Katzman, M. (2003) *The support of top graded local cohomology modules*

**arXiv.org**

http://arxiv.org/abs/math/0312333v1

# The support of top graded local cohomology modules

Mordechai Katzman, Department of Pure Mathematics, University of Sheffield, United Kingdom. *E-mail*: M.Katzman@sheffield.ac.uk

## 1 Introduction

Let $R_0$ be any domain, let $R = R_0[U_1, \dots, U_s]/I$, where $U_1, \dots, U_s$ are indeterminates of positive degrees $d_1, \dots, d_s$, and $I \subset R_0[U_1, \dots, U_s]$ is a homogeneous ideal.

The main theorem in this paper is Theorem 2.6, a generalization of Theorem 1.5 in [KS], which states that all the associated primes of $H := H^s_{R_+}(R)$ contain a certain non-zero ideal $\mathrm{c}(I)$ of $R_0$ called the "content" of $I$ (see Definition 2.4.) It follows that the support of $H$ is simply $\mathrm{V}(\mathrm{c}(I)R + R_+)$ (Corollary 1.8) and, in particular, $H$ vanishes if and only if $\mathrm{c}(I)$ is the unit ideal.

These results raise the question of whether local cohomology modules have finitely many minimal associated primes– this paper provides further evidence in favour of such a result (Theorem 2.10 and Remark 2.12.)

Finally, we give a very short proof of a weak version of the monomial conjecture based on Theorem 2.6.

## 2 The vanishing of top local cohomology modules

Throughout this section $R_0$ will denote an arbitrary commutative Noetherian domain. We set $S = R_0[U_1, \dots, U_s]$ where $U_1, \dots, U_s$ are indeterminates of degrees $d_1, \dots, d_s$, and $R = S/I$ where $I \subset R_0[U_1, \dots, U_s]$ is an homogeneous ideal. We define $\Delta = d_1 + \cdots + d_s$ and denote with $\mathcal{D}$ the sub-semi-group of $\mathbb{N}$ generated by $d_1, \dots, d_s$.

For $t \in \mathbb{Z}$, we shall denote by $(\bullet)(t)$ the $t$-th shift functor (on the category of graded $R$-modules and homogeneous homomorphisms).

For any multi-index $\lambda = (\lambda^{(1)}, \dots, \lambda^{(s)}) \in \mathbb{Z}^s$ we shall write $U^\lambda$ for $U_1^{\lambda^{(1)}} \dots U_s^{\lambda^{(s)}}$ and we shall set $|\lambda| = \lambda^{(1)} + \cdots + \lambda^{(s)}$.

Lemma 2.1 *Let $I$ be generated by homogeneous elements $f_1, \dots, f_r \in S$. Then there is an exact sequence of graded $S$-modules and homogeneous homomorphisms*

$$\bigoplus_{i=1}^{r} H^s_{S_+}(S)(-\deg f_i) \xrightarrow{(f_1,\dots,f_r)} H^s_{S_+}(S) \longrightarrow H^s_{R_+}(R) \longrightarrow 0.$$

**Proof:** The functor $H^s_{S_+}(\bullet)$ is right exact and the natural equivalence between $H^s_{S_+}(\bullet)$ and $(\bullet) \otimes_S H^s_{S_+}(S)$ (see [BS, 6.1.8 & 6.1.9]) actually yields a homogeneous $S$-isomorphism

$$H^s_{S_+}(S)/(f_1, \dots, f_r)H^s_{S_+}(S) \cong H^s_{S_+}(R).$$

To complete the proof, just note that there is an isomorphism of graded $S$-modules $H^s_{S_+}(R) \cong H^s_{R_+}(R)$, by the Graded Independence Theorem [BS, 13.1.6]. $\square$

We can realize $H^s_{S_+}(S)$ as the module $R_0[U_1^-, \dots, U_s^-]$ of inverse polynomials described in [BS, 12.4.1]: this graded $R$-module vanishes beyond degree $-\Delta$. More generally $R_0[U_1^-, \dots, U_s^-]_{-d} \neq 0$ if and only if $d \in \mathcal{D}$.

For each $d \in \mathcal{D}$, $R_0[U_1^-, \dots, U_s^-]_{-d}$ is a free $R_0$-module with base $\mathcal{B}(d) := \left(U^\lambda\right)_{-\lambda \in \mathbb{N}^s, |\lambda| = -d}$. We combine this realisation with the previous lemma to find a presentation of each homogeneous component of $H^s_{R_+}(R)$ as the cokernel of a matrix with entries in $R_0$.

Assume first that $I$ is generated by one homogeneous element $f$ of degree $\delta$. For any $d \in \mathcal{D}$ we have, in view of Lemma 2.1, a graded exact sequence

$$R_0[U_1^-, \dots, U_s^-]_{-d-\delta} \xrightarrow{\phi_d} R_0[U_1^-, \dots, U_s^-]_{-d} \longrightarrow H^s_{R_+}(R)_{-d} \longrightarrow 0.$$

The map of free $R_0$-modules $\phi_d$ is given by multiplication on the left by a $\#\mathcal{B}(d) \times \#\mathcal{B}(d + \delta)$ matrix which we shall denote later by $M(f; d)$.

In the general case, where $I$ is generated by homogeneous elements $f_1, \ldots, f_r \in S$, it follows from Lemma 2.1 that the $R_0$-module $H^s_{R_+}(R)_{-d}$ is the cokernel of a matrix $M(f_1, \ldots, f_r; d)$ whose columns consist of all the columns of $M(f_1, d), \ldots, M(f_r, d)$.

Consider a homogeneous $f \in S$ of degree $\delta$. We shall now describe the matrix $M(f; d)$ in more detail and to do so we start by ordering the bases of the source and target of $\phi_d$ as follows. For any $\lambda, \mu \in \mathbb{Z}^s$ with negative entries we declare that $U^\lambda < U^\mu$ if and only if $U^{-\lambda} <_{\mathrm{Lex}} U^{-\mu}$ where "$<_{\mathrm{Lex}}$" is the lexicographical term ordering in $S$ with $U_1 > \cdots > U_s$. We order the bases $\mathcal{B}(d)$, and by doing so also the columns and rows of $M(f; d)$, in ascending order. We notice that the entry in $M(f; d)$ in the $U^\alpha$ row and $U^\beta$ column is now the coefficient of $U^\alpha$ in $fU^\beta$.

Lemma 2.2 *Let $\nu \in \mathbb{Z}^s$ have negative entries and let $\lambda_1, \lambda_2 \in \mathbb{N}^s$. If $U^{\lambda_1} <_{\mathrm{Lex}} U^{\lambda_2}$ and $U^\nu U^{\lambda_1}$, $U^\nu U^{\lambda_2} \in R_0[U_1^-, \ldots, U_s^-]$ do not vanish then $U^\nu U^{\lambda_1} > U^\nu U^{\lambda_2}$.*

**Proof:** Let $j$ be the first coordinate in which $\lambda_1$ and $\lambda_2$ differ. Then $\lambda_1^{(j)} < \lambda_2^{(j)}$ and so also $-\nu^{(j)} - \lambda_1^{(j)} > -\nu^{(j)} - \lambda_2^{(j)}$; this implies that $U^{-\nu-\lambda_1} >_{\mathrm{Lex}} U^{-\nu-\lambda_2}$ and $U^{\nu+\lambda_1} > U^{\nu+\lambda_2}$. □

Lemma 2.3 *Let $f \neq 0$ be a homogeneous element in $S$. Then, for all $d \in \mathcal{D}$, the matrix $M(f; d)$ has maximal rank.*

**Proof:** We prove the lemma by producing a non-zero maximal minor of $M(f; d)$. Write $f = \sum_{\lambda \in \Lambda} a_\lambda U^\lambda$ where $a_\lambda \in R_0 \setminus \{0\}$ for all $\lambda \in \Lambda$ and let $\lambda_0$ be such that $U^{\lambda_0}$ is the minimal member of $\left\{U^\lambda : \lambda \in \Lambda\right\}$ with respect to the lexicographical term order in $S$.

Let $\delta$ be the degree of $f$. Each column of $M(f; d)$ corresponds to a monomial $U^\lambda \in \mathcal{B}(d + \delta)$; its $\rho$-th entry is the coefficient of $U^\rho$ in $fU^\lambda \in R_0[U_1^-, \ldots, U_s^-]_{-d}$.

Fix any $U^\nu \in \mathcal{B}(d)$ and consider the column $c_\nu$ corresponding to $U^{\nu-\lambda_0} \in \mathcal{B}(d + \delta)$. The $\nu$-th entry of $c_\nu$ is obviously $a_{\lambda_0}$.

By the previous lemma all entries in $c_\nu$ below the $\nu$th row vanish. Consider the square submatrix of $M(f; d)$ whose columns are the $c_\nu$ ($\nu \in \mathcal{B}(d)$); its determinant is clearly a power of $a_{\lambda_0}$ and hence is non-zero. □

Definition 2.4 *For any $f \in R_0[U_1, \ldots, U_s]$ write $f = \sum_{\lambda \in \Lambda} a_\lambda U^\lambda$ where $a_\lambda \in R_0$ for all $\lambda \in \Lambda$. For such an $f \in R_0[U_1, \ldots, U_s]$ we define the* content *$\mathrm{c}(f)$ of $f$ to be the ideal $\langle a_\lambda : \lambda \in \Lambda \rangle$ of $R_0$ generated by all the coefficients of $f$. If $J \subset R_0[U_1, \ldots, U_s]$ is an ideal, we define its* content *$\mathrm{c}(J)$ to be the ideal of $R_0$ generated by the contents of all the elements of $J$. It is easy to see that if $J$ is generated by $f_1, \ldots, f_r$, then $\mathrm{c}(J) = \mathrm{c}(f_1) + \cdots + \mathrm{c}(f_r)$.*

Lemma 2.5 *Suppose that $I$ is generated by homogeneous elements*

$f_1, \ldots, f_r \in S$. *Fix any* $d \in \mathcal{D}$. *Let* $t := \operatorname{rank} M(f_1, \ldots, f_r; d)$ *and let* $I_d$ *be the ideal generated by all* $t \times t$ *minors of* $M(f_1, \ldots, f_r; d)$. *Then* $\mathrm{c}(I) \subseteq \sqrt{I_d}$.

**Proof:** It is enough to prove the lemma when $r = 1$; let $f = f_1$. Write $f = \sum_{\lambda \in \Lambda} a_\lambda U^\lambda$ where $a_\lambda \in R_0 \setminus \{0\}$ for all $\lambda \in \Lambda$. Assume that $\mathrm{c}(I) \not\subseteq \sqrt{I_d}$ and pick $\lambda_0$ so that $U^{\lambda_0}$ is the minimal element in $\left\{U^\lambda : \lambda \in \Lambda\right\}$ (with respect to the lexicographical term order in $S$) for which $a_\lambda \notin \sqrt{I_d}$. Notice that the proof of Lemma 2.3 shows that $U^{\lambda_0}$ cannot be the minimal element of $\left\{U^\lambda : \lambda \in \Lambda\right\}$.

Fix any $U^\nu \in \mathcal{B}(d)$ and consider the column $c_\nu$ corresponding to $U^{\nu-\lambda_0} \in \mathcal{B}(d+\delta)$. The $\nu$-th entry of $c_\nu$ is obviously $a_{\lambda_0}$. Lemma 2.2 shows that, for any other $\lambda_1 \in \Lambda$ with $U^{\lambda_1} >_{\mathrm{Lex}} U^{\lambda_0}$, either $\nu - \lambda_0 + \lambda_1$ has a non-negative entry, in which case the corresponding term of $fU^{\nu-\lambda_0} \in R_0[U_1^-, \ldots, U_s^-]_{-d}$ is zero, or $U^\nu > U^{\nu-\lambda_0+\lambda_1}$.

Similarly, for any other $\lambda_1 \in \Lambda$ with $U^{\lambda_1} <_{\mathrm{Lex}} U^{\lambda_0}$, either $\nu - \lambda_0 + \lambda_1$ has a non-negative entry, in which case the corresponding term of $fU^{\nu-\lambda_0} \in R_0[U_1^-, \ldots, U_s^-]_{-d}$ is zero, or $U^\nu < U^{\nu-\lambda_0+\lambda_1}$.

We have shown that all the entries below the $\nu$-th row of $c_\nu$ are in $\sqrt{I_d}$. Consider the matrix $M$ whose columns are $c_\nu$ ($\nu \in \mathcal{B}(d)$) and let $\overline{\phantom{x}} : R_0 \to R_0/\sqrt{I_d}$ denote the quotient map. We have

$$0 = \overline{\det(M)} = \det(\overline{M}) = \overline{a_{\lambda_0}}^{\binom{d-1}{s-1}}$$

and, therefore, $a_{\lambda_0} \in \sqrt{I_d}$, a contradiction. $\square$

Theorem 2.6 *Suppose that* $I$ *is generated by homogeneous elements* $f_1, \ldots, f_r \in S$. *Fix any* $d \in \mathcal{D}$. *Then each associated prime of* $H^s_{R_+}(R)_{-d}$ *contains* $\mathrm{c}(I)$. *In particular* $H^s_{R_+}(R)_{-d} = 0$ *if and only if* $\mathrm{c}(I) = R_0$.

**Proof:** Recall that for any $p, q \in \mathbb{N}$ with $p \leq q$ and any $p \times q$ matrix $M$ of maximal rank with entries in any domain, $\operatorname{Coker} M = 0$ if and only if the ideal generated by the maximal minors of $M$ is the unit ideal. Let $M = M(f_1, \ldots, f_r; d)$, so that $H^s_{R_+}(R)_{-d} \cong \operatorname{Coker} M$.

In view of Lemmas 2.3 and 2.5, the ideal $\mathrm{c}(I)$ is contained in the radical of the ideal generated by the maximal minors of $M$; therefore, for each $x \in \mathrm{c}(I)$, the localization of $\operatorname{Coker} M$ at $x$ is zero; we deduce that $\mathrm{c}(I)$ is contained in all associated primes of $\operatorname{Coker} M$.

To prove the second statement, assume first that $\mathrm{c}(I)$ is not the unit ideal. Since all minors of $M$ are contained in $\mathrm{c}(I)$, these cannot generate the unit ideal and $\operatorname{Coker} M \neq 0$. If, on the other hand, $\mathrm{c}(I) = R_0$ then $\operatorname{Coker} M$ has no associated prime and $\operatorname{Coker} M = 0$. $\square$

Corollary 2.7 *Let the situation be as in* 2.6. *The following statements are equivalent:*

1. $\mathrm{c}(I) = R_0$;
2. $H^s_{R_+}(R)_{-d} = 0$ *for some* $d \in \mathcal{D}$;
3. $H^s_{R_+}(R)_{-d} = 0$ *for all* $d \in \mathcal{D}$.

*Consequently,* $H^s_{R_+}(R)$ *is asymptotically gap-free in the sense of* [BH, (4.1)].

Corollary 2.8 *The* $R$*-module* $H^s_{R_+}(R)$ *has finitely many minimal associated primes, and these are just the minimal primes of the ideal* $\mathrm{c}(I)R + R_+$.

**Proof:** Let $r \in \mathrm{c}(I)$. By Theorem 2.6, the localization of $H^s_{R_+}(R)$ at $r$ is zero. Hence each associated prime of $H^s_{R_+}(R)$ contains $\mathrm{c}(I)R$. Such an associated prime must contain $R_+$, since $H^s_{R_+}(R)$ is $R_+$-torsion.

On the other hand, $H^s_{R_+}(R)_{-\Delta} \cong R_0/\,\mathrm{c}(I)$ and $H^s_{R_+}(R)_i = 0$ for all $i > -\Delta$; therefore there is an element of the $(-\Delta)$-th component of $H^s_{R_+}(R)$ that has annihilator (over $R$) equal to $\mathrm{c}(I)R + R_+$. All the claims now follow from these observations. □

Remark 2.9 In [Hu, Conjecture 5.1], Craig Huneke conjectured that every local cohomology module (with respect to any ideal) of a finitely generated module over a local Noetherian ring has only finitely many associated primes. This conjecture was shown to be false (cf. [K, Corollary 1.3]) but Corollary 2.8 provides some evidence in support of the weaker conjecture that every local cohomology module (with respect to any ideal) of a finitely generated module over a local Noetherian ring has only finitely many *minimal* associated primes.

The following theorem due to Gennady Lyubeznik ([L]) gives further support for this conjecture:

Theorem 2.10 *Let* $R$ *be any Noetherian ring of prime characteristic* $p$ *and let* $I \subset R$ *be any ideal generated by* $f_1, \dots, f_s \in R$. *The support of* $H^s_I(R)$ *is Zariski closed.*

**Proof:** We first notice that the localization of $H^s_I(R)$ at a prime $P \subset R$ vanishes if and only if there exist positive integers $\alpha$ and $\beta$ such that

$$(f_1 \cdot \dots \cdot f_s)^\alpha \in \langle f_1^{\alpha+\beta}, \dots, f_s^{\alpha+\beta} \rangle$$

in the localization $R_P$. This is because if we can find such $\alpha$ and $\beta$ we can then take $q := p^e$ powers and obtain

$$(f_1 \cdot \dots \cdot f_s)^{q\alpha} \in \langle f_1^{q\alpha+q\beta}, \dots, f_s^{q\alpha+q\beta} \rangle$$

for all such $q$. This shows that all elements in the direct limit sequence

$$R/\langle f_1, \dots f_s\rangle \xrightarrow{f_1\cdot\ldots\cdot f_s} R/\langle f_1^2, \dots f_s^2\rangle \xrightarrow{f_1\cdot\ldots\cdot f_s} \dots$$

map to 0 in the direct limit and hence $H^s_I(R) = 0$.

But if

$$(f_1 \cdot \dots \cdot f_s)^\alpha \in \langle f_1^{\alpha+\beta}, \dots, f_s^{\alpha+\beta}\rangle$$

in $R_P$, we may clear denominators and deduce that this occurs on a Zariski open subset containing $P$.

Thus the complement of the support is a Zariski open subset. □

It may be reasonable to expect that non-top local cohomology modules might also have finitely many minimal associated primes; the only examples known to me of non-top local cohomology modules with infinitely many associated primes are the following: Let $k$ be any field, let $R_0 = k[x, y, s, t]$ and let $S$ be the localisation of $R_0[u, v, a_1, \dots, a_n]$ at the maximal ideal $\mathfrak{m}$ generated by $x, y, s, t, u, v, a_1, \dots, a_n$. Let $f = sx^2v^2-(t+s)xyuv+ty^2u^2 \in S$ and let $R = S/fS$. Denote by $I$ the ideal of $S$ generated by $u, v$ and by $A$ the ideal of $S$ generated by $a_1, \dots, a_n$.

Theorem 2.11 *Assume that $n \geq 2$. The local cohomology module $\mathrm{H}^2_{I\cap A}(R)$ has infinitely many associated primes and $\mathrm{H}^{n+1}_{I\cap A}(R) \neq 0$.*

**Proof:** Consider the following segment of the Mayer-Vietoris sequence

$$\dots \to \mathrm{H}^2_{I+A}(R) \to \mathrm{H}^2_I(R) \oplus \mathrm{H}^2_A(R) \to \mathrm{H}^2_{I\cap A}(R) \to \dots$$

Notice that $a_1, \dots, a_n, u$ form a regular sequence on $R$ so $\operatorname{depth}_{I+A} R \geq n+1 \geq 3$ and the leftmost module vanishes. Thus $\mathrm{H}^2_I(R)$ injects into $\mathrm{H}^2_{I\cap A}(R)$ and Corollary 1.3 in [K] shows that $\mathrm{H}^2_{I\cap A}(R)$ has infinitely many associated primes.

Consider now the following segment of the Mayer-Vietoris sequence

$$\dots \to \mathrm{H}^{n+1}_{I\cap A}(R) \to \mathrm{H}^{n+2}_{I+A}(R) \to \mathrm{H}^{n+2}_I(R) \oplus \mathrm{H}^{n+2}_A(R) \to \dots$$

The direct summands in the rightmost module vanish since both $I$ and $A$ can be generated by less than $n+2$ elements, so $\mathrm{H}^{n+1}_{I\cap A}(R)$ surjects onto $\mathrm{H}^{n+2}_{I+A}(R)$.

Now $\mathrm{c}(f)$ is the ideal of $R_0$ generated by $sx^2, -(t+s)xy$ and $ty^2$ so $\mathrm{c}(f) \subset \langle x, y\rangle \neq R_0$. Corollary 2.7 now shows that $\mathrm{H}^{n+2}_{I+A}(R)$ does not vanish and, therefore, nor does $\mathrm{H}^{n+1}_{I\cap A}(R)$. □

Remark 2.12 When $n \geq 3$, $\mathrm{H}^3_{I+A}(R) = 0$ and the argument above shows that $\mathrm{H}^2_I(R) \oplus \mathrm{H}^2_A(R) \cong \mathrm{H}^2_{I\cap A}(R)$. Corollary 2.8 implies that $\mathrm{H}^2_I(R)$ has finitely many minimal primes and since the only associated prime of $\mathrm{H}^2_A(R)$ is $A$, $\mathrm{H}^2_{I\cap A}(R)$ has finitely many minimal primes.

When $n = 2$ we obtain a short exact sequence

$$0 \to \mathrm{H}^2_I(R) \oplus \mathrm{H}^2_A(R) \to \mathrm{H}^2_{I\cap A}(R) \to \mathrm{H}^3_{I+A}(R) \to 0.$$

The short exact sequence

$$0 \to S \xrightarrow{f} S \to R \to 0$$

implies that $\mathrm{H}^3_{I+A}(R)$ injects into the local cohomology module $\mathrm{H}^4_{I+A}(S)$ whose only associated prime is $I + A$, so again we see that $\mathrm{H}^2_{I\cap A}(R)$ has finitely many minimal associated primes.

## 3 An application: a weak form of the Monomial Conjecture.

In [Ho] Mel Hochster suggested reducing the Monomial Conjecture to the problem of showing the vanishing of certain local cohomology modules which we now describe.

Let $C$ be either $\mathbb{Z}$ or a field of characteristic $p > 0$, let $R_0 = C[A_1, \ldots, A_s]$ where $A_1, \ldots, A_s$ are indeterminates, $S = R_0[U_s, \ldots, U_s]$ where $U_1, \ldots, U_s$ are indeterminates and $R = S/F_{s,t}S$ where

$$F_{s,t} = (U_1 \cdot \ldots \cdot U_s)^t - \sum_{i=1}^{s} A_i U_i^{t+1}.$$

Suppose that

$$H_{s,t} := H^s_{\langle U_1,\ldots,U_s\rangle}(R)$$

vanishes with $C = \mathbb{Z}$. If for some local ring $T$ we can find a system of parameters $x_1, \ldots, x_s$ so that $(x_1 \cdot \ldots \cdot x_s)^t \in \langle x_1^{t+1}, \ldots, x_s^{t+1}\rangle$, i.e., if there exist $a_1, \ldots, a_s \in T$ so that $(x_1 \cdot \ldots \cdot x_s)^t = \sum_{i=1}^t a_i x_i^{t+1}$ we can define an homomorphism $R \to T$ by mapping $A_i$ to $a_i$ and $U_i$ to $x_i$. We can view $T$ as an $R$-module and we have an isomorphism of $T$-modules

$$H^s_{\langle x_1,\ldots,x_s\rangle}(T) \cong H^s_{\langle U_1,\ldots,U_s\rangle}(R) \otimes_R T$$

and we deduce that

$$H^s_{\langle x_1,\ldots,x_s\rangle}(T) = 0$$

but this cannot happen since $x_1, \ldots, x_s$ form a system of parameters in $T$.

We have just shown that the vanishing of $H_{s,t}$ for all $t \geq 1$ implies the Monomial Conjecture in dimension $s$. In [Ho] Mel Hochster proved that these local cohomology modules vanish when $s = 2$ or when $C$ has characteristic $p > 0$, but in [R] Paul Roberts showed that, when $C = \mathbb{Z}$, $H_{3,2} \neq 0$, showing that Hochster's approach cannot be used for proving the Monomial Conjecture in dimension 3. This can be generalized further:

Proposition 3.1 *When $C = \mathbb{Z}$, $H_{s,2} \neq 0$ for all $s \geq 3$.*

**Proof:** We proceed by induction on $s$; the case $s = 3$ is proved in [R].

Assume that for some $s \geq 1$, $\alpha \geq 0$ and $\delta > \alpha$ the monomial $x_1^{\alpha} \ldots x_{s+1}^{\alpha}$ is in the ideal of $C[x_1, \ldots, x_{s+1}, a_1, \ldots, a_{s+1}]$ generated by $x_1^{\alpha+\beta}, \ldots, x_{s+1}^{\alpha+\beta}$ and $F_{s+1,t}$.

Define $G_{s+1,2}$ to be the result of substituting $a_{s+1} = 0$ in $F_{s+1,2}$, i.e.,

$$G_{s+1,2} = (x_1 \ldots x_{s+1})^2 - \sum_{i=1}^{s} a_i x_i^3.$$

If

$$x_1^{\alpha} \ldots x_{s+1}^{\alpha} \in \langle x_1^{\alpha+\beta}, \ldots, x_{s+1}^{\alpha+\beta}, F_{s+1,2} \rangle \tag{1}$$

then by setting $a_{s+1} = 0$ we see that

$$x_1^{\alpha} \ldots x_{s+1}^{\alpha} \in \langle x_1^{\alpha+\beta}, \ldots, x_{s+1}^{\alpha+\beta}, G_{s+1,2} \rangle.$$

If we assign degree 0 to $x_1, \ldots, x_s$, degree 1 to $x_{s+1}$ and degree 2 to $a_1, \ldots, a_s$ then the ideal $\langle x_1^{\alpha+\beta}, \ldots, x_{s+1}^{\alpha+\beta}, G_{s+1,2} \rangle$ is homogeneous and we must have

$$x_1^{\alpha} \ldots x_{s+1}^{\alpha} \in \langle x_1^{\alpha+\beta}, \ldots, x_s^{\alpha+\beta}, G_{s+1,2} \rangle.$$

If we now set $x_{s+1} = 1$ we obtain

$$x_1^{\alpha} \ldots x_s^{\alpha} \in \langle x_1^{\alpha+\beta}, \ldots, x_s^{\alpha+\beta}, F_{s,2} \rangle. \tag{2}$$

Now $H_{s+1,2} = 0$ if and only if for each $\beta \geq 1$ we can find an $\alpha \geq 0$ so that equation (1) holds and this implies that for each $\beta \geq 1$ we can find an $\alpha \geq 0$ so that equation (2) holds which is equivalent to $H_{s,2} = 0$. The induction hypothesis implies that $H_{s,2} \neq 0$ and so $H_{s+1,2} \neq 0$. □

The local cohomology modules $H_{s,t}$ are a good illustration for the failure of the methods of the previous section in the non-graded case. For example, one cannot decide whether $H_{s,t}$ is zero just by looking at $F_{s,t}$: the vanishing of $H_{s,t}$ depends on the characteristic of $C$! Compare this situation to the following graded problem.

Theorem 3.2 (A Weaker Monomial Conjecture) *Let $T$ be a local ring with system of parameters $x_1, \dots, x_s$. For all $t \geq 0$ we have*

$$(x_1 \cdot \ldots \cdot x_s)^t \notin \langle x_1^{st}, \dots, x_s^{st} \rangle.$$

**Proof:** Let $S = \mathbb{Z}[A_1, \dots, A_s][X_1, \dots, X_s]$ where $\deg A_i = 0$ and $\deg X_i = 1$ for all $1 \leq i \leq s$. Following Hochster's argument we reduce to the problem of showing that

$$H^s_{\langle X_1, \dots, X_s \rangle}(S/fS) = 0$$

where

$$f = (X_1 \cdot \ldots \cdot X_s)^t - \sum_{i=1}^{s} A_i X_i^{st}.$$

Since $f$ is homogeneous and $\mathrm{c}(f)$ is the unit ideal, the result follows from Theorem 2.6. □

## References


[BH] M. Brodmann and M. Hellus, *Cohomological patterns of coherent sheaves over projective schemes,* J. Pure Appl. Algebra 172 (2002)**2-3**, pp. 165–182.

[BS] M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications,* Cambridge University Press, 1998.

[Ho] M. Hochster, *Canonical elements in local cohomology modules and the direct summand conjecture,* J. Algebra **84** (2) (1983), pp. 503–553.

[Hu] C. Huneke, *Problems on local cohomology*, in *Free resolutions in commutative algebra and algebraic geometry, Sundance 90*, ed. D. Eisenbud and C. Huneke, Research Notes in Mathematics **2**, Jones and Bartlett Publishers, Boston, 1992, pp. 93–108.

[K] M. Katzman, *An example of an infinite set of associated primes of a local cohomology module,* J. Algebra **252**(1) (2002), pp. 161–166.

[KS] M. Katzman and R. Y. Sharp, *Some properties of top graded local cohomology modules,* J. Algebra **259**(2) (2003) , pp. 599–612.

[L] G. Lyubeznik, private communication.

[R] P. Roberts, *A computation of local cohomology,* in *Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992),* Contemp. Math., **159**, Amer. Math. Soc., Providence, RI, 1994, pp. 351–356.