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**Running Head: How Numbers Mean**

**How numbers mean: Comparing random walk models of numerical cognition varying both encoding processes and underlying quantity representations**

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**Abstract**

How do people derive meaning from numbers?  Here, we instantiate the primary theories of numerical representation in computational models and compare simulated performance to human data.  Specifically, we fit simulated data to the distributions for correct and incorrect responses, as well as the pattern of errors made, in a traditional "relative quantity" task.  The results reveal that no current theory of numerical representation can adequately account for the data without additional assumptions.  However, when we introduce repeated, error-prone sampling of the stimulus (e.g., Cohen, 2009) superior fits are achieved when the underlying representation of integers reflects linear spacing with constant variance.  These results provide new insights into (i) the detailed nature of mental numerical representation, and, (ii) general perceptual processes implemented by the human visual system.

How numbers mean: Comparing random walk models of numerical cognition varying both

encoding processes and underlying quantity representations

Understanding how numerical symbols and their associated quantities are represented and

used is a primary aspiration of those working in numerical cognition. Once we understand how

symbols are represented and assigned meaning, we will understand how best to frame questions

about how organisms become numerate (Cantlon, Platt, & Brannon, 2008; Feigenson, Dehaene,

& Spelke, 2004), how numeracy changes over time (Geary, 1994), how and why numerical

errors arise (Macaruso, McCloskey, & Aliminosa, 1993), how language and numeracy are

related (Hauser, Chomsky, & Fitch, 2002), whether or not numeracy is a cultural universal

(Dehaene, Isard, Spelke, & Pica, 2008), and, so on. Despite almost half a century of

experimental study, the underlying representation of numerical symbols remains hotly debated.

Here, we present a model of number symbol encoding, representation, and retrieval. We

instantiate this model in a random walk simulation and predict performance in the core paradigm

in the field of experimental numerical cognition: the relative quantity task. Although we

specifically simulate performance in the relative quantity task, the actual model informs more

generally about how a number conveys the quantity it denotes. Indeed, the processes that we use

to simulate performance in the task may reflect on more fundamental properties of the human

perceptual system. Such general implications of the work are examined in detail later.

**The relative quantity task**

The relative quantity task is a deceptively simple paradigm: On a typical trial, the

participant must assess, as quickly and accurately as possible, which of two digits is the larger.

The variations on the specific form of this task are numerous, ranging from the straightforward,

simultaneous presentation of two Arabic digits (Dehaene, Dupoux & Mehler, 1990; Hinrichs,

Yurko & Hu, 1981; Moyer & Landauer, 1967), to paradigms that include priming components

(Dehaene, Naccache, Le Clec, Koechlin, Mueller, Dehaene-Lambertz, et al., 1998; Ratinckx,

Brysbaert & Fias, 2005; Van Opstal, Gevers, De Moor & Verguts, 2008), Flanker components

(Notebaert & Verguts, 2006), multiple dimensions (e.g., in numerical Stroop tasks; Pavese &

Umiltà, 1998; Ratinckx & Brysbaert, 2002; Tzelgov, Yehene, Kotler, & Alon, 2000; Waldron &

Ashby, 2001), and, so on.  Nevertheless, the reaction time (RT) data from the relative quantity

task, and its variations, have revealed two effects that have proved foundational in all subsequent

theory development; these are the *numerical distance effect* and the *size effect* (see Figure 1A).

The numerical distance effect is characterized by RTs (for correct responses) that monotonically

decrease as the numerical distance between the two digits increases.  The size effect is that, for a

fixed difference between two digits, correct RTs increase monotonically as the size of the digits

increase.

Moyer and Landauer (1967) conducted the classic experiment that identified the

numerical distance effect and size effects.  The authors presented two Arabic digits, side-by-side,

and asked participants to identify as quickly and accurately as possible the numeral denoting the

larger quantity.  Moyer and Landauer (1967) explained the numerical distance effect and size

effect by proposing that numbers are represented as magnitudes that are similar to those in the

physical world and the discriminability of two perceived magnitudes is determined by the ratio

of the actual magnitudes (i.e., these representations obey Weber's law).

Since the original description of the numerical distance and size effects, various accounts

have been put forward to explain the psychological representations of numbers hypothesized by

Moyer and Landauer (1967).  All such accounts adopt a general Signal Detection Theory

framework.  That is, these accounts generally assume that the quantity associated with a given

digit is represented on an internal continuum (e.g., an internal "number line," Dehaene, 2003)

and there exists some perceptual variability (i.e., noise) associated with the placement of the digit

on this continuum. Take, for example, the number "5." All theories assume that each time an

observer experiences this symbol, the observer will have a slightly different "sense" of the

quantity associated with "5." So, sometimes the observer's sense is more than 5 and sometimes

less. Accordingly, each digit's quantity representation is captured by a distribution of values on

the continuum that we term a *psychological distribution of quantity* (PDQ) (see Figure 1B). The

PDQ captures the perceptual noise associated with one's understanding of the quantity associated

with a digit. The PDQs of successive digits are rank ordered and overlap (see Figure 2).

In Signal Detection Theory, the degree of difficulty in distinguishing between two stimuli

is determined by the amount of overlap between their corresponding perceptual distributions: the

greater the overlap, the more difficult it is to distinguish between the two stimuli. This premise

translates directly when discussing number representation. Most accounts explaining the

psychological representations of numbers, assume the difficulty in distinguishing between the

quantities of two numbers is determined, primarily, by the amount of overlap between their

PDQs (see Figure 2). 'Difficulty,' in this context, is defined by greater RTs and errors in the

relative quantity task.

The fundamental differences between the key theories lie with the assumptions they make

about the nature and spacing of the PDQs on the mental number line (see Figure 1). According

to the linear account, successive quantity representations are rank ordered at equal intervals and

the different PDQs have the same variance (we call this account the *Linear Theory*, see e.g.,

Cantlon, Cordes, Libertus & Brannon, 2009). The two other competing accounts of quantity

representation are the *Logarithmic Theory* (Dehaene 1992, 2003) and the *Scalar Variance*

*Theory* (see Church, Meck & Gibbon, 1983; Gallistel & Gelman, 1992; Meck & Church, 1983; Meck, Church & Gibbon, 1985). The Logarithmic Theory also posits that the different PDQs have the same variance, but claims that the means of the ordered PDQs are spaced on a logarithmic scale. As such, the means of successive PDQs get closer together as the numbers increase (see Figure 1B). In contrast, for the Scalar Variance Theory, the means of the ordered PDQs are spaced linearly but their variances scale linearly (i.e., increase) with quantity (see Figure 1B).

In the absence of providing a detailed model, intuitions about general patterns of performance suggest that the numerical distance effect can be accommodated by all three theories because the PDQs of numerals denoting adjacent quantities (e.g., "5" and "6") overlap more than the PDQs of numerals denoting distant quantities (e.g., "5" and "1"). It is similarly apparent that the Logarithmic Theory and Scalar Variance Theory accounts can accommodate the size effect. The size effect is hypothesized to result because, for a given quantity distance (e.g., "1") the PDQs of numerals denoting large quantities (e.g., "7" and "8") overlap more than the PDQs of numerals denoting small quantities (e.g., "2" and "3"). For the Logarithmic Theory this is true because the means of the PDQs for successive smaller quantities are farther apart than those for successive larger quantities. For the Scalar Variance Theory this is true because the SDs of the PDQs for successive smaller quantities are smaller than those for successive larger quantities. On first glance, though, the size effect sits less well with the Linear Theory.

This, in brief, is the current state of affairs with respect to explaining performance in the relative quantity task and the evidence relevant to discussion of the underlying representation of integers. Here, we assess the validity of the stated models and a new model; by (i) specifying the

details of models of this task from encoding to response, (ii) simulating data based on the

specified details, and, (iii) and assessing the model fits against human data.

**Modeling**

When assessing the internal representation of stimuli, perceptual, decisional, and

response processes influence the participant's behavior.  It is, therefore, vital to specify precisely

the foundational assumptions about how each of these might influence the data.  Without a

precise specification, one may erroneously conclude that particular patterns of data are the result

of internal representations, when in reality they are the result of encoding or decision processes

(e.g., Verguts, Fias, & Stevens, 2005).

Traditionally, the cognitive systems involved in completing a simple RT discrimination

task are described in terms of four broad stages, namely, Encoding, Comparison, Decision, and

Response (see Sternberg, 1998).  When explaining RT data, the researcher must ask critical

questions about each stage: for instance, does the time to complete this stage correlate

significantly with the variable of interest?  If the answer is "No," then the time resulting from

that stage is assumed to contribute merely a constant across all levels of the independent

variable, so the researcher can effectively ignore that stage when explaining the data.  If,

however, the answer is "Yes," then the researcher must include detailed discussion of the stage

in explaining the data.  Often, to simplify interpretation, researchers will assume that stages do

not correlate with their variable of interest.

In attempting to explain the numerical distance and size effects in the relative quantity

task, theorists have made the simplifying assumption that only the comparison process correlates

with the variable of interest: numerical distance.  Were any other stage to influence the relation

of the RTs to numerical distance, then the interpretations of the numerical distance effect and

size effect would no longer be as simple as described.  For example, suppose the response

function is logarithmic or that encoding times are related to numerical distance.  Such findings

would undermine the current interpretations of the numerical distance and size effects.  Thus,

whereas simplifying assumptions can make data interpretation relatively straightforward, they

can also lead researchers down a garden path.  Below, we describe the potential influences of

each of the four broad processing stages on performance in the relative quantity task.

**Encoding**

In the relative quantity task, encoding refers to the processes involved in converting and

identifying the numerical symbol.  As can be inferred from traditional explanations, encoding

time has been assumed to be unrelated to numerical distance.  Recently, however, Cohen (2009,

2010) has demonstrated that encoding of numerical symbols takes measurable time and this time

is related to numerical distance.

In an effort to assess whether numerical symbols automatically activate their quantity

representation, Cohen (2009) conducted a numerical same/different task.  Here, the participants

were presented with Arabic digits ranging from 1-9 and had to judge whether the digit presented

was a 5.  Theoretically, participants can complete the numerical same/different task based solely

on the physical features of the numeral. Nevertheless, previous research using the numerical

same/different task had revealed a function that correlated with numerical distance (Dehaene &

Akhavein, 1995; Ganor-Stern & Tzelgov, 2008).  Because numerical distance was task

irrelevant, researchers concluded that the numerical symbols automatically activated their

quantity representation (Dehaene & Akhavein, 1995; Ganor-Stern & Tzelgov, 2008). Cohen,

however, hypothesized that RT was a function of the physical similarity of the Arabic Digits to

the standard "5" rather than numerical distance from it.  Cohen developed an objective measure

of the physical similarity of Arabic digits ($PS_{digit}$) based on the seven-line segment, figure 8

structure of digits used on digital alarm clocks:

$$PS_{digit} = O/D \tag{1}$$

where $PS_{digit}$ is the measure of physical similarity, $O$ is the number of lines that the two integers

share, and $D$ is the number of remaining lines (see Table 1). Although the RTs were well fit by

measures of numerical distance, the Physical Similarity function predicted the data virtually

perfectly. Furthermore, when both numerical distance and physical similarity were entered into

the equation, numerical distance dropped out leaving only physical similarity as the significant

predictor. Cohen concluded that, although numerical distance is an important predictor of RTs

when quantity information is either required to complete the task (e.g., a relative quantity task) or

when quantity is an inherent part of the task (a numerical Stroop task), integers do not

automatically activate their quantity representation. Importantly, physical similarity correlated at

over .6 with numerical distance. Cohen (2009) explained this correlation as likely resulting from

the fact that Arabic digits evolved from analogue representations of the quantities themselves.

Therefore their physical form would be correlated with numerical distance. In a later paper,

Cohen (2010) showed that physical similarity was the primary predictor of RTs in a relative

quantity task that used decimals presented in Arabic notation. Since Cohen's work, others have

replicated the physical similarity effect with other languages and other paradigms (see García-

Orza, Perea, Mallouh, & Carreiras, 2012).

Cohen's (2009, 2010) work on the influence of encoding on RTs and their relation to

numerical distance provides evidence against the simplifying assumption that the encoding stage

does not materially influence RTs in the relative quantity task. Indeed, in recent work, Cohen,

Warren, and Blanc-Goldhammer (2013) have demonstrated that physical similarity effects

cannot be avoided by presenting numerical symbols in two different formats (e.g., "5" vs. "six"), because the numerical cognition system will automatically convert and compare symbols in a common representational system rather than comparing quantities.

**Comparison**

The comparison stage is the most well described stage of the cognitive processes involved in completing the relative quantity task. This stage requires a description of the internal representation of quantity so that the relevant features to be described are understood. As discussed above, all major theories accept a general signal detection framework, whereby the psychological quantity associated with this symbol is best described as a distribution with a mean perception and some variance around the mean (i.e., the PDQ). In turn, the similarity of the quantities denoted by symbols is described by some measure of the overlap of their respective PDQs. Signal detection theory provides numerous such measures, including $d'$, area under the curve, etc.

When comparing quantities, the time to complete a comparison is assumed to be some function of the overlap of the PDQs. Because the overlap of PDQs is determined solely by their relative mean placement and variance, these two features of the psychological representation of quantity are of critical importance. How best to characterize the internal number system in these terms, is the topic of a very heated debate. The debate rages over the relative strengths and weaknesses of the Linear Theory, Logarithmic Theory, and Scalar Variance Theory (although predominantly concern has been with the latter two, e.g., Cantlon et al., 2009; Dehaene et al., 2008). Having already provided a general description of these models, a more precise description is needed to appreciate them fully. Below we describe each proposed representation mathematically:

$$\text{Logarithmic Theory: } \psi_i = \log(\Theta_i) + e_i, \ e_i \sim N(0, s), \tag{2}$$

$$\text{Linear Theory: } \psi_i = \Theta_i + e_i, \ e_i \sim N(0, s), \tag{3}$$

$$\text{Scalar Variance Theory: } \psi_i = \Theta_i + e_i, \ e_i \sim N(0, \Theta_i * s) \tag{4}$$

where the subscript $i$ identifies the specific numerical symbol represented, $\psi$ is the psychological quantity representation, $\Theta$ is the quantity denoted by the numerical symbol, and $e$ represents the error variance.

The fixation with the Logarithmic Theory and Scalar Variance Theory in the literature results primarily from consideration of the size effect. That is, the data show that it takes longer to make a relative quantity judgment for larger numerals (e.g., 8 vs. 9) than for smaller numerals (e.g., 1 vs. 2). The Logarithmic Theory explains this effect by proposing that the means of the PDQs of successive integers are distributed logarithmically rather than linearly. So, the means of the PDQs of small numbers are farther apart than the means for the PDQs of large numbers. In contrast, the Scalar Variance Theory explains this effect by proposing that the variance associated with the PDQs for large numbers is larger than the variance associated with the PDQs for small numbers. In both cases, the PDQs overlap more for larger than for smaller numbers. These functions appear to accommodate the size effect.

The Linear Theory model does not accommodate the size effect in the comparison stage. This, however, is not necessarily a fatal problem. It is possible that the size effect is a manifestation of another stage (e.g., encoding or response), but such a hypothesis has not been seriously considered or tested by those advocating the Logarithmic Theory or Scalar Variance Theory. We return to this possibility later in the paper.

**Decision and Response**

Modeling the decision and response stages of an RT task requires an underlying theory of decision-making. Virtually all well described models of RTs are based on a Signal Detection Theory model of decision-making very similar to the one used to describe the comparison process (e.g., Ashby, 2000; Curtis, Paulos, & Rule 1973; McGill, 1963; Ratcliff, 1978; Thomas & Myers, 1972). Thus, we can borrow from what has been learned about the link between RTs and Signal Detection Theory in order to address issues about numerical cognition.

Many of the successful models linking Signal Detection Theory and RT make the RT-Distance assumption. For example, Thomas and Myers (1972) presented a mathematical analysis of RT on the assumption that RT is a monotonically decreasing function of the Euclidean distance between the percept and the criterion as described in Signal Detection Theory. Ashby and Maddox (1994) called this the *RT-Distance hypothesis* (see Ashby & Maddox, 1994). Thomas and Myers (1972) elaborated their account by (i) specifying the form of the RT distribution given that part of the variation in RT is the result of distance to criterion, and, (ii) by accepting that, for any fixed distance, RT is a random variable with a non-degenerative function. The authors continued by clarifying predictions on the form of the RT probability curve, variance, and mean of the RT distributions under various assumptions. Thomas and Myers (1972) concluded that the experimental data fit the predictions well. In following up on this work, Balakrishan and Ratcliff (1996) presented evidence that participants will use a distance to criterion rule when assigning confidence ratings even when the optimal decision rule is different. Furthermore, Zakay and Tuvia (1998) showed that confidence ratings and choice RT are negatively related. By extension, it is likely that observers' choice RTs would also support the distance-from-criterion model.

Most detailed computational models of the relation between RT and the underlying

psychological representation are variants of the Random Walk model (Link, 1990).  The random

walk model assumes a Signal Detection framework.  Although the Random Walk model can

generalize to more than two distributions, we will describe a simple two alternative forced choice

procedure.  Here, the participant is presented two stimuli and must identify one as the correct

choice with the push of a button.  The participant's RT is recorded.  Let us specify that the task is

a relative quantity task and the participant is to choose the number symbol that denotes the

largest quantity.  Here, the Random Walk model assumes that each number symbol (say "4" and

"5") activates separate PDQs.

To estimate RT, the model will repeatedly sample from both PDQs and find their

difference (see Figure 2).  So, for example, on the first sample, the model will randomly select a

value from the PDQ representing 4 and may retrieve a 3.5 (recall that the quantities associated

with the number "4" is noisy, so error occurs).  Similarly, the model will randomly select a value

from the PDQ representing 5 and may retrieve a 6.  Here, the difference between the selected

values is 2.5 (6-3.5 = 2.5).  So, on the first sample the model moves 2.5 units in the positive

direction.  On the next sample, the difference is added to the previous sample.  The Random

Walk model takes repeated samples until the sum of all the sample differences passes a pre-

determined threshold.  The positive threshold indicates that the participant responds that the "5"

is greater than the "4." The negative threshold indicates that the participant responds that the "4"

is greater than the "5."  Importantly, the number of samples required to pass the threshold is

taken as the surrogate for RT.

 As one may intuit, the greater the overlap of the PDQs, the more samples will be

required to pass the threshold.  In addition, greater overlap will lead to a higher probability of

erroneously passing the incorrect threshold, thus leading to an error. The number of samples

taken before encountering a boundary is a function of the shape, variance, and degree of overlap

of the perceptual distributions, and is assumed to be a monotone function of RT.

Buckley and Gillman (1974) were amongst the first to generate a random walk model of

performance in timed numerical comparison tasks. They accepted a standard Signal Detection

Theory model and assumed that the transformation from external stimulus to internal

representation was logarithmic. In one comparison task that they described, two digits (taken

from the set 1-9) were presented side-by-side and participants responded whether the left or right

was the larger. Buckley and Gillman (1974) conducted a standard random walk simulation and

stated that this basic model was successful in being able to capture scaled time measures of the

responses in the task. However, as pointed out by Link (1990), the simulated data were not the

actual condition mean RTs but rank ordered mean RTs. Moreover, in the actual experiment, any

trial that was responded to incorrectly was repeated until a correct response was collected. In this

respect, the simulated data were error-free. As Link (1990) remarked this was regrettable

because one of the strengths of random walk models is their ability to model error data.

In addressing these issues, Link (1990) explored the degree to which random walk

processes are able to simulate performance in variants of the speeded relative magnitude task. In

one case, two-digit numbers were presented sequentially and the participant had to respond

whether or not the current number was the larger or smaller than the immediate previous one. In

a second case, a standard of 55 was used and participants simply had to respond whether a singly

presented two-digit number was greater or less than the standard. Link (1990) carefully

discussed properties of random walk processes that were deemed necessary to model the RT

effects. However, rather than assessing the adequacy of the model fit, Link assumed the model's

validity and then used it to estimate model parameters that would mimic properties of the data in these tasks. Notably, Link accepted the Linear Theory representation but had to embellish his model with further *ad hoc* assumptions about (i) the nature of response bias, and, (ii) a numerical transformation in which the base 10 number system was mapped to a base 6 number system. Moreover, critically he did not assess the fit of competing theories of quantity representations on both correct and error data. In this regard, the work fails to provide further insights into how best to adjudicate between the key competing models as outlined above.

Two further modeling studies are notable. The first, by Poltrock (1989), examined how well random walk models could account for performance in a variant of the speeded magnitude estimation task (i.e., respond to the left or right digit that was the larger). In extending the experiments, participants were also tested under conditions where strict RT deadlines were imposed. Individual participant's RT and accuracy data were fit with a random walk model in which values of 10 free parameters were estimated. The model resulted in peculiar estimates when comparisons involved the digit "1". Essentially, the model predicted no distance effects if the digit "1" was included in the analysis. That is, if the data from the digit "1" was included in the analysis, the model estimated that the underlying quantity representations for all digits were equal distance from one another. However, if the digit "1" was excluded from the analysis, distance effects emerged that revealed that the estimated internal magnitudes of the remaining digits was approximately linear. No detailed account of this inconsistency was included in the paper.

In the second, Smith and Mewhort (1998) adopted the notion of random walk as a type of diffusion process (see e.g., Ratcliff, Van Zandt, & McKoon, 1999) and modeled performance in the more simple magnitude judgment task (i.e., respond as to whether a singly presented digit

was less than or greater than 5). Smith and Mewhort (1998) carried out extensive studies of

diffusion random walk models in which participants' performance was fit with two free

parameters that defined Gaussian and ex-Gaussian parts of participants' RT distributions.

Simulations provided estimates of these parameters and these were then compared with the

actual behavioral data. The models produced impressive fits with the correct RT data.

Comparisons were also reported between the models' error data and human accuracy. The

authors, however, did not model the error RT as a function of numerical distance. Rather, the

authors calculated a single, omnibus, mean RT for error data, and showed that by adding

variability to the start position of the walk they were also able to model how fast errors arise.

Thus, similar to previous researchers, Smith and Mewhort (1998) modeled correct RTs while

sidestepping the importance of modeling error RTs.

In sum, there is a strong tradition in which variants of a random walk process to model

performance in speeded magnitude estimation tasks have been explored (see e.g.,

Kamienkowski, Pashler, Dehaene & Sigman, 2011; Schwarz, 2001; Sigman & Dehaene, 2005).

The positive outcomes of this work indicate the utility of this approach. However, to date,

researchers have not provided comprehensive models that account for the distributions of correct

*and error* RTs, as well and the proportion of errors[1]. As will become clear, it is only by

attempting such a comprehensive exercise, that differences between the various models become

apparent and their relative strengths and weaknesses are laid bare.

**The Current Models**

Here we instantiated each of the three primary theories of quantity representation (Linear

Theory, Logarithmic Theory, and Scalar Variance Theory) as, respective, computational

models[2]. Random walk simulations were carried out in a bid to mimic RT performance in a

relative quantity task. The actual behavioral data were taken from a speeded relative quantity

task in which participants were asked to judge whether a visually presented Arabic digit denoted

a quantity greater than or less than five (described in detail below). An overarching aim was to

be able to model these data comprehensively. Consequently, each of the models was required to

produce estimates of both the speed and accuracy of response. The main objective was to go

beyond previous modeling attempts in being able to fit the skew of the RT distributions for both

the correct and incorrect response and the incidence of errors for each of the comparisons.

Furthermore, we compared the performance of models with and without encoding processes. In

addition, further analyses were directed towards comparing fits only for correct RTs and

comparing more comprehensive fits for both correct and error RTs. As such, we can clarify the

significance of modeling encoding processes and we underscore the importance of assessing fits

to complete data sets.

The behaviors of the Linear Theory, Logarithmic Theory, and Scalar Variance Theory

models were examined separately via random walk simulations. Initially perceptual distributions

were generated for each integer between 1 and 9 in a manner consistent with each of the models

(see Equations 2-4). Next we specified the actual walk process. Assuming a two-choice RT

task, information accumulates from a starting point, $z$, and drifts toward one of the two response

criteria, at which point the observer responds consistent with the encountered boundary (Figure

3A). In the current characterization, the evidence that accrues over time during this process is

information critical to making a response – Does the digit represent a quantity that is greater than

or less than five? At each time point following the presentation of the digit, evidence is sampled

at random from the distribution of the standard (i.e., "5") and from that of the probe digit (for

instance, "3"). The absolute magnitude of the difference between these two points determines

the size of the increment in the walk and the sign of this distance determines the direction of the walk.

To simulate a trial, one sample value was drawn from the perceptual distribution ($\psi$) of the standard and one from the probe and a running total of the difference between the two samples was logged, namely:

$$tot_j = tot_{j-1} + SQ_j - PQ_j, \tag{5}$$

where $tot_j$ is the running total for sample $j$, $SQ_j$ is the standard quantity drawn on sample $j$, and $PQ_j$ is the probe quantity drawn on sample $j$. A response was then initiated when $tot_j$ crossed a pre-specified decision threshold. If $tot_j$ crossed the positive decision threshold, the standard was identified as being larger than the probe. If $tot_j$ crossed the negative decision threshold, the probe was identified as being larger than the standard. The number of samples required (termed *NumSamp*) prior to crossing a decision threshold was the dependent measure taken to be analogous to RT.

In our random walk model, the decision threshold is identified by three free parameters, that instantiate two response biases. The first free parameter, $b_a$, identifies the intercept of the positive decision threshold. It's negative value, $-b_a$, identifies the intercept of the negative decision threshold. These values represent the initial amount of evidence that the observer requires before he or she responds. The farther away these points are from 0, the more evidence required. We also incorporated linear, time-varying boundaries (Smith, 2000; Zhang, Lee, Vandekerckhove, Maris & Wagenmakers, 2014). That is, we allowed each boundary to be angled toward the "0" evidence line (between 0 and -50 degrees). As will become clear, it proved critical to incorporate such time-varying boundaries, in order to obtain best fits. The second free parameter of the decision threshold, $b_\theta$, identifies the angle of the positive decision

threshold. Its negative value, $-b_\theta$, identifies the angle of the negative decision threshold.

Although we assume the positive and negative decision thresholds are symmetric, this assumption is not necessary. A functional consequence of incorporating these kinds of linear, time-varying boundaries is that a decrease in response evidence is required as time increases. In essence, this instantiates the increasing impulse to respond as time increases. Linear, time-varying boundaries have the effect of reducing the skew of the distributions, as well as influencing the relative means of error and correct RTs, as well as the proportion of errors. Finally, we assume between trial variance exists in the decision threshold angle. This third decision threshold free parameter, $b_\sigma$, captures this variance. Thus, for any simulated trial, we assume the threshold angle to be distributed as follows:

$$\theta \sim N(b_\theta, b_\sigma) \tag{6}$$

We ran two versions of the random walk model: one that included no encoding effects (termed the *Traditional Encoding* model) and one that included encoding effects (termed the *Encoding Errors* model). The Encoding Errors model assumes that the percept of the digit might be misperceived. This misperception will influence the perceived quantity and thus the participant's judgment. Importantly, the probability of this misperception is set *a priori* based on the *PS* metric derived in Cohen (2009) that defined the physical similarity of two digits (see Table 1). In Cohen (2010) that metric was revised to account for identical digits. Here, we use the 2010 formula to create a confusion matrix of the digits "1"-"9." Specifically, we calculated PS for each digit relative to every other digit. We then replaced the values of all identical numbers pairs (e.g., "1"-"1") with the average value of those pairs because of the need to have the same similarity value for identical pairs (i.e., here we assume "identity" is a constant and does not lie on a continuum). Finally, to calculate the probability of perceiving a particular

probe as a particular stimulus, we converted the similarity values of all the digits to proportions

by presented probe[3]. Table 1 shows these probabilities. The sum of the perceived stimulus

proportions for each presented probe (i.e., the rows) will sum to 1 (which is not the case when

computed column-wise).

The Traditional model assumes no encoding errors as described above.  The Encoding

Errors model is as follows. The Encoding Errors model assumes that the Arabic digit is encoded

before comparison, but the digit may be confused ("5" perceived as "6").  Here, prior to the

random walk, the perceived probe is selected on the basis of the probability structure presented

in Table 1.  For example, if the presented probe was "3" then there would be a 5.8% chance of

"8" being "perceived." Then the random walk simulation runs as if the perceived stimulus (e.g.,

8) was presented.

Conventionally, researchers assume that encoding occurs once (in an all-or-none fashion)

at the beginning of the process (e.g., Sternberg, 1998).  We call this the *initial encoding* of the

presented probe. We believe, however, that it is likely that the system continuously samples the

environment.  Such a system would, over time, correctly encode a stimulus that was initially mis-

encoded.  We included this potential correction process in our model. Thus, the Encoding Errors

model assumes the visual system repeatedly samples the environment, with the result that

slightly different impressions are derived at each time point.  This information is continually

made available to the magnitude comparison stage.  Critically, every time the environment is

sampled, the encoding of the probe is selected on the basis of the probability structure in Table 1

and a quantity is sampled from the encoded digit's PDQ.

We formalize the probability of sampling the environment within the Encoding Errors

account as an exponential decay function,

$$p(encode) = 100 \times k^n, \tag{7}$$

where $k$ is a free parameter ranging from 0-1 indicating the strength of the first encoding. If $k = 0$, then the initial encoding carries all the information and there is no further encoding. If $k = 1$, then there is 100% probability of a new encoding every sample of the Random Walk. Here, the perceived probe will change on every sample independent of the previous sample. If $0 < k < 1$, then the probability of a new encoding decreases exponentially with every step of the Random Walk. The second parameter, $n$, is the sample number of the Random Walk (starting at 0).

For steps in which the system does not interrogate the environment, the system must use a memory representation to identify the PDQ from which the Random Walk will draw the next sample. When $k = 0$, then that representation is the initial encoding. When $k > 0$, then the system uses the most frequently encoded integer since the initial encoding (i.e., the mode integer). So, if on trial $n$ there were four environmental sample encodings and three of them were "6," then the sample from memory would be a "6." In the rare occurrence where there are several mode integers in memory, the system would randomly choose between them. A system in which $k > 0$ will converge on the correct encoding over time because the correct encoding is the most likely integer to be encoded.

The $k$ parameter of this account is key to identifying the role of encoding errors. If parameter $k$ converges on 0, then the conventional account whereby the system only encodes the environment once is correct – that is, the system works essentially from memory. If the $k$ parameter converges on 1, then a memory free account is correct. In such a case, the system is "memory free" because it does not store an encoding representation of the probe and each updated sample is independent of the previous sample. Nonetheless, the history of samples does influence one's understanding of quantity. Specifically, one's perception of quantity associated

with viewing a particular integer at any moment of time is the average of all the "samples" up to that moment. Finally, if $0 < k < 1$, then the system has some encoding memory, but continues to update its description of the stimulus.

The number of samples in the Random Walk simulation (*NumSamp*) equates to RT. Because NumSamp is on a different scale as RT, we transformed NumSamp using the following formula:

$$RT = t_{er} + b_{NS}*NumSamp \tag{8}$$

Where $t_{er}$ is represents the ancillary processes unrelated to the comparison and decision processes being modeled (e.g., some encoding and response processes), and $b_{NS}$ scales *NumSamp* to milliseconds.

In summary, we simulated to the three primary quantity representation theories (Logarithmic, Scalar Variance, and Linear) in two random walk models (the Traditional model, and the Encoding Errors model). Table 2 has a summary of the free parameters in each random walk model.

## Methods

To assess the validity of each model, the data from the simulations were used to predict behavioral data from a typical relative quantity task.

### Experiment

**Participants.** One hundred and twenty-two undergraduate volunteers participated for class credit.

**Stimuli and Procedure**. The experiment was a timed relative quantity task in which participants were asked to judge, as quickly as possible, whether an Arabic digit denoted a quantity greater or less than the quantity denoted by a "5."

The experiment was computer controlled and stimuli were presented on a 24-inch LED color monitor with a 72-Hz refresh rates and a resolution of 1920 by 1200 pixels. Participants were tested individually in a small, dark room and given detailed task instructions. Participants sat approximately 30 inches away from the screen.

A trial consisted of a single integer subtending $1.33^o$ visual angle presented in the center of the screen. Each probe was selected randomly from the integers "1"-"9," excluding "5." All integers were presented in Ariel font. Half the participants were told to press the "D" key if an integer greater than a "5" was presented and the "K" key if an integer less than a "5" was presented. The keys were reversed for the remaining participants. RT in ms was recorded. The participants were instructed that speed was important, but accuracy was essential.

A trial was defined as a sequence consisting of the presentation of a stimulus followed by the participant's response. The stimulus remained on the screen until the participant responded and there was a 500 ms delay between trials. Scheduled breaks occurred after 160 trials. Each testing session comprised 16 (8 probes x 2) practice trials followed by 320 (8 probes x 40) experimental trials.

## Results

Prior to analysis, the RT data from the experiment and *NumSamp* from the simulation were trimmed to 5 SDs (across participants - maximum RT was 3700). By trimming the data so loosely, only true outliers were removed and this forced the simulations to accommodate the vast majority of the responses produced by participants. Furthermore, four participants were removed because their error rates were greater than 15%.

Figure 3B shows the behavioral data. Although it is generally preferable to analyze data on individual participants, we did not have enough data per participant to do so. Therefore, to

ensure that our averaged data represented the average behavior of individuals, we calculated our

summary statistics on individual participants and then averaged across those statistics.

Specifically, for correct responses, we identified the 25th, 50th, and 75th percentiles of each

individual's RT data and then calculated the average of each of these percentiles. However,

because individual participants did not produce enough errors per probe to provide robust RT

distributions, we only calculated the 50th percentile for the error RT data. Furthermore, it was

often that case that there were very few errors data per participant for a given probe and

sometimes only 1 error occurred. As such, it is unclear what position on the theoretical

distribution these data points occupy. We therefore followed the procedure of Ratcliff,

Thompson, and McKoon (2015) by trimming the error RT data based on the distributions of

correct RT data. That is, we only included median values of the error RT data that were within 5

SDs of the median values of the correct RT data. We note, however, that the results of the

simulations based on these data were, essentially, the same as the results of simulations based on

the pooled dataset (collapsed across subjects).

First, to ensure that the experimental data were consistent with published reports, we fit

the Welford function to the median correct RT data (i.e., RT= $a + k^*\log[L/(L - S)]$, whereby $L$ is

the larger integer to be compared and $S$ is the smaller integer to be compared; $a$ and $k$ are the

integer and slope respectively). The Welford function is the general function that fits the

combined numerical distance effect and size effect. The Welford function was a highly

significant predictor of the data, $F(1, 6) = 72$, $p < .001$, $r^2 = .92$. Thus, the data are representative

of performance in other timed relative quantity tasks. The error RT data, however, are not as

well fit by the Welford function, $F(1, 6) = 10$, $p = .02$, $r^2 = .63$. This is also consistent with

general findings, though error RT data are rarely reported.  Finally, the distance effect is present

in the proportion of errors mapped out as a function of probe.

We then ran six random walk models: three theories of quantity distributions for each of

the two simulation models (Traditional and Encoding Errors).  We fit each simulation,

simultaneously, to the $25^{th}$, $50^{th}$, and $75^{th}$ percentile score for each probe for the correct RTs, the

$50^{th}$ percentile score for the error RTs, and the proportion of errors.

To optimize the fit of the parameters, we implemented a semi-random grid search

method.  In preliminary runs of the simulation, we discovered that standard optimization

procedures were extremely sensitive to the start point of the parameter search.  Because we could

not assume we had valid start points, we explored new methods for optimization.  Systematic

grid-search methods are exponentially inefficient as the number of parameters increases.  It has

been demonstrated that random search methods are far more efficient and produce equally good

model fits (Bergstra, & Bengio, 2012).  We therefore programmed a semi-random grid-search

optimization method to identify initial start points to be used with existing optimization

procedures. However, after implementing our procedure, we discovered it produced superior

parameter estimates to existing procedures. Furthermore, in using our procedures' best-fit

parameter estimates as start points for more established optimization procedures, we discovered

that there were no further improvements in the fit statistics.  Therefore, we used our optimization

procedure exclusively. Our optimization procedure is as follows.

For each parameter, we set high and low bounds that were functionally outside the

likelihood of a good fit.  The optimization program then divided the space up into 25 intervals,

each representing a valid parameter value.  For each run of the simulation, a random value for

each parameter was chosen from those 25 available points.  A single run simulated 400 trials per

probe using the parameter values chosen. In first calibration, the optimization program ran 50

simulations. From these 50 simulations, the 15 best were identified.  For each parameter, the

high and low parameter values from this set of 15 were identified and used as the new high and

low boundaries.  The procedure was repeated, with the number of simulations runs increasing by

50% before each re-calibration, until the model fit did not increase three calibrations in a row.

At this point, a fine-grained procedure was implemented, whereby the high and low boundaries

were set at either 5% above and below the best-fit parameter or remained at the last value,

whichever was smaller.  Furthermore, the number of trials simulated per run was increased to

5000.  The procedure was repeated, with the number of simulations increasing by 50% with each

re-calibration, until the model fit did not increase three calibrations in a row.  At this point, the

best-fit parameters were identified as the final parameters, and a final simulation was run with

10,000 trials per probe.  We note that the quantities chosen in the simulation (e.g., 50 simulations

per calibration, 15 best runs, 400 trials per run, etc.), were those used to optimize the

performance of the system (i.e., its efficiency and ability to converge on the best fit parameter) in

preliminary testing of the optimization program.

　　　We used three model fit statistics to determine the superior model:  Bayesian Information

Criterion (BIC), $r^2$, and chi square.  BIC is a measure of model fit in which lower BICs indicate

the better model (all calculations were made in *R*, R Core Team, 2014).  The BIC penalizes

models as the number of parameters increase.  One drawback of the BIC is that an inferior model

that is able to provide fits for only a subset of points will have a lower BIC (because it will have

less absolute unexplained variance).  Because some of our models did not produce any error

trials for some probes, their BIC was artificially low.  Theoretically, this indicates that the model

failed and should result in an infinite BIC.  However, because these were the Traditional

simulations, we did not want to disqualify them. We therefore standardized all BIC measures by dividing the BIC by the number of points the model fit (termed $BIC_z$). We note here that the Error Encoding model has one extra parameter than the Traditional model. The Error Encoding model adds the $k$ parameter, which identifies the probability that the environment will be sampled. As the proportion of encoding errors is pre-specified in Table 1, error encoding in itself does not add a free parameter. We also calculated the chi square using a method similar to Ratcliff and Tuerlinckx (2002; see also Ratcliff & Childers, 2015). Because chi square, calculated in this way, is sensitive to the number of samples (we ran 10,000 samples per probe) and the number of conditions (we summed over 8 probes), the absolute value of the chi square is not meaningful. Nevertheless, the relative values of the chi square between models provides information about the relative fits of the models.

Because RT and error proportion were on different scales, we calculated $r^2$ separately for each. For some of the models, the simulations could not simultaneously fit the RT and error proportion. This resulted in an error proportion $r^2$ that was less than 0. When this was the case, we set the error proportion $r^2 = 0$. We used the average of the two $r^2$s as our fit statistic. Because, $r^2$ is the measure of the percent of variance in the data that is accounted for by the model, it does not provide a statistical advantage for fitting fewer points. However, $r^2$ does not penalize for added parameters.

Statistically, $r^2$ should be negatively correlated with both the $BIC_z$ and the chi square. We found significant negative correlations between the $BIC_z$ and $r^2$, $r(5) = -.98$, $p<.05$, and the chi square and $r^2$, $r(5) = -.99$, $p<.05$, fit statistics of our six models. There was also a positive correlation between the $BIC_z$ and chi square, $r(5) = .97$, $p<.05$. Because all three fit statistics were so highly correlated, and the conclusions are the same based on both results, we present $r^2$ in the

remainder of the analysis[4]. Figure 4 shows a plot of $r^2$ for the simulations of the 6 models we ran:

3 quantity representations (Linear Theory, Logarithmic Theory, and Scalar Variance Theory,

respectively) X 2 random walk models (Traditional Encoding and Error Encoding). Table 3

presents the best-fit parameter values for each model.  Figure 5 shows plots of the behavioral

data and the data simulated with the Traditional Encoding model for the three quantity

representations.  Figure 6 shows plots of the behavioral data and the data simulated with the

Encoding Errors model for the three quantity representations.

The simulations clearly show that the Encoding Errors model, regardless of

representation, fits the data better than the Traditional Encoding model.  The $r^2$ for the Encoding

Errors models ranged from .73-.88, whereas the $r^2$ for the Traditional Encoding models ranged

from .14-.27.  The Traditional Encoding model has the most difficulty fitting the error RT data

and the proportion of errors.  In fact, not one of the Traditional encoding models was able to

simultaneously fit all the RT data and match the behavioral error proportions.  In contrast, when

encoding errors were introduced with the Error Encoding models, all three representations

produced superior fits.  It is of some considerable importance to note that it is a trivial matter for

the Traditional Encoding model to fit the mean correct RT alone using any of the three quantity

representation theories.  It is only when the model is required to fit the correct and error RT as

well as the proportion of errors that it fails.

The second clear finding is that the Linear representation, regardless of model, fit the data

better than the Logarithmic or Scalar Variance models.  Although the Scalar Variance

representation performed the poorest overall, it performed comparably to the Logarithmic

representation when encoding errors were present.  The superior performance of the Linear

Representation is important considering the extensive acceptance of the Scalar Variance Theory

and Logarithmic Theory representations (Cantlon et al., 2009; Dehaene et al., 2008). This finding

demonstrates that the size effect is not necessarily a function of the compression in the

psychological representation of quantity (as is commonly believed).

All models converged on an angled decision threshold.  The angled threshold reduces the

amount of evidence required for a response as time increases.  The angle has the effect of

reducing the skew of the data as well as influencing the probability of an error.  The finding that

all models converged on an angled threshold is interesting because, in instances where one is

pressed for time, it has been traditional to assume the thresholds move closer to the "0" evidence

line.  Allowing the threshold to be angled provides another source for time pressure to influence

the data.  Specifically, the angle may steepen.  As Zhang et al. (2014) intimated, there are many

interesting and unexplored questions that arise when time-varying boundaries are incorporated

into sequential sampling models of human decision-making.

**Discussion**

The results of the simulations are strikingly clear: the Encoding Errors models produced

superior fits to the behavioral data from the relative quantity task (Figures 5 and 6) when

compared against the Traditional Encoding models.  The Encoding Errors models are based on

the key assumption that encoding processes are prone to error and that this error can be

quantified by the physical similarity of the digits (Cohen, 2009; Cohen, 2010).  Specifically, the

degree to which the encoding of presented digit resulted in the perception of another digit was

constrained by the measures of physical similarity between the two digits as set out in Table 1.

The success of the Encoding Errors models, relative to those based on error-less encoding,

indicate that without a proper consideration of perceptual mechanisms, our understanding of

recovering quantity from visual input will be incomplete.  The degrees of fit for these models are

striking (see Figure 5) considering the very strict constraints imposed on the modeling, namely,

(i) the distribution of the human correct and error data were fit simultaneously; (ii) only the most

extreme scores were removed from the data prior to the modeling; and (iii) the encoding

confusion probabilities were set *a priori* by the formula developed by Cohen (2009).

In addition to assessing the validity of the Encoding Error model vs. the Traditional

Encoding model, we compared the relative abilities of the three primary theories of quantity

representation (see Figure 1B) to fit the behavioral data in the relative quantity task. We

examined each of these quantity representations in terms of error-less (Traditional Encoding) and

error-prone encoding (Encoding Errors) mechanisms. When considered at this level of detail,

the data clearly show that the Linear version of the Encoding Errors model provides the best fit

to the data. Indeed, the finding that the Linear Theory models performed best stands in stark

contrast to how the extant models are currently perceived (Cantlon et al., 2009; Dehaene et al.,

2008). In the extant literature, key arguments have focused on the relative strengths and

weaknesses of the Logarithmic Theory and Scalar Variance Theory on the basis of the false

assumption that the Linear Theory does not adequately account for the size effect. It should be

noted that, although the Logarithmic Theory and Scalar Variance Theory have popular appeal

among researchers, most computer simulations of numerical cognition assume the Linear Theory

representation (e.g., Link, 1990). The attraction of the Logarithmic Theory and Scalar Variance

Theory accounts likely results, in part, from their intuitive ability to predict the correct RT data

in the relative quantity task. However, as our current simulations show, none of these models

can accommodate task performance when this is defined comprehensively, that is, it

encompasses both measures of speed and measures of accuracy (Figure 5). This is critically

important because the adequacy of the extant models has been based solely on various computer

simulations showing fits only to average correct RT in the task conditions (e.g., Verguts, et al., 2005).  We conclude therefore that it is no longer sensible to attempt to adjudicate between the models solely in terms of fits to average correct RTs.

**The Role of Perceptual Encoding**

Our simulation shows the importance of encoding processes to observers' behavior in the relative quantity task.  Specifically, the Linear Encoding Errors model assumes that (i) the numerical symbol that is physically present is continuously sampled, (ii) each sample has a probability of mis-encoding the symbol, and (iii) a perceived quantity is derived from each sample that contributes to the observer's cumulative understanding of the quantity associated with the numerical symbol.  As a result, the encoding process directly influences the observer's intuitive understanding of the quantity associated with each symbol.  For example, assume an observer is presented a "9." On Trial A, the observer samples the 9 as a "9" most often, but occasionally as a "4."  In contrast, on Trial B, the observer samples the 9 as a "9" most often, but occasionally as a "7."  In this situation, the observer's intuitive sense of quantity associated with the "9" would be lower on Trial A than on Trial B because of the influence of the error-prone encoding process.

This sampling process is encapsulated in the current simulations via the $k$ parameter. When $k = 0$, the presented digit is encoded once and its corresponding memory representation is never updated.  We have assumed that the encoding process is subject to noise hence we accept that the memory representation may not be veridical with respect to the input digit.  When $k = 1$, such sampling occurs continuously for every step in the random walk.  When $k$ falls between the extremes of 0 and 1 the probability of taking a new sample is more complex.  Specifically, the probability of re-sampling the input is given by an exponential decay function (see equation 7)

such that the likelihood of re-sampling decreases as the number of steps taken in the random walk increases.  When memory updating does not take place on a given step in the walk, the memory representation of the input digit is taken to be the most frequently sampled value. In this way, the system will converge on the correct encoding over time.  That is, because the correct digit is the most likely digit to be encoded, it will be the most frequently sampled value over time.  Thus, an Encoding Errors account as we described may be an optimal system:  it will be efficient because encoding and accessing meaning occur simultaneously, and any errors of mis-perception are corrected over time.  From this theoretical perspective, encoding, storage, and retention are intimately connected in ways that have not previously been examined. Indeed, at one level, the current work provides an existence proof that the sorts of ideas regarding error-prone perceptual sampling hold much potential.  Indeed, it seems important in the future to examine the relative explanatory power of the exponential decay function of equation 7 when compared against other plausible alternatives.

Given the novelty of Encoding Errors model it is only possible to speculate about the possible consequences for general theories of perception.  The Encoding Errors model accords with the fact that vision takes places within a constantly changing environment.  As a consequence, perceptual impressions of the world are in a constant state of flux.  Because repeated sampling of the stimulus provides subtly different impressions of the world over time, it may be more generally useful in coping with the fact that the perceptual world is in a constant state of flux.  In addition, Bayesian accounts of perception generally acknowledge that any one impression of the world is consistent with an indefinite number of possible states of the world (Norris & Kinoshita, 2008).  A remarkable feat of the human information processing system is its resilience in being able to recover a stable impression of the world from transient, noisy and

ambiguous visual signals (Ernst & Bülthoff, 2004). Repeatedly sampling the input averages out

random noise, thus helping reduce perceptual ambiguities. There is every reason to suppose that

the current notions of continuous encoding generalize to other aspects of cognition and

perception (Norris, 2006; see also Norris & Kinoshita, 2008).

We suspect that the *k* parameter is not fixed and may be under some cognitive control.

That is, when focal attention is applied and the task requires conscious identification of the

stimulus, the *k* parameter may tend towards 0 (but likely not reach 0). However, when scanning

the visual field quickly, where time is decreased and the amount of visual information is

increased, the *k* parameter may tend toward 1. Under these circumstances it is assumed that

there is no memory for the encoded identity of the stimulus. However, there is memory for the

semantic meaning of the stimulus. In this way, the visual system retains the gist of the visual

field without storing potentially unnecessary details. Such a system is efficient because the

environment contains the relevant information, which can be referred back to via a saccade,

which is not also duplicated as a memory representation. Such a system may provide a

mechanism for explaining change blindness (Simons & Ambinder, 2005).

To determine the average influence of the Encoding Errors model on one's intuitive sense

of quantity, we calculated the mean perceived quantity associated with each digit by running the

Random Walk model for the Encoding Errors model with a Linear Theory representation. Here,

we set $k = 1$, to bypass all encoding memory representations. The mean perceived quantity for

each digit was derived by averaging over all the samples for that digit. Importantly, the mean

perceived quantity for the Encoding Errors model includes samples from both accurately and

inaccurately encoded symbols. Figure 7 presents these results. The data reveals that the mean

perceived quantities generated from the Encoding Errors model produce a negatively

decelerating function similar to the log.

The negatively decelerating function in Figure 7 describing the intuitive sense of amount

denoted by numerical symbols is of central importance to linking the present simulation to

previous research.  Specifically, much of the previous research has concluded that the quantity

associated with numerical symbols was represented on a logarithmic scale because of the size

effect.  Our simulation revealed that the Logarithmic Theory representation does not actually fit

the behavioral data very well.  Rather, we concluded that the current evidence favors the Linear

Theory and this falsifies the intuition that the model cannot accommodate the basic size effect.

The important question therefore is, 'How does the Linear Theory accommodate the size effect?'

The negatively decelerating function in Figure 7 suggests that the size effect manifests primarily

from encoding processes, rather than the underlying quantity representation. This contention is

supported by the fact that none of the three primary quantity representations fit the size effect

well with the Traditional Encoding account, despite the fact that both the Logarithmic and Scalar

Variance accounts have negatively decelerating functions.  It is only after encoding errors are

added that the size effect is fit adequately by all of the quantity representation variations.

The role of perceptual encoding in the relative quantity task also has the potential to

explain the general finding that the size effect is much stronger when pairs of numbers are

presented side-by-side (e.g., Banks, Fum & Kayra-Stuart, 1976; Schwarz & Stein,1998) than

when only a single number is presented in isolation. When numbers are presented side-by-side,

uncertainty about the identity of both the standard and the probe is present.  The influence of this

uncertainty in the Encoding Errors account will manifest as a stronger size effect (because there

are two uncertain symbols, rather than just one).

**Further insights and implications**

      One notable aspect of the random walk models examined here concerns the nature of the response thresholds. As in all previous incarnations of random walk models, the walk takes place in a 2-D Euclidian plane. The start position is a point on the *y*-axis and the walk proceeds until it reaches one of two decision boundaries. In the majority of previous random walk models the boundaries are fixed and are perpendicular to the *y*-axis.  In the current modeling we have retained the idea that the y intercepts of the boundaries are fixed but have defined their angle of intersection (b) with the *y*-axis as a free parameter (see Figure 3A). We have provided another concrete example of where the incorporation of time-varying boundaries in sequential sampling models enhances their explanatory power (cf. Zhang et al., 2014). A simplifying assumption has been that the angle of incidence for both smaller and larger decision boundaries is constrained to be the same but it would be desirable that future work examines the consequences of relaxing this symmetry assumption. Nonetheless a key feature of the current account is that the angle of incidence of the response boundaries is not fixed.

      In allowing the angle of incidence to vary we have been able to discover which value provides the best fit to the data. The simulations reveal that the best-fit decision boundaries are angled towards one another. Angled decision boundaries require less evidence to make a response as the walk proceeds. Critically, angled decision boundaries provide a mechanism for the model to accommodate the particular skewed nature RT distributions, as well as influence both the proportion of errors and the relative speed of error and correct responses. As such, angled boundaries (with trial-by-trial variability in the angle) provide a unique degree of flexibility in random walk and diffusion models.  We take it that this is a general truth about

random walk models and, in agreement with Zhang et al. (2014), we suggest that this could be examined more generally when using such models in future.

**$z$ - the start point of the walk**

A central feature of our simulations was the fact that we modeled correct and error RT as well as the proportion of errors for every probe simultaneously. Although other researchers have suggested that simulating the error proportions obtained from the behavioral data is trivial (Verguts et al., 2005), our simulations suggest that they are, in fact, one of the key characteristics of the data that can be used to distinguish the different models. In particular, although the numerical distance effect is clearly present in the error proportions, there is a key inconsistency with this pattern and the pattern predicted by compressed quantity representations such as the Logarithmic and Scalar Variance accounts. The compressed quantity representations predict that the errors produced in response to the larger numbers (when compared to a standard of 5) will be greater than their symmetrical counterparts. So, for example, the errors produced in response to a "6" should be greater than those produced in response to a "4." We see in the behavioral data, however, that this is not the case for the "4" and the "6." This is again a reason to question the explanatory adequacy of the Logarithmic and Scalar Variances models.

Our simulations account for this "backwards" data by shifting the start point (the $z$ parameter) towards the "smaller" threshold: z is negative in all cases (see Table 3). Indeed, for the Error Encoding models (i.e., the only models that fit well), the larger the compression of the original quantity representation on which the model was simulated, the larger the shift of the start point. Because the Linear Theory is not compressed, it did not require the simulations to compensate with a large start point shift. We confirmed this hypothesis by running simulations whereby we held the start point fixed at "0." As expected, the simulations of the compressed

quantity representations produced patterns in which the "4" had fewer errors than the "6." Here again therefore is an example of the utility of having an explicit computational model. In this case it has provided an effective tool that can be used to test intuitions about both data and theory.

**Of Models and Modeling**

The present work emphasizes the contribution that models and modeling can make to theory development and assessment.  We have shown how a relatively simple task translates into a fairly complicated set of cognitive processes.  However, when these processes are stated precisely, then different theories can be weighed against one another via the relative fits they provide to the behavioral data.  Although we have focused on understanding performance in the relative magnitude task, we feel that the implications of the work go further than this and do speak to issues that have arisen with other numerical tasks. Perhaps the most prominent amongst these is the cross modal matching paradigm in the form of the number line task (see e.g., Berteletti, Lucangeli, Piazza, Dehaene, & Zorzi, 2010; Booth & Siegler, 2006; Cohen & Blanc-Goldhammer, 2011; Dehaene, Isard, Spelke, & Pica, 2008; Geary, Hoard, Nugent & Bryd-Craven, 2008; Siegler & Booth, 2004).  The number line task requires the encoding and processing of both digits and lines, as well as relatively complex cross modal matching.  These complications are often glossed over in the literature and the processes involved with completing the task are rarely modeled.

We note that we have developed models based on Signal Detection Theory and random walk simulations. Others have developed models of numerical cognition that comprise neural network architectures and distributed processing (see, e.g., Verguts et al., 2005; and Zorzi & Butterworth, 1999).  Here is not the place to offer an in-depth analysis of this work, as we have

not attempted to compare random walk models with neural network counterparts: this would be a major and different undertaking altogether.  However, we do note that the neural network models of numerical cognition appear wanting when assessed against the hard constraints discussed here.

For instance, the network models described by Verguts et al. (2005) do not address the full complement of the kinds of behavioral data considered here, namely, correct and error RT distributions, and error proportions. Verguts et al (2005) primarily focused on their model's ability to fit mean correct RT in various number tasks and respectable fits are reported via this level of analysis.  Although they offer a promissory note about an "additional stochastic component," and claim that the error data do not "pose a significant challenge to the model" (p. 78), it remains to be seen whether their model can account for these data. Indeed, our data reveal that modeling mean RTs proves no challenge to any of the three key theories of number representation.  It is only when the full complement of data is considered is it possible to discriminate between the explanatory power of the different models.  On these grounds, it will be interesting to see how neural network models fair when the full complement of data is taken into account.

**Prospects for future work**

Provision of the current computer simulations is an important step in understanding the numerical cognition system.  Nevertheless, this work is not without some limitations.  First, we identified the Error Encoding models as proving superior to the Traditional Encoding models, yet we have to explore fully and test the Error Encoding model's predictions.  For example, the Error Encoding model appears to predict that adding visual noise to the presented numerals will influence the confusions and thus the response latencies and error rates in a predictable way.

Furthermore, limiting encoding time should have a predictable effect.  These, and other,

predictions require testing.  In addition, it is unclear how the Error Encoding model will account

for the SNARC effect (Dehaene, Bossini & Giraux, 1993).  In the current work, we

counterbalanced response button, so the SNARC effect should not influence the results in a

meaningful way.  Most models of numerical cognition adopt a post hoc explanation for the

SNARC effect, and we will consider possible explanations for the effect in future versions.

A second limitation of the current model is that it was run as a computer simulation of a

discrete random walk model and this is distinctive against a background of the other work that

has considered random walk models based on diffusion processes (e.g., Smith & Mewhort,

1998).  Clearly future work might be directed to extending the modeling to examine the

consequences of adopting a continuous rather than discrete random walk. We adopted the present

methods because the different quantity representations (logarithmic distributions, unequal

variances) and the Encoding Errors model (simultaneous encoding and comparisons) pose

considerable mathematical challenges to the derivation of an analytic solution.  Addressing these

challenges could form the basis of a primarily mathematical rather than psychological exercise

and is simply beyond the scope of this present paper.

Other interesting questions remain about the generality of the approach when other forms

of input numbers are used, for example, decimals, double-digit numbers, etc. Despite such

challenges, we stand by the general framework for thinking about performance in relative

quantity judgments that we have put forward here; namely, that a proper understanding of task

performance can only be achieved when due consideration is given to both perceptual and

cognitive processes. We have been able to make considerable progress in the current work

because of the very clear constraints provided by the single digit similarity measures (Cohen,

2009). We assume further progress will be made once details about the encoding of more complex numbers are uncovered.

Some previous work with double-digit integers (Cohen, 2010) has revealed that factors associated with encoding do play a determining role in quantity judgments.  In one of the experiments reported by Cohen (2010) speeded quantitative judgments were made in cases where participants judged whether a probe (in the range 1 – 99) was greater/ less than 55. Although effects of numeral distance were present in the data, best fits to the data depended on including factors concerning physical similarity of the decade of the double digits.

Evidence from the other experiments concerning decimals (Cohen, 2010) however was quite different and showed very strong effects due to physical similarity.  Indeed, in one experiment no effects of numerical distance on task performance were observed when physical similarity was taken into account.  In these cases, the decimals (.01 - .99) were judged relative to the standard .55.  In the latter experiment participants were forced to attend to the position of the decimal point by varying the rounding of the numbers across trials.  When all decimals were presented to the same level of precision participants may simply have ignored the decimal point and treated the numbers as being integers.  Clearly, therefore, when attention is focused more broadly on a range of number formats other than single digits more complex accounts of performance are needed.  We accept that these will demand a proper consideration of encoding processes.

## Conclusions

In sum, the current simulations lead to the conclusion that an adequate account of task performance cannot be achieved without consideration of both (i) perceptual encoding and, (ii) the representation of quantity information. In attempting to infer the underlying representation of

quantity from behavioral data, it is important to consider very carefully the influences of encoding, decision, and response execution as well as other, more salient task related processes such as comparison, on the participants' responses. We have instantiated the primary models of numerical cognition in computational models of the relative quantity task. Our data reveal that (i) encoding processes influence performance in non-negligible ways, (ii) quantities are represented as perceptual distributions that are equally spaced and have equal variance, (iii) the perceptual system repeatedly samples the stimulus in an error-prone fashion, and, (iv) the recovery of number meaning proceeds in parallel with, and is continuously influenced by, stimulus encoding. Together, these findings represent a new and comprehensive understanding of the perceptual and cognitive mechanisms that underpin human number comparison. We feel that consideration of these ideas gives rise to more wide-reaching implications for thinking about how the human perceptual system operates in general.

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Table 1

*The Arabic Digit confusion matrix for the digits "1"-"9."*

| Presented Probe | Perceived Stimulus | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 0.854 | 0.006 | 0.019 | 0.026 | 0.006 | 0.005 | 0.052 | 0.012 | 0.019 |
| 2 | 0.005 | 0.783 | 0.048 | 0.011 | 0.020 | 0.035 | 0.013 | 0.065 | 0.020 |
| 3 | 0.016 | 0.043 | 0.707 | 0.022 | 0.043 | 0.032 | 0.035 | 0.058 | 0.043 |
| 4 | 0.024 | 0.010 | 0.023 | 0.757 | 0.023 | 0.019 | 0.017 | 0.034 | 0.093 |
| 5 | 0.005 | 0.018 | 0.043 | 0.021 | 0.696 | 0.106 | 0.011 | 0.057 | 0.043 |
| 6 | 0.004 | 0.030 | 0.030 | 0.017 | 0.101 | 0.659 | 0.009 | 0.121 | 0.030 |
| 7 | 0.049 | 0.013 | 0.034 | 0.018 | 0.013 | 0.011 | 0.797 | 0.020 | 0.040 |
| 8 | 0.009 | 0.052 | 0.052 | 0.028 | 0.052 | 0.115 | 0.016 | 0.626 | 0.052 |
| 9 | 0.015 | 0.018 | 0.042 | 0.083 | 0.042 | 0.031 | 0.035 | 0.056 | 0.680 |

*Note.* The probability for digit pair j-k (e.g., 1-1) is given by: $p(PS_{j-k}) = \frac{PS_{j-k}}{\Sigma_{i=1}^{9} PS_{j-i}}$

Table 2

The free parameters in the Traditional Encoding and Encoding Errors Random walk models.

| Parameter | | Included in Random Walk Model | |
| Name | Description | Traditional Encoding | Encoding Errors |
|---|---|---|---|
| $s$ | SD of the PDQ (see Equations 1-3) | Yes | Yes |
| $d_a$ | Decision threshold intercept | Yes | Yes |
| $d_\theta$ | Decision threshold angle | Yes | Yes |
| $d_s$ | Decision threshold angle SD. This is expressed as a proportion of d✓ | Yes | Yes |
| $z$ | Start point: This is expressed as a proportion of $d_a$. The sign indicates the direction of bias. | Yes | Yes |
| $k$ | Probability of sampling the environment | No | Yes |
| $t_{er}$ | Intercept representing ancillary processes unrelated to decision process | Yes | Yes |
| $b_{NS}$ | Slope to transform the number of random walk samples into RT | Yes | Yes |

Table 3

The best fit parameter estimates for the six random walk simulations.

| | $s$ | $d_a$ | $d_\theta$ | $d_\sigma$ | $z$ | $k$ | $t_{er}$ | $b_{NS}$ |
|---|---|---|---|---|---|---|---|---|
| | | | Traditional Encoding | | | | | |
| Representation | | | | | | | | |
| Logarithmic | 0.63 | 170.8 | -26.1 | .44 | -.03 | NA | 376 | 0.73 |
| Scalar Variance | 0.88 | 112.8 | -41.2 | .41 | -.09 | NA | 428 | 1.79 |
| Linear | 5.12 | 92.25 | -14.9 | .24 | -.06 | NA | 429 | 1.77 |
| | | | Error Encoding | | | | | |
| Logarithmic | 1.08 | 29.69 | -7.3 | .38 | -.11 | 0.61 | 384 | 3.48 |
| Scalar Variance | 1.79 | 136.4 | -16.5 | .22 | -.10 | .41 | 395 | 2.34 |
| Linear | 8.89 | 98.3 | -22.6 | .05 | -.02 | .58 | 375 | 4.30 |
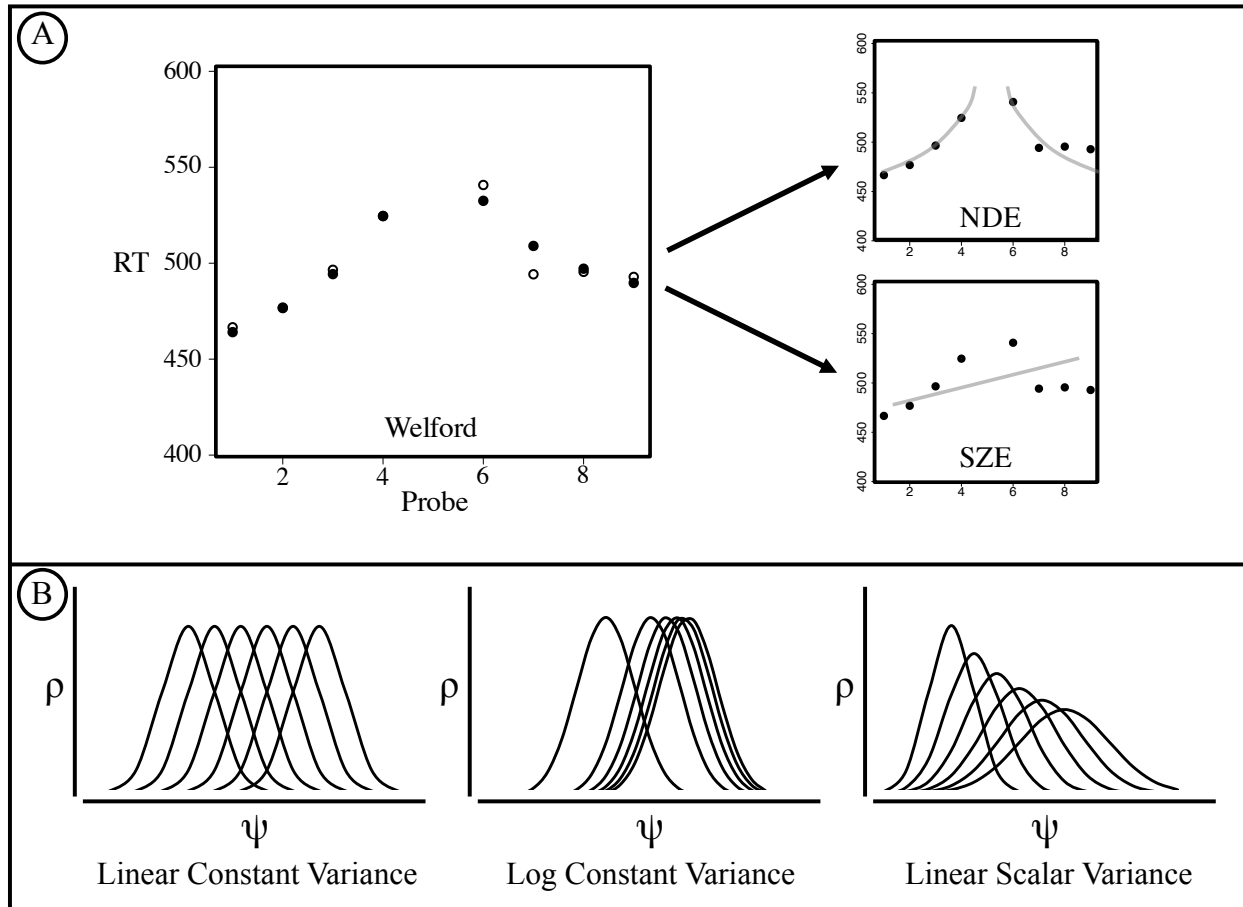
*Figure 1.* Basic data and key models. (**A**) Summary data from a typical relative quantity task in which the quantity denoted by a probe digit is compared with "5." RT is mean reaction time in ms. Open circles indicate human data and '+'s indicate the fit provided by the Welford function (Welford, 1960). Two robust effects are (i) the numerical distance effect (NDE) such that RTs are an inverse function of the numerical distance between the two numbers presented and (ii) the size effect (SZE) reflects a monotonically increasing function relating RTs and the quantity denoted by the probe. (**B**) The three alternative models of the representation of numerical quantities. The x-axis represents the psychological representation of quantity (from small to large); the y-axis represents density. The graphs, from left to right, describe the linear, logarithmic, and scalar variance theories. Please see the text for detailed descriptions.
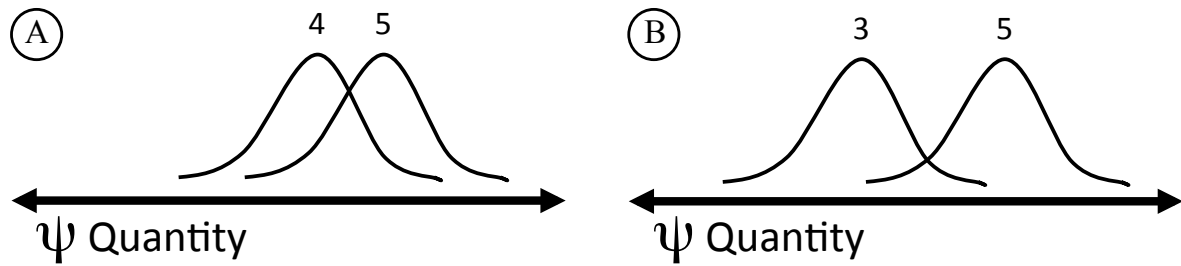
*Figure 2.* A visualization of the PDQs for the number symbols 3, 4, and 5. The *x*-axis is the

psychological representation of quantity (e.g., the mental number-line). The distributions

represent the frequencies that each symbol activates a particular psychological quantity. The

overlap of the distributions determines the difficulty of distinguishing the quantities of the two

symbols. The PDQ for 4 and 5 (**A**) overlap more than those of 3 and 5 (**B**). Therefore, 4 is more

difficult to distinguish from 5 than 3.
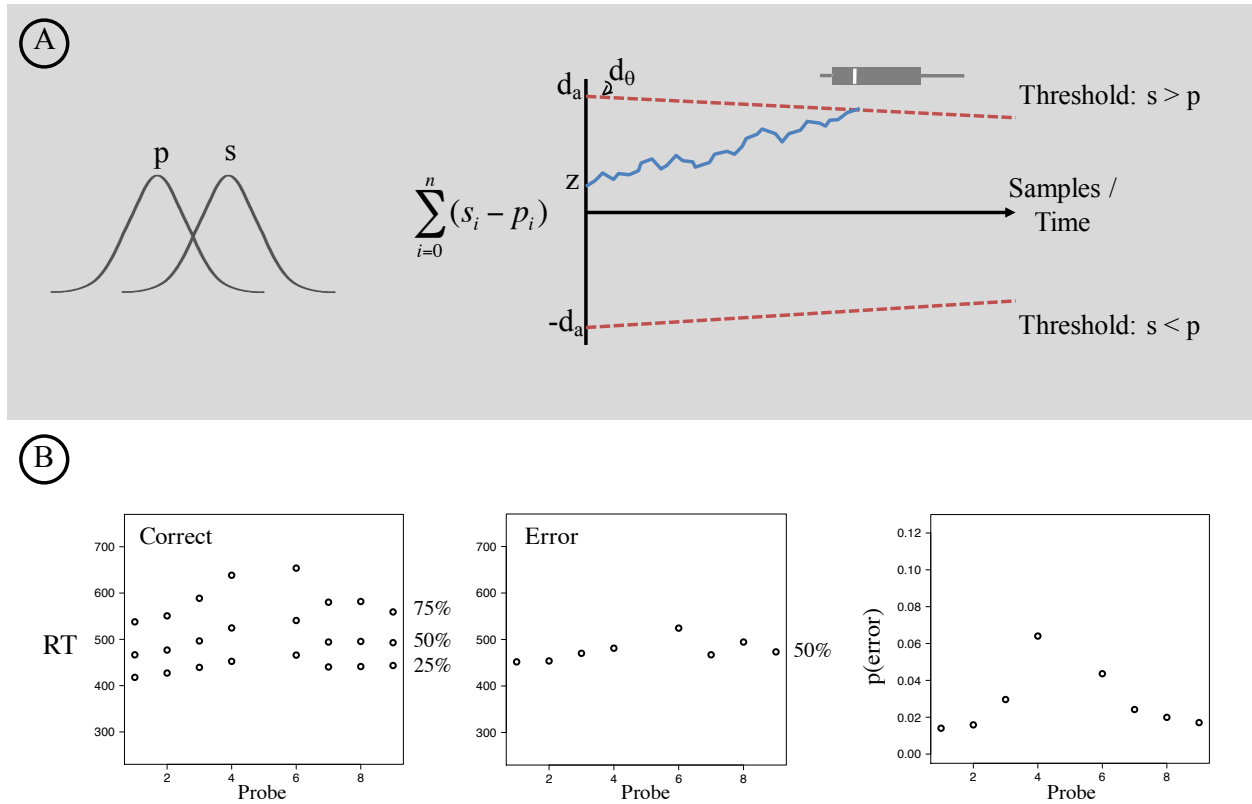
*Figure 3*. The random walk process and details of the behavioral data. (**A**) Schematic

representation of the random walk process. The account assumes that both the standard ("S" e.g.,

"5") and the probe digit "P" are represented in terms of corresponding Gaussian PDQs (left hand

side of the figure). At each step in the walk the information associated with the probe is assessed

relative to the distribution of the standard.  Evidence accumulates once the stimulus is presented

and a decision is made once either the upper or the lower threshold is reached. (**B**) The left and

center panels display the 25$^{th}$ percentiles of the human data from the relative quantity task. The

right-most panel displays the error count for each probe
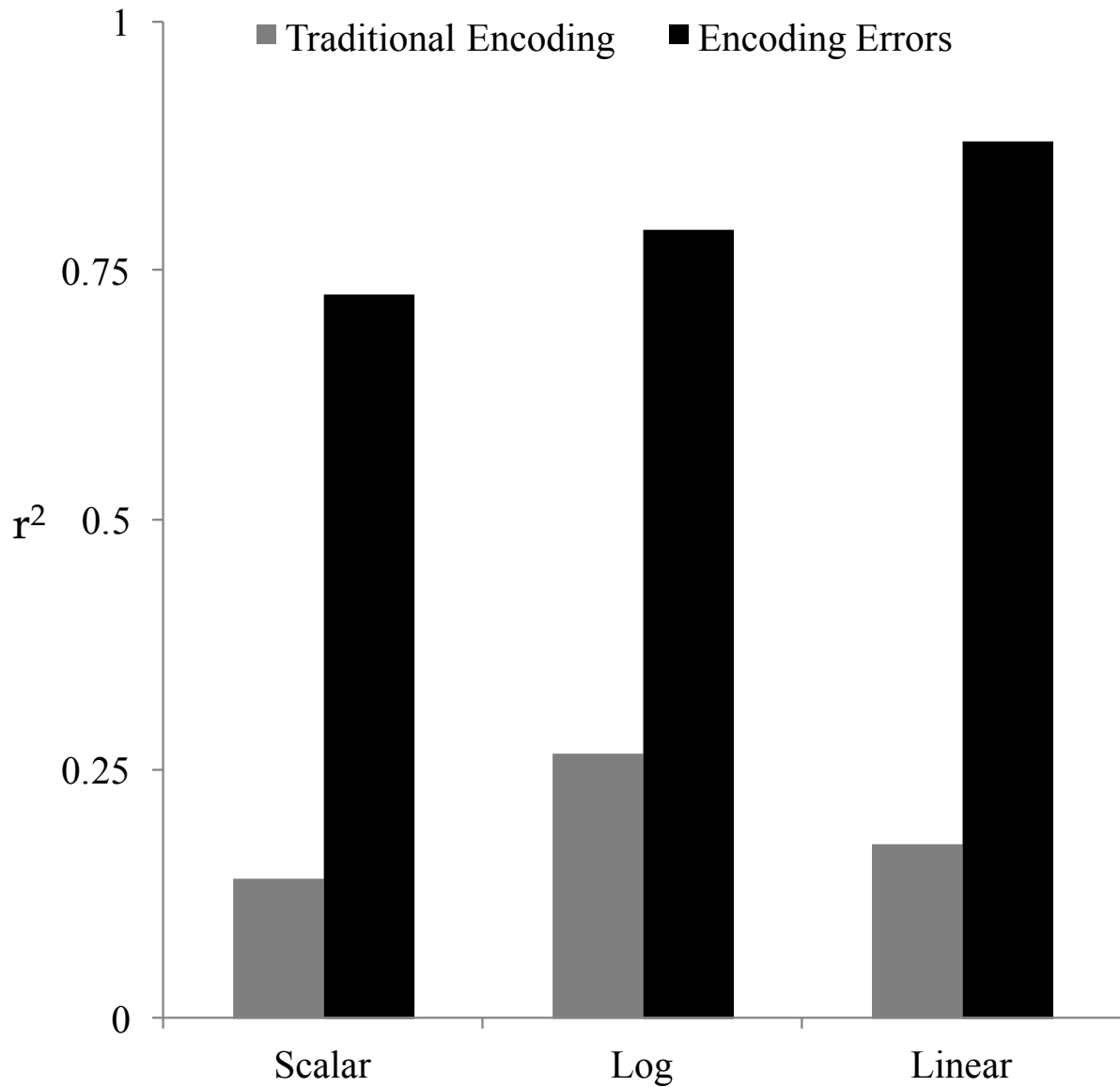
*Figure 4.* The fit statistic, $r^2$ (BIC$_Z$ and chi square provides the same results), of the Traditional

Encoding and Encoding Error models for the three primary quantity representations.  The

simulations were simultaneously fit to the correct and error RTs as well as the proportion of error

for each probe. The Encoding Error models outperformed the Tradition Encoding models and the

Linear Encoding Error Model out performs all other models.

# Traditional Encoding
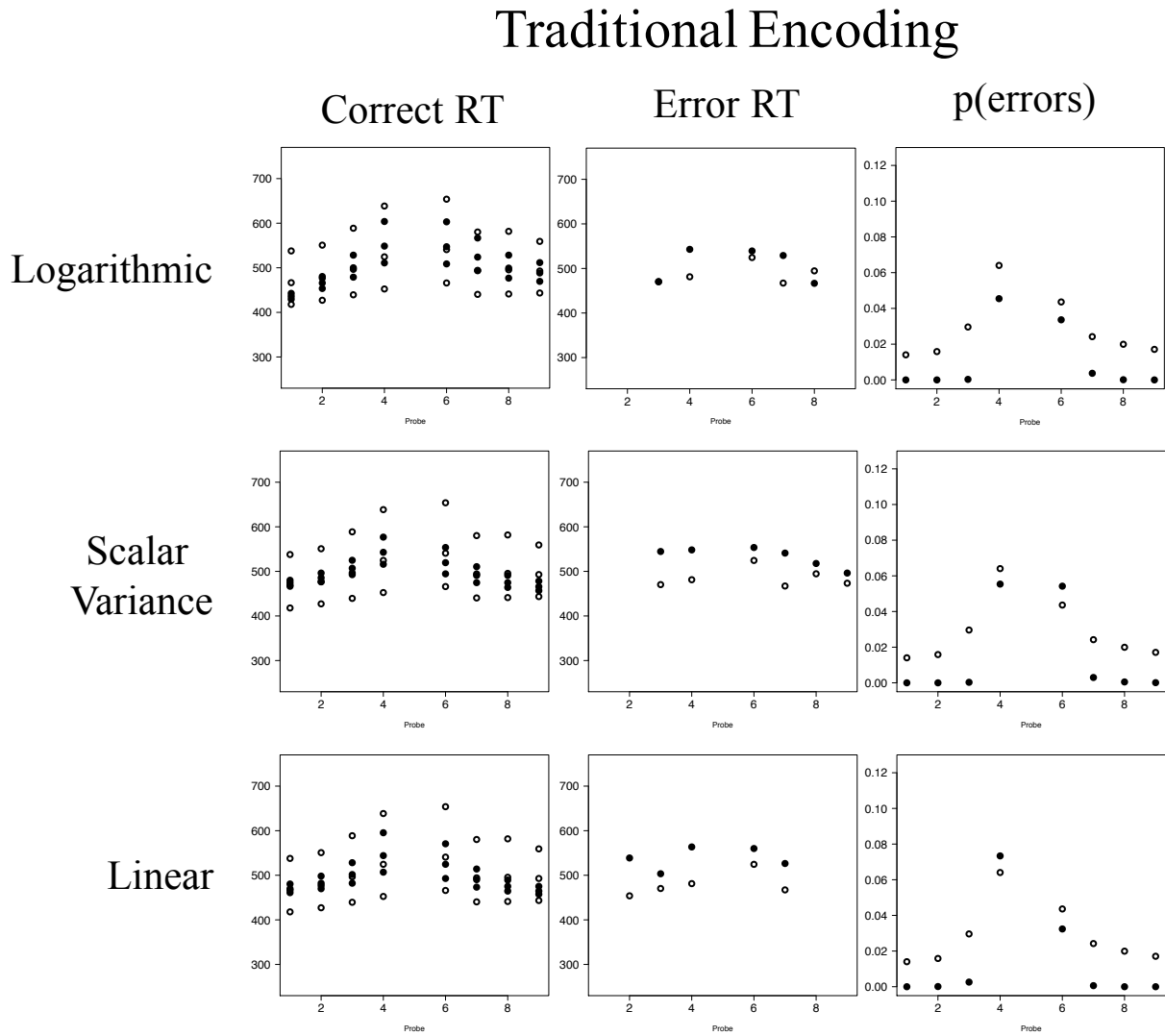


*Figure 5.* The summary behavioral data broken down according to the fits of the Traditional

Encoding model for the three quantity representations. Open circles indicate human data and

filled circles indicate the fit provided by the model. No model fares well when simultaneously fit

to the correct RT, error RT, and proportion of errors for each probe.
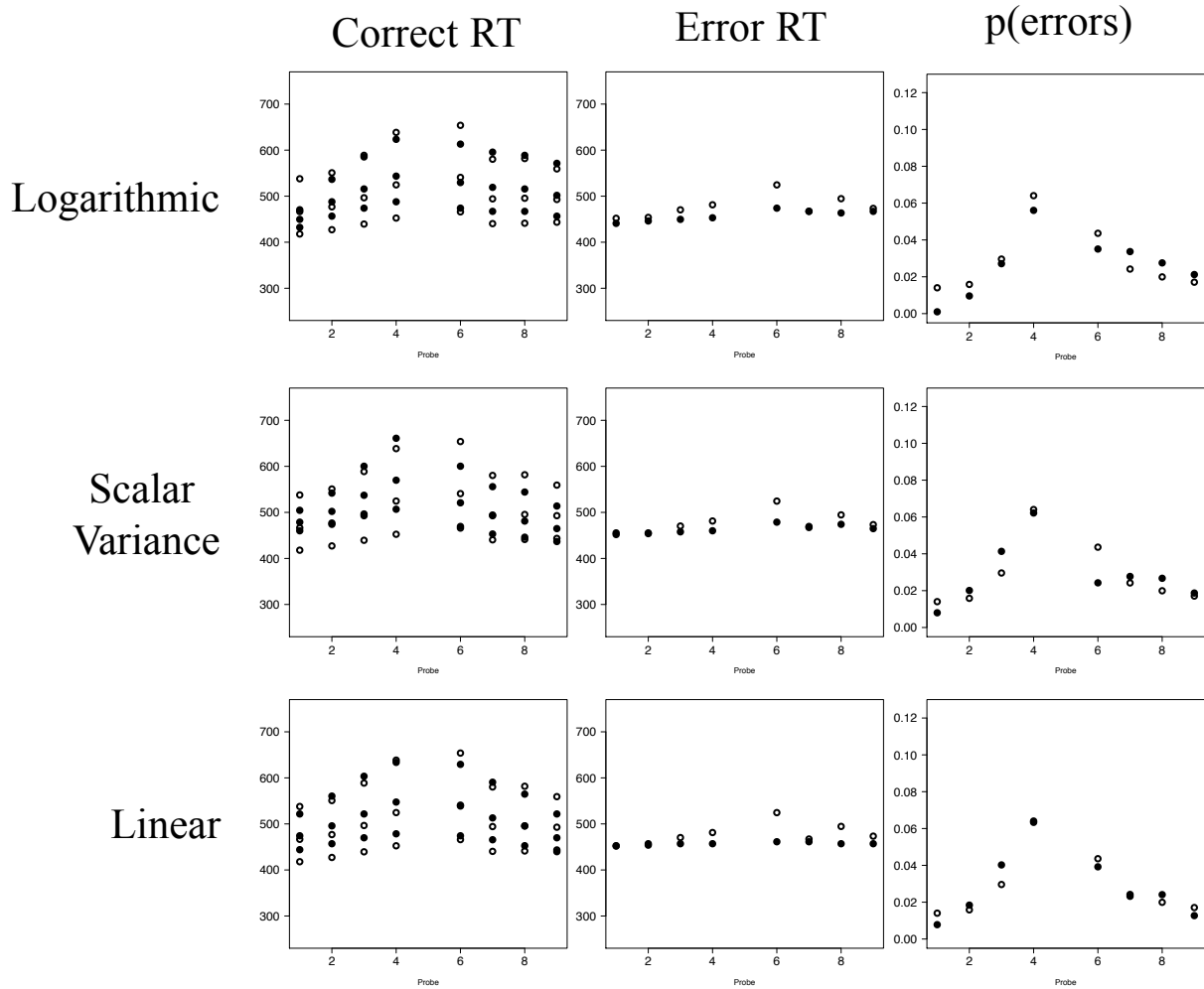
*Figure 6.* The summary behavioral data broken down according to the fits of the Error Encoding model for the three quantity representations. Open circles indicate human data and filled circles indicate the fit provided by the model. All models fare well when simultaneously fit to the correct RT, error RT, and proportion of errors for each probe. The Linear Error Encoding model out performs all other models with an $r^2 = .86$.
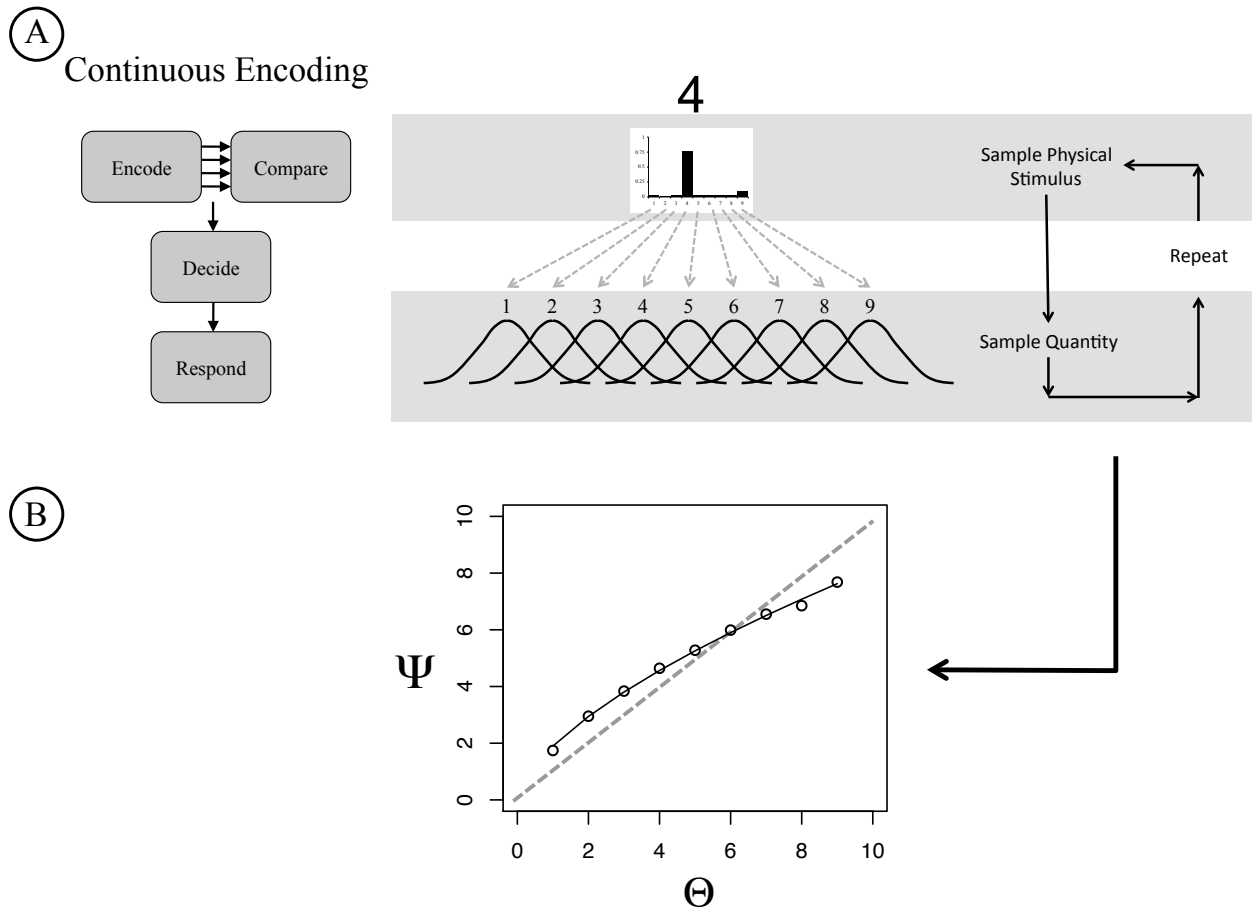
*Figure 7.* The importance of encoding in the size effect. (**A**) Left most panel. Schematic

representation of the general framework for thinking about the derivation of number meaning.

Encoding works in tandem with the comparison process and continually influences this process.

Continuous encoding as shown in the rightmost panel is fundamental to the operation of the

linear hybrid account. (**B**) The mean psychological sense of quantity ($\Psi$) by the actual quantity

($\Theta$) for digits 1-9 stemming from the underlying Linear Theory representation and the confusions

resulting from continuous encoding.  The latter influence causes the shift from linearity to the

negatively decelerating function present.

Footnotes

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[1] Ratcliff, Thompson, and McKoon, (2015) also modeled number discrimination using a diffusion model. We do not review it here because the authors did not attempt to model the numerical distance effect with a single set of parameters. Rather, the authors assessed two-digit numbers, grouped digits in intervals of five units from the standard, and then fit separate models to each of these groups. It was neither the authors' intent, nor were the models appropriate for, adjudicating between different underlying quantity representations.

[2] Although one may conceive of other models, our intent here is to assess the major models published in the extant literature.

[3] It appears that the PS function has changed over the course of Cohen's papers because the 2010 paper introduced a constant in the denominator so that the formula produced a metric for identical numerals (e.g., 5 vs. 5). Furthermore, the present paper converts the PS function into probabilities. These changes, however, are superficial. Specifically, the three versions of the PS metrics mentioned correlate with one and other at the 0.99 level. As such, the current PS function is essentially identical to that introduced in the first 2009 paper.

[4] We note here that $r^2$ will always be higher when the number of parameters is increased. Nevertheless, BICz, chi square, and $r^2$ reveal the same pattern in our data. This is undoubtedly because of the vastly improved fit of the Error Encoding models, which only add a single parameter. We chose to report the $r^2$, rather than the other two fit statistics, because (1) $r^2$ is readily understood by most researchers, and (2) $r^2$ has the added benefit of providing information about variance accounted for by each model.