mathrm{GKdim}_{\mathbb{K}}(\hat{T}) \geq \mathrm{GKdim}_{\mathbb{K}}(\hat{T}/P) + \mathrm{ht}(P) = \mathrm{GKdim}_{\mathbb{K}}(\hat{S}^\theta) + \mathrm{ht}(P) = 2 + \mathrm{ht}(P).$$

Hence $\mathrm{ht}(P) \leq 1$. As $P \neq 0$ and $\hat{T}$ is a domain, $\mathrm{ht}(P) = 1$. Also $\ker \eta = P \cap T$ so $\ker \eta$ is a prime ideal of T of height one by [21, Proposition 2.1.16(vii)]. By Proposition 6.4, T/gT is a domain so, as $g \in \ker \eta$, it follows that $\ker \eta = gT$ and hence that $T/gT \simeq S^\theta$. □

The following result, identifying the invariants for the reversing automorphism of the quantum torus in Example 2.5 rather than the generic quantum torus, can be proved either by adapting the methods above, with simplification due to the replacement of the invertible indeterminate Q by the non-zero scalar q, or by applying the results above together with Corollary 2.12. The algebra T_q defined in the statement is isomorphic to $T/(Q - q)T$ and p is the image of g in T_q.

Proposition 6.6. *Let $S_q = W_q$ and the reversing automorphism θ_q be as in Example 2.5(i). Let T_q be the $\mathbb{F}$-algebra generated by x_1, x_2 and x_3 subject to the relations*

$$x_1 x_2 - q x_2 x_1 = (1 - q^2) x_3, \tag{6.18}$$

$$x_2 x_3 - q x_3 x_2 = (q^{-1} - q) x_1, \tag{6.19}$$

$$x_3 x_1 - q x_1 x_3 = (q^{-1} - q) x_2. \tag{6.20}$$

(i) T_q has a PBW basis $\{x_3^i x_2^j x_1^k : i, j, k \geq 0\}$.

(ii) T_q has a filtration for which $\deg x_1 = 0$ and $\deg x_2 = \deg x_3 = 1$ and the associated graded ring is an iterated skew polynomial ring over $\mathbb{F}$ generated by y_1, y_2 and y_3 subject to the relations

$$y_1 y_2 - q y_2 y_1 = (1 - q^2) y_3, \quad y_2 y_3 - q y_3 y_2 = 0, \quad y_3 y_1 - q y_1 y_3 = (q^{-1} - q) y_2.$$

(iii) T_q has a filtration for which $\deg x_1 = \deg x_2 = \deg x_3 = 1$ and the associated graded ring is an iteration skew polynomial ring over $\mathbb{F}$ generated by y_1, y_2 and y_3

subject to the relations

$$y_1y_2 = qy_2y_1, \quad y_2y_3 = qy_3y_2, \quad y_3y_1 = qy_1y_3.$$

(iv) Let $p = x_3x_2x_1 - qx_3^2 - q^{-2}x_2^2 - x_1^2 + 2(1+q^{-2})$. *Then* p *is a central element of* T_q *and* T_q/pT_q *is a domain.*

(v) $S_q^{\theta_q}$ *is isomorphic to* T_q/pT_q.

Remark 6.7. Setting $t = Q - 1$, which is a central regular non-unit, $T/tT \simeq \mathbb{F}[x_1, x_2, x_3]$, the commutative polynomial algebra. It is a routine matter to confirm that, in accordance with the discussion in Example 2.5(ii), T is a quantization of the Poisson algebra $A = \mathbb{F}[x_1, x_2, x_3]$ with

$$\{x_1, x_2\} = x_1x_2 - 2x_3, \quad \{x_2, x_3\} = x_2x_3 - 2x_1, \quad \{x_3, x_1\} = x_1x_3 - 2x_2,$$

that is with the exact Poisson bracket determined by $f = x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2 + 4$. Also T/gT is a quantization of the Poisson algebra A/fA.

Remark 6.8. The associated graded rings in Lemma 6.2 are also quantizations of A with appropriate exact Poisson brackets $\{-,-\}_f$. In Lemma 6.2(i) $f = x_1x_2x_3 - x_2^2 - x_3^2$ while in Lemma 6.2(ii), $f = x_1x_2x_3$. In both cases $\mathrm{gr}(T/gT)$ is a quantization of the Poisson algebra A/fA, with $\mathrm{gr}(T_q/pT_q)$ as a deformation. Note the cyclic form of the relations in Lemma 6.2(ii) and Proposition 6.6(iii). In the latter, this distinguishes $\mathrm{gr}(T_q)$ from quantum affine space $\mathcal{O}_q$ as defined in [3, p. 15].

Another filtration giving rise to an associated graded ring that is a quantization of A with an exact Poisson bracket is obtained by taking Q and Q^{-1} to have degree 0, x_2 and x_3 to have degree 1 but x_1 to have degree 2. For this filtration, $\mathrm{gr}(T)$ is generated by $Q^{\pm 1}$ and y_i, $i = 1, 2, 3$, subject to the relations

$$y_iQ = Qy_i,\ y_iQ^{-1} = Q^{-1}y_i \text{ for } i = 1, 2, 3,\ QQ^{-1} = Q^{-1}Q = 1,$$
$$y_1y_2 = Qy_2y_1, \quad y_2y_3 = Qy_3y_2 + (Q^{-1} - Q)y_1, \quad y_1y_3 = Q^{-1}y_3y_1.$$

There is a corresponding filtration of T_q with $\mathrm{gr}(T_q) \simeq \mathrm{gr}(T)/(Q - q)\,\mathrm{gr}(T)$. Here $\mathrm{gr}(T)$ and $\mathrm{gr}(T_q)$ are respectively a quantization and deformation of A for the exact Poisson bracket $\{-,-\}_f$ where $f = x_1x_2x_3 - x_1^2$. For each $q \in \mathbb{F}$, $\mathrm{gr}\, T_q$ is an iterated skew polynomial ring over $\mathbb{F}$ and is an ambiskew polynomial ring in the sense of [15] and, by the results of [15], it is a down-up algebra in the sense of [2].

Note that, for two of these filtrations, the polynomial f is reducible and $\mathrm{gr}(T/gT)$ and $\mathrm{gr}(T_q/pT_q)$ are not domains.

Remark 6.9. There is some flexibility in the defining relations for T_q. Let $a, b, c \in \mathbb{F}\backslash\{0\}$ and let $T_q(a, b, c)$ denote the $\mathbb{F}$-algebra generated by x_1, x_2 and x_3 subject to the relations

$$x_1x_2 - qx_2x_1 = ax_3, \quad x_2x_3 - qx_3x_2 = bx_1, \quad x_3x_1 - qx_1x_3 = cx_2.$$

Thus $T_q = T_q(1-q^2, q^{-1}-q, q^{-1}-q)$. It can be checked that there exist $\lambda_i \in \mathbb{F}\backslash\{0\}$, $i = 1, 2, 3$, such that there is an isomorphism $\theta : T_q(1, 1, 1) \to T_q(a, b, c)$ with $\theta(x_i) = \lambda_i x_i$. Algebras isomorphic to $T_q(1, 1, 1)$ have been considered, sometimes with q^2 in place of q, in the mathematical physics literature [8, 11, 12] and in the literature on Leonard triples and Askey-Wilson relations, for example [28, 29]. The

Askey-Wilson relations for T_q are

$$(1+q^2)x_2x_1x_2 - qx_2^2x_1 - qx_1x_2^2 + q^2x_2x_1x_2 = q^{-1}(1-q^2)^2x_1 \text{ and}$$
$$(1+q^2)x_1x_2x_1 - qx_1^2x_2 - qx_2x_1^2 + q^2x_1x_2x_1 = q^{-1}(1-q^2)^2x_2.$$

and are obtained by using (6.18) to substitute for x_3, in terms of x_1 and x_2, in (6.19) and (6.20).

Let us extend our definitions of quantization and deformations to noncommutative algebras by defining a quantization of a noncommutative algebra A to be an algebra B with a central, regular nonunit t such that $B/tB \simeq A$ and a deformation of A to be an algebra of the form $B/(t-\lambda)B$ for some quantization B and some $\lambda \in \mathbb{F}$ for which $t-\lambda$ is a nonunit. Then $T_q(1,1,1)$ is a deformation of the enveloping algebra U of the Lie algebra so_3, the appropriate quantization being the $\mathbb{F}$-algebra generated by $Q^{\pm 1}$, x_1, x_2 and x_3 subject to the relations

$$x_iQ = Qx_i,\ x_iQ^{-1} = Q^{-1}x_i \text{ for } i = 1,2,3,\ QQ^{-1} = Q^{-1}Q = 1,$$
$$x_1x_2 - Qx_2x_1 = x_3, \quad x_2x_3 - Qx_3x_2 = x_1, \quad x_3x_1 - Qx_1x_3 = x_2.$$

Note that if $q^2 \neq 1$, $T_q = T_q(1-q^2, q^{-1}-q, q^{-1}-q)$ is a deformation of the commutative polynomial algebra $\mathbb{F}[x_1,x_2,x_3]$ and, being isomorphic to $T_q(1,1,1)$, it is also a deformation of U. This gives rise to a dichotomy in its behaviour. For example, the trivial representation, with x_1, x_2, x_3 each acting as 0, can be viewed as a deformation of the unique one-dimensional representation of U. On the other hand, if G is the Klein 4-group $\{e, a_1, a_2, a_3\}$, there is a surjection $\phi : T_q(1-q, 1-q, 1-q) \to \mathbb{F}G$ with $\phi(x_i) = a_i$, $i = 1,2,3$. It follows that T_q has a further four 1-dimensional representations. This typifies the finite-dimensional simple representations of T_q. For each $d \geq 1$, Fairlie [8] constructed a d-dimensional simple representation of T_q while Havlicek, Klimyk and Posta constructed another four in [11] and, in [12], Havlicek and Posta showed that there were no more. The finite-dimensional simple representations of T_q are also classified, by an independent method, in [26].

Remark 6.10. When T_q is viewed as a deformation of the Poisson algebra $A = \mathbb{F}[x_1,x_2,x_3]$ in Example 2.5(ii), the central element p is a deformation of the element $x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2 + 4$ of $\mathrm{PZ}(A)$. On the other hand, when T_q is viewed as a deformation of the enveloping algebra U of the Lie algebra so_3, the cubic term of p has a coefficient of the form $g(q)$ where $g(Q) \in \mathbb{F}[Q^{\pm 1}]$ and $g(1) = 0$ so, although p is cubic, it is a deformation of a quadratic element of U which, up to scalar multiplication and translation, is the Casimir element of U.

Remark 6.11. In common with T in Remark 5.6, the $\mathbb{F}$-algebra T_q in Proposition 6.6 is determined by a noncommutative potential. Here it is a variation on one of the basic examples of such an algebra, see [9, Example 1.3.6], $T_q = \mathfrak{U}(\mathbb{F}\langle x_1,x_2,x_3\rangle, \Pi_q)$ where

$$\Pi_q = x_1x_2x_3 - qx_3x_2x_1 + \tfrac{1}{2}(q-q^{-1})(x_1^2 + x_2^2 + qx_3^2).$$

Remark 6.12. By Proposition 6.6(v), the quotient division ring $Q(T_q/pT_q)$ is isomorphic to $Q(W_q^{\theta_q})$ which, by [21, Theorem 10.5.19(v)], is equal to the ring of invariants $Q(W_q)^{\theta_q}$ for the induced action of θ_q on the quotient division ring $Q(W_q)$ of the quantum plane. In [27, 13.6], Stafford and Van den Bergh have shown that, if q is not a root of unity, $Q(W_q)^{\theta} \simeq Q(W_q)$ and have presented, without details of the calculation, a pair of generators f, g for $Q(W_q)^{\theta_q}$, as a division algebra, satisfying $fg = qgf$. The elements of $Q(T_q/pT_q)$ corresponding to f and g are $(a_3a_2 - 2a_1)(a_2^2 - 4)^{-1}$ and $(2a_3 - a_1a_2)(a_2^2 - 4)^{-1}$ respectively, where each a_i is the image of x_i in T_q/pT_q.

References

[1] J. Alev and D. R. Farkas, *Finite group actions on Poisson algebras*, The orbit method in geometry and physics (Marseille, 2000), Progr. Math., **213**, 9–28, Birkhäuser, Boston, 2003.

[2] G. Benkart and T. Roby, *Down-up algebras*, J. Algebra **209** (1998), 305–344.

[3] K. A. Brown and K. R. Goodearl, Lectures on Algebraic quantum groups, Birkhäuser (Advanced Courses in Mathematics CRM Barcelona), Basel-Boston-Berlin, 2002.

[4] R. Devaney, *Reversible diffeomorphisms and flows*, Trans. Amer. Math. Soc. **218** (1976), 89–113.

[5] R. Devaney, An Introduction to Chaotic Dynamical Systems, 2nd edition, Addison Wesley, Redwood City (1989).

[6] F. Dumas and D. A. Jordan, *The* 2×2 *quantum matrix Weyl algebra*, Communications in Algebra, **24(4)**, (1996), 1409–1434.

[7] J. Donin and L. Makar-Limanov, *Quantization of quadratic Poisson brackets on a polynomial algebra of three variables*, Journal of Pure and Applied Algebra **129** (1998), 247–261.

[8] D. B. Fairlie, *Quantum deformations of* SU(2), J. Phys. A **23** (1990), no. 5, L183–L187.

[9] V. Ginzburg, *Calabi-Yau algebras*, arXiv:math.AG/0612139v2.

[10] K. R. Goodearl and R. B. Warfield, An Introduction to Noncommutative Noetherian Rings, Second Edition, London Math. Soc. Student Texts 61, Cambridge, 2004.

[11] M. Havlicek, A. V. Klimyk and S. Posta, *Representations of the cyclically q-deformed algebra* $so_q(3)$, J. Math. Phys., **40**, (1999), 2135–2161.

[12] M. Havlicek and S. Posta, *On the classification of irreducible finite-dimensional representations of* $U_q'(so_3)$ *algebra*, J. Math. Phys., **42** (2001), 472–500.

[13] W. Jarczyk, *Reversible interval homeomorphisms*, J. Math. Anal. Appl. **272** (2002), 473–479.

[14] C. R. Jordan, D. A. Jordan, and J.H. Jordan, *Reversible complex Hénon maps*, Experimental Mathematics **11** (2002), 339–347.

[15] D. A. Jordan, *Down-up algebras and ambiskew polynomial rings*, J. Algebra, **228** (2000), 311–346.

[16] D. A. Jordan, *Normal elements of degree one in Ore extensions*, Communications in Algebra, **30(2)** (2002), 803–807.

[17] D. A. Jordan, *Poisson brackets and spectra in polynomial algebras*, in preparation.

[18] D. A. Jordan, *Reversing automorphisms and deformations of Poisson automorphisms*, in preparation.

[19] G. R. Krause and T. H. Lenagan, Growth of algebras and Gelfand-Kirillov Dimension, revised edition, Grad. Studies in Maths., Amer. Math. Soc., 2000.

[20] M.Lorenz, Multiplicative invariant theory, Encyclopaedia of Mathematical Sciences, Vol. 135 Springer-Verlag, Berlin (2005).

[21] J. C. McConnell and J. C. Robson, Noncommutative Noetherian rings, Wiley, Chichester (1987).

[22] A. V. Odesskii and V. N. Rubtsov, *Polynomial Poisson algebras with a regular structure of symplectic leaves*, Theoret. and Math. Phys. 133 (2002), no. 1, 1321–1337 (translation of Teoret. Mat. Fiz. 133 (2002), no. 1, 3–23).

[23] A. G. O'Farrell, *Conjugacy, involutions, and reversibility for real homeomorphisms*, Irish Math. Soc. Bull. No. 54 (2004), 41–52.

[24] D. S. Passman, Infinite crossed products, Academic Press, San Diego, London, 1989.

[25] A. Pichereau, *Poisson (co)homology and isolated singularities*, J. Algebra 299 (2006) 747–777.

[26] N. Sasom, *Reversible skew Laurent polynomial rings, rings of invariants and related rings*, PhD thesis, University of Sheffield (2006).

[27] J. T. Stafford and M. Van den Bergh, *Noncommutative curves and noncommutative surfaces*, Bull. Amer. Math. Soc., 38 (2001), 171–216.

[28] P. Terwilliger, *Two linear transformations each tridiagonal with respect to an eigenbasis of the other; the TD-D canonical form and the LB-UB canonical form*, J. Algebra 291 (2005) 1–45.

[29] P. Terwilliger and R. Vidunas, *Leonard pairs and the Askey-Wilson relations*, J. Algebra Appl. 3 (2004) 411–426.

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