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Real option valuation for reserve capacity

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Abstract

Motivated by the potential use of electricity storage to smooth fluctuations in supply and demand, we study the problem of writing American-type call options when the holder's exercise strategy is of threshold type (so that the time of exercise is known, but random). The writer must provide physical cover by buying and storing the asset *before* selling the option. We optimise the writer's strategy for a single option and for an infinite sequence of options, these two strategies being different. The latter is motivated by the lifetime valuation of an energy storage unit when used as reserve capacity in a power system. Our stochastic process is a Brownian motion representing the real-time system imbalance, and which we rescale to represent an imbalance price. The single option leads to an optimal stopping problem in which the principle of smooth fit may be violated and the stopping region may be disconnected. The lifetime analysis uses techniques and results for the single option to construct a certain fixed point characterising the value function.

Keywords: Applied probability, OR in energy, real option, power system balancing, capacity market

1. Introduction

In an electrical power system, unexpected variations in both generation and load give rise to imbalance which is costly to correct. Various solutions to this challenge have been proposed, including dynamic consumer pricing mechanisms (Tsitsiklis and Xu 2015). This paper is motivated by an alternative or complementary solution using electricity storage to smooth such variations. In particular we study the design of financial-type option contracts on this ancillary service.

It is common for the writer of a call option to hedge their position and in the Black-Scholes world this may be achieved by *dynamic delta hedging* (Hull 2006), whereby more of the underlying stock is bought as the stock price rises,

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and vice versa. Where trading activity has an impact on the market price, the "buy high and sell low" nature of dynamic hedging may therefore exacerbate extreme fluctuations of market prices. If the buyer of a call option wishes to help stabilise or *balance* the market price, dynamic hedging therefore conflicts with this objective.

"Buying low and selling high", a reverse of delta hedging, is a fundamental investment strategy which can have the advantage of stabilising the market. Its design and optimisation has received recent attention in the context of quantitative finance under a variety of asset price models (see, for example, Zhang (2001), Zhang and Zhang (2008), Zervos et al. (2013)). Suppose now that the underlying asset must be bought and stored *before* a call option can be written on it: in other words, that the option must be *physically covered*. This incentivises the option writer to buy and store the underlying when it is cheap, while the call option itself can deliver the asset when it is expensive. This requirement therefore leads to an alternative hedging strategy for the call option which is compatible with market balancing. Since the asset purchase can be timed flexibly, our study falls within the scope of real options analysis (Boomsma et al. 2012). This approach can be contrasted with studies which look at the impact of storage on price formation through direct trading in energy (see, for example, Gast et al. (2013)).

A certain agent is deemed to require supply of an asset when a stochastic process, which represents its price, first lies above a pre-determined threshold. At that point a second agent must provide one unit of the asset to the first agent, who pays a reward in exchange. The first agent may also pay an initial premium which is additional to the reward. Optimally timing both the purchase of the underlying asset and the writing of the option is an optimal stopping problem that the second agent solves.

This problem may be interpreted as the second agent writing a call option of American style on the underlying asset. Valuing such contracts by optimising the option holder's strategy is a classical application of optimal stopping theory (Peskir and Shiryaev 2006, Chapter VII); we reverse this setup, fixing the holder's strategy and using optimal stopping theory to optimise the writer's actions. Our motivation comes from a problem of providing *reserve capacity* in power systems (see, for example, Just and Weber (2008)) and from assuming that the underlying asset is electricity in an *imbalance market*. In this motivating problem we model the price as a function of the instantaneous level of *imbalance* in the power system, that is supply minus load; the first agent is the *network operator* who uses the option (among other interventions) to keep the imbalance close to zero, and the second agent is the operator of an electricity storage facility such as a grid-scale battery. While the storage operator is concerned with maximising profit (the expected net present value of the cashflows described in the problem), the network operator is assumed to be concerned primarily with the physical stability of the power system. It is for this reason that the network operator's exercise strategy is assumed to be specified exogenously, rather than resulting from an economic optimisation.

Our setup leads to an optimal stopping problem in which the principle of smooth fit (see, for example, Peskir and Shiryaev (2006)) may be violated and the stopping region may be disconnected. The methodological approach we take for the single option is similar to that in Carmona and Dayanik (2008), which considers finite sequences of American type options. By adding a fixed point argument we go further, obtaining the optimal strategy and *lifetime* valuation when an arbitrarily large number of exercises is permitted. The fixed point is constructed using techniques and results for the single option case. Our stochastic process is a Brownian motion representing the real-time system imbalance, and which we rescale to represent an imbalance price. This lifetime analysis may be regarded as a single project valuation model for an electricity store (cf. Hach et al. (2016) and references therein).

The paper is organised as follows. Subsection 1.2 introduces the model with main findings summarised in Subsection 1.3. Section 2 analyses the single option, and in Section 3 we perform the lifetime analysis. Appendices contain auxiliary results and detailed proofs for the single option setup.

1.1. Real Options approach

We employ the *Real Options* approach (see, e.g., Brennan and Schwartz (1985), Guthrie (2009)) in which the dynamics of the underlying stochastic process are under the physical (real) measure. The alternative of pricing under a martingale measure leads to delta hedging strategies of the type "buy high and sell low" (Hull 2006), which exacerbate extreme market prices. In our motivating application, an electricity balancing market trades real-time adjustments to generation and load and the market price should be driven by a model of the system imbalance process $(X_t)_{t\geq 0}$. Through the prediction of load (see, for example, Hahn et al. (2009) and references therein), the imbalance process should have zero mean at all times and following Gast et al. (2013) we model X as a Brownian motion. In addition to being an approximation to other zero-mean diffusion processes over short time intervals, the choice of Brownian motion enables the explicit analysis which follows in this paper. We note here that the diffusion X has natural boundaries at positive and negative infinity, which plays a role in the methodology of Section 2.2.

1.2. Balancing markets

In order to move between the modelling of imbalance and the related question of price modelling we consider the UK Balancing Mechanism, which exists to equalise electricity supply and demand close to real time. In this market parties submit offers to increase generation or decrease consumption, and bids to decrease generation or increase consumption. National Grid, the network operator in the UK, seeks to correct the prevailing imbalance at least cost by taking the lowest-priced offers or accepting the highest-priced bids, subject to system constraints (see, for example, Elexon Limited (2015)). Figure A.1 in the Appendix provides a histogram of the main system price obtained in this way, over a two and a half year period. This distribution of prices is not centred and is heavily skewed to the right. We take account of these empirical features in a straightforward manner by applying a convex transformation f to the imbalance process X, where

$$f(x) := D + de^{-bx}, \tag{1}$$

and $b, d > 0$. Thus our model has an 'imbalance price' process $(f(X_t))_{t\geq 0}$ which is a shifted exponential Brownian motion, and is simply a rescaling of the physical imbalance process. Here the minus sign in the exponent means that positive values of the imbalance correspond to the oversupply of the asset, and vice versa. We note that the price process $f(X_t)$ has a natural lower boundary at D, in the terminology of Borodin and Salminen (2012, Chapter 2), i.e., the price cannot reach it.

1.3. Main results

We derive the option writer's optimal policy under the above setup and the corresponding option value in both the single and lifetime problems. These two policies are different in general and the single option, in addition to being a 'basic unit', is of independent interest as it exhibits three different types of optimal stopping region. We show that the possible types are:

(a) a half-line,

(b) a bounded interval,

(c) a union of two disjoint intervals.

The smooth fit property may not hold at the (finite) boundaries of the optimal stopping region in each of the above cases. This variety of solution types, which is rather unusual in the literature on one-dimensional optimal stopping problems, can be anticipated: lower price levels mean a lower cost for purchasing the asset but also a longer time until the reward is received. The parameter-dependent interplay between these opposing considerations therefore determines the precise form of the solution.

In our motivating problem, energy is stored in a single battery of unit capacity and so in order to ensure delivery of the energy when needed, a second option may be written only after the first option has been exercised and the battery has been replenished. The investment value of the battery when used as reserve capacity is therefore equal to the value of an infinite sequence of such real options, which we call the 'lifetime valuation' (a somewhat related study may be found in Carmona and Ludkovski (2010), where the value of gas storage units used for price arbitrage is derived using a numerical approach). We show that the lifetime value is finite when the sum of the single contract payments (initial premium and reward) is strictly less than the imbalance price upon exercise. We obtain the writer's optimal policy and the corresponding lifetime value.

2. Single call option

In this section we formulate the writer's optimal strategy as an optimal stopping problem (Section 2.1), present the solution method and geometric analysis of the obstacle (Sections 2.2 and 2.3) and give an overview of the set of optimal strategies depending on the parameter values, from which the option value follows directly. The detailed case by case analysis is given in the appendices (Sections C.1 to C.3). We conclude by discussing whether the contract, once optimised from its writer's point of view, indeed stabilises the imbalance process $(X_t)_{t\geq 0}$, or equivalently the price process $(f(X_t)_{t\geq 0})$.

2.1. An optimal stopping problem

For any $x \in \mathbb{R}$, take a Brownian motion $X = (X_t)_{t\geq 0}$ which starts at x and is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}^x)$ with $\mathbb{E}^x$ denoting the expectation with respect to $\mathbb{P}^x$. Following Borodin and Salminen (2012, p. 18 and p. 163), the hitting time of the point $y \in \mathbb{R}$ by X has the Laplace transform

$$\mathbb{E}^x\{e^{-r\tau_y}\} = e^{-a|y-x|}, \qquad r \geq 0, \tag{2}$$

where $a = \sqrt{2r}$ and $\tau_y = \inf\{t \geq 0 : X_t = y\}$. We define $\phi(x) = e^{-ax}$ and $\psi(x) = e^{ax}$ as the decreasing and increasing solutions respectively of the ordinary differential equation $\frac{1}{2}u'' = ru$.

Recalling the sign convention in (1), the time of exercise by the first agent is

$$\hat{\tau}_e = \inf\{t \geq 0 : X_t \leq x^*\}$$

where x^* is a fixed threshold. Writing $K_c \geq 0$ for the reward and taking $r > 0$ as the continuously compounded interest rate, the expected net present value of the reward is

$$h_c(x) = \mathbb{E}^x\{e^{-r\hat{\tau}_e} K_c\} = \begin{cases} K_c, & x < x^*, \\ K_c e^{-a(x-x^*)}, & x \geq x^*. \end{cases} \tag{3}$$

When the contract is entered the writer receives the *premium* $p_c \geq 0$. If the writer has already purchased the asset, their problem of optimal entry into the single option contract is

$$w_c(x) = \sup_{\tau} \mathbb{E}^x\{e^{-r\tau}\big(p_c + h_c(X_\tau)\big)\}.$$

We observe immediately that the solution to this problem is trivial: for any $\omega \in \Omega$, taking $\tau(\omega) = 0$ is optimal for the sample path $X(\omega) = (X_t(\omega))_{t\geq 0}$ since (i) the option premium p_c is received immediately, and (ii) the reward K_c is received at the earliest opportunity (which is the first hitting time by $X(\omega)$ of $(-\infty, x^*]$). The stopping time $\tau \equiv 0$ is therefore optimal: if the asset has already been purchased, the option is written immediately and its value is

$$w_c(x) = p_c + h_c(x).$$

In our setup the asset has not initially been purchased. The writer's problem of optimally timing the purchase is therefore the following optimal stopping problem, which is solved in the remainder of this section:

$$V_c(x) \quad = \quad \sup_{\tau} \mathbb{E}^x\{e^{-r\tau} h(X_\tau)\}, \tag{4}$$

where h is the value of the contract net the purchase price of the asset:

$$h(x) = -f(x) + w_c(x) = -f(x) + p_c + h_c(x). \tag{5}$$

Remark. Henceforth, we use the term 'stopping' as shorthand for 'purchasing the asset and immediately writing the option contract'.

2.2. Solution method and terminology

As discussed in the Introduction, the optimal stopping problem (4) is non-standard. In particular the parametric nature of the threshold x^* means that the value function may not be differentiable at x^* or, in the terminology of optimal stopping, the smooth fit property may not hold at x^*. Instead of applying particular conditions such as smooth fit we therefore choose instead a constructive solution technique, namely the characterisation of the value function through the smallest concave majorant of a modified payoff function, appealing to results originating from Dynkin (1965) and Dynkin and Yushkevich (1969). Since the payoff h of the optimal stopping problem (4) is bounded on compact sets and we have

$$0 \leq \limsup_{x\to\infty} \frac{h^+(x)}{\psi(x)} \leq \limsup_{x\to\infty} \frac{p_c + K_c e^{-a(x-x^*)}}{e^{ax}} = 0$$

and

$$\limsup_{x\to-\infty} \frac{h^+(x)}{\phi(x)} = 0$$

we may specialise Proposition 5.12 of Dayanik and Karatzas (2003) to give

Proposition 2.1. *Let $F(x) := \psi(x)/\phi(x) = e^{2ax}$ and let $W : [0, +\infty) \to \mathbb{R}$ be the smallest nonnegative concave majorant of*

$$H(y) := \begin{cases} \frac{h(F^{-1}(y))}{\phi(F^{-1}(y))}, & y > 0 \\ 0, & y = 0. \end{cases} \tag{6}$$

Then $V_c(x) = \phi(x)W(F(x))$ and the optimal stopping time is $\tau^* := \inf\{t \geq 0 : X_t \in \Gamma\}$, where $\Gamma = \{x \in \mathbb{R} : V_c(x) = h(x)\}$, and $\Gamma = F^{-1}(\hat{\Gamma})$ where $\hat{\Gamma} = \{y > 0 : W(x) = H(x)\}$.

Because of the duality between V_c and W in Proposition 2.1 we shall refer to both as *value functions*; similarly, Γ and $\hat{\Gamma}$ will be called *stopping regions*. Due to the continuity of the payoff functions H, h and value functions W, V_c, both sets are closed. For readability, values in the original scale of the process X_t will be denoted by x whereas points in the transformed scale (through the mapping F) will be denoted by y.

2.3. Geometry of the obstacle

To find the majorant in Proposition 2.1 we partition the space of parameters $(x^*, D, d, b, r, K_c, p_c)$ according to the geometry of the obstacle H in equation (6), making it straightforward to construct the majorant W on each parameter set in the partition. In this subsection we perform preliminary calculations which will be used repeatedly, collecting results in Table 1 for convenience. Noting that $F^{-1}(y) = \ln(y)/(2a)$, $\phi(F^{-1}(y)) = y^{-\frac{1}{2}}$ and putting $y^* = F(x^*)$ gives

$$H(y) = \frac{p_c + h_c(F^{-1}(y)) - f(F^{-1}(y))}{\phi(F^{-1}(y))} = \begin{cases} g(y), & y \geq y^*, \\ \hat{g}(y), & 0 < y < y^*, \\ 0, & y = 0, \end{cases}$$

where

$$\begin{aligned} g(y) &= y^{\frac{1}{2}}\left[p_c - D - d\, y^{-\frac{b}{2a}}\right] + K_c\sqrt{y^*}, \\ \hat{g}(y) &= y^{\frac{1}{2}}\left[K_c + p_c - D - d\, y^{-\frac{b}{2a}}\right]. \end{aligned} \tag{7}$$

Define also

$$\eta(y) = y^{\frac{1}{2}}\left[p_c - D - d\left(1 + \frac{b}{a}\right)y^{-\frac{b}{2a}}\right] + 2K_c\sqrt{y^*},$$

so that $\frac{1}{2}\eta(y) = g(y) - yg'(y)$ is the intercept at the vertical axis for the tangent to g at y. So the equation $\eta(y) = 0$ is equivalent to the tangent at y passing through the origin, i.e., to $g(y)/y = g'(y)$. Define the following values in $[0, \infty]$ (these values play a key role in Section 3 and in Section C in the Appendix, with the convention that $1/0 = \infty$):

$$Y_m = \left(\left[\frac{p_c - D}{d\left(1 - \frac{b}{a}\right)}\right]^+\right)^{-2a/b}, \qquad Y_c = \left(\left[\frac{p_c - D}{d\left(1 - \frac{b^2}{a^2}\right)}\right]^+\right)^{-2a/b},$$

$$\hat{Y}_m = \left(\left[\frac{K_c + p_c - D}{d\left(1 - \frac{b}{a}\right)}\right]^+\right)^{-2a/b}, \qquad \hat{Y}_c = \left(\left[\frac{K_c + p_c - D}{d\left(1 - \frac{b^2}{a^2}\right)}\right]^+\right)^{-2a/b},$$

so that Y_m locates the turning point for g and $Y_c \geq Y_m$ (the inequality is strict when Y_m is finite) locates the change between convexity and concavity for g with corresponding relationships for the hats $\hat{g}, \hat{Y}_m, \hat{Y}_c$; Y_c also locates the turning point of η (see Table 1). Note also that $Y_c \geq \hat{Y}_c$ and $Y_m \geq \hat{Y}_m$ when $b < a$ and that the opposite inequalities hold when $b > a$.

The smallest nonnegative concave majorant W is equal to H in the stopping region, while outside the stopping region it lies above H and is linear. Whenever such a straight line segment is tangential to H at $(y, H(y))$ we shall say that the *smooth fit* condition holds at y. For investigation of the smooth fit condition for

Table 1: Summary of the monotonicity and convexity properties of functions $g, \hat{g}, \eta$ and their limits at 0 and ∞. We define $\mathrm{sgn}(0) \cdot \infty$ differently depending on the relation between b and a: when $b < a$ we put $\mathrm{sgn}(0) \cdot \infty = -\infty$; otherwise, $\mathrm{sgn}(0) \cdot \infty = 0$.

	$\mathbf{b < a}$		$\mathbf{b > a}$	
	to Left	**to Right**	**to Left**	**to Right**
Y_m	g decreasing	g increasing	g increasing	g decreasing
Y_c	g convex	g concave	g concave	g convex
	η decreasing	η increasing	η increasing	η decreasing
$\hat{Y}_m$	$\hat{g}$ decreasing	$\hat{g}$ increasing	$\hat{g}$ increasing	$\hat{g}$ decreasing
$\hat{Y}_c$	$\hat{g}$ convex	$\hat{g}$ concave	$\hat{g}$ concave	$\hat{g}$ convex
Limits	**at** 0	**at** ∞	**at** 0	**at** ∞
g	$K_c\sqrt{y^*}$	$\mathrm{sgn}(p_c - D) \cdot \infty$	$-\infty$	$\mathrm{sgn}(p_c - D) \cdot \infty + K_c\sqrt{y^*}$
$\hat{g}$	0	$\mathrm{sgn}(K_c + p_c - D) \cdot \infty$	$-\infty$	$\mathrm{sgn}(K_c + p_c - D) \cdot \infty$
η	$2K_c\sqrt{y^*}$	$\mathrm{sgn}(p_c - D) \cdot \infty$	$-\infty$	$\mathrm{sgn}(p_c - D) \cdot \infty + 2K_c\sqrt{y^*}$

$y < y^*$, let $\hat{y}_b$ be the solution to $\hat{g}(y)/y = \hat{g}'(y)$:

$$\hat{y}_b = \left(\left[\frac{K_c + p_c - D}{d(1 + \frac{b}{a})} \right]^+ \right)^{-2a/b}$$

with the convention that $\hat{y}_b = \infty$ when the solution does not exist. If $\hat{y}_b < \infty$ then

$$\hat{g}'(\hat{y}_b) = (K_c + p_c - D)\frac{2b}{a+b}, \qquad \hat{g}''(\hat{y}_b) = -(K_c + p_c - D)\frac{b}{a},$$

so that when $K_c + p_c > D$, the function $\hat{g}$ is strictly increasing, concave and positive at $\hat{y}_b$. For investigation of the smooth fit condition for $y > y^*$ we note the following Lemma.

Lemma 2.2. *If $b \neq a$ then there is at most one root y_b of η in $(0, \infty)$ satisfying $g''(y_b) \leq 0$.*

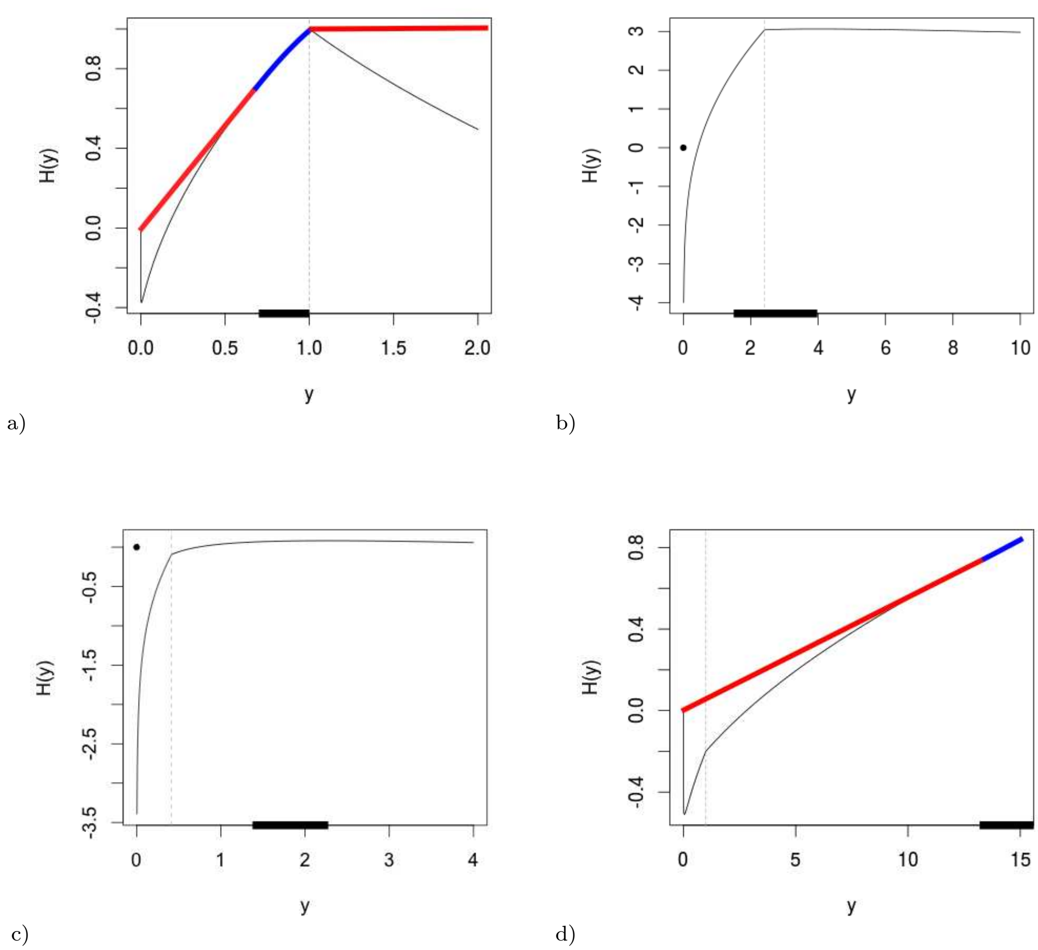
Proof. The half-line $(0, \infty)$ partitions into two intervals I_1, I_2 such that g is strictly convex on the interior of I_1 and is strictly concave on the interior of I_2 (see Table 1). Since η is also monotone on each of these intervals, there can be at most one root y_b of η in the concavity interval I_2. □

Remark. Although there is no closed formula for the unique root in the above lemma, if it exists then it may be found quickly by Newton-Raphson iteration or a bijection method using the monotonicity of η on the interval I_2.

2.4. Optimal stopping solutions and real option valuation

A detailed analysis of the single option is presented in Appendix C. The stopping regions are of one of four types depending on the parameter combination. In six different parameter blocks the stopping region is empty. In the remaining parameter blocks the stopping region is of one of the following forms:

Figure 1: Illustrative plots for the single option obstacle and stopping region (thick horizontal line). The dashed vertical lines mark y^*. In panels a) and d) the least nonnegative concave majorant W is shown in blue (where W coincides with H) and red (otherwise).



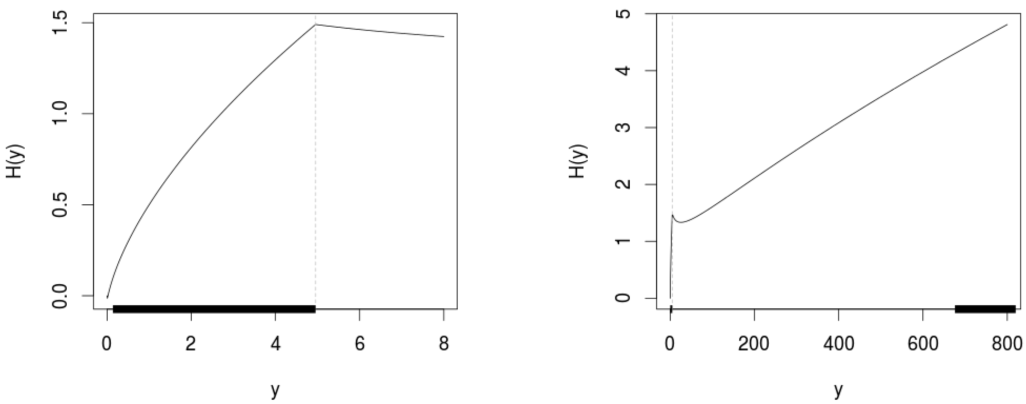


Figure 2: An illustrative plot of H for the single option when $b < a$, in case A, with $y^* < Y_c$ and with a non-degenerate left part to the stopping region (the same graph is depicted in both panels).

(a) a half-line, (b) a bounded interval, (c) a union of two disjoint intervals, with combinations of smooth or non smooth fit at the finite boundaries of these intervals. Figures 1 and 2 show how each of the above forms of stopping region arises as a result of the geometry of the obstacle H. Panels (a)-(c) of Figure 1 depict the case of a bounded interval, together with the different possibilities for its positioning around the kink of the obstacle at y^*. A half-line stopping region is shown in Panel (d). A case of two disjoint intervals, unusual in the literature on optimal stopping problems, is presented in Figure 2. Here it is optimal to stop in an interval to the left of x^*, where the reward is both positive and immediate. To the right of x^* (see the right panel) the reward is not immediate and depends on future randomness: one prefers to wait until either the positive immediate reward is again available (at $x = x^*$), or alternatively the imbalance x is sufficiently high. In the latter case one purchases power at a low price, loads the storage and issues the option, leading to a sufficiently large expected profit.

In all cases the stopping region is strictly separated from the point $y = 0$ and W is linear between 0 at 0 and H at the left-most point of the stopping region. By Proposition 2.1, the option value at point $x = F^{-1}(y)$ in the original scale of the process $(X_t)_{t\geq 0}$ is $V_c(x) = \phi(x)W(F(x))$. The principle of smooth fit holds at all finite boundary points except y^*: since H is non-differentiable at y^*, if it is a boundary point then the principle of smooth fit cannot hold there.

2.5. Qualitative remarks on the parameter ranges

In this section we give interpretations of the main parameter regimes, namely the relationships between p_c and D and between a and b. We begin with the easy observation that the contract is worthless to the seller if the total payment $p_c + K_c$ does not exceed the initial purchase price of the underlying asset, so this sum must be strictly greater than D in order to obtain a positive real option value.

2.5.1. Relation between p_c and D

When $p_c < D$ the initial cashflow $p_c - f(x)$ is negative (equation (1)) and as x increases to infinity the expected discounted value of the reward decreases to zero (equation (3)) so the gain $h(x)$ from stopping eventually becomes negative (equation (5)); the stopping region is therefore bounded in all cases (Figure 1, panels a to c). Conversely, when $p_c > D$ the stopping region is unbounded (Table 1, panel d), c.f. Tables C.1 - C.3 in the Appendix.

2.5.2. Relation between a and b

More generally, when $p_c < D$ the balance between the negative initial cashflow $p_c - f(x)$ and the positive future reward becomes crucial. Indeed, stopping at $x > x^*$ is optimal if this negative initial cashflow is sufficiently compensated by the expected value of the reward K_c received when the process X falls to the level x^*. This is in turn determined by the relation between $a = \sqrt{2r}$ and b, r being the rate at which future payoffs are discounted, while b governs the speed at which the price process moves. In particular, if $a > b$ then it is never optimal to wait for a favourable movement of the price process: this means it is not optimal to enter the contract when $x > x^*$. In this case the stopping region is either empty (when $K_c + p_c \leq f(x^*)$) or lies to the left of x^*, where stopping means that the reward is received immediately (Figure 1, panel a). When $a < b$ the stopping region may contain points to the right of x^*, depending again on the balance between the initial loss and the future reward (Figure 1, panels b and c).

2.6. From stopping policies to power system balancing

We now examine the implications of these optimal stopping regions for our original problem of option hedging in the context of balancing a power system. In our model there are two physical transfers: (a) power is drawn from the network by the storage operator (at the 'stopping time' studied above), and (b) power is delivered to the network by the storage operator (when the imbalance next falls below the level x^*). Assuming that the store is small compared to the power system, these physical transfers will be balancing in nature if (a) takes place when the imbalance is strictly positive and (b) takes place when the imbalance is strictly negative. Clearly the network operator will choose the parameter $x^* < 0$. The storage operator optimally chooses their strategy for transfer (a) given the problem parameters, and in principle this could occur at negative imbalance values. It is therefore desirable for balancing that the left endpoint x_Γ of the stopping region be strictly positive. In the transformed coordinates of Figure 1, this corresponds to stopping regions which are either a bounded interval (panel c) or half-line (panel d) with left endpoint $y_\Gamma > f(0) = 1$.

Having established which parameter-dependent hedging strategies are desirable to the network operator, in Section 3 we now return to the perspective of the storage operator. In particular we next derive the lifetime value of such options. This analysis may be used to inform assessments of the commercial viability of electricity storage capital infrastructure.

3. Lifetime valuation

In this section we obtain the optimal strategy in our setup when the second agent writes an infinite sequence of call options by repeating the cycle presented above. We keep the same notation as Section 2.

3.1. Fixed point method: operators $\mathcal{T}$ and $\tilde{\mathcal{T}}$

We first define two related operators (one for each scale) for which we seek fixed points. The operator $\mathcal{T}$ (whose domain will be specified later, see Lemma 3.1) is defined by a string of operations. Take a non-negative function $\xi(x)$ which should be thought of as the continuation value after a contract has been exercised. As before, the option is exercised at the stopping time $\hat{\tau}_e = \inf\{t \geq 0 : X_t \leq x^*\}$ when the imbalance process reaches x^* or below and the payoff is now $K_c + \xi(X_{\hat{\tau}_e})$, making the expected payoff equal to

$$h_c^\xi(x) := \mathbb{E}^x\big\{e^{-r\hat{\tau}_e}\big(K_c + \xi(X_{\hat{\tau}_e})\big)\big\} = \begin{cases} \big[K_c + \xi(x^*)\big]\frac{\phi(x)}{\phi(x^*)}, & x > x^*, \\ K_c + \xi(x), & x \leq x^*. \end{cases}$$

When the continuation value ξ is included, the optimal stopping problem (4) therefore becomes

$$\mathcal{T}\xi(x) = \sup_\tau \mathbb{E}^x\{e^{-r\tau}h^\xi(X_\tau)\}$$

where

$$h^\xi(x) = -f(x) + p_c + h_c^\xi(x).$$

Lemma 3.1. *Assume that ξ is non-negative and $\limsup_{x\to-\infty}\xi(x)e^{ax} < \infty$. Then $\mathcal{T}\xi(x)$ is non-negative and finite for any $x \in \mathbb{R}$. Moreover, $\mathcal{T}\xi(x) = \phi(x)W(F(x))$, where W is the smallest non-negative concave majorant of*

$$\bar{H}(y) = \sqrt{y}(-D - de^{-(b/2a)\ln(y)} + p_c) + \begin{cases} \big[K_c + \xi(x^*)\big]\sqrt{y^*}, & y > y^*, \\ \big[K_c + \xi(F^{-1}(y))\big]\sqrt{y}, & y \leq y^*, \end{cases}$$

and

$$\lim_{x\to-\infty} e^{ax}\mathcal{T}\xi(x) = \limsup_{x\to-\infty}\xi(x)e^{ax}, \qquad \lim_{x\to\infty} e^{-ax}\mathcal{T}\xi(x) = 0.$$

Proof. First we verify the conditions of Dayanik and Karatzas (2003, Proposition 5.12), i.e., that the following limits are finite:

$$\limsup_{x\to-\infty}\frac{(h^\xi)^+(x)}{\phi(x)}, \qquad \limsup_{x\to\infty}\frac{(h^\xi)^+(x)}{\psi(x)}.$$

Indeed,

$$\begin{aligned} 0 \le \limsup_{x\to\infty} \frac{(h^\xi)^+(x)}{\psi(x)} &\le \limsup_{x\to\infty} \frac{1}{\psi(x)} \left(p_c + \big[K_c + \xi(x^*)\big] \frac{\phi(x)}{\phi(x^*)} \right) \\ &= \limsup_{x\to\infty} \frac{p_c}{\psi(x)} + \frac{K_c + \xi(x^*)}{\phi(x^*)} \limsup_{x\to\infty} \frac{1}{F(x)} = 0 \end{aligned}$$

and

$$0 \le \limsup_{x\to-\infty} \frac{(h^\xi)^+(x)}{\phi(x)} \le \limsup_{x\to-\infty} \frac{\xi(x)}{\phi(x)} = \limsup_{x\to-\infty} \xi(x) e^{ax} < \infty$$

by the assumptions of the lemma. Under the above conditions the value function is finite, i.e., $\mathcal{T}\xi(x) < \infty$ for any $x \in \mathbb{R}$ and $\mathcal{T}\xi(x) = \phi(x) W(F(x))$, where W is the smallest non-negative concave majorant of $\frac{(h^\xi)(F^{-1}(y))}{\phi(F^{-1}(y))}$. This quotient simplifies to $\bar{H}$ in the lemma as follows:

$$\frac{(h^\xi)(F^{-1}(y))}{\phi(F^{-1}(y))} = \sqrt{y}(-D - de^{-bF^{-1}(y)} + p_c) + \frac{(h^\xi_c)(F^{-1}(y))}{\phi(F^{-1}(y))}.$$

We also use the equality $1/\phi(x^*) = 1/\phi(F^{-1}(y^*)) = \sqrt{y^*}$. □

Define operators $\mathcal{M}$ and $\tilde{\mathcal{T}}$ by

$$\mathcal{M}\nu(y) = \frac{\nu(F^{-1}(y))}{\phi(F^{-1}(y))}, \tag{8}$$

$$\tilde{\mathcal{T}} = \mathcal{M} \circ \mathcal{T} \circ \mathcal{M}^{-1}. \tag{9}$$

The operator $\tilde{\mathcal{T}}$ is on the y scale, so it acts on the space of transformed obstacles for which the solution of an optimal stopping problem is given by finding the smallest non-negative concave majorant. This will be useful in finding a fixed point of $\tilde{\mathcal{T}}$ and hence, equivalently, of $\mathcal{T}$. (To verify the formulas (8) and (9) note that $\mathcal{M}^{-1}\nu(x) = \phi(x)\nu(F(x))$.)

Lemma 3.2. *Assume that ζ is non-negative and $\limsup_{y\downarrow 0} \zeta(y) < \infty$. Then $\tilde{\mathcal{T}}\zeta(y)$ is non-negative and finite for any $y > 0$. Moreover, $\tilde{\mathcal{T}}\zeta$ is the smallest non-negative concave majorant of*

$$\bar{H}(y) = \sqrt{y}(-D - dy^{-b/2a} + p_c) + \begin{cases} K_c\sqrt{y^*} + \zeta(y^*), & y > y^*, \\ K_c\sqrt{y} + \zeta(y), & y \le y^*. \end{cases} \tag{10}$$

Proof. The proof relies on inserting $\xi(x) = \phi(x)\zeta(F(x))$ in Lemma 3.1 and then simplifying the expression for $\bar{H}$. □

The operator $\mathcal{T}$ is monotone in the sense that if $\xi_1 \le \xi_2$ then $\mathcal{T}\xi_1 \le \mathcal{T}\xi_2$. Moreover, $\mathcal{T}\xi \ge 0$ for any continuous function ξ satisfying the assumptions of Lemma 3.1 so $\mathcal{T}\mathbf{0} \ge 0$. These properties imply that

$$\mathcal{T}^{n+1}\mathbf{0} = \mathcal{T}^n\big(\mathcal{T}\mathbf{0}\big) \ge \mathcal{T}^n\mathbf{0}$$

and, consequently, the sequence $(\mathcal{T}^n\mathbf{0})_{n\geq 1}$ is non-decreasing and the limit

$$V := \lim_{n\to\infty} \mathcal{T}^n\mathbf{0}$$

is well defined (although it may be infinite). We will call it the *lifetime value function*. Indeed, $\mathcal{T}^n\mathbf{0}$ is the value function for the problem with at most n options sold. In the limit, one obtains the value attainable by an option writer who does not face any constraints on the number of options written. Our plan for the solution of the lifetime problem is the following:

Objectives 3.3.
1. find a fixed point ζ^* of the operator $\tilde{\mathcal{T}}$, then $\xi^* = \mathcal{M}^{-1}\zeta^*$ is a fixed point of $\mathcal{T}$,
2. show that the corresponding optimal stopping time is the first hitting time of a set Γ whose left endpoint is strictly greater than x^*,
3. use step 2 to prove that $\lim_{n\to\infty} \mathcal{T}^n\mathbf{0} = \xi^*$.

The rest of this section is organised as follows. We address step 3 in Subsection 3.2, in Subsection 3.3 we establish a sufficient condition for the finiteness of the lifetime value function, and in Subsection 3.4 we carry out steps 1 and 2.

3.2. Convergence to the lifetime value function

We begin by addressing step 3 of Objectives 3.3. Recalling that $\mathcal{T}\mathbf{0} = V_c$, the value function for the single option, we conclude that $V \neq 0$ (and hence that the perpetual regime is profitable) if and only if $V_c \neq 0$. Indeed, if $V_c = 0$ then trivially $\lim_{n\to\infty} \mathcal{T}^n\mathbf{0} = \mathbf{0}$. On the other hand, if $\mathcal{T}\mathbf{0} \neq \mathbf{0}$ (i.e., it is strictly positive for some arguments), then the limit $\lim_{n\to\infty} \mathcal{T}^n\mathbf{0} \neq \mathbf{0}$ since the sequence $(\mathcal{T}^n\mathbf{0})$ is non-decreasing.

Lemma 3.4. *Let ξ, ξ' be two continuous non-negative functions with ξ satisfying the assumptions of Lemma 3.1. Assume that $\xi \geq \xi'$ and that the stopping region Γ for $\mathcal{T}\xi$ satisfies $\Gamma \subset [x', \infty)$ for some $x' > x^*$. Then*

$$\|\mathcal{T}\xi - \mathcal{T}\xi'\|_{\#} \leq \rho\|\xi - \xi'\|_{\#},$$

where $\rho = e^{2a(x^*-x')} < 1$ and $\|f\|_{\#} = |f(x^*)|$ is a seminorm on the space of continuous functions. Moreover we have the following uniform estimate: for all $x \in \mathbb{R}$,

$$0 \leq \mathcal{T}\xi(x) - \mathcal{T}\xi'(x) \leq \|\xi - \xi'\|_{\#}. \tag{11}$$

PROOF. Let $\tau^* = \inf\{t \geq 0 :\ X_t \in \Gamma\}$. Then, by monotonicity of $\mathcal{T}$, we have

$$\begin{aligned}
0 &\leq \mathcal{T}\xi(x) - \mathcal{T}\xi'(x) \\
&\leq \mathbb{E}^x\Big\{e^{-r\tau^*}\Big(-D - de^{-bX_{\tau^*}} + p_c + \big[K_c + \xi(x^*)\big]\frac{\phi(X_{\tau^*})}{\phi(x^*)}\Big)\Big\} \\
&\quad - \mathbb{E}^x\Big\{e^{-r\tau^*}\Big(-D - de^{-bX_{\tau^*}} + p_c + \big[K_c + \xi'(x^*)\big]\frac{\phi(X_{\tau^*})}{\phi(x^*)}\Big)\Big\} \\
&= \mathbb{E}^x\Big\{e^{-r\tau^*}\Big(\big[\xi(x^*) - \xi'(x^*)\big]\frac{\phi(X_{\tau^*})}{\phi(x^*)}\Big)\Big\} \\
&= \|\xi - \xi'\|_{\#}\, \mathbb{E}^x\Big\{e^{-r\tau^*}\frac{\phi(X_{\tau^*})}{\phi(x^*)}\Big\}.
\end{aligned}$$

This proves (11). Let $x_\Gamma = \min\Gamma$. By assumption we have $x_\Gamma > x^*$, so for $x \leq x^*$ it follows that

$$\mathbb{E}^x\Big\{e^{-r\tau^*}\frac{\phi(X_{\tau^*})}{\phi(x^*)}\Big\} = \frac{\psi(x)}{\psi(x_\Gamma)}\frac{\phi(x_\Gamma)}{\phi(x^*)} = e^{a(x+x^*-2x_\Gamma)} \leq \rho. \qquad \square$$

The main assumption in the proof of the above lemma is the strict separation of the stopping region for $\mathcal{T}\xi$ from x^* by $x' > x^*$. This will be a crucial assumption in the following lemma which establishes conditions under which a fixed point of $\mathcal{T}$ is the lifetime value function.

Lemma 3.5. *Assume that there exists a fixed point ξ^* of $\mathcal{T}$ in the space of continuous non-negative functions and the corresponding optimal stopping time is a first hitting time of a closed set $\Gamma \subset (x^*, \infty)$. Then there is a constant $\rho < 1$ such that $\|\xi^* - \mathcal{T}^n\mathbf{0}\|_{\#} \leq \rho^n\|\xi^*\|_{\#}$ and $\|\xi^* - \mathcal{T}^n\mathbf{0}\|_\infty \leq \rho^{n-1}\|\xi^*\|_{\#}$, where $\|\cdot\|_\infty$ is the supremum norm.*

PROOF. Let $x_\Gamma = \min\Gamma$. Clearly, $\|\xi^* - \mathbf{0}\|_{\#} < \infty$. By virtue of Lemma 3.4 with $x' = x_\Gamma$, we have $\|\mathcal{T}^n\mathbf{0} - \xi^*\|_{\#} \leq \rho^n\|\mathbf{0} - \xi^*\|_{\#}$ for $\rho = e^{2a(x^*-x_\Gamma)} < 1$. Hence, $\mathcal{T}^n\mathbf{0}$ converges exponentially fast to ξ^* in the seminorm $\|\cdot\|_{\#}$. Using (11)

$$\|\xi^* - \mathcal{T}^n\mathbf{0}\|_\infty = \|\mathcal{T}\xi^* - \mathcal{T}\circ\mathcal{T}^{n-1}\mathbf{0}\|_\infty \leq \rho^{n-1}\|\xi^*\|_{\#}. \qquad \square$$

We have, therefore, demonstrated in Lemma 3.5 that if there exists a fixed point ξ^* of $\mathcal{T}$ with a corresponding optimal stopping time given by a closed stopping set Γ and contained in (x^*, ∞) then ξ^* is the lifetime value function. Moreover, the convergence of $\mathcal{T}^n\mathbf{0}$ to ξ^* is exponential in the supremum norm.

3.3. A bound on the contract payments

In order to address the first step in Objectives 3.3 we henceforth assume that

$$p_c + K_c < f(x^*). \tag{12}$$

This assumption means that the contract offers a discount relative to the market: the total cost of the contract (ignoring the time value of money) is strictly less than the price of the asset at the exercise time. This can be seen as a natural condition in our setup, as follows. If $p_c + K_c > f(x^*)$ then buying the asset in the market for the price $f(x^*)$ is cheaper than paying $p_c + K_c$ for the delivery of one unit of the asset under the contract. Thus when $X_t = x^*$, the contract is attractive to the second agent but is not attractive to the first agent. Alternatively when $p_c + K_c = f(x^*)$ the second agent can only make a profit by purchasing the asset and selling the option when $X_t > x^*$ (and so $f(X_t) < f(x^*) = p_c + K_c$). In this case the contract is not attractive to the first agent, since the alternative of depositing p_c in the bank and purchasing the asset in the market when its price reaches x^* is guaranteed to be cheaper (or has identical cost if $p_c = 0$).

The following lemma also shows that condition (12) is sufficient for the finiteness of the lifetime value function.

Lemma 3.6. *Assume condition* (12) *and that* $\mathcal{T}\mathbf{0} \neq \mathbf{0}$. *Then*

1. $\mathcal{T}\mathbf{0} > \mathbf{0}$,
2. $\Gamma^n \cap (-\infty, x^*] = \emptyset$, *where* Γ^n *is the stopping region for* $\mathcal{T}^n\mathbf{0} = \mathcal{T}(\mathcal{T}^{n-1}\mathbf{0})$,
3. *the functions* $\mathcal{T}^n\mathbf{0}$ *are strictly positive and uniformly bounded in* n,
4. *the limit* $V = \lim_{n\to\infty} \mathcal{T}^n\mathbf{0}$ *exists and is a strictly positive bounded function.*

Proof. 1. By assumption there is x such that $\mathcal{T}\mathbf{0}(x) > 0$. For any other $\hat{x}$ consider the following strategy: wait until the process (X_t) hits x and then proceed optimally. This results in a strictly positive expected value: $\mathcal{T}\mathbf{0}(\hat{x}) > 0$. By the arbitrariness of $\hat{x}$ we have $\mathcal{T}\mathbf{0} > \mathbf{0}$.

2. Assume that $x \in \Gamma^n \cap (-\infty, x^*]$. Then in the optimal stopping problem $\mathcal{T}^n\mathbf{0}$, if the process starts from x then the asset is bought immediately and the option is both written and exercised immediately, resulting in the payoffs

$$\mathcal{T}(\mathcal{T}^{n-1}\mathbf{0})(x) = -f(x) + p_c + K_c + \mathcal{T}^{n-1}\mathbf{0}(x) < \mathcal{T}^{n-1}\mathbf{0}(x)$$

by recalling the condition (12) and $f(x) > f(x^*)$. Consequently, the above inequality contradicts that $\mathcal{T}^n\mathbf{0}$, $n \geq 0$, is a non-decreasing sequence of functions, which we have established previously.

3. The monotonicity of $\mathcal{T}$ and statement 1 guarantee that if $\mathcal{T}\mathbf{0} > \mathbf{0}$ then $\mathcal{T}^n\mathbf{0} > \mathbf{0}$ for every n. Inequality (12) implies that there is $x' > x^*$ such that $p_c + K_c < f(x')$. We claim that the stopping region for every iteration $\mathcal{T}^n\mathbf{0}$ must be included in (x', ∞). Assume the opposite, i.e., that for some n this is not true. Let Γ be the optimal stopping region for $\mathcal{T}\xi$, where $\xi = \mathcal{T}^{n-1}\mathbf{0}$. Let $x_\Gamma = \min \Gamma \leq x'$ (the set Γ is closed). Recall that $x_\Gamma > x^*$ by statement 2. Then

$$\mathcal{T}\xi(x^*) = \mathbb{E}^{x^*}\Big\{e^{-r\tau_{x_\Gamma}}\Big(p_c - f(x_\Gamma) + \big(K_c + \xi(x^*)\big)\mathbb{E}^{x_\Gamma}\{e^{-r\tau_{x^*}}\}\Big)\Big\} < \xi(x^*),$$

as $f(x_\Gamma) \geq f(x') > p_c + K_c$, where $\tau_z := \inf\{t \geq 0 : \; X_t = z\}$. This contradicts the monotonicity of the sequence $\mathcal{T}^n\mathbf{0}$. Consequently, the stopping region Γ for each iteration $\mathcal{T}^n\mathbf{0}$ has an empty intersection with the interval $(-\infty, x']$. Using this fact we obtain

$$\begin{aligned}\mathcal{T}^{n+1}\mathbf{0}(x^*) &= \mathbb{E}^{x^*}\Big\{e^{-r\tau_{x_\Gamma}}\Big(p_c - f(x_\Gamma) + \big(K_c + \mathcal{T}^n\mathbf{0}(x^*)\big)\mathbb{E}^{x_\Gamma}\{e^{-r\tau_{x^*}}\}\Big)\Big\}\\ &\le \mathbb{E}^{x^*}\big\{e^{-r\tau_{x_\Gamma}}(p_c + K_c + \mathcal{T}^n\mathbf{0}(x^*))\big\} \le e^{-a(x'-x^*)}\Big(p_c + K_c + \mathcal{T}^n\mathbf{0}(x^*)\Big),\end{aligned}$$

(using equations (2)), where $x_\Gamma > x'$ is the left end of the stopping region for $\mathcal{T}(\mathcal{T}^n\mathbf{0})$. Therefore, $\mathcal{T}^{n+1}\mathbf{0}(x^*) \le (p_c + K_c)\frac{\rho}{1-\rho} =: M$. Also $\mathcal{T}^{n+1}\mathbf{0}(x) \le p_c + K_c + \mathcal{T}^n\mathbf{0}(x^*) \le p_c + K_c + M$ for any x.

4. By the monotonicity of $\mathcal{T}^n\mathbf{0}$ there exists a limit V which is bounded from above by $p_c + K_c + M$. □

3.4. Solutions for the lifetime problem

We now complete Objectives 3.3 by carrying out steps 1 and 2. Finding a fixed point of $\mathcal{T}$ is equivalent to finding a fixed point of $\tilde{\mathcal{T}}$. In Section 2 we have studied the particular problem of evaluating $\mathcal{T}\mathbf{0}$ or, equivalently, $\tilde{\mathcal{T}}\mathbf{0}$. Our results (see also Appendix C) show that when the stopping region Γ of the single option has empty intersection with $(-\infty, x^*]$ as specified in Section 3.3, the set $\hat{\Gamma}$ (the coincidence set of W and H which we also call the stopping region) is of the form $[A, B]$, where $B \in \mathbb{R} \cup \{\infty\}$ and $A > y^*$. We concluded that in such cases the smooth fit principle holds at A and the value function $\tilde{\mathcal{T}}\mathbf{0}$ is linear between 0 at 0 and $H(A)$ at A. We will now show that a similar form for the value function persists with iterations of the operator $\tilde{\mathcal{T}}$.

Assume that $\tilde{\mathcal{T}}\mathbf{0} \neq 0$ (otherwise, the perpetual value function is zero). Denote by γ_1 the slope of $\tilde{\mathcal{T}}\mathbf{0}$ at 0, i.e., $\tilde{\mathcal{T}}\mathbf{0}(y) = \gamma_1 y$ for $y \in (0, y^*]$. By equations (10), when $\tilde{\mathcal{T}}$ is again applied to $\zeta_1 = \tilde{\mathcal{T}}\mathbf{0}$ only the 'continuation value' ζ_1 on $(0, y^*]$ plays a role and so the form of ζ_1 outside the interval $(0, y^*]$ is irrelevant for the operator $\tilde{\mathcal{T}}$. Let $\bar{H}_1$ be the obstacle $\bar{H}$ in (10) when we set $\zeta = \zeta_1$, i.e.,

$$\bar{H}_1(y) = H(y) + \gamma_1 \min(y, y^*),$$

where H is the obstacle in the one-option case (see the beginning of Section 2.3). In the proof of Lemma 3.6 we showed that the stopping region corresponding to $\tilde{\mathcal{T}}\zeta_1$ has an empty intersection with $(0, y^*]$. Therefore $\tilde{\mathcal{T}}\zeta_1$, being the smallest non-negative concave majorant of $\bar{H}_1$, is a straight line between 0 and the left end of the stopping region, in particular, for $y \in (0, y^*]$. Since $\bar{H}_1(y)$ is non-positive in a sufficiently small neighbourhood of 0 (see the properties of $\hat{g}$, c.f. Table 1 and equations (C.1) and (C.2) in the Appendix) we conclude that this straight line has zero intercept (it passes through 0 at 0). Denote its slope by γ_2. Iterating this argument we obtain that $\tilde{\mathcal{T}}^n\mathbf{0}(y) = \gamma_n y$ on $(0, y^*]$ for a sequence $(\gamma_n)_{n\ge1}$ of positive numbers. The functions $\tilde{\mathcal{T}}^n\mathbf{0}$ increase to $\zeta^* = \mathcal{M}V$, cf. Lemma 3.6. Therefore, $\zeta^*(y) = \gamma^* y$ for $y \in (0, y^*]$ with $\gamma^* = \lim_{n\to\infty}\gamma_n$. We summarise our findings in the following lemma.

Lemma 3.7. *If* (12) *holds then* $\zeta_n = \tilde{\mathcal{T}}^n\mathbf{0}$ *is linear on* $(0, y^*]$*:* $\zeta_n(y) = \gamma_n y$ *for* $y \le y^*$*. Further, the limit* $\zeta^* = \lim_{n\to\infty}\tilde{\mathcal{T}}^n\mathbf{0}$ *is linear on* $(0, y^*]$*:* $\zeta^*(y) = \gamma^* y$ *for* $y \le y^*$*.*

In order to address step 2 of Objectives 3.3 we proceed now to finding explicitly the initial slope γ^* of the function ζ^* and the stopping region $\hat{\Gamma}$ of $\tilde{\mathcal{T}}\zeta^*$. Let $A = \min \hat{\Gamma}$ be the left end of the stopping region corresponding to the obstacle $\bar{H}$. The following lemma, which echoes the second part of Lemma 3.6, implies that $A > y^*$:

Lemma 3.8. *Assume* (12) *and* $V \neq \mathbf{0}$. *Then the stopping region* Γ *for* $\mathcal{T}V$ *must have an empty intersection with* $(-\infty, x^*]$. *Or, equivalently, the stopping region* $\hat{\Gamma}$ *corresponding to* $\tilde{\mathcal{T}}\zeta^*$ *must have an empty intersection with* $(0, y^*]$.

Proof. Otherwise, for any $x \in \Gamma \cap (-\infty, x^*]$ using (12) and Lemma 3.7 we have $\tilde{\mathcal{T}}\zeta^*(y) = p_c + K_c - f(x) + \zeta^*(y) \leq \gamma^* y - \delta$ with $y = F(x)$ and $\delta = f(x^*) - p_c - K_c > 0$. On the other hand, for any n we have $\tilde{\mathcal{T}}\zeta^*(y) \geq \tilde{\mathcal{T}}\tilde{\mathcal{T}}^n\mathbf{0}(y) = \tilde{\mathcal{T}}^{n+1}\mathbf{0}(y) = \gamma_{n+1}y$. Hence $\gamma^* y - \delta \geq \gamma_{n+1}y$ for $n \geq 0$. This contradicts that γ^* is the limit of γ_n. □

If, as we hope, ζ^* is the fixed point of $\tilde{\mathcal{T}}$ sought in step 1 of Objectives 3.3 then it is also the smallest concave majorant of $\bar{H}$ (with $\zeta = \zeta^*$). In this case ζ^* is linear at least on $(0, A)$. Further, since Γ is closed we have from Lemma 3.8 that $A > y^*$ and so, since $\bar{H}$ is differentiable at A, the principle of smooth fit would hold there. We will therefore search for A such that the straight line $y \mapsto \gamma^* y$ is tangent to $\bar{H}$ at A. This leads to the following characterisation of A:

$$\begin{aligned} &\bar{H}(y) < \gamma^* y \quad \text{on } (0, A), \\ &\bar{H}(A) = \gamma^* A, \\ &\bar{H}'(A) = \gamma^*. \end{aligned} \tag{13}$$

We will use this characterisation to explicitly find A and γ^*.

Substituting (10) (with $\zeta = \zeta^*$ and using Lemma 3.7) into the last two equations of (13) and eliminating γ^* we obtain

$$\bar{\eta}(A) := (p_c - D - dA^{-\frac{b}{2a}})\Big(\sqrt{A} + \frac{1}{\sqrt{A}}y^*\Big) - \frac{bd}{a}A^{\frac{1}{2}-\frac{b}{2a}}(1 - A^{-1}y^*) + 2K_c\sqrt{y^*} = 0.$$

We conclude this section with the case by case analysis, which is summarised in Table 2, with the graphical form of solutions presented in Table 1. In each case we verify that equations (13) are satisfied. Using equations (7) we define for any $\gamma > 0$:

$$g_\gamma(y) = g(y) + \gamma y^*, \qquad \hat{g}_\gamma(y) = \hat{g}(y) + \gamma y.$$

3.4.1. Case $b > a$

We have $\lim_{y \downarrow 0} \hat{g}_\gamma(y) = -\infty$, $\hat{g}_\gamma$ is increasing and strictly concave, and $g'_\gamma(y) < \hat{g}'_\gamma(y)$ for $\gamma > 0$ (see equation C.1 in Appendix and Table 1).

Case $p_c \geq D$: In this case g_γ is concave on $(0, \infty)$. By Lemma 3.8 the stopping region is contained in (y^*, ∞). Let us find the tangency point, i.e., search

Table 2: Stopping regions for the lifetime problem.

Parameters			Stopping region	Figure 1
$b > a$	$p_c \geq D$		$\hat{\Gamma} = [A, \infty)$	d
	$p_c < D$	$\bar{\eta}(Y_m) \geq 0$ and $Y_m > y^*$	$\hat{\Gamma} = [A, Y_m]$	c
		otherwise	$V_c = 0$	
$b = a$	$p_c > D$		$\hat{\Gamma} = [A, \infty)$	d
	$p_c \leq D$		$V_c = 0$	
$b < a$	$p_c > D$		$\hat{\Gamma} = [A, \infty)$	d
	$p_c \leq D$		$V_c = 0$	

for a root $\bar{\eta}(A) = 0$. Notice that $\bar{\eta}(y^*) = 2\sqrt{y^*}(p_c + K_c - f(x^*)) < 0$ and $\lim_{y\to\infty} \bar{\eta}(y) = \infty$. Since

$$\bar{\eta}'(y) = \frac{1}{2}(1 - y^{-1}y^*)y^{-\frac{1}{2}}\left(dy^{-\frac{b}{2a}}\left(\frac{b^2}{a^2} - 1\right) + p_c - D\right) > 0 \qquad \text{on } (y^*, \infty), \tag{14}$$

there is exactly one point $A > y^*$ such that $\bar{\eta}(A) = 0$. Moreover, $\gamma^* = g'_\gamma(A) = g'(A) > 0$ since $g' > 0$ on $(0, \infty)$ (see Table 1 and recall that $Y_m = \infty$). Noting that $\bar{H}$ is strictly concave now and the line $y \mapsto \gamma^* y$ is tangent to $\bar{H}$ at A, it dominates $\bar{H}$ strictly everywhere apart from A satisfying therefore the inequality in (13). In view of the previous discussion, A and γ^* uniquely determine ζ^*. Indeed, knowing ζ^* on $(0, y^*]$ is sufficient for the operator $\tilde{\mathcal{T}}$ and we extend ζ^* to the whole domain by applying $\tilde{\mathcal{T}}$. Consequently, the perpetual value function $V = \mathcal{M}^{-1}\zeta^*$ (c.f. Lemma 3.5) and the stopping region corresponding to $\tilde{\mathcal{T}}\zeta^*$ is $\hat{\Gamma} = [A, \infty)$.

Case $p_c < D$**:** Recall the definitions of Y_c, Y_m and that $Y_c > Y_m$.

For any $\gamma > 0$, the function g_γ is strictly increasing on $(0, Y_m)$ and strictly decreasing on (Y_m, ∞). It is strictly concave for $y < Y_c$ and strictly convex for $y > Y_c$. A fixed point of $\tilde{\mathcal{T}}$ is characterised by the system (13), whose solution can be found by considering the zeros of the function $\bar{\eta}$. Clearly, $\bar{\eta}(y^*) < 0$, but contrary to the previous case we have $\lim_{y\to\infty} \bar{\eta}(y) = -\infty$. Since $\bar{\eta}$ is strictly increasing on (y^*, Y_c) and strictly decreasing on (Y_c, ∞) (see the derivative equation (14)), the equation $\bar{\eta}(y) = 0$ has a solution iff $\bar{\eta}(Y_c) \geq 0$. Any solution $A > Y_m$ violates the condition $\gamma^* = g'_\gamma(A) > 0$. Hence, the operator $\tilde{\mathcal{T}}$ has a fixed point determined by (13) iff $\bar{\eta}(Y_m) \geq 0$ and $Y_m > y^*$. This solution can be efficiently found numerically due to the monotonicity of $\bar{\eta}$ on $[y^*, Y_m]$. It is again clear by the strict concavity of $\bar{H}(A)$ on $(0, Y_m)$ that the inequality in (13) is satisfied. The stopping region corresponding to this fixed point is $[A, Y_m]$. We show below that other combinations of parameters lead to a trivial value function for the perpetual regime.

$Y_m \leq y^*$: Using the notation from Section 2, the function H is strictly decreasing on (y^*, ∞). Hence the stopping region $\hat{\Gamma}$ for the one-option case lies in the interval $(0, y^*]$, so from Section 3.3 it must therefore be empty and $V_c = V = 0$.

$Y_m > y^*$ and $\bar{\eta}(Y_m) < 0$: Assume that $\mathcal{T}\mathbf{0} \neq \mathbf{0}$. By Lemma 3.6 the perpetual value function V is well-defined, finite and strictly positive. Hence the assumptions of Lemma 3.2 are satisfied and the stopping problem $\mathcal{T}\zeta^*$, where $\zeta^* = \mathcal{M}V$, has a non-trivial solution given by a stopping region $\hat{\Gamma}$. By Lemma 3.8 its stopping set $\hat{\Gamma}$ must have an empty intersection with $(0, y^*]$ and $y_\Gamma := \min \hat{\Gamma} \in (y^*, \infty)$. The smallest non-negative concave majorant of $\bar{H}$ must, therefore, meet $\bar{H}$ smoothly at y_Γ and the characterisation of equations (13) must hold. This contradicts the assumption $\bar{\eta}(Y_m) < 0$. Consequently, $\mathcal{T}\mathbf{0} = \mathbf{0}$ and the perpetual value function is zero.

3.4.2. Case $b = a$

When $p_c > D$, the solution follows identical lines as for $b > a$. When $p_c \leq D$, we have $V = 0$. Indeed, the condition $p_c + K_c < f(x^*)$ from Section 3.3 is expanded as $p_c + K_c < D + d(y^*)^{-1/2}$. This implies that $g(y^*) = (p_c - D)\sqrt{y^*} + K_c\sqrt{y^*} - d < 0$, which, together with g being non-increasing (see Subsections C.2.3 and C.2.4 in the Appendix), gives $\tilde{\mathcal{T}}\mathbf{0} = \mathbf{0}$.

3.4.3. Case $b < a$

If $p_c \leq D$ then (12) yields $g(y^*) = (p_c + K_c - f(x^*))\sqrt{y^*} < 0$ and $\tilde{\mathcal{T}}\mathbf{0} = \mathbf{0}$ (see Subsection C.1.1 in the Appendix) so that $V = 0$.

If $p_c > D$ then $V_c \neq 0$ from Section C.1.2 in the Appendix and hence $V \neq 0$. Let $\hat{\Gamma}$ be the stopping region corresponding to $\tilde{\mathcal{T}}\zeta^*$ with $\zeta^* = \mathcal{M}V$ and $y_\Gamma = \min \hat{\Gamma} \in (y^*, \infty)$. We will show that $y_\Gamma \geq Y_c$ and $\bar{H}'(y_\Gamma) = \gamma^*$ with γ^* being such that $\zeta^*(y) = \gamma^* y$ for $y \in (0, y_\Gamma]$. Recall that g_{γ^*} shares convexity properties with g, i.e., it is strictly convex on $(0, Y_c)$ and strictly concave on (Y_c, ∞). A concave majorant of a strictly convex function can coincide with this function only at the ends of the domain interval. Since g_{γ^*} is strictly convex on (y^*, Y_c) so $y_\Gamma \notin (y^*, Y_c)$. By concavity of $\bar{H} = g_{\gamma^*}$ on $[Y_c, \infty)$ the smallest concave majorant of $\bar{H}$ coincides with $\bar{H}$ on $\hat{\Gamma} = [y_\Gamma, \infty)$. Appealing to continuous differentiability of g_{γ^*} proves that there must be a smooth fit at y_Γ, i.e., the derivatives of $\bar{H}$ and its smallest concave majorant ζ^* must be equal: $\bar{H}'(y_\Gamma) = \gamma^*$ (recall that ζ^* is a fixed point of $\tilde{\mathcal{T}}$).

It remains to find y_Γ explicitly. Above discussion shows that $A = y_\Gamma > y^*$ and $\gamma^* > 0$ satisfy the system of two equations in (13). We have $\bar{\eta}(y^*) < 0$ and $\lim_{A \to \infty} \bar{\eta}(A) = \infty$. Since $\bar{\eta}$ is decreasing on $(0, Y_c)$ and increasing on (Y_c, ∞), there is a unique root A of $\bar{\eta}$ on $(\max(y^*, Y_c), \infty)$. Since $Y_m < Y_c$ we have $\gamma^* = g'(A) = g'_\gamma(A) > 0$.

Summarising, there is a unique solution of the system of equations (13) with constraints $A > y^*$ and $\gamma^* > 0$. This corresponds to a fixed point of $\tilde{\mathcal{T}}$ with the stopping region $\hat{\Gamma} = [A, \infty)$.

4. Discussion and conclusions

The real-time balancing of electrical power systems is a challenging problem which has intertwined technical and economic perspectives. In this context we have studied the possible introduction of American-style call options in an imbalance market, physically hedged by using a storage device. We have observed that if the option writer purchases the underlying when the imbalance is already negative, this would have the clearly unintended consequence of instantaneously worsening the imbalance. In particular, delta hedging, which is a classical approach in mathematical finance, would have this effect.

In this paper we have shown that by requiring physical cover, such options may be designed without this unintended consequence. In particular we have identified explicit conditions under which the storage operator has a positive expected economic profit from the option, and, further, the underlying is only bought when the imbalance is positive. This analysis is relevant to parties involved in the design of contracts for power system balancing. Further we have computed analytically the value to the storage operator of one such contract and the lifetime value of a sequence of such contracts. This analysis is relevant to commercial operators of electricity storage, to establish the commercial viability of capital investments under such contracts.

It would be interesting to study a quantitative measure of the contribution of such options to the balancing challenge. This would involve significant extra complexity beyond the scope of the present work as we do not model price impact. Since emerging electricity storage technologies are often embedded within the electricity distribution network, the network power flow equations would play a role. The latter study would essentially be one of time-domain power system simulation under uncertainty in the context of power systems engineering. In contrast, our requirement in this paper is merely that the introduction of option contracts should not have the unintended consequence of exacerbating (rather than reducing) imbalance and should also provide a commercial opportunity to the storage operator, provided that the capital cost of the store is sufficiently competitive.

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Appendix A Plots of UK system prices

Figure A.1 provides a histogram of the UK main system price, which is used in the imbalance mechanism, between 2nd June 2013 and 12th January 2016. The boxplot is displayed in Figure A.2. The data was obtained from the *ELEXON Portal*.

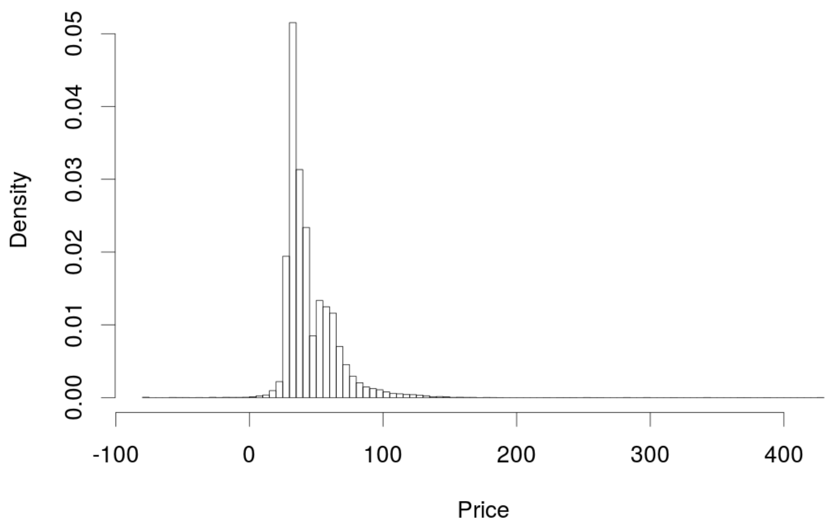


Figure A.1: Histogram of main UK system price, 2nd June 2013 to 12th January 2016.

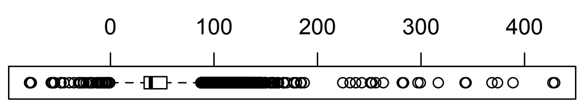


Figure A.2: Boxplot of main UK system price, 2nd June 2013 to 12th January 2016.

Appendix B Auxiliary results for smooth fit

This appendix provides several results concerning the existence of points of smooth fit.

Lemma B.1. *Let* $h : [x, z] \to \mathbb{R}$ *for some* $x > 0$ *and* $z \in (x, \infty]$ *satisfy* $h(x) = 0$ *and* $h'(y) \geq h(y)/y$ *for* $y \in (x, z)$. *Then* $h \geq 0$.

Proof. Let $g = -h$. Then $g'(y) \leq g(y)/y$ and Gronwall's lemma yields $g(y) \leq g(x)e^{\int_x^y u^{-1}du} = 0$.

□

Lemma B.2. *Let* f *be a continuously differentiable function on* $[0, A)$ *for* $A \in (0, \infty]$. *Assume that* $f(y) = 0$ *for some* $0 < y < A$, $\lim_{y \to A} f(y) > 0$ *and*

$\lim_{y\to A} f'(y) \le 0$. *Then there is a point* $z \in [y, A)$ *such that* $f(z)/z = f'(z) \ge 0$. *Moreover, there is at most one such point on each interval of strict concavity of* f *(concavity is sufficient at the ends of the interval).*

PROOF. Let $y_1 < A$ be the largest root of f (its existence is guaranteed by $\lim_{y\to A} f(y) > 0$). Then $y_1 > 0$ and $f(y) > 0$ on (y_1, A). Define $\xi(y) = f(y) - f'(y)y$. Clearly, $\xi(y_1) \le 0$. If there is $y_2 > y_1$ such that $\xi(y_2) > 0$ then by continuity ξ must have a root z between y_1 and y_2. Since $f(z) \ge 0$ then $f'(z) \ge 0$.

Assume, for a contradiction, that $\xi(y) \le 0$ on $[y_1, A)$ and take any $x \in (y_1, A)$. Let g be the solution to the ODE: $g(y) - g'(y)y = 0$ for $y \in [x, A)$, $g(x) = f(x)$, i.e., $g(y) = yf(x)/x \ge f(x)$. Let $h = f - g$. By Lemma B.1, $h \ge 0$, i.e., $f \ge g$ on $[x, A)$. When $A < \infty$ then since $\lim_{y\to A} f'(y) \le 0$ we have $\lim_{y\to A} \xi(y) \ge \lim_{y\to A} f(y) \ge \lim_{y\to A} g(y) \ge f(x) > 0$, a contradiction. Otherwise $A = \infty$ and $\lim_{y\to\infty} h'(y) \le -f(x)/x < 0$, which contradicts the positivity of h.

Assume further that f is concave on $[a, b]$ and strictly concave inside of this interval. Roots of ξ define tangents to f of the form $x \mapsto x\, f(y)/y$. Due to concavity the function f is majorised by its tangents. Hence, if there are two roots y_1, y_2 of ξ then these tangents have to coincide. This is impossible due to strict concavity. □

Corollary B.3. *Point* z *in the above lemma can be chosen such that* f *is not strictly convex in its neighbourhood.*

PROOF. Assume that f is strictly convex on (l, r). This implies $f(y_1) > f(y_2) + f'(y_2)(y_1 - y_2)$ for any $y_1, y_2 \in (l, r)$ and $y_1 < y_2$. Rearranging the terms yields

$$\xi(y_2) = f(y_2) - f'(y_2)y_2 < f(y_1) - f'(y_2)y_1 < f(y_1) - f'(y_1)y_1 = \xi(y_1),$$

where we used the fact that $f'(y_1) < f'(y_2)$ following from strict convexity. Hence, ξ is strictly decreasing on intervals of strict convexity of f. Similarly, ξ is non-increasing on intervals of convexity of f and non-decreasing on the intervals of concavity of f.

Let z be the point constructed in the proof of Lemma B.2. Assume that f is strictly convex in the neighbourhood (l, r) of z. Then $\xi(y) < 0$ on $(z, r]$. This implies that there is a root of ξ on (r, A). Let $\hat{z}$ be the root on (r, A) closest to r. Then $\xi < 0$ on $(z, \hat{z})$ and if f were strictly convex around $\hat{z}$ then f would decrease to 0 at $\hat{z}$, a contradiction. This implies that f is not stricty convex around $\hat{z}$. □

Corollary B.4. *Assume that* f *is continuously differentiable on* $[0, \infty)$ *and strictly convex on* $(0, r)$. *If* $f(0) = 0$, $\lim_{y\to\infty} f(y) > 0$ *and* $\lim_{y\to\infty} f'(y) \le 0$ *then there exists* $z > 0$ *such that* $f(z)/z = f'(z) \ge 0$.

PROOF. If there is $y > 0$ such that $f(y) = 0$ then the result follows from Lemma B.2. Otherwise, assume that $f > 0$ on $(0, \infty)$. Define $\xi(y) = f(y) - f'(y)y$.

Then $\xi(0) = 0$ and ξ is decreasing on the interval of strict convexity $(0, r)$. Hence $\xi(r) < 0$. Arguments from the proof of Lemma B.2 imply that there is $y_2 > r$ such that $\xi(y_2) > 0$. This combined with the continuity of ξ yields that there is a root z of ξ on (r, y_2). Recalling that $f(z) > 0$ we obtain $f'(z) > 0$. □

Lemma B.5. *Assume that function f is continuously differentiable on $[0, \infty)$, convex on $[0, \bar{y}]$ and increasing on $[\bar{y}, \infty)$ for some $\bar{y} > 0$ and that the following hold:*

$$\begin{aligned} f(0) &= 0, \\ \lim_{y\to\infty} f(y) &> 0, \\ \lim_{y\to\infty} f'(y) &= 0. \end{aligned}$$

Then there is a point $y \in [\bar{y}, \infty)$ such that $f(y)/y = f'(y)$. Moreover, if f is strictly concave on $(\bar{y}, \infty)$ then this point is unique.

Proof. If there is $y \geq \bar{y}$ such that $f(y) \leq 0$, then the result follows from Lemma B.2. Otherwise, $f > 0$ on $[\bar{y}, \infty)$. Define $\xi(y) = f(y) - f'(y)y$. By convexity of f, $f(0) \geq f(\bar{y}) + f'(\bar{y})(0 - \bar{y})$. Hence, $\xi(\bar{y}) \leq 0$. Existence of y such that $\xi(y) > 0$ completes the proof due to continuity of ξ. Assume, by contradiction, that $\xi \leq 0$ on $[\bar{y}, \infty)$. Let g be the solution to the ODE: $g(y) - g'(y)y = 0, g(\bar{y}) = f(\bar{y})$, i.e., $g(y) = yf(\bar{y})/\bar{y}$. Let $h = f - g$. By Lemma B.1, $h \geq 0$, i.e., $f \geq g$ on $[\bar{y}, \infty)$. But then $\lim_{y\to\infty} f'(y) \geq f(\bar{y})/\bar{y} > 0$, a contradiction. Uniqueness is proved identically as in Lemma B.2. □

Appendix C Single option: case-by-case analysis

Notice that $g(y^*) = \sqrt{y^*}\big(p_c + K_c - f(x^*)\big)$, hence its sign is determined by the relation between $p_c + K_c$ and $f(x^*)$. This will be useful in interpreting the conditions arising in the analysis below.

For the convenience of the reader we state the derivatives of $g, \hat{g}$ and η:

$$\begin{aligned}
\eta'(y) &= \frac{1}{2} y^{-\frac{1}{2}} \left[p_c - D - d\left(1 - \frac{b^2}{a^2}\right) y^{-\frac{b}{2a}} \right], \\
g'(y) &= \frac{1}{2} y^{-\frac{1}{2}} \left[p_c - D - d\left(1 - \frac{b}{a}\right) y^{-\frac{b}{2a}} \right], \\
\hat{g}'(y) &= \frac{1}{2} y^{-\frac{1}{2}} \left[K_c + p_c - D - d\left(1 - \frac{b}{a}\right) y^{-\frac{b}{2a}} \right], \\
g''(y) &= \frac{1}{4} y^{-\frac{3}{2}} \left[D - p_c + d\left(1 - \frac{b^2}{a^2}\right) y^{-\frac{b}{2a}} \right], \\
\hat{g}''(y) &= \frac{1}{4} y^{-\frac{3}{2}} \left[D - K_c - p_c + d\left(1 - \frac{b^2}{a^2}\right) y^{-\frac{b}{2a}} \right].
\end{aligned} \tag{C.1}$$

A summary of the results for each case is collected in Tables C.1-C.3. Graphs showing the shape of the obstacle and related stopping regions are located in

Table C.1: Stopping regions for the single option when $b < a$. Whenever the stopping region is trivial we write $V_c = 0$.

Stopping regions in the case $b < a$.					
Parameter range				Stopping region	Figure C.2
$p_c \leq D$	$K_c + p_c \leq D$			$V_c = 0$	
	$K_c + p_c > D$		$K_c + p_c \leq f(x^*)$	$V_c = 0$	
			$K_c + p_c > f(x^*)$	$\hat{\Gamma} = [\min\{\hat{y}_b, y^*\}, y^*]$	a
$p_c > D$	$K_c + p_c \leq f(x^*)$			$\hat{\Gamma} = [y_b, \infty)$	d
	$K_c + p_c > f(x^*)$	Case A	$y^* \geq Y_c$	$\hat{\Gamma} = [\min(\hat{y}_b, y^*), \infty)$	e
			$y^* < Y_c$	$\hat{\Gamma} = [\min(\hat{y}_b, y^*), y^*] \cup [y_b^{(1)}, \infty)$	f & Fig. 2
		Case A^c		$\hat{\Gamma} = [y_b, \infty)$	d

Figures C.1 and C.2 with links in the last column of the aforementioned tables for guidance. For futher clarity, the graphs in Figure C.1 display the smallest concave majorant of the obstacle in red and blue. The blue region, which is where the majorant coincides with the obstacle, defines the stopping region.

C.1 Solutions in the case $b < a$

A summary of the results of this subsection is presented in Table C.1.

C.1.1 Case $p_c \leq D$

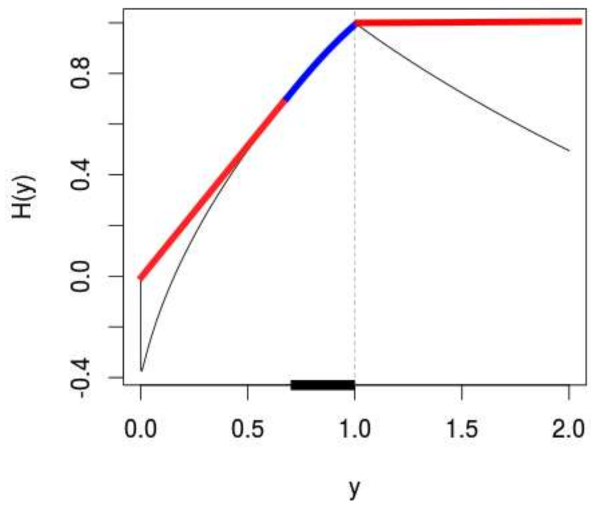
When $K_c + p_c \leq D$: Each of $Y_m, Y_c, \hat{Y}_m, \hat{Y}_c$ are equal to positive infinity, hence g and $\hat{g}$ are decreasing on $(0, \infty)$. Combining this with $\hat{g}(0) = 0$ makes H non-positive and W zero everywhere, giving $V_c = 0$.
When $K_c + p_c > D$: $\hat{g}$ is 0 at 0, then decreases and, if $\hat{Y}_m < y^*$, later increases, to meet g at y^*. Also g is decreasing everywhere as $Y_m = \infty$. Hence, $g(y^*) \leq 0$ makes H non-positive and W zero everywhere, giving $V_c = 0$. When $g(y^*) > 0$ we have $V_c \neq 0$, the stopping region $\hat{\Gamma}$ has right endpoint y^* and exercise is profitable at y^*. We have $\hat{\Gamma} = [\min\{y^*, \hat{y}_b\}, y^*]$ (see panel (a) in Figure C.1). Note that the smooth fit condition never holds at the right end of the stopping region, and holds at the left end only if $\hat{y}_b < y^*$.

C.1.2 Case $p_c > D$

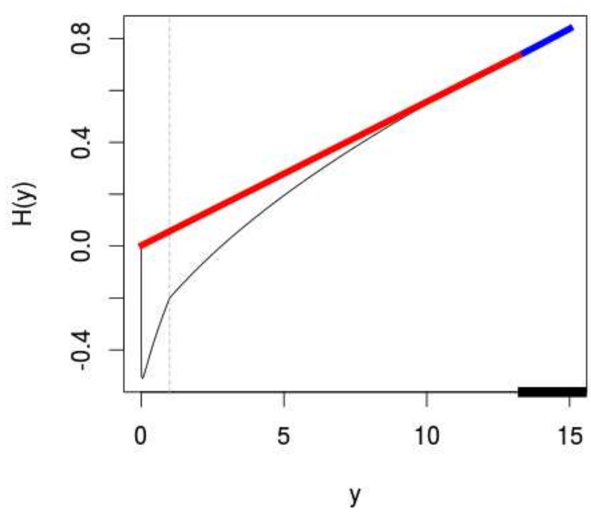
Both functions g and $\hat{g}$ are convex close to 0 (decreasing then increasing) and then concave, increasing without bound, and so the majorant W is nonzero and $V_c \neq 0$.

$g(y^*) \leq 0$: There exists $y_0 \geq y^*$ such that g is nonpositive on $(0, y_0]$ and positive on (y_0, ∞). Since $g(y)$ grows to infinity as $y \to \infty$ and $\lim_{y\to\infty} g'(y) = 0$, by Lemma B.2 in the Appendix there exists a point $y_b \geq y_0$ such that the tangent to g at y_b crosses the origin, i.e., $g'(y_b) = g(y_b)/y_b$ and $g'(y_b) > 0$ since $g(y_b) > 0$. This point can be taken on the concave part of g (so that $y_b \geq Y_c$) by Corollary B.3. Then it is unique by Lemma 2.2 and we conclude that $\hat{\Gamma} = [y_b, \infty)$.

Figure C.1: Illustrative plots for the single option obstacle and stopping region (thick horizontal line). The dashed vertical lines mark y^*. The least nonnegative concave majorant W is shown in blue (where W coincides with H) and red (otherwise).

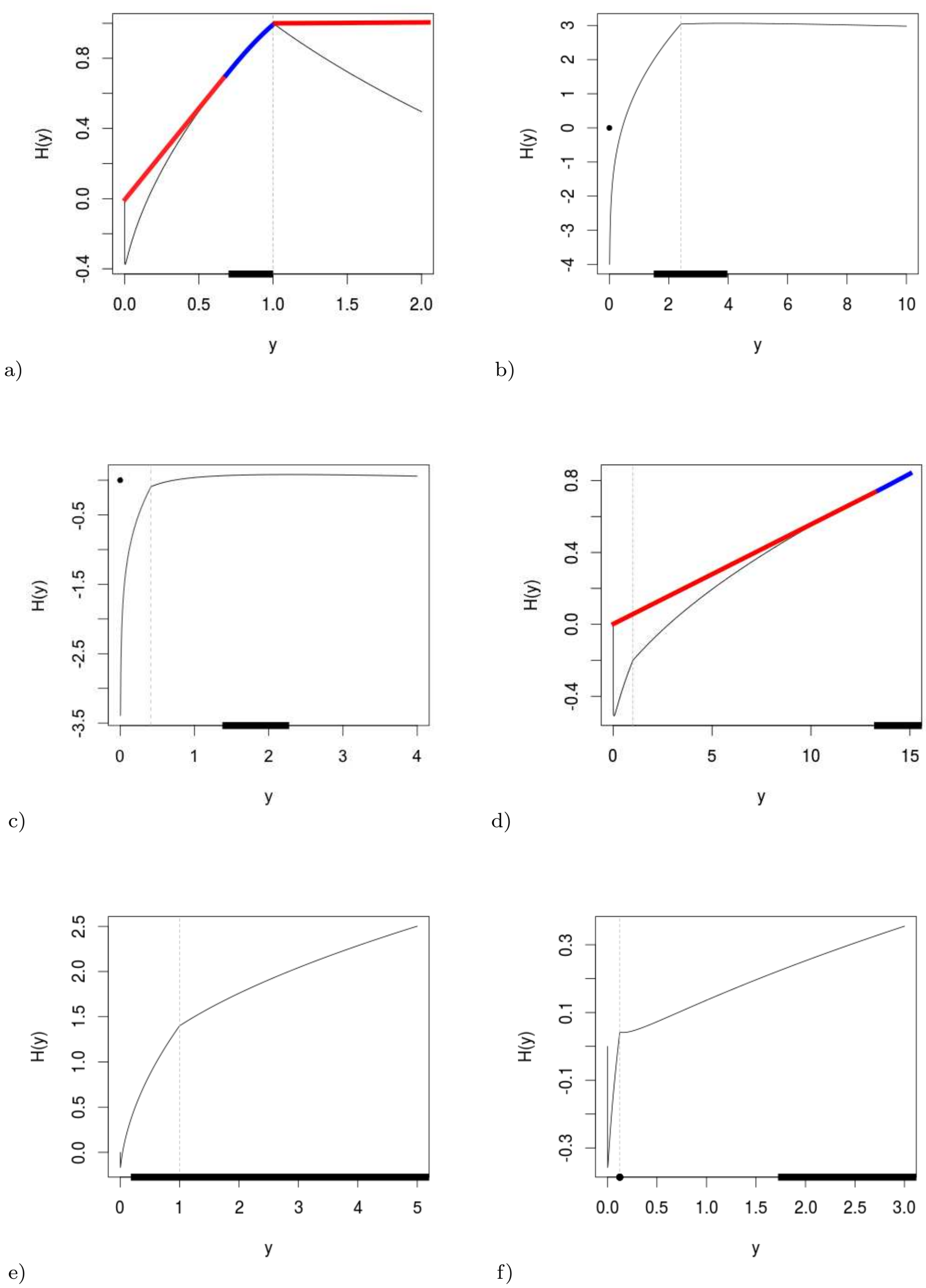


a)



b)

Figure C.2: Illustrative plots for the single option obstacle and stopping region (thick horizontal line). The dashed vertical lines mark y^*.



When $g(y^*) > 0$ let $y' := \min(\hat{y}_b, y^*)$. We distinguish between the following two cases:

- Case A: $g(y)/y \le \hat{g}(y')/y'$ for all $y > y^*$ and hence the majorant coincides with H at y' (note that $\hat{g}$ is concave at $\hat{y}_b$)
- Case A^C: there exists $\tilde{y} > y^*$ with $g(\tilde{y})/\tilde{y} > \hat{g}(y')/y'$ and so the majorant does not coincide with H anywhere on $[0, y^*]$. If the majorant touches H then it must do so to the right of y^* and then smooth fit holds.

Lemma C.1. *Case A^C holds if and only if both of the following conditions hold:*

1. *η has a root $y_b > \max(y^*, Y_c)$,*
2. *$g'(y_b) > \hat{g}(y')/y'$.*

Proof. Suppose first that case A^C holds. For condition 1, apply Lemma B.5 to the function $g^{(1)}(y) = g(y^*+y) - g(y^*)$ on $[0, \infty)$ (taking $\bar{y} = \max(0, Y_c - y^*)$) to establish the existence of a smooth fit point $\tilde{y}_b^{(1)}$. Let $y_b^{(1)} = \tilde{y}_b^{(1)} + y^*$. Since we are in case A^C, it is easy to see that the tangent to g at $y_b^{(1)}$ has a negative intercept at the vertical axis, i.e., $\eta(y_b^{(1)}) < 0$ and since $\eta \to \infty$ as $y \to \infty$ it follows that η has a root y_b in $[y_b^{(1)}, \infty)$. Since the tangent at y_b must strictly dominate H on $[0, y^*]$, condition 2 follows.

Conversely, if conditions 1 and 2 hold then the conclusion follows by the definition of η. □

Case A and $y^* \ge Y_c$: H is concave at every point in $[y', \infty)$ (because $\hat{g}$ is steeper at y^* than g and both are concave there) so $\hat{\Gamma} = [y', \infty)$.

Case A and $y^* < Y_c$: Then H is convex on (y^*, Y_c). The problem decomposes into (i) finding the smallest non-negative concave majorant of $\hat{g}$ on $[0, y^*]$ and (ii) finding the smallest non-negative concave majorant of the function $g^{(1)}(y) = g(y^*+y) - g(y^*)$ on $[0, \infty)$. The majorant in (i) coincides with $\hat{g}$ on $[y', y^*]$ and is linear on $(0, y')$. Since $\lim_{y\to\infty} g^{(1)}(y) = \infty$ and the derivative converges to 0 as $y \to \infty$, there exists a unique point z_b such that $g^{(1)}$ and its smallest nonnegative concave majorant coincide exactly on $[z_b, \infty)$ (apply Corollaries B.3 and B.4 in the Appendix and recall that $Y_c > y^*$). Note that $z_b > 0$ since $g^{(1)}$ is strictly convex on $(0, Y_c - y^*)$. Clearly, $y_b^{(1)} := y^* + z_b$ is a unique solution of $\frac{g(y)-g(y^*)}{y-y^*} = g'(y) > 0$. We will show that the smallest nonnegative concave majorant of H is given by

$$W(y) = \begin{cases} \frac{\hat{g}(y')}{y'} y, & y < y', \\ \hat{g}(y), & y' \le y \le y^*, \\ g(y^*) + g'(y_b^{(1)})(y - y^*), & y^* < y < y_b^{(1)}, \\ g(y), & y_b^{(1)} < y. \end{cases}$$

If $y' < y^*$, then $\hat{g}$ is concave on $[y', y^*]$ and lies below the tangent at y^*. We infer the concavity of W at y^* from this and the fact that $\hat{g}$ majorises H. When

Table C.2: Stopping regions for the single option when $b > a$. When the stopping region is trivial we simply write $V_c = 0$.

<table>
<tr><th colspan="4">Stopping regions in the case $b > a$.</th><th></th><th></th></tr>
<tr><th colspan="4">Parameter range</th><th>Stopping region</th><th>Figure C.2</th></tr>
<tr><td rowspan="2">$p_c \geq D$</td><td colspan="3">$\hat{y}_b \leq y^*$</td><td>$\hat{\Gamma} = [\hat{y}_b, \infty)$</td><td>e</td></tr>
<tr><td colspan="3">$\hat{y}_b > y^*$</td><td>$\hat{\Gamma} = [\max(y_b, y^*), \infty)$</td><td>d</td></tr>
<tr><td rowspan="5">$p_c < D$</td><td colspan="3">$K_c + p_c < D$</td><td>$V_c = 0$</td><td></td></tr>
<tr><td rowspan="4">$K_c + p_c \geq D$</td><td rowspan="2">$K_c + p_c \leq f(x^*)$</td><td>$Y_m \leq y^*$ or $g(Y_m) \leq 0$</td><td>$V_c = 0$</td><td></td></tr>
<tr><td>$Y_m > y^*$ and $g(Y_m) > 0$</td><td>$\hat{\Gamma} = [y_b, Y_m]$</td><td>c</td></tr>
<tr><td rowspan="2">$K_c + p_c > f(x^*)$</td><td>$g'(y^*) > g(y^*)/y^*$</td><td>$\hat{\Gamma} = [y_b, Y_m]$</td><td>c</td></tr>
<tr><td>$g'(y^*) \leq g(y^*)/y^*$</td><td>$\hat{\Gamma} = [\min(\hat{y}_b, y^*), \max(Y_m, y^*)]$</td><td>b</td></tr>
</table>

$y' = y^*$ the concavity at y^* follows from the condition A; concavity at other points is trivial. Finally, we conclude that $\hat{\Gamma} = [y', y^*] \cup [y_b^{(1)}, \infty)$, see panel (b) in Figure C.1. The principle of smooth fit fails at y^*.

Case A^C: By Lemma C.1 y_b lies in $(\max(y^*, Y_c), \infty)$, a region in which H is equal to g, concave, and increasing. We conclude that $\Gamma = [y_b, \infty)$.

C.2 Solutions in the case $b > a$

A summary of the results of this subsection is presented in Table C.2.

C.2.1 Case $p_c \geq D$

In this case each of $Y_m, Y_c, \hat{Y}_m, \hat{Y}_c$ are equal to positive infinity. Noting that $\hat{g}'(y^*) > g'(y^*)$, H is concave and increasing without bound so that $V_c \neq 0$. The tangency points $\hat{y}_b$ and y_b are uniquely defined. Then $\hat{\Gamma} = [A, \infty)$ where $A = \hat{y}_b$ if $\hat{y}_b \leq y^*$ and $A = \max(y_b, y^*)$ otherwise. Note that there is no smooth fit at A when $y_b \leq y^* \leq \hat{y}_b$.

C.2.2 Case $p_c < D$

When $K_c + p_c \geq D$: $\hat{Y}_c, \hat{Y_m}$ are equal to $+\infty$ while Y_c, Y_m lie in $(0, \infty)$ so that $\hat{g}$ is increasing and concave, making H concave on $(0, \max(y^*, Y_c))$ and both convex and decreasing on $(\max(y^*, Y_c), \infty)$. Notice that if $Y_m \leq y^*$ then the stopping region $\hat{\Gamma}$ has an empty intersection with (y^*, ∞) and the value function W is constant on $[y^*, \infty)$.

If $\hat{g}(y^*) = g(y^*) \leq 0$ then the problem reduces to finding a non-negative concave majorant of g. In this case, if $Y_m \leq y^*$ or $g(Y_m) \leq 0$ then $V_c = 0$. Otherwise, $V_c > 0$ and there exists a unique solution y_b of $\eta(y) = 0$ on (y^*, Y_m) such that $g'(y_b) > 0$ (uniqueness follows from Lemma 2.2, existence is easy). The stopping region has the form $\hat{\Gamma} = [y_b, Y_m]$.

If $\hat{g}(y^*) = g(y^*) > 0$ and $g'(y^*) > g(y^*)/y^*$ then $Y_m > y^*$. Since $\eta(y) = 2(g(y) - g'(y)y)$ we have $\eta(y^*) < 0$ and $\eta(Y_m) > 0$, so by the continuity and monotonicity of η(recall that $Y_c > Y_m$) there exists a unique solution y_b of

Table C.3: Stopping regions for the single option when $a = b$. Whenever the stopping region is trivial we write $V_c = 0$.

Stopping regions in the case $b = a$.			
Parameter range		Stopping region	Figure C.2
$p_c > D$	$y_b \geq y^*$	$\hat{\Gamma} = [y_b, \infty)$	d
	$\hat{y}_b < y^*$	$\hat{\Gamma} = [\hat{y}_b, \infty)$	e
	$y_b < y^*, \hat{y}_b \geq y^*$	$\hat{\Gamma} = [y^*, \infty)$	
$p_c = D$	$K_c\sqrt{y^*} - d > 0$	$\hat{\Gamma} = [\min(\hat{y}_b, y^*), \infty)$	e
	$K_c\sqrt{y^*} - d \leq 0$	$V_c = 0$	
$p_c < D$	$K_c + p_c \leq f(x^*)$	$V_c = 0$	
	$K_c + p_c > f(x^*)$	$\hat{\Gamma} = [\min\{\hat{y}_b, y^*\}, y^*]$	a

$\eta(y) = 0$ on (y^*, Y_m), and we have $g'(y_b) > 0$. It follows also that the tangent at y_b goes through 0 (has a null vertical intercept). By concavity of H on $(0, y_b)$ it majorises H there. Hence, the stopping region is $\hat{\Gamma} = [y_b, Y_m]$.

Alternatively, suppose that both $g(y^*) > 0$ and $g'(y^*) \leq g(y^*)/y^*$. Then since $\hat{g}'(y^*) > g'(y^*)$, the problem decomposes into (i) finding the smallest non-negative concave majorant of $\hat{g}$ on $[0, y^*]$ and (ii) finding the smallest non-negative concave majorant of the function $g^{(1)}(y) = g(y^* + y) - g(y^*)$ on $[0, \infty)$. The majorant in (i) coincides with $\hat{g}$ on $[\min(\hat{y}_b, y^*), y^*]$, whereas the majorant in (ii) coincides with $g^{(1)}$ on $[0, \max(Y_m - y^*, 0)]$. The overall stopping region and majorant are then recovered by adjoining these parts, so that $\hat{\Gamma} = [\min(\hat{y}_b, y^*), \max(Y_m, y^*)]$. Notice that when $\hat{y}_b > y^*$ there is no smooth fit at the left end of the interval $\hat{\Gamma}$.

$K_c + p_c < D$: we have $\hat{g} < 0$ on $(0, \infty)$ and $g < \hat{g}$ on (y^*, ∞), so that $H \leq 0$ and $V_c = 0$.

C.3 Solutions in the case $b = a$

A summary of the results of this subsection is presented in Table C.3. Although there does not seem to be any economic rationale behind this border case, the analysis simplifies:

$$\begin{aligned}
g(y) &= (p_c - D)\sqrt{y} + K_c\sqrt{y^*} - d, \\
g'(y) &= \frac{1}{2} y^{-\frac{1}{2}} (p_c - D), \\
g''(y) &= \frac{1}{4} y^{-\frac{3}{2}} (D - p_c), \\
\hat{g}(y) &= (K_c + p_c - D)\sqrt{y} - d, \\
\hat{g}'(y) &= \frac{1}{2} y^{-\frac{1}{2}} (K_c + p_c - D), \\
\hat{g}''(y) &= \frac{1}{4} y^{-\frac{3}{2}} (D - K_c - p_c).
\end{aligned} \tag{C.2}$$

C.3.1 Case $p_c > D$

Here g, $\hat{g}$ and hence also H are strictly concave and increasing without bound so $V_c \neq 0$ and the stopping region $\hat{\Gamma}$ will be of the form $[A, \infty)$. We have

$$y_b = 4\left(\frac{d - K_c\sqrt{y^*}}{p_c - D}\right)^2, \qquad \hat{y}_b = 4\left(\frac{d}{K_c + p_c - D}\right)^2.$$

If $y_b \geq y^*$ then $A = y_b$ and smooth fit holds; otherwise, if $\hat{y}_b < y^*$ then $A = \hat{y}_b$ and smooth fit holds. If both $y_b < y^*$ and $\hat{y}_b \geq y^*$ then smooth fit does not hold and $A = y^*$.

C.3.2 Case $p_c = D$

The function g is constant and $\hat{g}$ is increasing and concave. Hence $V_c \neq 0$ precisely when $K_c\sqrt{y^*} - d > 0$, in which case $\hat{\Gamma} = [A, \infty)$ with $A = \min(\hat{y}_b, y^*)$.

C.3.3 Case $p_c < D$

In this case g is strictly convex and strictly decreasing and also $\hat{g}(0) < 0$, so $V_c \neq 0$ if and only if $g(y^*) > 0$. In this case the stopping region is $\hat{\Gamma} = [\min\{\hat{y}_b, y^*\}, y^*]$.