THE EVALUATION OF MULTIPLE YEAR GAS SALES AGREEMENT WITH REGIME SWITCHING

CARL CHIARELLA$^\sharp$, LES CLEWLOW* AND BODA KANG†

ABSTRACT. A typical gas sales agreement (GSA), also called a gas swing contract, is an agreement between a supplier and a purchaser for the delivery of variable daily quantities of gas, between specified minimum and maximum daily limits, over a certain number of years at a specified set of contract prices. The main constraint of such an agreement that makes them difficult to value is that in each gas year there is a minimum volume of gas (termed take-or-pay or minimum bill) for which the buyer will be charged at the end of the year (or penalty date), regardless of the actual quantity of gas taken. We propose a framework for pricing such swing contracts for an underlying gas forward price curve that follows a regime-switching process in order to better capture the volatility behaviour in such markets. With the help of a recombining pentanomial tree, we are able to efficiently evaluate the prices of the swing contracts, find optimal daily decisions and optimal yearly use of both the make-up bank and the carry forward bank at different regimes. We also show how the change of regime will affect the decisions.

Keywords: gas sales agreement, swing contract, take-or-pay, make-up, carry forward, forward price curve, regime switching volatility, recombing pentanomial tree.

1. INTRODUCTION

In today's challenging energy business environment, senior management and company shareholders are demanding ever greater financial scrutiny of any assets that offer flexibility of operation, and thus contain embedded value. In the natural gas markets, there is an increasing focus on swing contracts and gas storage assets as sources of hidden, untapped flexibility. This makes their accurate valuation, operation, and optimisation more important than ever before.

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The best practice accountancy and management of flexible gas assets now require a most thorough understanding of the underlying gas market fundamentals, and the range of supporting mathematical techniques for the assets' valuation and optimisation. An inadequate understanding of these issues could result in the sub-optimal performance of flexible assets, in both financial and physical terms. In this paper we mainly concentrate on the evaluation of the gas swing contracts.

There are a number of papers that discuss the valuation of more general swing contracts, with the earliest being that of Thompson (1995) in which a lattice (tree) method is introduced and applied to take-or-pay gas contracts and mortgage-backed securities. Clewlow, Strickland & Kaminski (2001*a,b*) discuss the risk analysis and the properties of the optimal exercise strategies with the help of a trinomial tree method. Ibanez (2004) uses a simulation approach and seeks to determine an approximate optimal strategy before pricing swing options by implementing another simulation. Barrera-Esteve et al. (2006) develop a stochastic programming algorithm to evaluate swing options with penalty. Bardou et al. (2009) use the so called optimal quantization method to price swing options with the spot price following a mean reverting process.

Most recently, Wahab & Lee (2011) implement a pentanomial lattice approach to evaluate swing options in gas markets under the assumption that the spot price follows a regime switching Geometric Brownian Motion where the volatility can switch between different values based on the state of a hidden Markov chain. In Wahab, Yin & Edirisinghe (2010), the authors develop a heptanomial lattice approach to price swing options in the electricity market with the spot price switching between a mean-reverting processes and a Geometric Brownian Motion. Chiarella, Clewlow & Kang (2012) implemented a pentanomial tree approach to price a one year swing contract with penalty including regime switching forward price curve. However all of the above contributions only discuss the single year contracts without make-up and carry forward provisions, which are quite different from the multiple year GSA that we consider in this paper.

Edoli, Fiorenzani, Ravelli & Vargiolu (2013) discussed one type of make-up clause in a gas swing contract. The authors built an algorithm to price and optimally manage the make-up gas allocation among the years and the gas taking in the swing subperiods within the years: they proved that this problem has a quadratic complexity with respect to the number of years. The algorithm can be adapted to different instances of make-up clauses as well as to some forms of carry-forward clauses. However a multiple year gas sales agreement with penalty is not discussed in the paper.

Basei, Cesaroni & Vargiolu (2014) characterized the value of swing contracts in continuous time as the unique viscosity solution of a Hamilton-Jacobi-Bellman equation with suitable boundary con- ditions. The authors discussed both the case of contracts with penalties and the case of contracts with strict constraints in which case a penalty method has been proposed. In the case of swing contracts with strict constraints, the authors characterized the value function as the unique viscosity solution with polynomial growth of the HJB equation subject to appropriate boundary conditions. The paper also only discuss a one year contract without make-up and carry forward provisions.

Breslin, Clewlow, Strickland & van der Zee (2008a) introduced the definition and explained many basic features of a typical gas swing contract, which is an agreement between a supplier and a purchaser for the delivery of variable daily quantities of gas - between specified minimum and maximum daily limits - over a certain number of years at a specified set of contract prices. While swing contracts have been used for many years to manage the inherent uncertainty of gas supply and demand, it is only in recent years with the deregulation of energy markets that there has been an interest in understanding and valuing the optionality contained in these contracts. In the model of Breslin et al. (2008a) the volatility is a deterministic function of both the current time and the time-to-maturity, however there is a great deal of evidence indicating that the the volatility is stochastic in gas markets and we argue that a regime switching model is better able to capture such random features. The main contribution of this paper is to evaluate the multiple year GSA introduced in Breslin et al. (2008a), but with a regime switching forward price curve and over multiple years.

The remainder of the paper is structured as follows. In Section 2, we propose a one factor regime switching model for the gas forward price curve and we build a recombining pentanomial tree to approximate the gas spot price process derived from the forward price curve model. We introduce the basic features and the detailed evaluation procedures of the multiple year gas sales agreement with make-up and carry forward provisions in Section 3. In Section 4, we provide several numerical examples to demonstrate the properties of both the decision surfaces and value surfaces of these contracts and also show how the change of regime will affect the decisions. We draw some conclusions in Section 5

2. REGIME SWITCHING FORWARD PRICE CURVE AND A TREE

The stochastic or random nature of commodity prices plays a central role in models for valuing contingent claims on commodities, and in procedures for evaluating investments

to extract or produce the commodity. There are currently two approaches to modelling forward price dynamics in the literature. The first starts from a stochastic representation of the energy spot asset and other key variables, such as the convenience yield on the asset and interest rates (see for example Gibson & Schwartz (1990) and Schwartz (1997)), and then derives the prices of energy contingent claims consistent with the spot process. However, one of the problems in implementing such models is that often the state variables are unobservable - even the spot price is hard to obtain, with the problem being exacerbated if the convenience yield has to be jointly estimated.

The second stream of literature models the evolution of the forward curve. Forward contracts are widely traded on many exchanges with prices easily observed - often the nearest maturity forward price is used as a proxy for the spot price with longer dated contracts used to imply the convenience yield. Clewlow & Strickland (1999a) work in this second class of models, simultaneously modelling the evolution of the entire forward curve conditional on the initially observed forward curve and so setup a unified approach to the pricing and risk management of a portfolio of energy derivative positions. In this paper we follow the second approach to model the forward curve or the volatility functions of the forward curve directly.

Since the forward price curve shares similar patterns with forward interest rate dynamics, in particular the Heath et al. (1992) (HJM) model, many of these ideas of modelling the interest rate term structure could potentially work in modelling forward energy prices. Clewlow & Strickland (1999b) develop a general framework with a multi-factor model for the risk management of energy derivatives. This framework is designed to be consistent not only with the market observable forward price curve but also the volatilities and correlations of forward prices. Breslin et al. (2008) further generalize this framework to accommodate a more general multi-factor, multi-commodity (MFMC) model and also describe a process for estimating parameters from historical data.

The instantaneous forward price volatility is not directly observable. However the observed implied volatility (obtained from the prices of various derivative contracts) is closely related to it and indicates that the market is also changing its belief about the instantaneous volatilities discontinuously. In deterministic volatility HJM-type models, such as the one in Breslin et al. (2008), the volatility curve is fixed and the volatility of a specific forward rate can change deterministically only with maturity. In order to properly describe the actual evolution of the volatility curve, one needs a process consisting of both deterministic factors and random factors. The drawback of diffusion models is that they cannot generate

sudden and sufficiently large shifts of the volatility curve. If one augments that feature by adding traditional type jump processes, for example Poisson jumps, one finds that the frequency of the jumps is too large while the magnitude of the jumps is too small.

It seems that the class of piecewise-deterministic processes provide an appropriate framework for modelling the dynamics of the term structure of volatilities since they allow volatility to follow an almost deterministic process between two random jump times. Davis (1984) claims that this class covers almost all important non-diffusion applications. The simplest process in this class is the continuous-time homogeneous Markov chain with a finite number of jump times. Modelling with such a process approximates the actual jumps in volatility with jumps over a finite set of values but allows the well-developed stochastic calculus for continuous Markov chains (Elliott et al. (1995) and Aggoun & Elliott (2005)) to be used.

Vo (2009) use a stochastic volatility model with regime switching to model the return series in the crude oil market. The author modeled the volatility of oil return as a stochastic volatility process whose mean is subject to shifts in regime. The shift is governed by a two-state first-order Markov process. The author found clear evidence of regime-switching in the oil market. The author also finds that incorporating regime-switching into the SV framework significantly enhances the forecasting power of the SV model.

2.1. **Forward price curve with regime switching volatility.** Deterministic volatility functions cannot capture the complicated movements of the forward curves. Hence we propose a stochastic volatility model under which we price a multiple year GSA. Volatility of the forward curve is stochastic due to a hidden Markov Chain that causes it to switch between "high volatility load" and "low volatility load" states. Chiarella et al. (2009) have found that a regime switching model captures quite well the stochastic nature of the volatility function in the gas market and they implement an MCMC approach to estimate the parameters of the model.

We consider a one factor regime switching forward curve model:

$$\frac{dF(t,T)}{F(t,T)} = <\sigma, X_t> c(t) \cdot e^{-\alpha(T-t)} dW_t. \tag{1}$$

In this paper, the dynamics of the forward price is defined in the risk neutral world. We can certainly define the forward price in the market dynamics but as we show in the later sections, the drift term in the stochastic differential equation for either futures prices or spot prices is not used in any way in constructing the lattice for pricing the gas sales

agreement. Instead, we match the level of the pentanomial lattice to the market forward price curve. Hence we believe it is proper to assume risk neutral dynamics here since the regime switching volatility function mainly matters in this pricing context. However we will introduce the market price of risk associated with the Brownian motion and associated with the Markov Chain if we are more interested in estimating the drift of the return.

In Equation (1), $F(t, T)$ is the price of the gas forward at time t with a maturity at time T. W_t is a standard Brownian Motion. The time varying term

$$c(t) = c + \sum_{j=1}^{M} (d_j(1 + \sin(f_j + 2\pi jt))) \tag{2}$$

captures the seasonal effect.

X_t is a finite state Markov chain with state space $E = \{e_1, e_2, \cdots, e_N\}$ and e_i is a vector of length N and equal to 1 at the i−th position and 0 elsewhere, that is

$$e_i = (0, \ldots, 0, 1, 0, \ldots, 0)' \in \mathbb{R}^N \tag{3}$$

where $'$ indicates the transpose operator. $P = (p_{ij})_{N \times N}$ is the transition probability matrix of the Markov Chain X_t. For all $i = 1, \ldots, N, j = 1, \ldots, N$, p_{ij} is the conditional probability that the Markov Chain X_t transits from state e_i at current time t to state e_j at the next time $t + \Delta t$, that is,

$$p_{ij} = \Pr(X_{t+\Delta t} = e_j | X_t = e_i). \tag{4}$$

In Equation (1), $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ are the different values of the volatilities which evolve following the rule of the Markov Chain X_t and $< \cdot, \cdot >$ denotes the scalar product in $\mathbb{R}^N$,

$$< \sigma, X_t >= \sum_{i=1}^{N} \sigma_i \mathbb{1}_{(X_t=e_i)}; \tag{5}$$

where the indicator function is

$$\mathbb{1}_{(X_t=e_i)} = \begin{cases} 1, & \text{if } X_t = e_i; \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

This scalar product let the spot volatility of the forward price curve switch among different values σ_i randomly depending on the state of the Markov Chain X_t.

We also know that for $F(t,T)$ satisfying (1) the spot price $S(t) = F(t,t)$ is given by (see e.g. Breslin et al. (2008a))

$$S(t) = F(0,t) \cdot \exp \left(\int_0^t < \sigma, X_s > c(s) \cdot e^{-\alpha(t-s)} dW_s - \frac{1}{2} \Lambda_t^2 \right), \qquad (7)$$

where $\Lambda_t^2 = \int_0^t (< \sigma, X_s > c(s) \cdot e^{-\alpha(t-s)})^2 ds$.

2.2. **Pentanomial tree construction.**

The spot price dynamics in (7) is rather complicated since it involves path dependence of the history of the hidden Markov chain, which makes it hard to construct a recombining discrete grid to approximate the continuous spot price process. The multiple year GSA that we are trying to evaluate has several features and also can be early exercised multiple times during the life of the contract. The complexity of evaluating these contracts with simulation methods, for instance using that of Ibanez (2004), is quite high and not really possible for practical use. We are also aware that the Least-Squares Monte Carlo method (LSMC) has been used to evaluate both the gas storage contract, see e.g. Boogert & Jong (2008) and Carmona & Ludkovski (2010) and the gas swing contract without penalty, see Kiesel et al. (2010). However the penalty at the end of the gas year introduces a discontinuity in the first derivative of the value surface not only in the spot price dimension but also in the volume taken dimension and this aspect the LSMC does not handle well. Furthermore, the complexity of this multi-year swing contract with features such as make-up and carry-forward can not be handled easily by LSMC because of the additional discontinuities. Holden et al. (2011) in their paper considered the carry forward but not the make-up feature but in the gas sales agreement we do have both carry forward and make-up which should be taken into account. Also they considered the contract with a fixed number of swing rights and fixed number of carry forward rights which is different from our contract specification in the gas sales agreement where we have decisions on a daily basis and the decisions on each day will depend not only on the gas price, volume taken but also on both carry forward and make-up.

We have found that lattice approaches are widely used because of their computational simplicity and flexibility. Bollen (1998) constructed a pentanomial lattice to approximate a regime switching Geometric Brownian Motion. Wahab & Lee (2011) extended the pentanomial lattice to a multinomial tree and studied the price of swing options under regime switching dynamics. However from the contract point of view, those researchers study a different version of the contract which has multiple early exercise opportunities without

penalty. However the penalty usually will be applied at the end of each gas year for a multi-year contract and the above mentioned discontinuity introduced by the penalty makes the contract difficult to evaluate.

Wahab & Lee (2011) did discuss pricing a swing option under a regime switching model but there are two main differences between their paper and ours. Firstly, Wahab & Lee (2011) proposed a Geometric Brownian Motion process for the gasoline price where the volatility can switch between different regimes according to a Markov chain. However the mean reverting process in our paper is more appropriate in capturing the behavior of the gas price and the process of building a pentanomial tree for a regime-switching mean reverting process is different to that for GBM. Moreover, very important feature of the gas swing contract is that there will be a penalty at the end of the gas year if the volume taken in the year did not meet the minimal bill, hence both the penalty and the volume taken makes the contracts more complicated to evaluate, however all the contracts in the numerical examples in Wahab & Lee (2011) neither have a penalty nor take the volume taken into consideration which makes the features of the contract essentially different from what we have in our paper.

In this paper, in order to construct a discrete lattice that approximates the spot price process $S(t)$, we let

$$Y_t = \int_0^t < \sigma, X_s > c(s) \cdot e^{-\alpha(t-s)} dW_s, \tag{8}$$

so that

$$dY_t = -\alpha Y_t dt + < \sigma, X_t > c(t) dW_t, \tag{9}$$

and we build a discrete lattice to approximate Y_t first. Then at each time step we add an adjustment term to the nodes on the lattice for Y_t so that the lattice obtained for the spot price is consistent with the observed market forward price curve. (as followed below)

2.2.1. *Nodes.* We assume that there are only two regimes ($N = 2$) for the volatility, instead of σ_1, σ_2, we use σ_L when $X_t = L$ for the low volatility regime and σ_H when $X_t = H$ for the high volatility regime. In the one stage pentanomial tree in Figure 1, each regime is represented by a trinomial tree with one branch being shared by both regimes. In order to minimize the number of nodes in the tree, the nodes from both regimes are merged by setting the step sizes of both regimes at a $1 : 2$ ratio which is the only ratio to make the tree recombine when we have two regimes.

In this paper, we choose $N = 2$. However if one chooses more than two states of the volatility, a recombined multinomial lattice can be built to approximate the (multi) regime

switching forward curve model. Wahab & Lee (2011) has a more detailed discussion on how to build a multinomial lattice with more than two regimes under GBM however the approach discussed in this paper can be generalised in a similar way to build a multinomial lattice to approximate a mean reverting process with more than two volatility regimes. The ratio $1 : 2$ should also be adjusted accordingly in this case, also see Wahab & Lee (2011) for more details.

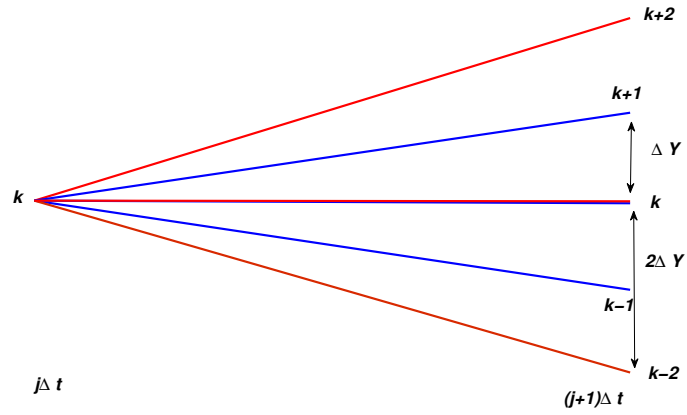Figure 2 demonstrates the recombing feature of the tree.

FIGURE 1. One step of a pentanomial tree. The outer two branches together with the middle branch represent the regime with high volatility and the inner two branches together with the middle branch represent the regime with low volatility.

The time values represented in the tree are equally spaced and have the form $t_j = j\Delta t$ where j is a non-negative integer and Δt is the time step, usually one day in our context. The values of Y at time t_j are equally spaced and have the form $Y_{j,k} = k\Delta Y$ where ΔY is the space step and k determines the level of the variable in the tree. Any node in the tree can therefore be referenced by a pair of integers, that is the node at the $j-$th time step and $k-$th level we refer to as node (j, k). From stability and convergence considerations, a reasonable choice for the relationship between the space step ΔY and the time step Δt suggested by Wahab & Lee (2011) is given by

$$\Delta Y = \begin{cases} \sigma_L\sqrt{3\Delta t}, & \sigma_L \geq \frac{1}{2}\sigma_H, \\ \frac{1}{2}\sigma_H\sqrt{3\Delta t}, & \sigma_L < \frac{1}{2}\sigma_H. \end{cases} \tag{10}$$

The trinomial branching process and the associated probabilities are chosen to be consistent with the drift and volatility of the process. The three nodes that can be reached by the
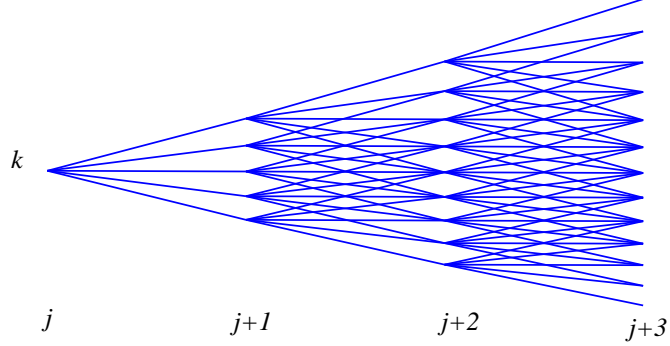
FIGURE 2. The recombining nature of a pentanomial tree. At time step t, the number of nodes is $4t - 3$. At each node, two sets (outer and inner) of transition probabilities are worked out to match the first two moments implied by the tree and those implied by Eq. (9)

branches emanating from node (j, k) are $(j + 1, l - 1)$, $(j + 1, l)$, and $(j + 1, l + 1)$ for the low volatility regime and $(j + 1, l - 2)$, $(j + 1, l)$, and $(j + 1, l + 2)$ for the high volatility regime. Here l is chosen so that the value of Y reached by the middle branch is as close as possible to the expected value of Y at time t_{j+1}. From the Euler discretization of equation (9), the expected value of Y at time t_{j+1} conditional on $Y = Y_{j,k}$ is $Y_{j,k} - \alpha Y_{j,k} \Delta t$.

2.2.2. *Transition probabilities.* For either regime $x = L$ or H, let $p^x_{u,j,k}, p^x_{m,j,k}$ and $p^x_{d,j,k}$ define the probabilities associated with the upper, middle and lower branches emanating from node (j, k) respectively. These probabilities can be calculated as follows. When the volatility is in the low regime, $\sigma = \sigma_L$, looking at the inner trinomial tree we need to match

$$E[\Delta Y] = -\alpha Y_{j,k} \Delta t, \text{ and } E[\Delta Y^2] = \sigma_L^2 c(t_j) \Delta t + E[\Delta Y]^2. \tag{11}$$

Therefore equating the first and second moments of ΔY in the tree with the above values we obtain

$$p^L_{u,j,k}((l + 1) - k) + p^L_{m,j,k}(l - k) + p^L_{d,j,k}((l - 1) - k) = -\alpha Y_{j,k} \frac{\Delta t}{\Delta Y}, \tag{12}$$

and

$$p^L_{u,j,k}((l + 1) - k)^2 + p^L_{m,j,k}(l - k)^2 + p^L_{d,j,k}((l - 1) - k)^2 = (\sigma_L^2 c(t_j) \Delta t + (-\alpha Y_{j,k} \Delta t)^2)/\Delta Y^2. \tag{13}$$

Solving equations (12) and (13) together with conditions that $p^L_{u,j,k} + p^L_{m,j,k} + p^L_{d,j,k} = 1$ we obtain

$$p^L_{u,j,k} = \frac{1}{2}\left[\frac{\sigma^2_L c(t_j)\Delta t + \alpha^2 Y^2_{j,k}\Delta t^2}{\Delta Y^2} + (l-k)^2 - \frac{\alpha Y_{j,k}\Delta t}{\Delta Y}(1 - 2(l-k)) - (l-k)\right], \quad (14)$$

$$p^L_{d,j,k} = \frac{1}{2}\left[\frac{\sigma^2_L c(t_j)\Delta t + \alpha^2 Y^2_{j,k}\Delta t^2}{\Delta Y^2} + (l-k)^2 + \frac{\alpha Y_{j,k}\Delta t}{\Delta Y}(1 + 2(l-k)) + (l-k)\right], \quad (15)$$

and

$$p^L_{m,j,k} = 1 - p^L_{u,j,k} - p^L_{d,j,k}. \quad (16)$$

When the volatility is in high regime, $\sigma = \sigma_H$, looking at the outer trinomial tree and applying a similar procedure, we find that

$$p^H_{u,j,k} = \frac{1}{8}\left[\frac{\sigma^2_H c(t_j)\Delta t + \alpha^2 Y^2_{j,k}\Delta t^2}{\Delta Y^2} + (l-k)^2 - \frac{\alpha Y_{j,k}\Delta t}{\Delta Y}(2 - 2(l-k)) - 2(l-k)\right], \quad (17)$$

$$p^H_{d,j,k} = \frac{1}{8}\left[\frac{\sigma^2_H c(t_j)\Delta t + \alpha^2 Y^2_{j,k}\Delta t^2}{\Delta Y^2} + (l-k)^2 + \frac{\alpha Y_{j,k}\Delta t}{\Delta Y}(2 + 2(l-k)) + 2(l-k)\right], \quad (18)$$

and

$$p^H_{m,j,k} = 1 - p^H_{u,j,k} - p^H_{d,j,k}. \quad (19)$$

2.2.3. *State prices for both regimes.* Following a similar approach to that in Chapter 7 of Clewlow & Strickland (2000), we displace the nodes in the above simplified tree by adding the drifts a_i which are consistent with the observed forward prices.

In fact, since we have two regimes, for $x = L, H$ we define state (or Arrow-Debreu) prices $Q^x_{j,k}$ as the present value of a security that pays off \$1 if $Y = k\Delta Y$ and $X_{j\Delta t} = x$ at time $j\Delta t$ and zero otherwise. The $Q^x_{j,k}$ are in fact the state prices that accumulate according to

$$Q^L_{j+1,k} = \sum_{k'}(Q^L_{j,k'}p_{L,L} + Q^H_{j,k'}p_{H,L})p^L_{k',k}B(j\Delta t, (j+1)\Delta t), \quad (20)$$

$$Q^H_{j+1,k} = \sum_{k'}(Q^L_{j,k'}p_{L,H} + Q^H_{j,k'}p_{H,H})p^H_{k',k}B(j\Delta t, (j+1)\Delta t), \quad (21)$$

with the initial values for the lower and higher volatility regimes being $Q_{0,0}^L = 1$, $Q_{0,0}^H = 0$ and $Q_{0,0}^L = 0$, $Q_{0,0}^H = 1$, respectively.

In Equations (20) - (21), $p_{x,x'}$ is the probability the Markov Chain transits from the state x to the state x' and $p_{k',k}^L$ and $p_{k',k}^H$ are the probabilities the spot price transits from k' to k but arriving at low and high volatility regime respectively. $B(j\Delta t, (j+1)\Delta t)$ denotes the price at time $j\Delta t$ of the pure discount bond maturing at time $(j+1)\Delta t$.

We see that Arrow-Debreu securities are the building blocks of all securities; in particular when we have j time steps in the tree, the price today, $C(0)$, of any European claim with payoff function $C(S)$ at time step j in the tree is given by

$$C(0) = \sum_k (Q_{j,k}^L + Q_{j,k}^H) C(S_{j,k}), \tag{22}$$

where $S_{j,k}$ is the time t_j spot price at level k and the summation takes place across all of the nodes k at time j.

In order to use the state prices to match the forward curve we use the special case of equation (22) that values the initial forward curve, namely

$$B(0, j\Delta t) F(0, j\Delta t) = \sum_k (Q_{j,k}^L + Q_{j,k}^H) S_{j,k}. \tag{23}$$

By the definition of a_j we have $S_{j,k} = e^{Y_{j,k} + a_j}$, then the term a_j is needed to ensure that the tree correctly returns the observed futures curve is given by

$$a_j = \ln \left(\frac{B(0, j\Delta t) F(0, j\Delta t)}{\sum_k (Q_{j,k}^L + Q_{j,k}^H) e^{Y_{j,k}}} \right). \tag{24}$$

In fact, inserting $S_{j,k} = e^{Y_{j,k} + a_j}$ into equation (23) we have

$$B(0, j\Delta t) F(0, j\Delta t) = \sum_k (Q_{j,k}^L + Q_{j,k}^H) e^{Y_{j,k} + a_j} = e^{a_j} \sum_k (Q_{j,k}^L + Q_{j,k}^H) e^{Y_{j,k}}. \tag{25}$$

Hence we have

$$e^{a_j} = \frac{B(0, j\Delta t) F(0, j\Delta t)}{\sum_k (Q_{j,k}^L + Q_{j,k}^H) e^{Y_{j,k}}}, \tag{26}$$

then equation (24) follows immediately.

The upper panel of Figure 3 demonstrates an example of a pentanomial tree which has been constructed to be consistent with the seasonal gas forward prices shown in the lower panel of Figure 3.
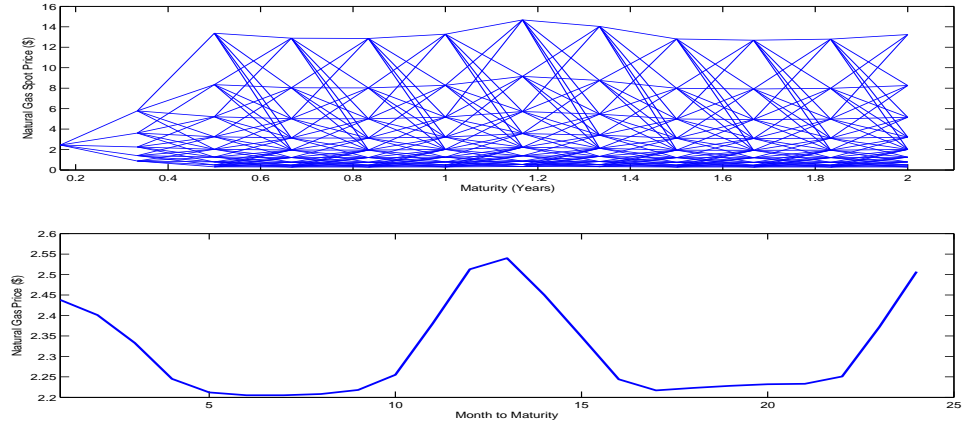
FIGURE 3. The evolution of spot price tree constructed by the procedure described in Section 2.2 (upper panel) fitted to a seasonal market gas forward price curve (lower panel).

3. MULTIPLE YEAR GAS SALES AGREEMENT WITH MAKE UP AND CARRY FORWARD PROVISIONS

A gas sales agreement is an agreement between a supplier and a purchaser for the delivery of variable daily quantities of gas, between specified minimum and maximum daily limits, over a certain number of years at a specified set of contract prices. The main features of these contracts that make them difficult to value and risk manage are the constraints on the quantity of gas which can be taken. The main constraint is that in each gas year, there is a minimum volume of gas (termed take-or-pay or minimum bill) for which the buyer will be charged at the end of the year (or penalty date), regardless of the actual quantity of gas taken. Typically, there is also a maximum annual quantity which can be taken. The minimum bill or take-or-pay level is usually defined as a percentage of the notional annual quantity which is called the annual contract quantity (ACQ).

These agreements usually last for ten or twenty years and there are two more features embedded in those contracts, namely the make-up and carry forward. In years where the gas taken is less than the Minimum Bill the shortfall (paid for in the current year) is added to the *Make-Up Bank* (M_{T_i}). In later years where the gas taken is greater than some reference level (typically Minimum Bill or ACQ) additional gas can be taken from the Make-Up Bank and a refund paid.

In years where the gas taken is greater than some reference level (typically ACQ) the excess gas is added to the *Carry Forward Bank* (C_{T_i}). In later years Carry Forward Bank gas can be used to reduce the Minimum Bill for that year.

With the help of the pentanomial tree that we have constructed, we are able to evaluate the prices of the above swing contract. The value of the contract at maturity (the final purchase date) can be computed first. The final decision is simple because the penalty amount is known with certainty. Then we step back through the pentanomial tree computing the discounted expectations of the contract value at each node for both low and high volatility regimes and computing the optimal purchase decision at the purchase dates for both regime as well. The optimal purchase decision at each node and for each value of the remaining volume and for each regime can be computed by searching over the range of possible purchase volumes for the volume which maximises the sum of the discounted expectation averaged by the transition probabilities of the hidden Markov Chain on different regimes and the value of the current purchase.

3.1. **Input and Notation.** In this section, we introduce some notations for calculating the multiple year gas sales agreement with both make-up and carry forward provisions. In the following, we assume that the economy is in regime $x = L, H$ at the particular time depending on the evolution of the hidden Markov chain.

The buy of a multiple year swing contract may face a penalty at the end of each year and both the make-up bank and the carry forward bank will possibly start to accumulate from the end of the first year of the contract. The contract will span I years and let $T_i, i = 1, \ldots, I$ denote the end of each year i. Also assume that there are J periods within each year and usually $J = 365$ for daily decisions and transactions.

Let $V_{t_{ij}}^*(x)$ denote the value of the swing contract at day $t_{ij}(T_{i-1} < t_{ij} \leq T_i)$, given $(T_I \cdot J - t_{ij})$ periods to maturity and $q_{t_{ij}}(x) (\in [q_{\min}, q_{\max}])$ denote the amount of gas taken in period t_{ij} and the corresponding single period (daily) constraints.

The volume taken $Q_{t_{ij}}$ is the cumulative amount of gas taken up to time t_{ij} in year T_i and is given by $Q_{t_{ij}} = \sum_{k=0}^{j-1} q_{t_{ik}}$ and set $Q_{T_i} = Q_{t_{iJ}}$ which is the total amount of gas taken during the year i. MB_{T_i} is the minimal bill for the year i, namely the total amount of gas that should be taken to avoid a penalty at time T_i, the end of year i.

In terms of make-up bank, $M_{T_i}(x)$ is the amount of gas available in the make-up bank within the year $T_i(i = 2, \ldots, I)$, which is a consequence of both the balance of the previous

years and the decision of the current year. MRL_{T_i} is the make-up bank recovery limit which is the maximal amount of gas allowed to be recovered in year i.

In terms of carry forward bank, CB_{T_i} is the carry forward base for the year i. The surplus, if the volume taken exceeds the carry forward base, will be added into the carry forward bank. This level could equal MB_{T_i} or be higher. $C_{T_i}(x)$ is the amount of gas available in the carry-forward bank within the year T_i. It is derived from both the balance of the previous years and the decision of the current year. CRL_{T_i} is the carry forward bank recovery limit which is the maximal amount of gas allowed to be recovered from the carry bank in year i.

$S_{t_{ij}}(x)$ is the current spot price at time t_{ij} and K_i is the purchase price in year i. The penalty at the end of each year will be with $\eta \in [0, 1]$:

$$\eta \cdot \min\{Q_{T_1} - MB_{T_1}, 0\} \cdot K_1, \tag{27}$$

for the first year and with β_i being the percentage usage of the carry forward bank at T_i:

$$\eta \cdot \min\{Q_{T_i} - (MB_{T_i} - \beta_i C_{T_i}), 0\} \cdot K_i, \tag{28}$$

for later years.

3.2. **Decisions.** The buyers of the swing contract should take decisions so that their total expected discounted payoffs are maximized. In the following, we will give a detailed analysis on the optimal decisions on the last day of the contract. Then the dynamic programming principle will be implemented to work out both the optimal decisions and the optimal values of the swing contract at each day.

Generally speaking, in the first year of the contract, the buyer decides on each possible trading day whether to exercise one swing right or not, and the amount $q_{t_{ij}}(x)$ taken upon exercise. From the second year, the buyer makes decisions following analogous rules to those in the first year before the last day of the year but must make a joint decision on *exercise, carry forward* and *make-up* on the *last day* of that year. In the following discussion, $\beta_i(x)$ and $\gamma_i(x)$ are the decisions on the percentage usage of the carry forward bank and make-up bank at the end of each year i, respectively.

At the last day of each gas year, the buyer should decide on: first, how much gas $(q_{t_{iJ}}(x))$ to buy and then, how much in the carry forward bank $(\beta_i(x) \cdot C_{T_i}(x))$ should be used to lower the minimal bill if possible and finally, how much gas in the make-up bank $(\gamma_i(x) \cdot M_{T_i}(x))$ will be taken free.

Denote the decision vector at time t_{ij} by $d_{ij}(x) = (q_{t_{ij}}(x), \beta_i(x), \gamma_i(x)), \forall i, j$ with $\beta_1(x) = 0$ and $\gamma_1(x) = 0$ since both make-up bank and carry forward bank are empty when the contract initiates. Each decision will depend on the state variables in a given year, namely, the underlying spot price $(S(x))$, the cumulative gas taken $(Q(x))$, the amount in the carry forward bank $(C(x))$, the amount in the make-up bank $(M(x))$ and the regime of the economy (x).

At the end of each year i, the buyer would face the following possible cash flow: first, the pay off

$$q_{t_{iJ}}(x)(S_{t_{iJ}}(x) - K_i) \tag{29}$$

from the decision to take gas and then, the possible penalty when the total gas taken in year i is less than the new minimal bill which is adjusted by using the fraction $\beta_i(x)$ of the carry forward bank

$$\eta K_i \min\{Q_{t_{iJ}} + q_{t_{iJ}}(x) - (MB_{T_i} - \beta_i(x)C_{T_i}(x)), 0\}; \tag{30}$$

and finally, the possible refund from using the fraction $\gamma_i(x)$ of the make-up bank when the total gas taken in year i is more than the adjusted minimal bill which is adjusted by using the fraction $\beta_i(x)$ of the carry forward bank

$$K_{i-1} \min\{\gamma_i(x)M_{T_i}(x), \max\{Q_{t_{iJ}} + q_{t_{iJ}}(x) - (MB_{T_i} + \beta_i(x)C_{T_i}(x)), 0\}\}. \tag{31}$$

The evolution of the carry forward bank may be written

$$C_{T_i}(x) = (1 - \beta_{i-1}(x))C_{T_{i-1}}(x) + \max\{Q_i(x) - CB_{T_i}, 0\}, \tag{32}$$

namely, in year i, the balance of the carry forward bank is the balance in year $(i-1)$ plus the additional gas when the total gas taken in year i exceeds the carry forward base.

The balance in the carry-forward bank can be used to reduce the minimal bill

$$MB^{(1)}_{T_i}(x) = MB_{T_i} - \beta_i(x)C_{T_i}(x);$$

after which the evolution of the make-up bank is

$$M_{T_i}(x) = (1 - \gamma_{i-1}(x))M_{T_{i-1}}(x) + \max(MB^{(1)}_{T_i}(x) - Q_{T_i}(x), 0), \tag{33}$$

namely, in year i, the balance of the make-up bank is the balance in year $(i-1)$ plus the shortfall, if the total gas taken in year i is less than the reduced minimal bill $MB^{(1)}_{T_i}(x)$.

3.3. **The Value of Swing Contract – Objective Functions.** The total expected discounted payoff at the end of the contract with $S_{t_{10}} = S, Q_{t_{10}} = Q, X_{t_{10}} = x$ is given

by[1]

$$
V_{I,J}(S, Q, C, M, x, q, \beta, \gamma) = \sum_{i=1}^{I} \Bigg[\sum_{j=0}^{J} e^{-rt_{ij}} q_{t_{ij}}(X_{t_{ij}})(S_{t_{ij}}(X_{t_{ij}}) - K_i)
$$

$$
+ \eta K_i \min\{Q_{t_{iJ}} + q_{t_{iJ}}(X_{t_{iJ}}) - (MB_{T_i} - (\beta_i C_{T_i})(X_{t_{ij}})), 0\} \tag{34}
$$

$$
+ K_{i-1} \min\{(\gamma_i M_{T_i})(X_{t_{ij}}), \max\{Q_{t_{iJ}} + q_{t_{iJ}}(X_{t_{iJ}}) - (MB_{T_i} + (\beta_i C_{T_i})(X_{t_{ij}})), 0\}\} \Bigg]
$$

$$
= \sum_{i=1}^{I} [\text{Payoff}_i - \text{Penalty}_i + \text{Refund}_i].
$$

Here $q = (q_{t_{ij}}(X_{t_{ij}}))$, $\beta = (\beta_i(X_{t_{ij}}))$, $\gamma = (\gamma_i(X_{t_{ij}}))$ and we have for $i \geq 2$, with the evolutions of both carry forward bank and make-up bank[2].

3.4. **The Terminal Condition** − **the Initial Step.** We first consider the decision and the value of the contract at the last day and then step backwards to find the decisions and values at each day of the swing option. We also assume that there are no differences in the decisions and values on the last day between two regimes.

Hence in either regime, the following rule should apply. At the last day of the contract, we have to decide how much gas (q_{T_I}) to take, how much to use from both the carry forward bank (β_I) and the make-up bank (γ_I). Since this is the last day of the contract, we should use as much of the balance in both the make-up bank and carry forward bank as possible, hence,

$$
\beta_I^* = \gamma_I^* = 1.
$$

Next we need to compute the optimal quantity for this last day: if $S_{T_I} > K_I$, then the payoff is strictly increasing in the volume purchased and the maximum quantity of gas $q_{\max}$ should be purchased; if $(1 - \eta)K_I \leq S_{T_I} < K_I$ then the optimal choice is to purchase a quantity up to that required to avoid the penalty or the maximum possible, whichever is smaller. Since the loss on the purchase of the energy is more than compensated by the reduction in the penalty payment; if $S_{T_I} < (1 - \eta)K_I$ then the purchase of zero gas is

---

[1]In the following discussions, for the sake of brevity, we use the notation $(\beta_i C_{T_i})(X_{t_{ij}})$ instead of $\beta_i(X_{t_{ij}})C_{T_i}(X_{t_{ij}})$, meaning that both β_i and C_{T_i} depend on $X_{t_{ij}}$.
[2]Please note that the second term in equation (34) is non-positive, hence we put a minus (−) sign in front of the Penalty term.

optimal. Summarizing the above, we have $q_{T_I}(= q_{t_{Ij}})$ is equal to

$$
q_{T_I}^* = \begin{cases} q_{\max}, & \text{for} & S_{T_I} \geq K_I; \\ f(Q_{T_{I-1}}, MB_{T_I}, M_{T_I}, C_{T_I}), & \text{for} & (1-\eta)K_I \leq S_{T_I} < K_I; \\ 0, & \text{for} & 0 \leq S_{T_I} < (1-\eta)K_I; \end{cases} \tag{35}
$$

In fact, when $(1-\eta)K_I \leq S_{T_I} < K_I$, after taking into consideration the possible values of M_{T_I} and C_{T_I}, we find the optimal decisions $q_{T_I}^*$ as follows and consequently we know the detail of the function $f(Q_{T_{I-1}}, MB_{T_I}, M_{T_I}, C_{T_I})$.

If $M_{T_I} = 0$, which means that the make-up bank is empty, then the optimal decision will be to use all carry forward bank to reduce the minimal bill and then take maximal allowed to avoid the penalty: $q_{T_I}^* = \min(\max(MB_{T_I} - Q_{t_{I(J-1)}} - C_{T_I}, 0), q_{\max})$. Otherwise if $M_{T_I} > 0$, the make-up bank is not empty, then all carry forward bank will still be used to lower the minimal bill and if the value taken is greater than the adjusted minimal bill, namely, $Q_{t_{I(J-1)}} - MB_{T_I} - C_{T_I} \geq 0$, there are two cases then, one is that the additional gas is not greater than the make-up bank, $Q_{t_{I(J-1)}} - MB_{T_I} - C_{T_I} \leq M_{T_I}$, then the optimal decision should be to take up to maximal allowed to reach the minimal bill adjusted by both make-up bank and carry forward bank which leads to $q_{T_I}^* = \min(M_{T_I} + MB_{T_I} + C_{T_I} - Q_{t_{I(J-1)}}, q_{\max})$; while the other case being that the additional gas is greater than the make-up bank, $Q_{t_{I(J-1)}} - MB_{T_I} - C_{T_I} \geq M_{T_I}$, then the optimal decision will be to take nothing, $q_{T_I}^* = 0$.

When the make-up bank is not empty, $M_{T_I} > 0$ but the value taken doesn't meet the adjusted minimal bill: $Q_{t_{I(J-1)}} - MB_{T_I} - C_{T_I} < 0$, then we have to compare the shortfall $MB_{T_I} + C_{T_I} - Q_{t_{I(J-1)}}$ and $q_{\max}$. If the shortfall is greater than the maximal daily take, $MB_{T_I} + C_{T_I} - Q_{t_{I(J-1)}} \geq q_{\max}$, then the optimal decision will be to take up to maximal allowed to minimise the shortfall: $q_{T_I}^* = \min(\max(MB_{T_I} - Q_{t_{I(J-1)}} - C_{T_I}, 0), q_{\max})$. Otherwise, the shortfall is less than the maximal daily take, $MB_{T_I} + C_{T_I} - Q_{t_{I(J-1)}} < q_{\max}$, then we have to take into consideration of both contract price and spot price to compare the value $(MB_{T_I} + C_{T_I} - Q_{t_{I(J-1)}}) * K_{T_I}/S_{T_I}$ and $q_{\max}$: if $(MB_{T_I} + C_{T_I} - Q_{t_{I(J-1)}}) * K_{T_I}/S_{T_I} > q_{\max}$, then $q_{T_I}^* = 0$; otherwise $q_{T_I}^* = q_{\max}$.

The terminal payoff for either regime L or H including possible penalty is

$$
\mathcal{P}(S_{T_I}, Q_{T_I}, C_{T_I}, M_{T_I}) = q_{T_I}^*(S_{T_I} - K_I) + \eta K_I \min\{Q_{T_I} + q_{T_I}^* - (MB_{T_I} - C_{T_I}), 0\}
$$
$$
+ K_{I-1} \min\{M_{T_I}, \max\{Q_{T_I} + q_{T_I}^* - (MB_{T_I} + C_{T_I}), 0\}\}. \tag{36}
$$

In fact, it is a direct consequence of the penalty and refund form in equations (30) and (31).

3.5. **The General Step.** The objective function $V_{I,J}(S, Q, C, M, x, q, \beta, \gamma)$ at the beginning of the contract can be rewritten as

$$
V_{I,J}(S, Q, C, M, x, q, \beta, \gamma) = \sum_{i=1}^{I-1} \left[\sum_{j=0}^{J} e^{-rt_{ij}} q_{t_{ij}}(X_{t_{ij}})(S_{t_{ij}}(X_{t_{ij}}) - K_i) \right.
$$

$$
+ \eta K_i \min\{Q_{t_{ij}} + q_{t_{ij}}(X_{t_{ij}}) - (MB_{T_i} - (\beta_i C_{T_i})(X_{t_{ij}})), 0\}
$$

$$
+ K_{i-1} \min\{(\gamma_i M_{T_i})(X_{t_{ij}}), \max\{Q_{t_{ij}} + q_{t_{ij}}(X_{t_{ij}}) - (MB_{T_i} + (\beta_i C_{T_i})(X_{t_{ij}})), 0\}\} \Big]
$$

$$
+ \sum_{j=0}^{J-1} e^{-rt_{Ij}} q_{t_{Ij}}(X_{t_{Ij}})(S_{t_{Ij}}(X_{t_{Ij}}) - K_I) + \mathcal{P}(S_{T_I}, Q_{T_I}, C_{T_I}, M_{T_I}). \tag{37}
$$

The value of a swing contract $V_{I,J}^*(S, Q, C, M, x)$ with both make-up and carry forward provisions is determined by

$$
V_{I,J}^*(S, Q, C, M, x) = \max_{q,\beta,\gamma} \mathbb{E} V_{I,J}(S, Q, C, M, x, q, \beta, \gamma), \tag{38}
$$

where q is a sequence of daily decisions and β and γ are sequences of yearly decisions.

3.6. **Evaluation using Dynamic Programming.** We use $V(S, Q, C, M, x, q, \beta, \gamma, t_{ij})$ to denote the cost-to-go function of the total payoff $V_{I,J}(S, Q, C, M, x, q, \beta, \gamma)$, that is the value of the payoff from time t_{ij} onwards up to maturity. Let

$$
V^*(S, Q, C, M, x, t_{ij}) = \max_{q,\beta,\gamma} \mathbb{E} V(S, Q, C, M, x, q, \beta, \gamma, t_{ij}) \tag{39}
$$

denote the optimal cost-to-go value function at time t_{ij}. Obviously

$$
V_{I,J}^*(S, Q, C, M, x) = V^*(S, Q, C, M, x, t_{10}). \tag{40}
$$

With the help of the dynamic programming principle, we are able to show that at the end of the contract, the optimal value function for any $x = L, H$ follows

$$
V^*(S, Q, C, M, x, T_I) = \mathcal{P}(S, Q, C, M), \tag{41}
$$

where we recall that the function $\mathcal{P}$ is defined by Equ (36).

At the end of each gas day within a gas year, we should choose the optimal quantity $q_{t_{ij}}^*$ according to

$$V^*(S, Q, C, M, x, t_{ij}) =$$

$$\max_{q_{t_{ij}}} \left[q_{t_{ij}}(S - K_i) + \sum_{x'=L}^{H} p_{xx'} \mathbb{E}[V^*(S_{t_{i(j+1)}}, Q + q, C, M, x', t_{i(j+1)}) | S_{t_{ij}} = S, X_{t_{ij}} = x] \right].$$

$$(42)$$

$$q^*(S, Q, C, M, x, t_{ij}) =$$

$$\text{argmax}_{q_{t_{ij}}} \left[q_{t_{ij}}(S - K_i) + \sum_{x'=L}^{H} p_{xx'} \mathbb{E}[V^*(S_{t_{i(j+1)}}, Q + q, C, M, x', t_{i(j+1)}) | S_{t_{ij}} = S, X_{t_{ij}} = x] \right].$$

$$(43)$$

for $i = 1, 2, \cdots, I, j = 0, 1, \ldots, J - 1$.

Here, both the optimal value and optimal quantity depend on the regime x at a particular time. In practice, the owner of the contract can work out the implied volatilities from options on natural gas futures contracts and he or she can determine if the system is in high or low volatility regime based on the level of those implied volatilities.

However, at the last day of each year, we should choose the optimal quantity q_i^*, the fraction taken from the carry forward bank (β_i^*) and the fraction taken from the make-up bank (γ_i^*) according to:

$$V^*(S, Q, C, M, x, T_i) =$$

$$\max_{q_i, \beta_i, \gamma_i} \left[q_i(S - K_i) + \mathcal{P}_i(q, S, Q, \beta_i C, \gamma_i M, x) + \right. \tag{44}$$

$$\left. \sum_{x'=L}^{H} p_{xx'} \mathbb{E}[V^*(S_{t_{(i+1)0}}, Q + q, C_{T_{i+1}}, M_{T_{i+1}}, x', t_{(i+1)0}) | S_{T_i} = S, X_{T_i} = x] \right]$$

$$(q_i^*, \beta_i^*, \gamma_i^*)(S, Q, C, M, x, T_i) =$$

$$\text{argmax}_{q_i, \beta_i, \gamma_i} \left[q_i(S - K_i) + \mathcal{P}_i(q, S, Q, \beta_i C, \gamma_i M, x) + \right. \tag{45}$$

$$\left. \sum_{x'=L}^{H} p_{xx'} \mathbb{E}[V^*(S_{t_{(i+1)0}}, Q + q, C_{T_{i+1}}, M_{T_{i+1}}, x', t_{(i+1)0}) | S_{T_i} = S, X_{T_i} = x] \right]$$

for $i = 1, 2, \ldots, I - 1$ and $C_{T_i} = C, M_{T_i} = M$. The evolutions of both make-up bank and carry forward bank follow Equations (32) and (33) respectively. Also $\mathcal{P}_i$ is the possible penalty or refund after taking actions at the end of year i:

$$
\begin{aligned}
\mathcal{P}_i(q, S, Q, C, M, x) &= \eta K_i \min\{Q + q(x) - (MB_{T_i} - C(x)), 0\} \\
&+ K_{i-1} \min\{M(x), \max\{Q + q(x) - (MB_{T_i} + C(x)), 0\}\}. \quad (46)
\end{aligned}
$$

The nodes and transition probabilities of the pentanomial tree constructed in the previous section can be used to calculate the conditional expectation $\mathbb{E}[\cdot|\cdot]$.
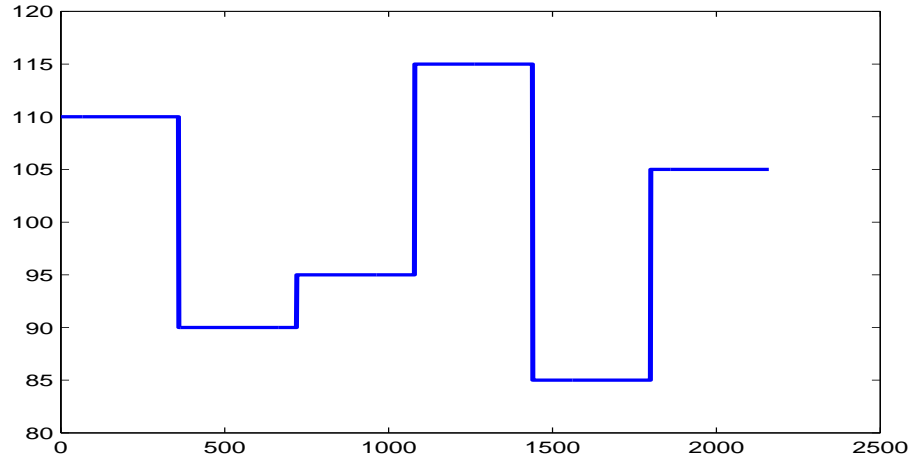

FIGURE 4. A typical gas forward price curve defined in Equ.(47) which evolves flat within each year but with different levels between years.

4. NUMERICAL EXAMPLES

In this section, we provide a numerical example to demonstrate how we evaluate the multiple year contracts and how we calculated the optimal decisions on the amount of daily gas consumption and accumulation from the make-up and carry forward bank.

4.1. **Value surfaces and decision surfaces.** In the following, we evaluate a six-year gas sales agreement according to the following parameter settings. Assume that volatilities in different regimes are $\sigma_L = 0.5, \sigma_H = 1.0$; and mean reversion rate is $\alpha = 5$.
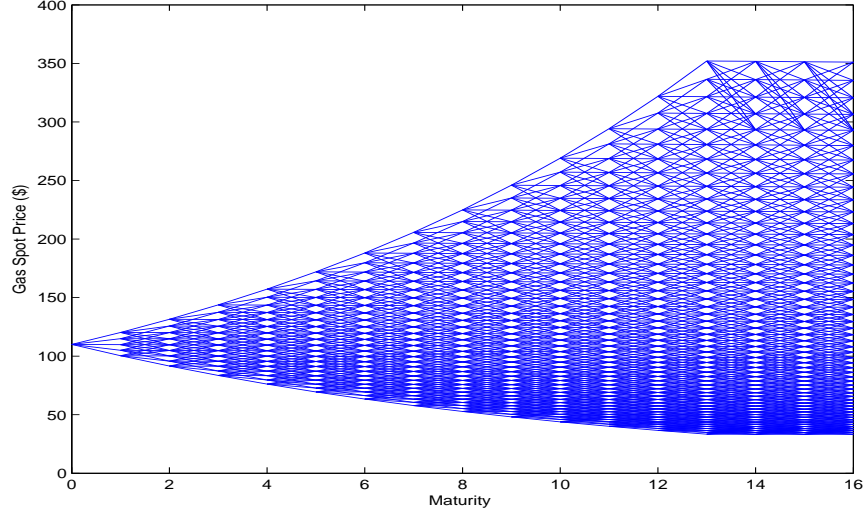
FIGURE 5. Part of the Pentanomail tree based on the forward price curve in Equ.(47) of Figure 4. The tree is constructed based on the the parameters given in Section 4.1 and the procedures described in Section 2.2.

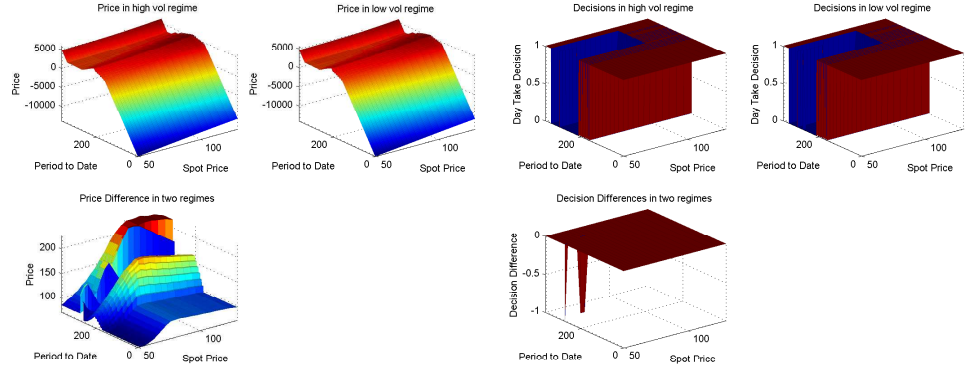We also assume the interest rate $r = 0$ and the gas forward curve in Figure 4 behaves as:

$$F(0,t) = \begin{cases} 110, & 0 \le t \le 365, \\ 90, & 366 \le t \le 730, \\ 95, & 731 \le t \le 1095, \\ 115, & 1096 \le t \le 1460, \\ 85, & 1461 \le t \le 1825, \\ 105, & 1826 \le t \le 2190. \end{cases} \tag{47}$$

In terms of the gas sales agreement, we have the contract price: $K = 100$; daily take limit: $q_{\min} = 0$ and $q_{\max} = 1$; maturity time: $T = 365 \times 6 = 2190$; and the minimal Bill: $MB = 365 \times 75\% = 273$. The carry Base: $CB = 365 \times 80\% = 292$; the recovery limits of the make-up bank and the carry forward bank are $MRL = 365 \times 20\% = 73$ and $CRL = 365 \times 20\% = 73$, respectively.
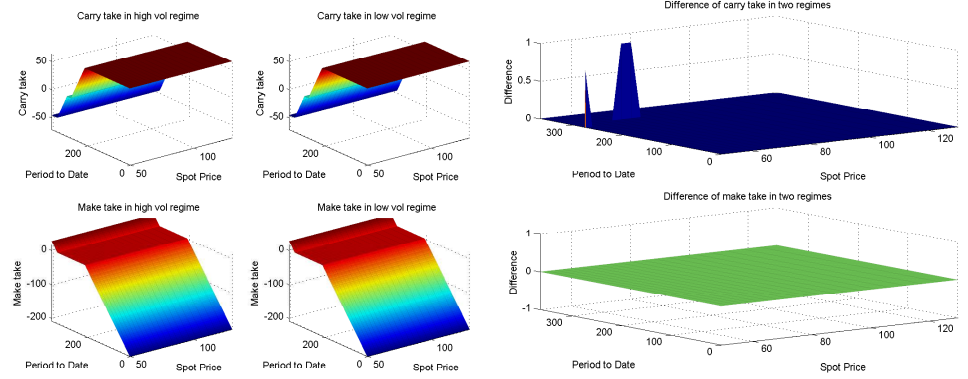
We assume the transition matrix of the hidden Markov Chain:

$$P = \begin{bmatrix} 0.99 & 0.01 \\ 0.01 & 0.99 \end{bmatrix}. \tag{48}$$

Following the detailed procedures described in Section 2, we build a pentanomial lattice part of which is shown in Figure 5. It is consistent with the forward price curve shown in

(a) Day 1825 value differences in two different regimes.

(b) Day 1825 decision differences in two different regimes.

(c) Day 1825 make and carry in two different regimes.

(d) Day 1825 make and carry differences in two different regimes.

FIGURE 6. (a), (b), (c) and (d) are Day 1825 value, decision and make up and carry forward surfaces with 32 in the make up bank and 64 in the carry forward bank.

Figure 4. In the panels of Figure 6 we select a number of value surfaces, decision surfaces, make-take surfaces and carry take surfaces in both regimes and the differences between two regimes at different days when there are different units remaining in the make up bank and carry forward bank. Our algorithm is very efficient; it takes less than 5 minutes to evaluate such a six-year contract and produce the surfaces of the optimal values, day take decisions, decisions on make-up and carry forward takes.

4.2. **How the change of regime affect the decisions.** In this section, we want to assess how different regimes affect the decisions on day take, carry take and make take and also the influences of both regimes on the volume taken. We simulate a path of the Brownian motion first and then for this given path, we simulate a number of different realizations of the Markov Chain X_t and the corresponding spot prices and then we make decisions

based on the optimal decision surface we calculated in the previous section. Figures 7 and 8 demonstrate how decisions on day take, carry take and make take change when the realizations of the Markov Chain are different.

In the first year of the example the forward curve is above the contract price (we refer to this as in-the-money). Assuming that the spot price follows the forward curve (we call this the intrinsic strategy) a simulation of the spot price is represented by the second panel in both Figures 7 and 8 and the optimal strategy would then be to take the maximum possible (365) and create 73 units of carry forward (365-292). However, the simulated spot price in Figure 7 is sometimes below the contract price and so the simulated take is 342, which creates 23 units of carry forward. While in Figure 8, since the volatility remains in the high vol regime for an extended period, there are more spot prices below the contract price and so the simulated take is 265, which even creates 8 (273 - 265) units of make up.

In the second year, the forward curve is below the contract price (out-of-the-money) and so the optimal intrinsic strategy would be to use the carry-forward bank to reduce the MB to 273 - 73 = 200 and then take 127 to create a makeup bank of 73 (the maximum that can be recovered). The simulated spot price in Figure 7 is sometimes above the strike price, so the simulated take is 207, creating a make-up bank of 35 (273 - 23 - 225). However in the case of Figure 8, there is zero balance in the carry-forward bank in the previous year, and the simulated spot price in this case is sometime above the strike price as well, so the simulated take is 221 , creating a make-up bank of 60 (8 + (273 - 221)).

The third year is also out-of-the-money, so the intrinsic strategy is to take the MB plus the amount of gas in the make-up bank that will be free, giving a take of 273 + 73 = 346. With the simulated spot price in Figure 7 the actual strategy is to take 271 and increase the make-up bank to 37 (35 + (273 - 271)). However there are more simulated spot prices in Figure 8 above the strike price and the actual strategy is to take 285 and use 12 (285-273) from make-up bank for free and decrease the make-up bank to 48 (60 - 12).

In the fourth year the contract is in-the-money, so the optimal intrinsic strategy is to take the maximum quantity of gas (365) and create another 73 units of carry forward. In the simulation of Figure 7 the spot price is below the contract price for a few days, so the take is only 349 and the full 37 units of the make-up bank are taken as free gas and also create 39 (349 - (273 + 37)) units of the carry-forward bank. While more simulated spot prices in Figure 8 are above the strike price and the actual strategy is to take 363 and the full 48 units of the make-up bank are taken as free gas and also create 42 (363 - (273 + 48)) units of the carry-forward bank.

In the fifth year the contract is out-of-the-money, so the intrinsic strategy would be the same as that seen in year two. In the simulation in Figure 7, some days are actually in-the-money, so the take is 222 and a full 39 units of the carry-forward bank are taken to lower the MB to 234 (273 - 39) which creates 12 (234 - 222) units of make up. However in the simulation in Figure 8 the volatility stays in the lower regime for an extended period, more days are in-the-money, so the take is 260 and part 13 units of the carry-forward bank are taken to lower the MB to 260 (273 - 13) and hence there are 29 (42 - 13) units left in the carry-forward bank.

In the final year the contract is in-the-money, so the intrinsic strategy would be to take the maximum possible with 73 units being taken free from the make-up bank. The simulated price in Figure 7 is below the contract price for a significant part of the year, so the simulated optimal strategy is to take 330, of which the 12 in the make-up bank is taken for free and creates 38 (330 - 292) units of the carry forward bank. However most of the spot price in the simulation in Figure 8 are above the strike price, hence the take is 362 and creates 70 (362 - 292) more units of the carry forward bank which makes total 99 (29 + 70) units in the carry forward bank.

In summary, different realizations of the hidden Markov Chain will have a significant impact on the day take decisions and the evolution of both the carry-bank and make-bank.
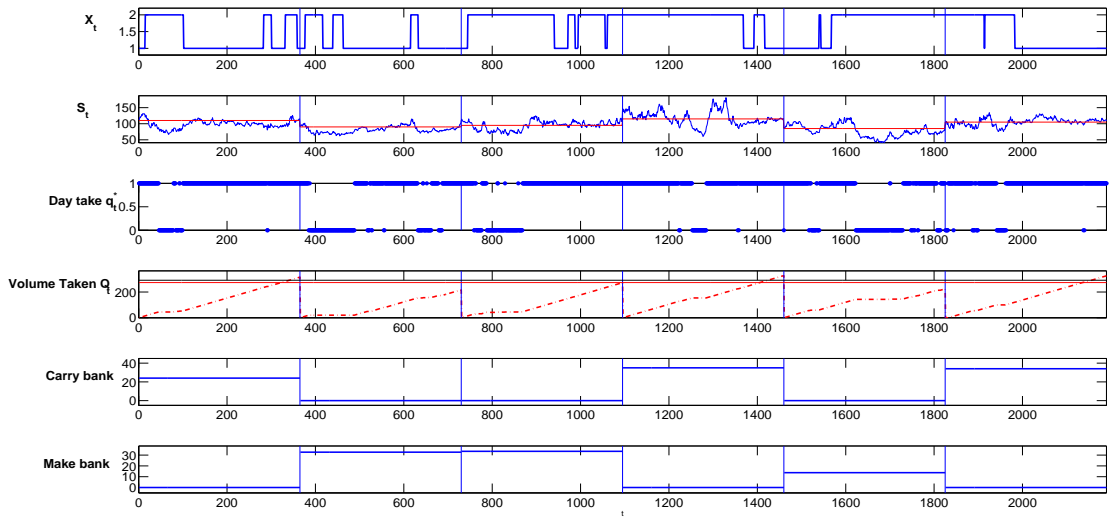

FIGURE 7. One realization of the Markov Chain and the corresponding spot prices, optimal day takes, volume taken, the evolution of both Carry bank and Make bank.
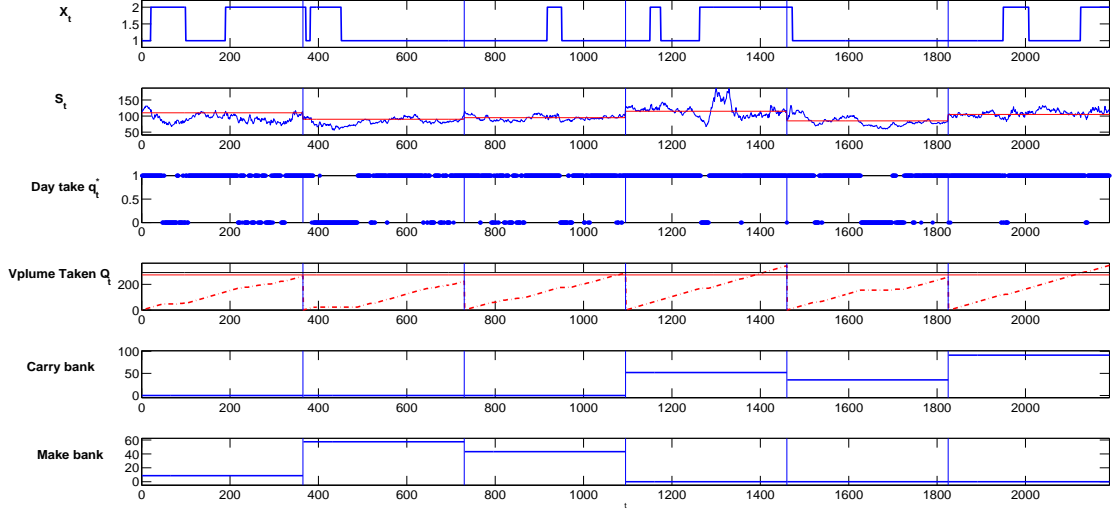
FIGURE 8. Another realization of the Markov Chain and the corresponding spot prices, optimal day takes, volume taken, the evolution of both Carry bank and Make bank.

5. CONCLUSIONS

In this paper, we have proposed a pentanomial tree framework for pricing multiple year gas sales agreements (GSAs) with make-up and carry forward provisions for an underlying gas forward price curve that follows a regime-switching process. The GSAs are complicated because the buyers can exercise their rights in a daily manner and make decisions on the make-up bank and carry forward bank on a yearly basis. Hence in the evaluation we need to keep track of multi-variables on a daily basis lasting for multiple years. Those complexities, along with the regime switching uncertainty of the daily price, require efficient numerical procedures to value these contracts and have been the main contribution of this paper.

With the help of a recombining pentanomial tree, we are able to efficiently evaluate the prices of the contracts, find optimal daily decisions and optimal yearly use of both the make-up bank and carry forward bank in different regimes. We also demonstrate how different regimes are able to affect the decisions on make-up and carry forward takes.

Breslin et al. (2008*b*) discuss the risks and hedging of swing contracts with the features we have discussed in this paper. Hence an important task of future research will be to find the risks and the hedging strategies for these contracts when the underlying forward curve

follows regime switching dynamics. The computational tools developed in this paper may play an important role in this research.

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