Article:

Bernar, Rafael P, Crispino, Luis C B and Higuchi, Atsushi orcid.org/0000-0002-3703-7021 (2016) Graviton two-point function in 3+1 static de Sitter spacetime. International Journal of Modern Physics D. 1641016. ISSN 0218-2718

https://doi.org/10.1142/S0218271816410169

International Journal of Modern Physics D

Graviton Two-point Function in 3 + 1 Static de Sitter Spacetime

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Received Day Month Year
Revised Day Month Year

In Ref. 1 we investigated gravitational perturbations in the background of de Sitter spacetime in arbitrary dimensions. More specifically, we used a gauge-invariant formalism to describe the perturbations inside the cosmological horizon, i.e. in the static patch of de Sitter spacetime. After a gauge-fixed quantization procedure, the two-point function in the Bunch-Davies-like vacuum state was shown to be infrared finite and invariant under time translation. In this work, we give details of the calculations to obtain the graviton two-point function in $3 + 1$ dimensions.

Keywords: graviton two-point function; static de Sitter spacetime; infrared behavior.

PACS numbers: 04.60.-m, 04.62.+v, 04.50.-h, 04.25.Nx, 04.60.Gw, 11.25.Db

1. Introduction

Linearized gravity treated as a quantum field propagating in a background spacetime can describe several interesting phenomena. Since observations[2,3] indicate an accelerated rate of the Universe's expansion, de Sitter spacetime can be regarded as an accurate model for our Universe, at least in its phases of exponential rate of expansion. Moreover, linear quantum gravity in the background of de Sitter spacetime is particularly important for the inflationary cosmology. Linearized gravitons can induce fluctuations in the cosmic microwave background, for example (See Refs. 4, 5 for the more general setting of Robertson-Walker spacetimes). However, there have been some controversies about the infrared (IR) properties of the graviton two-point function. The main source of these controversies is that the graviton mode functions natural to the Poincaré patch of de Sitter spacetime behave in a manner similar to those for the minimally-coupled massless scalar field,[5] which allows no de Sitter-invariant vacuum state because of IR divergences.[6,7] However, due to the gauge invariance of linearized gravity, it is important to determine if these IR divergences are physical.[a] In Ref. 1, we analyzed the graviton two-point function in the static patch of de Sitter spacetime and found it to be finite in the infrared. Moreover, it is, by construction, time-translation invariant. In this paper, we give more details about the construction of this graviton two-point function in $(3+1)$-dimensional de Sitter spacetime. In Section 2, we present the gravitational perturbations in the background of $(3+1)$-dimensional de Sitter spacetime in the static patch. The perturbations are related to master variables, as described in Ref. 8. These master variables satisfy wave-like equations, which are also presented together with their solutions. In Section 3, we present our main result, that is, a graviton two-point function finite in the infrared limit. We also write it in a form similar to the two-point function of the conformally-coupled massless scalar field in de Sitter spacetime.

2. Gravitational Perturbations in Static de Sitter Spacetime

We write $\mathcal{M}$, 4-dimensional de Sitter spacetime in the static patch, locally, as the product:

$$\mathcal{M} = \mathcal{O} \times S^2 \ni (y^a, \hat{x}^i) = (x^\mu), \tag{1}$$

where S^2 is the 2-sphere and $\mathcal{O}$ is the so-called orbit spacetime, a Lorentzian space given in coordinates t and r. The de Sitter spacetime in the static patch can be described by the following line element:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -(1-r^2)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-r^2} + r^2 \mathrm{d}\Omega^2, \tag{2}$$

where

$$\mathrm{d}\Omega^2 = \gamma_{ij}\mathrm{d}\hat{x}^i \mathrm{d}\hat{x}^j = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2 \tag{3}$$

is the line element on the 2-sphere S^2 and $0 \le r \le 1$. We are using units such that the cosmological horizon is located at $r = 1$. Thus, we set the Hubble parameter $H = 1$. The orbit spacetime has a line element given by

$$\mathrm{d}s^2_{orb} = g_{ab}\mathrm{d}y^a \mathrm{d}y^b = -(1-r^2)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-r^2}. \tag{4}$$

The equations above establish the notation used. Greek indices are spacetime indices running from 0 to 3. The letters a and b are used as the indices for the t and r components, while the letters $i, j, k, ...$ are used as the indices for the θ and φ components. We denote the covariant derivatives compatible with the metrics $\mathrm{d}s^2$, $\mathrm{d}s^2_{orb}$ and $\mathrm{d}\Omega^2$, respectively, by ∇_μ, D_a and $\hat{D}_i$.

The metric perturbations can be expanded in terms of harmonic tensors of rank 0, 1 and 2. These are called scalar-, vector- and tensor-type perturbations, respectively. The scalar-type perturbations are expanded in terms of scalar spherical harmonics $\mathbb{S}$, which satisfy

$$\hat{D}_i\hat{D}^i\mathbb{S} + l(l+1)\mathbb{S} = 0, \quad l = 0, 1, 2.... \tag{5}$$

[a]See Ref. 1 and references therein for a overview of the discussion.

The metric perturbations of the scalar-type read

$$h_{ab} = f_{ab}(t,r)\mathbb{S}, \tag{6}$$

$$h_{ai} = f_a(t,r)\mathbb{S}_i, \tag{7}$$

$$h_{ij} = 2r^2\left[H_L(t,r)\gamma_{ij}\mathbb{S} + H_T(t,r)\mathbb{S}_{ij}\right], \tag{8}$$

where $\mathbb{S}_i$ and $\mathbb{S}_{ij}$ are tensors constructed from covariant derivatives of $\mathbb{S}$, that is

$$\mathbb{S}_i = -\frac{\hat{D}_i\mathbb{S}}{\sqrt{l(l+1)}} \tag{9}$$

and

$$\mathbb{S}_{ij} = \frac{\hat{D}_i\hat{D}_j\mathbb{S}}{l(l+1)} + \frac{1}{2}\gamma_{ij}\mathbb{S}. \tag{10}$$

We are omitting the labels (l,m) on the harmonic expansion for convenience. But we point out that all coefficients, which are functions of t and r, can depend only on the l label. The vector-type perturbations are expanded in terms of divergenceless harmonic vectors $\mathbb{V}_i$ satisfying

$$\hat{D}_j\hat{D}^j\mathbb{V}_i + [l(l+1)-1]\mathbb{V}_i = 0, \quad \hat{D}_i\mathbb{V}^i = 0, \tag{11}$$

with $l = 1, 2, ...$ The vector-type metric perturbations are written as

$$h_{ai} = g_a(t,r)\mathbb{V}_i, \tag{12}$$

$$h_{ij} = 2r^2 G_T(t,r)\mathbb{V}_{ij}, \tag{13}$$

where

$$\mathbb{V}_{ij} = -\frac{1}{\sqrt{l(l+1)-1}}\left(\hat{D}_i\mathbb{V}_j + \hat{D}_j\mathbb{V}_i\right). \tag{14}$$

For perturbations of the tensor type we would need harmonic tensors of rank 2, which are traceless and have zero divergence. However, there are no solutions for rank-2 harmonic tensors of this type on the unit 2-sphere.[9,10] Thus we do not have perturbations of the tensor type in four dimensions.

It can be shown that these gravitational perturbations in the background de Sitter spacetime can be related to master variables, by writing gauge-invariant quantities for each type of perturbation. Each master variable satisfies, in general, a distinct wave-like equation.[8,11] The usefulness of this method is apparent from the fact that the perturbations decouple from each other. Moreover, the dynamics is contained in the master variables, which satisfy 2-dimensional equations in the orbit spacetime. For the case of 4-dimensional de Sitter spacetime, both the scalar- and vector-type master variables satisfy the same equation, namely:

$$\Box\Phi_P^{(l)} - \frac{V(r)}{1-r^2}\Phi_P^{(l)} = 0, \tag{15}$$

where $P = S$ or V stands for scalar- or vector-type perturbations, respectively. The differential operator $\Box$ is the d'Alembertian in the orbit spacetime. The effective potential $V(r)$ in Eq. (15) is given by

$$V(r) = l(l+1)\frac{1-r^2}{r^2}. \tag{16}$$

By using a Fourier decomposition with respect to the time variable, one can write positive-frequency solutions regular at the origin to Eq. (15) as

$$\Phi_P^{(\omega l)}(t,r) = A_P^{(\omega l)} e^{-i\omega t} r^{l+1}(1-r^2)^{i\omega/2} \times F\left(\frac{1}{2}(i\omega+l+1), \frac{1}{2}(i\omega+l+2); l+\frac{3}{2}; r^2\right), \tag{17}$$

where $F(\alpha,\beta;\gamma;z)$ is Gauss's hypergeometric function.[12] The $A_P^{(\omega l)}$ are normalization constants. The scalar-type gravitational perturbations with $l = 0$ and $l = 1$ need to be considered separately.[8] The $l = 0$ mode is a spherically symmetric perturbation and, by Birkhoff's theorem, is locally isometric to the linearized version of Schwarzschild-de Sitter solution. For a *vacuum* perturbation, this corresponds to a change in the background metric by the introduction of a small black hole at the origin (see Appendix A for more details). The $l = 1$ mode can always be set to zero by a gauge transformation. Therefore, we exclude these cases. As for the vector-type perturbations, the $l = 1$ modes — there are no $l = 0$ vector harmonics — are physical[b] only if there is a black hole of nonzero mass at the center and corresponds to the addition of angular momentum to this black hole. These modes, $l = 0$ scalar-type and $l = 1$ vector-type, are both non-radiative modes. Thus, we exclude them and consider both mode functions $\Phi_S^{(\omega l)}$ and $\Phi_V^{(\omega l)}$ starting at $l = 2$.

3. Graviton Two-point function in static de Sitter spacetime

As we pointed out before, there are no tensor-type modes in $3+1$ dimensions. The scalar-type modes will be given in terms of the usual scalar spherical harmonics $Y^{(l,m)}(\theta,\phi)$. The relation between the metric perturbations $h_{\mu\nu}$ and the master variables actually depends on the choice of gauge, that is, on some conditions the metric perturbations are required to satisfy. We have chosen a gauge[c] such that

[b]Here, the expression "the modes are physical" means that the modes cannot be eliminated by a gauge transformation.
[c]See Ref. 1 for more details.

perturbations of the scalar type read

$$h_{ai}^{(S;\omega lm)} = 0, \tag{18}$$

$$h_{tt}^{(S;\omega lm)} = \frac{Y^{(l,m)}(\theta,\phi)}{2}\left[\partial_t^2 + (1-r^2)^2\partial_r^2\right](r\Phi_S^{(\omega l)}), \tag{19}$$

$$h_{rr}^{(S;\omega lm)} = \frac{Y^{(l,m)}(\theta,\phi)}{2}\left[\partial_r^2 + \frac{1}{(1-r^2)^2}\partial_t^2\right](r\Phi_S^{(\omega l)}), \tag{20}$$

$$h_{tr}^{(S;\omega lm)} = Y^{(l,m)}(\theta,\phi)\left[\partial_r\partial_t + \frac{r}{1-r^2}\partial_t\right](r\Phi_S^{(\omega l)}), \tag{21}$$

$$h_{ij}^{(S;\omega lm)} = \frac{r^2 Y^{(l,m)}(\theta,\phi)}{2}\gamma_{ij}(\Box + 2)(r\Phi_S^{(\omega l)}). \tag{22}$$

Note that we are reinstating the labels l and m. Vector spherical harmonics, which are divergenceless on S^2, can be written as:[9,13]

$$Y_i^{(l,m)}(\theta,\phi) = \frac{\epsilon_{ij}}{\sqrt{l(l+1)}}\partial^j Y^{(l,m)}(\theta,\phi), \tag{23}$$

where ϵ_{ij} is the Levi-Civita tensor on S^2. In the gauge we have chosen, the vector-type perturbations are given by

$$h_{ti}^{(V;\omega lm)} = Y_i^{(l,m)}(\theta,\phi)(1-r^2)\partial_r\left(r\Phi_V^{(\omega l)}\right), \tag{24}$$

$$h_{ri}^{(V;\omega lm)} = \frac{Y_i^{(l,m)}(\theta,\phi)}{1-r^2}\partial_t\left(r\Phi_V^{(\omega l)}\right), \tag{25}$$

with all other components vanishing.

We expand the graviton field $\hat{h}_{\mu\nu}(x)$ as

$$\hat{h}_{\mu\nu}(x) = \sum_{P=S,V}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\int_0^{\infty} \mathrm{d}\omega\left[\hat{a}_P^{(lm)}(\omega)h_{\mu\nu}^{(P;\omega lm)}(x) + \hat{a}_P^{(lm)\dagger}(\omega)\overline{h_{\mu\nu}^{(P;\omega lm)}}(x)\right], \tag{26}$$

where the overbar denotes complex conjugation. The $h_{\mu\nu}^{(P;\omega lm)}(x)$ are gravitational perturbations given in terms of the classical solutions of Eq. (17), by Eqs. (18)-(22) and Eqs. (24) and (25). We can write a Lagrangian density for the field $h_{\mu\nu}$ in the background de Sitter geometry, given by

$$\mathcal{L} = \sqrt{-g}\left[\nabla_\mu h^{\mu\nu}\nabla^\rho h_{\rho\nu} - \frac{1}{2}\nabla_\rho h_{\mu\nu}\nabla^\rho h^{\mu\nu} + \frac{\nabla_\mu h}{2}\left(\nabla^\mu h - 2\nabla_\rho h^{\rho\mu}\right) - H^2\left(h_{\mu\nu}h^{\mu\nu} - \frac{h^2}{2}\right)\right], \tag{27}$$

where $h = g^{\mu\nu}h_{\mu\nu}$ is the trace of the perturbation. We define an inner product between two solutions $h_{\mu\nu}^{(m)}$ and $h_{\mu\nu}^{(n)}$ by

$$\langle h^{(m)}, h^{(n)}\rangle := -i\int_\Sigma \mathrm{d}\Sigma n_\lambda\left(\overline{h_{\mu\nu}^{(m)}}p^{(n)\lambda\mu\nu} - h_{\mu\nu}^{(n)}\overline{p^{(m)\lambda\mu\nu}}\right), \tag{28}$$

where Σ is a Cauchy hypersurface in the static patch of de Sitter spacetime and $p^{\lambda\mu\nu}$ is the conjugate momentum defined by

$$p^{\lambda\mu\nu} := \frac{1}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial \left(\nabla_\lambda h_{\mu\nu}\right)}. \tag{29}$$

By requiring the field $h_{\mu\nu}$ to be normalized with respect to the inner product above, the normalization constants have to satisfy

$$|A_S^{(\omega l)}|^2 = \frac{\sinh \pi\omega \left|\Gamma(\frac{1}{2}(i\omega + l + 2))\Gamma(\frac{1}{2}(i\omega + l + 1))\right|^2}{2\pi^2 (l-1)l(l+1)(l+2)\left|\Gamma(l+\frac{3}{2})\right|^2} \tag{30}$$

and

$$|A_V^{(\omega l)}|^2 = \frac{\sinh \pi\omega \left|\Gamma(\frac{1}{2}(i\omega + l + 1))\Gamma(\frac{1}{2}(i\omega + l + 2))\right|^2}{8\pi^2 (l-1)(l+2)\left|\Gamma(l+\frac{3}{2})\right|^2}. \tag{31}$$

Then the operators $\hat{a}_P^{(lm)}(\omega)$ and $\hat{a}_P^{(lm)\dagger}(\omega)$ will satisfy the commutation relations:

$$\left[\hat{a}_P^{(lm)}(\omega), \hat{a}_{P'}^{(l'm')\dagger}(\omega')\right] = \delta^{PP'}\delta^{ll'}\delta^{mm'}\delta(\omega - \omega'). \tag{32}$$

In the de Sitter invariant Bunch-Davies vacuum state, which is a thermal state with temperature $H/2\pi$, when probed only in the static region, one can write the graviton two-point function as

$$\begin{aligned}\left\langle \hat{h}_{\mu\nu}(x)\hat{h}_{\mu'\nu'}(x')\right\rangle = \sum_{P=S,V}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\int_0^{\infty} d\omega \left\{ \frac{1}{e^{2\pi\omega}-1}\overline{h_{\mu\nu}^{(P;\omega lm)}}(x) h_{\mu'\nu'}^{(P;\omega lm)}(x') \right. \\ \left. + \frac{1}{1-e^{-2\pi\omega}} h_{\mu\nu}^{(P;\omega lm)}(x)\overline{h_{\mu'\nu'}^{(P;\omega lm)}}(x') \right\}, \end{aligned} \tag{33}$$

in which $H = 1$ due to our choice of units. Let us analyze the low-ω behavior of this two-point function. The low-ω behavior of the solutions $h_{\mu\nu}^{(S;\omega lm)}(x)$ and $h_{\mu\nu}^{(V;\omega lm)}(x)$ coincides with the behavior of the normalization constants $A_S^{(\omega l)}$ and $A_V^{(\omega l)}$, respectively. These constants tend to zero like $\omega^{1/2}$ in the limit $\omega \to 0$, which means that the integrand of the two-point function have the following ω behavior:

$$\text{integrand of } \left\langle \hat{h}_{\mu\nu}(x)\hat{h}_{\mu'\nu'}(x')\right\rangle \sim \frac{\omega}{e^{2\pi\omega}-1} + \frac{\omega}{1-e^{-2\pi\omega}}, \tag{34}$$

hence it is finite in the infrared. As a consequence of the fact that the two-point function is constructed in the static patch using the stationary modes, it is time-translation invariant as well.

We can simplify the graviton two-point function given by Eq. (33). Let us consider first the contribution to the graviton two-point function from the scalar-type modes. We define the following tensor differential operators:

$$\mathcal{D}^{(S)}_{tt} = \frac{1}{2}[\partial_t^2 + (1-r^2)^2\partial_r^2], \tag{35}$$

$$\mathcal{D}^{(S)}_{rr} = \frac{1}{2}\left[\partial_r^2 + \frac{1}{(1-r^2)^2}\partial_t^2\right], \tag{36}$$

$$\mathcal{D}^{(S)}_{tr} = \partial_r\partial_t + \frac{r}{1-r^2}\partial_t, \tag{37}$$

$$\mathcal{D}^{(S)}_{ij} = \frac{r^2}{2}\gamma_{ij}(\Box + 2), \tag{38}$$

with all other components vanishing. We choose $\theta' = 0$ in $x' = (t', r', \theta', \phi')$ without loss of generality. Then the contribution to the graviton two-point function (33) reads

$$\Delta^{(S)}_{\mu\nu\mu'\nu'}(y,y') = \mathcal{D}^{(S)}_{\mu\nu}\mathcal{D}^{(S)}_{\mu'\nu'}G(y,y'), \tag{39}$$

where

$$G(y,y') = \sum_{l=2}^{\infty} Y^{(l,0)}(0,\phi')Y^{(l,0)}(\theta,\phi)\int_0^\infty \mathrm{d}\omega \left[\frac{\overline{r\Phi_S^{(\omega l)}}(t,r)r'\Phi_S^{(\omega l)}(t',r')}{e^{2\pi\omega}-1} + \frac{r\Phi_S^{(\omega l)}(t,r)\overline{r'\Phi_S^{(\omega l)}}(t',r')}{1-e^{-2\pi\omega}}\right], \tag{40}$$

since $Y^{(l,m)}(\theta' = 0, \phi') = 0$, unless $m = 0$.

We can write the function $G(y, y')$ in a way similar to the well known[14] two-point function for the conformally-coupled massless scalar field, namely

$$\Delta^{(c)}(y,y') = \frac{1}{8\pi^2(1-\cos\mu(y,y') + i\epsilon(t-t'))}, \tag{41}$$

where $\mu(y, y')$ is the geodesic distance between the two points $y = (t, r, \theta, \phi)$ and $y' = (t', r', \theta', \phi')$, if they are spacelike separated. If the separation between these points is timelike, we make the substitution $\cos\mu(y, y') \to \cosh\mu_T(y, y')$, where $\mu_T(y, y')$ is the timelike geodesic distance of the two points. The term $i\epsilon(t-t')$, where ϵ is an infinitesimal positive number, indicates how the singularity at $\mu(y, y') = 0$ is avoided. We can write the two-point function $\Delta^{(c)}(y, y')$ in the static patch as follows:[1]

$$\Delta^{(c)}(y,y') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \overline{Y^{(l,m)}}(\theta,\phi)Y^{(l,m)}(\theta',\phi')\int_0^\infty \mathrm{d}\omega |N^{(\omega l)}|^2 R_{\omega l}(r)R_{\omega l}(r') \times\left[\frac{e^{i\omega(t-t')}}{e^{2\pi\omega}-1} + \frac{e^{-i\omega(t-t')}}{1-e^{-2\pi\omega}}\right], \tag{42}$$

where

$$|N^{(\omega l)}|^2 = \frac{\sinh \pi\omega}{4\pi^2} \frac{\left|\Gamma(\frac{1}{2}(i\omega + l + 1))\Gamma(\frac{1}{2}(i\omega + l + 2))\right|^2}{\left|\Gamma(l + \frac{3}{2})\right|^2}, \tag{43}$$

$$R_{\omega l}(r) = r^l (1 - r^2)^{i\omega/2} F\left(\frac{1}{2}(i\omega + l + 1), \frac{1}{2}(i\omega + l + 2); l + \frac{3}{2}; r^2\right). \tag{44}$$

We have used the fact that $R_{\omega l}(r)$ and $\sum_{m=-l}^{l} \overline{Y^{(l,m)}}(\theta,\phi)Y^{(l,m)}(\theta',\phi')$ are both real. Notice that by Eq. (30) we have

$$|A_S^{(\omega l)}|^2 = \frac{2|N^{(\omega l)}|^2}{(l-1)l(l+1)(l+2)}. \tag{45}$$

We multiply Eq. (42) by $Y^{(l,0)}(\theta',\phi')$ and integrate over S^2. Using Eq. (45) we find by the orthonormality of the spherical harmonics

$$\begin{aligned} &\frac{rr'}{4\pi^2(l-1)l(l+1)(l+2)} \int_{S^2} \mathrm{d}\phi' \mathrm{d}\theta' \sin\theta' \frac{Y^{(l,0)}(\theta',\phi')}{1 - \cos\mu(y,y') + i\epsilon(t-t')} \\ &= \int_0^\infty \mathrm{d}\omega \left[\frac{\overline{\Phi_S^{(\omega l)}}(t,r)\Phi_S^{(\omega l)}(t',r')}{e^{2\pi\omega} - 1} + \frac{\Phi_S^{(\omega l)}(t,r)\overline{\Phi_S^{(\omega l)}}(t',r')}{1 - e^{-2\pi\omega}} \right] Y^{(l,0)}(\theta,\phi). \end{aligned} \tag{46}$$

Using the formula

$$Y^{(l,0)}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}} \mathrm{P}_l(\cos\theta), \tag{47}$$

we obtain

$$\begin{aligned} &\frac{(2l+1)rr'}{16\pi^3(l-1)l(l+1)(l+2)} \int_{S^2} \mathrm{d}\phi' \mathrm{d}\theta' \sin\theta' \frac{\mathrm{P}_l(\cos\theta')}{1 - \cos\mu(y,y') + i\epsilon(t-t')} \\ &= \int_0^\infty \mathrm{d}\omega \left[\frac{\overline{\Phi_S^{(\omega l)}}(t,r)\Phi_S^{(\omega l)}(t',r')}{e^{2\pi\omega} - 1} + \frac{\Phi_S^{(\omega l)}(t,r)\overline{\Phi_S^{(\omega l)}}(t',r')}{1 - e^{-2\pi\omega}} \right] Y^{(l,0)}(0,\phi')Y^{(l,0)}(\theta,\phi). \end{aligned} \tag{48}$$

We compare this expression with Eq. (40) to find

$$G(y,y') = r^2 r'^2 \int_{S^2} \mathrm{d}\phi' \mathrm{d}\theta' \sin\theta' \frac{Q(\theta')}{1 - \cos\mu(y,y') + i\epsilon(t-t')}, \tag{49}$$

where

$$Q(\theta') = \frac{1}{16\pi^3} \sum_{l=2}^{\infty} \frac{2l+1}{(l-1)l(l+1)(l+2)} \mathrm{P}_l(\cos\theta'). \tag{50}$$

We have $|\mathrm{P}_l(\cos\theta')| \leq 1$ as is well known. Hence, this series converges for all θ' because the coefficient tends to zero like $1/l^3$ as $l \to \infty$.

Next, let us examine the contribution of the vector-type modes to the two-point function. We can show that, for $\theta'=0$, the only terms of the vector spherical harmonics $Y_i^{(l,m)}$ (and $\overline{Y_i^{(l,m)}}$ as well) that will contribute are the ones with $|m|=1$. We note first that

$$\sum_{m=\pm 1}\overline{Y^{(l,m)}}(\theta,\phi)Y^{(l,m)}(\theta',\phi')=\frac{2l+1}{2\pi l(l+1)}\mathrm{P}_l^1(\cos\theta)\mathrm{P}_l^1(\cos\theta')\cos(\phi-\phi'). \tag{51}$$

We choose $\phi'=0$. (At the end of the calculation, the result will be independent of this choice). This means that the θ'-direction and ϕ'-direction are identified with the x'- and y'-directions, respectively, in the cartesian coordinates. We denote the unit vectors in the x'- and y'-directions by $\hat{e}_{i'}^{(x)}$ and $\hat{e}_{i'}^{(y)}$, respectively. For small θ' we have[15]

$$P_l^1(\cos\theta')\approx-\frac{l(l+1)}{2}\sin\theta'. \tag{52}$$

Then, for $\theta'\to 0$ and $\phi'\to 0$ we find

$$\epsilon_{i'j'}\partial^{j'}[P_l^1(\cos\theta')\cos(\phi-\phi')]\to\frac{l(l+1)}{2}e_{i'}^{(\phi)}, \tag{53}$$

where

$$e_{i'}^{(\phi)}=-e_{i'}^{(x)}\sin\phi+e_{i'}^{(y)}\cos\phi. \tag{54}$$

Hence, by Eq. (23) we obtain

$$\begin{aligned}\sum_{m=\pm 1}\overline{Y_i^{(l,m)}}(\theta,\phi)Y_{i'}^{(l,m)}(0,0)&=\frac{2l+1}{4\pi l(l+1)}\epsilon_{ij}\partial^j\left[\mathrm{P}_l^1(\cos\theta)\hat{e}_{i'}^{(\phi)}\right]\\&=\frac{1}{l(l+1)}\epsilon_{ij}\partial^j\frac{\partial}{\partial\theta}\left[Y^{(l,0)}(\theta,\phi)Y^{(l,0)}(0,0)\hat{e}_{i'}^{(\phi)}\right].\end{aligned} \tag{55}$$

We now define the following differential operators:

$$\mathcal{D}_t^{(V)}=(1-r^2)\partial_r, \tag{56}$$

$$\mathcal{D}_r^{(V)}=\frac{1}{1-r^2}\partial_t. \tag{57}$$

Thus, the contribution of the vector-type modes to our graviton two-point function can be given as

$$\Delta_{aia'i'}^{(V)}(y,y')=\mathcal{D}_a^{(V)}\mathcal{D}_{a'}^{(V)}F_{ii'}(y,y') \tag{58}$$

with all other components vanishing, where

$$\begin{aligned}F_{ii'}(y,y')=\sum_{l=2}^{\infty}\sum_{m=\pm 1}\overline{Y_i^{(l,m)}}(\theta,\phi)Y_{i'}^{(l,m)}(0,0)\int_0^{\infty}\mathrm{d}\omega\,&\left[\frac{r\overline{\Phi_V^{(\omega l)}}(t,r)r'\Phi_V^{(\omega l)}(t',r')}{e^{2\pi\omega}-1}\right.\\&\left.+\frac{r\Phi_V^{(\omega l)}(t,r)r'\overline{\Phi_V^{(\omega l)}}(t',r')}{1-e^{-2\pi\omega}}\right].\end{aligned} \tag{59}$$

We note the relation

$$\Phi_V^{(\omega l)} = \sqrt{\frac{l(l+1)}{4}}\Phi_S^{(\omega l)}. \tag{60}$$

Using the relation given by Eq. (60) into Eq. (59), using Eq. (55) and then using the definition (40) of $G(y,y')$, we obtain

$$F_{ii'}(y,y') = \frac{1}{4}\epsilon_{ij}\partial^j\left[\partial_\theta G(y,y')\hat{e}_{i'}^{(\phi)}\right]. \tag{61}$$

By writing $F_{ii'}(y,y')$ we find that as a function of the unit vectors $e_i^{(x)}$ and $e_i^{(y)}$ of the cartesian coordinate system, the result does not depend on the particular choice of ϕ'. In summary, with the choice $\theta' = 0$ in $y' = (t', r', \theta', \phi')$ our graviton two-point function in $3+1$ dimensions can be written as

$$\Delta_{\mu\nu\mu'\nu'}(y,y') = \mathcal{D}_{\mu\nu}^{(S)}\mathcal{D}_{\mu'\nu'}^{(S)}G(y,y') + \delta^a_{\{\mu}\delta^i_{\nu\}}\delta^{a'}_{\{\mu'}\delta^{i'}_{\nu'\}}\mathcal{D}_a^{(V)}\mathcal{D}_{a'}^{(V)}\epsilon_{ij}\partial^j\left[\partial_\theta G(y,y')\hat{e}_{i'}^{(\phi)}\right], \tag{62}$$

where $\{...\}$ indicates symmetrization.

4. Discussion

We have studied the gravitational perturbations in the static patch of $(3+1)$-dimensional de Sitter spacetime. Expanding the metric perturbations in terms of harmonic tensors, it is possible to relate them to master variables satisfying the same wave-like equation. We find an inner product for the metric perturbations solutions with the gauge degrees of freedom fully fixed. Then, the graviton two-point function in the de Sitter invariant Bunch-Davies vaccuum state can be constructed in a straightforward manner. Since the normalized modes $h_{\mu\nu}^{(P;\omega lm)}$ all behave like $\omega^{1/2}$ as $\omega \to 0$, this two-point function is finite in the infrared. This graviton two-point function can be written in a way similar to the conformally-coupled scalar field two-point function. We note also that the graviton two-point function is similar to the IR divergent two-point function of the *minimally* coupled scalar field. However, the graviton two-point function lacks a contribution to the integrand similar to the $l = 0$ mode of the minimally coupled scalar field, which diverges like ω^{-2} as $\omega \to 0$.[1] Additionally, the graviton two-point function in the static patch does not grow as a function of time, due to its time-translation invariance, in contrast to the IR-divergent two-point function in the Poincaré patch and the IR-finite two-point function in the global patch[d].

[d]For the two-point function in the global patch, it grows as a function of time if the two points are kept at a fixed physical distance.

Acknowledgments

We would like to acknowledge Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), Fundação Amazônia de Amparo a Estudos e Pesquisas do Pará (FAPESPA) and Marie Curie action NRHEP-295189- FP7-PEOPLE-2011-IRSES for partial financial support. A. H. and L. C. also acknowledge partial support from the Abdus Salam International Centre for Theoretical Physics through the Visiting Scholar/Consultant Programme and Associates Scheme, respectively. A. H. thanks the Universidade Federal do Pará (UFPA) in Belém for the kind hospitality.

Appendix A. Monopole ($l = 0$) scalar-type mode

We can solve the perturbed Einstein's equation directly for the $l = 0$ scalar-type mode. For $l = 0$, $\mathbb{S}_i$ and $\mathbb{S}_{ij}$ are not defined and $\mathbb{S}$ is just a constant. As is well-known, linearized gravity is invariant under the gauge transformation $h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu \Lambda_\nu + \nabla_\nu \Lambda_\mu$. Using a gauge vector of the form

$$\Lambda_a = \xi_a(t,r), \quad \Lambda_i = 0, \tag{A.1}$$

we can set to zero both h_{tr} and h_{ij}. The only remaining components of the metric perturbations are h_{tt} and h_{rr}, which are functions only of t and r. The vacuum perturbed Einstein's equations[e] are given by

$$\begin{aligned}\frac{1}{2} & \left(\nabla_\alpha \nabla^\alpha h_{\mu\nu} - \nabla^\alpha \nabla_\mu h_{\nu\alpha} - \nabla^\alpha \nabla_\nu h_{\mu\alpha} + \nabla_\mu \nabla_\nu h\right) \\ & + \frac{1}{2} g_{\mu\nu} \left(\nabla^\alpha \nabla^\beta h_{\alpha\beta} - \nabla_\alpha \nabla^\alpha h\right) + 3H^2 \left(h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h\right) = 0, \end{aligned} \tag{A.3}$$

[e] Recall that we are considering a cosmological background. In other words, we take the *vacuum* Einstein's equation to be

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}(R - 2\Lambda) g_{\mu\nu} = 0, \quad \Lambda = 3H^2. \tag{A.2}$$

where we do not set $H = 1$. We insert the $l = 0$ scalar-type metric perturbation in this equation to get the following set of partial differential equations:

$$\frac{\partial h_2}{\partial t} = 0, \qquad \text{(A.4)}$$

$$\left(1 - 5H^2r^2\right) h_2 + r\left(1 - H^2r^2\right)\frac{\partial h_2}{\partial r} = 0, \qquad \text{(A.5)}$$

$$(1 - H^2r^2)\left[r\frac{\partial h_0}{\partial r} + (1 - 4H^2r^2 + 3H^4r^4)h_2\right] + 2H^2r^2h_0 = 0, \qquad \text{(A.6)}$$

$$\begin{aligned} &2H^2r\left[\left(2 - H^2r^2\right)h_0 - (1 - H^2r^2)(4 - 9H^2r^2 + 5H^4r^4)h_2\right] \\ &+(1 - H^2r^2)\left[\frac{\partial h_0}{\partial r} + (1 - H^2r^2)(1 - 3H^2r^2 + 2H^4r^4)\frac{\partial h_2}{\partial r}\right. \\ &\left. + r(1 - H^2r^2)\left(\frac{\partial^2 h_0}{\partial r^2} + \frac{\partial^2 h_2}{\partial t^2}\right)\right] = 0, \qquad \text{(A.7)} \end{aligned}$$

where $h_{tt} = h_0(t, r)$ and $h_{rr} = h_2(t, r)$. Solutions to Eqs. (A.4), (A.5) and (A.6) above are given by

$$h_2(r) = \frac{c}{r} \qquad \text{(A.8)}$$

and

$$h_0(t, r) = \frac{c}{r} + (1 - H^2r^2)f(t), \qquad \text{(A.9)}$$

where c is a constant of integration and $f(t)$ is an arbitrary function. Eq. (A.7) is identically satisfied by the solutions given above. However, by a further gauge transformation of the form

$$\Lambda_t = -\frac{1}{2}(1 - H^2r^2)\int f(t)dt, \quad \Lambda_r = \Lambda_i = 0, \qquad \text{(A.10)}$$

we can eliminate $f(t)$. Thus, since we are considering first order perturbations, we have, for the $l = 0$ mode,

$$g_{tt} + h_{tt} \approx -\left(1 - \frac{c}{r} - H^2r^2\right), \qquad \text{(A.11)}$$

$$g_{rr} + h_{rr} \approx \left(1 - \frac{c}{r} - H^2r^2\right)^{-1}. \qquad \text{(A.12)}$$

Hence, in this approximation, these solutions correspond to the Schwarzschild-de Sitter family of solutions.

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