UNIVERSITY of York

This is a repository copy of *Invariant measure for the stochastic Navier-Stokes equations in unbounded 2D domains*.

White Rose Research Online URL for this paper:
https://eprints.whiterose.ac.uk/104995/

Version: Accepted Version

Article:
Brzezniak, Zdzislaw orcid.org/0000-0001-8731-6523, Motyl, Elżbieta and Ondrejat, Martin (2017) Invariant measure for the stochastic Navier-Stokes equations in unbounded 2D domains. Annals of Probability. AOP1133. pp. 3145-3201. ISSN 0091-1798

https://doi.org/10.1214/16-AOP1133

Reuse
Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

Takedown
If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.

White Rose university consortium
Universities of Leeds, Sheffield & York

eprints@whiterose.ac.uk
https://eprints.whiterose.ac.uk/

Submitted to the Annals of Probability

INVARIANT MEASURE FOR THE STOCHASTIC NAVIER-STOKES EQUATIONS IN UNBOUNDED 2D DOMAINS*

By Zdzisław Brzeźniak† , Elżbieta Motyl and Martin Ondrejat

University York, University of Łódź and Czech Academy of Sciences

Building upon a recent work by two of the authours and J. Seidler on bw-Feller property for stochastic nonlinear beam and wave equations, we prove the existence of an invariant measure to stochastic 2-D Navier-Stokes (with multiplicative noise) equations in unbounded domains. This answers an open question left after the first authour and Y. Li proved a corresponding result in the case of an additive noise.

1. Introduction. A classical method of proving the existence of an invariant measure for a Markov proceess is the celebrated Krylov-Bogoliubov method. Originally it was used for Markov processes with values in locally compact state spaces, e. g. finite dimensional Euclidean spaces, see e.g. [36] and [46]. In the recent years it has been successfully generalised to Markov processes with non-locally compact state spaces, e.g. infinite dimensional

*The research of the first and second named authours was partially supported by the Leverhulme grant RPG-2012-514. The research of the third named authour was supported by the GAČR grant 15-08819S.

†Corresponding authour.

MSC 2010 subject classifications: Primary 60H15, 35Q30, 37L40; secondary 76M35, 60J25

Keywords and phrases: invariant measure, bw-Feller semigroup, stochastic Navier-Stokes equations

Hilbert and Banach spaces, see for instance the books by Da Prato and Zabczyk [26, 27] and a fundamental paper by Flandoli [30] for the case of 2 dimensional Navier-Stokes equations with additive noise. One should also mention here a somehow reverse problem, found for instance in the stochastic quantisation approach of Parisi and Wu [48], of constructing a Markov process with certain properties given an 'a priori invariant measure'. In the context of Stochastic Partial Differential Equations, this approach has been successfully implemented by Da Prato and Debussche for 2 dimensional Navier-Stokes equations with periodic boundary conditions driven by space time white noise in [24] and for the 2-D stochastic quantization equation in [25].

The latter method is related to the approach by Dirichlet forms as for instance in [2]. In the field of deterministic dynamical systems the so called Avez method, see [3] and [38], is also popular. It seems that the first of these methods when used in order to prove the existence of an invariant measure for Markov processes generated by SPDEs one requires the existence of an auxiliary set which is compactly embedded into the state space and in which the Markov process eventually lives. Thus, it has so far been restricted to SPDEs of parabolic type (giving necessary conditions with smoothing effect) and in bounded domains (providing the needed compactness via the Rellich Theorem).

On the other hand, as a byproduct of results obtained by Yuhong Li and the 1st named authour in [12], about the existence of a compact absorbing set for stochastic 2 dimensional Navier-Stokes equations with additive noise in a certain class of unbounded domains, there exists an invariant measure

for the Markov process generated by such equations. This, to the best of the authours knowledge, provides the first example of a nontrivial SPDEs without the previously required compactness assumption possessing an invariant measure. A posteriori, one can see that behind the proof is the continuity of the corresponding solution flow with respect to the **weak topologies**, see Example 1.1.

It is has been discovered in [42, Proposition 3.1] that a bw-Feller semigroup has an invariant probability measure provided the set

$$\left\{ \frac{1}{T_n} \int_0^{T_n} P_s^* \nu \, ds; \ n \geq 1 \right\} \tag{1.1}$$

is tight on (H, bw). However, it is far from straightforward to identify stochastic PDEs for which the associated transition semigroups are bw-Feller. This has been recently done for SPDEs of hyperbolic type (i.e. second order in time) such as beam and nonlinear wave equations in [19]. The aim of this work is to show that the general approach proposed in that paper is also applicable to stochastic Navier-Stokes equations in unbounded domains. In the case of bounded domains, the first such a result has been obtained by Flandoli in the celebrated paper [30]. A similarity between the equations studied in [19] and the current paper is that the linear generator has no compact resolvent. However, in the current situation, the generator is sectorial contrary to the former case. However, the smoothing of the semigroup is rather used to counterweight the non-smoothness of the nonlinearity.

On the other hand, in [42] Maslowski and Seidler proposed to use the of weak topologies to the proof of the existence of invariant measures but the applications of the proposed theory had limited scope.

These two papers, i.e. [42] and [12] have inspired us to investigate this

matter further.

Moreover, while working on the existence of solutions to geometric wave equations it has become apparent to us that the methods of using very fine techniques in order to overcome the difficulty arising from having only weak a'priori estimates should also allow one to prove the sequentially weak Feller property required by the Maslowski and Seidler approach. This made it possible to prove the existence of invariant measure for SPDES of hyperbolic type as for instance wave and beam, see the recent paper [19] by the Seidler and the 1st and 3rd authours.

The aim of the current work is to show that the approach worked out in [19] combined with the method of proving the existence of Stochastic Navier-Stokes Equations in general domains developed recently by 1st and 2nd authours, see for instance [16], indeed can lead to a proof of the existence of an invariant measure for stochastic 2 dimensional Navier-Stokes equations with multiplicative noise (and additive as well) in unbounded domains and thus generalizing the previously mentioned result [12].

Let us stress that the general result proved in Sections 5-10 of [19] does no apply directly to Stochastic NSEs. Instead we propose a scheme which is general enough that it should be applicable to other equations. Let us describe it in more detail. In a domain $\mathcal{O} \subset \mathbb{R}^2$ satisfying the Poincaré inequality we consider the following stochastic Navier-Stokes equations in the functional form

(1.2)
$$\begin{cases} du(t) + Au(t)\,dt + B\big(u(t),u(t)\big)\,dt = f\,dt + G\big(u(t)\big)\,dW(t), \quad t \in [0,T], \\ u(0) = u_0, \end{cases}$$

where A is the Stokes operator, $u_0 \in \mathrm{H}$, $f \in \mathrm{V}'$ and we use the standard notation, see the parts of the paper around equation (3.2). In particular, $W = \big(W(t)\big)_{t\geq 0}$ is a cylindrical Wiener process on a separable Hilbert space K defined on a ceratin probability space and the nonlinear diffusion coefficient G satisfy some natural assumptions. It is known (but we provide an independent proof of this fact) that the above problem has a unique global solution $u(t;u_0)$, $t \geq 0$. The corresponding semigroup $(P_t)_{t\geq 0}$ is Markov, see Proposition 6.1. This semigroup is defined by the formula, see (6.2),

$$(P_t\varphi)(u_0) = \mathbb{E}[\varphi(u(t;u_0))], \quad t \geq 0, \quad u_0 \in \mathrm{H}, \tag{1.3}$$

for any bounded Borel function $\varphi \in \mathcal{B}_b(\mathrm{H})$. Then, see Proposition 6.2, we prove that this semigroup is bw-Feller, i.e. for every $t > 0$ and every bounded sequentially weakly continuous function $\phi : \mathrm{H} \to \mathbb{R}$, the function $P_t\phi : \mathrm{H} \to \mathbb{R}$ is also bounded sequentially weakly continuous.

The idea of the proof of the last result can be traced to recent papers by all three of us in which we proved the existence of weak martingale solutions to the stochastic geometric wave and Navier-Stokes and equations developed respectively in [17, 18] and [16].

Finally, our main result, i.e. Theorem 6.5 about the existence of an invariant measure for the semigroup $(P_t)_{t\geq 0}$, follows provided some natural assumptions, as inequality (G2) holds with $\lambda_0 = 0$, i.e. for some[1] $\rho \geq 0$,

$$|G(u)|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})} \leq (2-\eta)\|u\|^2 + \rho, \qquad u \in \mathrm{V}, \tag{1.4}$$

guaranteeing the uniform boundedness in probability, are satisfied, see Corollary 6.4.

[1]Throughout the whole paper we use the symbol $\mathcal{T}_2$ to denote the space of Hilbert-Schmidt operators between corresponding Hilbert spaces.

In proving Proposition 6.2 the continuity/stability result contained in Theorem 5.9 plays an essential rôle.

We will present now the earlier promised example based on the paper [12].

EXAMPLE 1.1. *If $\varphi = (\varphi_t)_{t\geq 0}$ is a deterministic dynamical system on a Hilbert space* H*, then one can define the corresponding Markov semigroup by*

$$[P_t(f)](x) := f(\varphi_t(x)), \quad t \geq 0, \quad x \in \mathrm{H}. \tag{1.5}$$

Suppose that the semiflow is sequentially weakly continuous in the following sense.

(1.6)

If $t_n \to t \in \mathbb{R}_+$, $x_n \to x$ weakly in H *then $\varphi_{t_n}(x_n) \to \varphi_t(x)$ weakly in* H.

Note that the above condition is satisfied for the deterministic 2-d Navier-Stokes equations, see [52] and also [12, Lemma 7.2].

Then, the assertion of Theorem 9.4 in [19] holds. Indeed, let us choose and fix a bounded sequentially weakly continuous function $f : \mathrm{H} \to \mathbb{R}$, a sequence $(t_n) \to t$ and a sequence (x_n) such that $x_n \to x$ weakly in H*. Then by assumption (1.6) $\varphi_{t_n}(x_n) \to \varphi_t(x)$ weakly in* H *and since f is sequentially weakly continuous we infer that*

$$[P_{t_n}(f)](x_n) = f(\varphi_{t_n}(x_n)) \to f(\varphi_t(x)) = P_t f(x).$$

The condition guaranteeing the existence of an invariant measure, see [19, Theorem 10.1], now reads as follows. There exists $x \in \mathrm{H}$ such for every $\varepsilon > 0$, there exists $R > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t 1_{|\varphi_s(x)|_{\mathrm{H}} \geq R}\, ds \leq \varepsilon \tag{1.7}$$

which is obviously satisfied provided the dynamical system $\varphi = (\varphi_t)_{t\geq 0}$ *is bounded at infinity, i.e. there exists* $x \in \mathrm{H}$ *and* $R > 0$ *such that* $|\varphi_s(x)|_{\mathrm{H}} \leq R$ *for all* $s \geq 0$. *It is well known that this condition holds for the deterministic 2-d Navier-Stokes equations in a Poincaré domain (as well as for the damped Navier-Stokes Equations in the whole space* $\mathbb{R}^2$. *Thus we conclude, that in those cases, there exists an invariant measure. Of course, these are known results, the purpose of this Example is only to elucidate our paper by showing that it is also applicable to these cases.*

Let us point out that [12, Lemma 7.2] played an important rôle in that paper.

We believe that the result described in this Example holds also for the Random dynamical system from [12]. In this way, we will get an alternative proof of the result existence of an invariant measure proved in that paper.

The weak continuity property (1.6) *has also been investigated [4, 52, 12, 23]. In the first three of these references the weak to weak continuity is an important tool in proving the existence of an attractor for deterministic 2D Navier-Stokes Equations in unbounded domains, where, as we pointed out earlier, the compactness of the embedding from the Sobolev space* H^1 *to* L^2 *does not hold. A similar type of continuity (weak to strong), is encountered in the proof of the large deviation principle for SPDES, see for instance [9, Lemma 6.3] for the case of Stochastic Landau-Lifshitz Equations. It might be interesting to understand in the relationship between these two types of continuity.*

Let us finish the Introduction with a brief description of the content of the

paper. Section 2 is devoted to recalling some basic notation and information. In section 3 we recall the fundamental facts about Navier-Stokes Equations. This section is based on a similar presentation in [16], however, in the present paper, we make some modifications. In section 4 we formulate and prove the convergence result for a sequence of martingale solutions of the Stochastic NSEs, see for instance Theorems 4.9 and 4.11. The results of section 4 hold both in 2 and 3-dimensional possibly unbounded domains. Let us stress this again, these two results are for sequence of martingale solutions of the Stochastic NSEs. In the case when these are replaced by strong solutions of the corresponding Galerkin approximations, the corresponding results have been proved in [16], see also Theorem 4.8 in the present paper. In section 5 we recall the main results from [16] in the special case of 2-dimensional domains. Besides, we prove Theorem 5.9, needed in the main section, and being the counterpart of Theorem 4.11 for the 2-dimensional case. Theorems 4.9, 4.11 and 5.9 generalise [12, Lemmata 7.1 and 7.2]. In section 6 we state and proof the main result of this paper, i.e. the existence of invariant measures for Stochastic Navier-Stokes equations in 2-dimensional Poincaré, possibly unbounded, domains with multiplicative noise.

Acknowledgements. The authours would like to thank an anonymous referee for careful reading of the manuscript and useful remarks.

The second named authour would like to thank the Department of Mathematics of the University of York, where part of this research started, for the hospitality.

2. Preliminaries. The following introductory section is for the reader convenience and hence relies heavily on paper [16] by the firts two named authours.

Let $\mathcal{O} \subset \mathbb{R}^d$, where $d = 2, 3$, be an open connected subset with smooth boundary $\partial\mathcal{O}$. For $p \in [1, \infty)$ by $L^p(\mathcal{O}, \mathbb{R}^d)$ we denote the Banach space of (equivalence classes) of Lebesgue measurable $\mathbb{R}^d$-valued p-th power integrable functions on the set $\mathcal{O}$. The norm in $L^p(\mathcal{O}, \mathbb{R}^d)$ is given by

$$|u|_{L^p} := \left(\int_{\mathcal{O}} |u(x)|^p \, dx \right)^{\frac{1}{p}}, \qquad u \in L^p(\mathcal{O}, \mathbb{R}^d).$$

By $L^\infty(\mathcal{O}, \mathbb{R}^d)$ we denote the Banach space of Lebesgue measurable essentially bounded $\mathbb{R}^d$-valued functions defined on $\mathcal{O}$ with the norm defined by

$$|u|_{L^\infty} := \operatorname{esssup} \{|u(x)|, \ x \in \mathcal{O}\}, \qquad u \in L^\infty(\mathcal{O}, \mathbb{R}^d).$$

If $p = 2$, then $L^2(\mathcal{O}, \mathbb{R}^d)$ is a Hilbert space with the inner product given by

$$\big(u, \mathrm{v}\big)_{L^2} := \int_{\mathcal{O}} u(x) \cdot \mathrm{v}(x) \, dx, \qquad u, \mathrm{v} \in L^2(\mathcal{O}, \mathbb{R}^d).$$

By $H^1(\mathcal{O}, \mathbb{R}^d) = H^{1,2}(\mathcal{O}, \mathbb{R}^d)$ we will denote the Sobolev space consisting of all $u \in L^2(\mathcal{O}, \mathbb{R}^d)$ for which there exist weak derivatives $D_i u \in L^2(\mathcal{O}, \mathbb{R}^d)$, $i = 1, \cdots, d$. It is a Hilbert space with the inner product given by

$$\big(u, \mathrm{v}\big)_{H^1} := \big(u, \mathrm{v}\big)_{L^2} + \big(\nabla u, \nabla \mathrm{v}\big)_{L^2}, \qquad u, \mathrm{v} \in H^1(\mathcal{O}, \mathbb{R}^d),$$

where $\big(\nabla u, \nabla \mathrm{v}\big)_{L^2} := \sum_{i=1}^d \int_{\mathcal{O}} D_i u(x) \cdot D_i \mathrm{v}(x) \, dx$. Let $\mathcal{C}_c^\infty(\mathcal{O}, \mathbb{R}^d)$ denote the space of all $\mathbb{R}^d$-valued functions of class $\mathcal{C}^\infty$ with compact supports contained

in $\mathcal{O}$. We will use the following classical spaces

$$\mathcal{V} := \{u \in \mathcal{C}_c^\infty(\mathcal{O}, \mathbb{R}^d) : \ \operatorname{div} u = 0\},$$
$$\mathrm{H} := \text{the closure of } \mathcal{V} \text{ in } L^2(\mathcal{O}, \mathbb{R}^d),$$
$$\mathrm{V} := \text{the closure of } \mathcal{V} \text{ in } H^1(\mathcal{O}, \mathbb{R}^d).$$

In the space H we consider the inner product and the norm inherited from $L^2(\mathcal{O}, \mathbb{R}^d)$ and denote them by $\big(\cdot, \cdot\big)_{\mathrm{H}}$ and $|\cdot|_{\mathrm{H}}$, respectively, i.e.

$$\big(u, \mathrm{v}\big)_{\mathrm{H}} := \big(u, \mathrm{v}\big)_{L^2}, \qquad |u|_{\mathrm{H}} := |u|_{L^2(\mathcal{O})}, \qquad u, \mathrm{v} \in \mathrm{H}.$$

In the space V we consider the inner product inherited from $H^1(\mathcal{O}, \mathbb{R}^d)$, i.e.

$$\big(u, \mathrm{v}\big)_{\mathrm{V}} := \big(u, \mathrm{v}\big)_{L^2} + \big(\!\big(u, \mathrm{v}\big)\!\big), \tag{2.1}$$

where

$$\big(\!\big(u, \mathrm{v}\big)\!\big) := \big(\nabla u, \nabla \mathrm{v}\big)_{L^2}, \qquad u, \mathrm{v} \in \mathrm{V}. \tag{2.2}$$

Note that the norm in V satisfies

$$|u|_{\mathrm{V}}^2 := |u|^2 + |\nabla u|_{L^2}^2, \qquad \mathrm{v} \in \mathrm{V}. \tag{2.3}$$

We will often use the notation $\|\cdot\|$ for the seminorm

$$\|u\|^2 := \big(\!\big(u, u\big)\!\big) = \big(\nabla u, \nabla u\big)_{L^2}, \qquad u \in \mathrm{V}.$$

A domain $\mathcal{O}$ satisfying the Poincaré inequality, i.e. there exists a constant $C > 0$ such that

$$C \int_{\mathcal{O}} \varphi^2 \, d\xi \le \int_{\mathcal{O}} |\nabla \varphi|^2 \, d\xi \quad \text{for all } \varphi \in H_0^1(\mathcal{O}) \tag{2.4}$$

will be called a Poincaré domain. It is well known that, in the case when $\mathcal{O}$ is a Poincaré domain, the inner product in the space V inherited from $H^1(\mathcal{O},\mathbb{R}^d)$, i.e. $\big(u,\mathrm{v}\big)_{\mathrm{V}} := \big(u,\mathrm{v}\big)_{L^2} + \big(\!\big(u,\mathrm{v}\big)\!\big)$ is equivalent to the following one:

$$\big(u,\mathrm{v}\big)_{\mathcal{P}} := \big(\!\big(u,\mathrm{v}\big)\!\big), \qquad u,\mathrm{v}\in \mathrm{V}. \tag{2.5}$$

In the sequel, if $\mathcal{O}$ is a Poincaré domain, then in the space V *we consider the inner product* $\big(\!\big(\cdot,\cdot\big)\!\big)$ *given by* (2.2) *and the corresponding norm* $\|\cdot\|$.

Denoting by $\langle\cdot,\cdot\rangle$ the dual pairing between V and V′, i.e. $\langle\cdot,\cdot\rangle := {}_{\mathrm{V}'}\langle\cdot,\cdot\rangle_{\mathrm{V}}$, by the Lax-Milgram Theorem, there exists a unique bounded linear operator $\mathcal{A}:\mathrm{V}\to\mathrm{V}'$ such that we have the following equality

$$\langle \mathcal{A}u,\mathrm{v}\rangle = \big(\!\big(u,\mathrm{v}\big)\!\big), \qquad u,\mathrm{v}\in \mathrm{V}. \tag{2.6}$$

The operator $\mathcal{A}$ is closely related to the Stokes operator A defined by

$$\begin{aligned} D(\mathrm{A}) &= \{u\in \mathrm{V}:\mathcal{A}u\in \mathrm{H}\},\\ \mathrm{A}u &= \mathcal{A}u, \text{ if } u\in D(\mathrm{A}). \end{aligned} \tag{2.7}$$

The Stokes operator A is a non-negative self-adjoint operator in H. Moreover, if $\mathcal{O}$ is a 2D or 3D Poincaré domain, see (4.11) below, then A is strictly positive. We will not use the Stokes operator as in this paper we will be concerned only with the weak solutions to the stochastic Navier-Stokes equations, which in particular do not take values in the domain $D(\mathrm{A})$ of A.

Let us consider the following tri-linear form

$$b(u,w,\mathrm{v}) = \int_{\mathcal{O}} \big(u\cdot\nabla w\big)\mathrm{v}\,dx. \tag{2.8}$$

We will recall fundamental properties of the form b. By the Sobolev embedding Theorem (or Gagliardo-Nirenberg Inequality) we have, see for instance [54, Lemmata III.3.3 and III.3.5],

$$|u|_{L^4(\mathcal{O})} \leq 2^{1/4}|u|_{L^2(\mathcal{O})}^{1-\frac{d}{4}}|\nabla u|_{L^2(\mathcal{O})}^{\frac{d}{4}}, \quad u \in H_0^{1,2}(\mathcal{O}), \qquad \text{for } d = 2, 3. \tag{2.9}$$

by applying the Hölder inequality, we obtain the following estimates

$$|b(u, w, \mathrm{v})| = |b(u, \mathrm{v}, w)| \leq |u|_{L^4}|w|_{L^4}|\nabla \mathrm{v}|_{L^2} \tag{2.10}$$

$$\leq c|u|_{\mathrm{V}}\|w\|_{\mathrm{V}}\|\mathrm{v}\|_{\mathrm{V}}, \qquad u, w, \mathrm{v} \in \mathrm{V} \tag{2.11}$$

for some positive constant c. Thus the form b is continuous on V, see also [54]. Moreover, if we define a bilinear map B by $B(u, w) := b(u, w, \cdot)$, then by inequality (2.11) we infer that $B(u, w) \in \mathrm{V}'$ for all $u, w \in \mathrm{V}$ and, by the Gagliardo-Nirenberg Inequality (2.9)) that the following inequality holds, for $d = 2, 3$,

$$\begin{aligned} |B(u, w)|_{\mathrm{V}'} \leq c_1|u|_{L^4}|w|_{L^4} &\leq c_2|u|_{L^2}^{1-\frac{d}{4}}|\nabla u|_{L^2}^{\frac{d}{4}}|w|_{L^2}^{1-\frac{d}{4}}|\nabla w|_{L^2}^{\frac{d}{4}}, \\ &\leq c_3\|u\|_{\mathrm{V}}\|w\|_{\mathrm{V}}, \qquad u, w \in \mathrm{V}. \end{aligned}$$

In particular, the mapping $B : \mathrm{V} \times \mathrm{V} \to \mathrm{V}'$ is bilinear and continuous.

Let us also recall the following properties of the form b, see Temam [54], Lemma II.1.3,

$$b(u, w, \mathrm{v}) = -b(u, \mathrm{v}, w), \qquad u, \mathrm{w}, \mathrm{v} \in \mathrm{V}. \tag{2.12}$$

In particular,

$$\langle B(u, \mathrm{v}), \mathrm{v})\rangle = b(u, \mathrm{v}, \mathrm{v}) = 0 \qquad u, \mathrm{v} \in \mathrm{V}. \tag{2.13}$$

We will need the following Fréchet topologies.

DEFINITION 2.1. By $L^2_{\text{loc}}(\mathcal{O},\mathbb{R}^d) = \mathbb{L}^2_{\text{loc}}$ we denote the space of all Lebesgue measurable $\mathbb{R}^d$-valued functions v such that $\int_K |\mathrm{v}(x)|^2\,dx < \infty$ for every compact subset $K \subset \mathcal{O}$. In this space we consider the Fréchet topology generated by the family of seminorms

$$p_R := \Big(\int_{\mathcal{O}_R} |\mathrm{v}(x)|^2\,dx\Big)^{\frac{1}{2}}, \qquad R \in \mathbb{N},$$

where $(\mathcal{O}_R)_{R\in\mathbb{N}}$ is an increasing sequence of open bounded subsets of $\mathcal{O}$ with smooth boundaries and such that $\bigcup_{R\in\mathbb{N}} \mathcal{O}_R = \mathcal{O}$. [2]

By H_{loc} we denote the space H endowed with the Fréchet topology inherited from the space $L^2_{\text{loc}}(\mathcal{O},\mathbb{R}^d)$.

Let us, for any $s > 0$ define the following standard scale of Hilbert spaces

$$\mathrm{V}_s := \text{the closure of } \mathcal{V} \text{ in } H^s(\mathcal{O},\mathbb{R}^d).$$

If $s > \frac{d}{2} + 1$ then by the Sobolev embedding Theorem,

$$H^{s-1}(\mathcal{O},\mathbb{R}^d) \hookrightarrow \mathcal{C}_b(\mathcal{O},\mathbb{R}^d) \hookrightarrow L^\infty(\mathcal{O},\mathbb{R}^d).$$

Here $\mathcal{C}_b(\mathcal{O},\mathbb{R}^d)$ denotes the space of continuous and bounded $\mathbb{R}^d$-valued functions defined on $\mathcal{O}$. If $u, w \in \mathrm{V}$ and $\mathrm{v} \in \mathrm{V}_s$ with $s > \frac{d}{2} + 1$, then for some constant $c > 0$,

$$|b(u,w,\mathrm{v})| \quad = \quad |b(u,\mathrm{v},w)| \le |u|_{L^2}|w|_{L^2}|\nabla \mathrm{v}|_{L^\infty} \le c|u|_{L^2}|w|_{L^2}|\mathrm{v}|_{\mathrm{V}_s}.$$

We have the following well know result used in the proof of [16, Lemma 5.4].

[2] Such sequence $(\mathcal{O}_R)_{R\in\mathbb{N}}$ always exist since it is sufficient to consider as $\mathcal{O}_R$ a smoothed out version of the set $\mathcal{O} \cap B(0,R)$, see for instance [56] and references therein.

LEMMA 2.2. *Assume that $s > \frac{d}{2}+1$. Then there exists a constant $C>0$ such that*

$$|B(u,\mathrm{v})|_{\mathrm{V}'_s} \leq C|u|_{\mathrm{H}}|\mathrm{v}|_{\mathrm{H}}, \quad u,\mathrm{v}\in \mathrm{V}. \tag{2.14}$$

Hence, in particular, there exists a unique bilinear and bounded map $\tilde{B}$: $\mathrm{H}\times\mathrm{H}\to \mathrm{V}'_s$ such that $B(u,\mathrm{v})=\tilde{B}(u,\mathrm{v})$ for all $u,\mathrm{v}\in\mathrm{V}$.
In what follows, the map $\tilde{B}$ will be denoted by B as well.

3. Stochastic Navier-Stokes equations. We begin this section with listing all the main assumptions.

ASSUMPTION 3.1. *We assume that the following objects are given.*

(H.1) *A separable Hilbert space* K*;*

(H.2) *a map $G:\mathrm{V}\to\mathcal{T}_2(\mathrm{K},\mathrm{H})$ that*

(i) is Lipschitz continuous, i.e. there exists a constant $L>0$ such that

$$|G(u_1)-G(u_2)|_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}\leq L\|u_1-u_2\|_{\mathrm{V}}, \quad u_1,u_2\in\mathrm{V}, \tag{G1}$$

(ii) for some constants λ_0, ρ and $\eta\in(0,2]$,

$$|G(u)|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}\leq (2-\eta)\|u\|^2+\lambda_0|u|^2_{\mathrm{H}}+\rho, \qquad u\in\mathrm{V}, \tag{G2}$$

(iii) extends to a measurable map $G:\mathrm{H}\to\mathcal{T}_2(\mathrm{K},\mathrm{V}')$ such that for some $C>0$

$$\|G(u)\|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{V}')}\leq C(1+|u|^2_{\mathrm{H}}), \qquad u\in\mathrm{H}. \tag{G3}$$

(iv) and, for every $\psi\in\mathcal{V}$ the function

$$\psi^{**}G:\mathrm{H}_{\mathrm{loc}}\ni u\mapsto \Big\{\mathrm{K}\ni y\mapsto {}_{\mathrm{V}'}\langle G(u)y,\psi\rangle_{\mathrm{V}}\in\mathbb{R}\Big\}\in\mathrm{K}' \tag{G4}$$

is continuous.

(H.3) *A real number p such that*

$$p \in \big[2, 2 + \frac{\eta}{2-\eta}\big), \tag{3.1}$$

where we put $\frac{\eta}{2-\eta} = \infty$ when $\eta = 2$.;

(H.4) *a Borel probability measure μ_0 on H such that $\int_H |x|^p \mu_0(dx) < \infty$ is given.*

(H.5) *an linear operator $\mathcal{A} : \mathrm{V} \to \mathrm{V}'$ satisfying equality (2.6).*

Now we state definition of a martingale solution of equation (3.2). We really need to consider the infinite time interval, i.e. $[0, \infty)$, however, we need also to state some of the results on the interval $[0, T]$, where $T > 0$ is fixed. Thus, in the following definition we distinguish between the two cases of solution on a finite interval $[0, T]$ and on $[0, \infty)$.

Definition 3.2. Let us assume Assumption 3.1. Let $T > 0$ be fixed. We say that there exists a **martingale solution** of the following stochastic Navier-Stokes Equations (in an abstract form) on the interval $[0, T]$

$$\begin{cases} du(t) + & \mathcal{A}u(t)\,dt + B\big(u(t), u(t)\big)\,dt \\ & = f(t)\,dt + G\big(u(t)\big)\,dW(t), \quad t \geq 0, \\ & \mathcal{L}(u(0)) = \mu_0, \end{cases} \tag{3.2}$$

iff there exist

- a stochastic basis $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ with a complete filtration $\hat{\mathbb{F}} = \{\hat{\mathcal{F}}_t\}_{t\in[0,T]}$,
- a K-cylindrical Wiener process $\hat{W} = \big(\hat{W}\big)_{t\in[0,T]}$

- and an $\hat{\mathbb{F}}$-progressively measurable process $u : [0,T] \times \hat{\Omega} \to \mathrm{H}$ with $\hat{\mathbb{P}}$-a.e. paths satisfying

$$u(\cdot,\omega) \in \mathcal{C}\big([0,T],\mathrm{H}_w\big) \cap L^2(0,T;\mathrm{V}) \tag{3.3}$$

such that

$$\text{the law on H of } u(0) \text{ is equal to } \mu_0$$

and, for all $t \in [0,T]$ and all $v \in \mathcal{V}$,

(3.4)

$$\begin{aligned} &\big(u(t),v\big)_{\mathrm{H}} + \int_0^t \langle \mathcal{A}u(s),v\rangle\,ds + \int_0^t \langle B(u(s)),v\rangle\,ds \\ &\quad = \big(u(0),v\big)_{\mathrm{H}} + \int_0^t \langle f(s),v\rangle\,ds + \Big\langle \int_0^t G(u(s))\,d\hat{W}(s), v\Big\rangle, \quad \hat{\mathbb{P}}\text{-a.s.} \end{aligned}$$

and

$$\hat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |u(t)|_{\mathrm{H}}^2 + \int_0^T |\nabla u(t)|^2\,dt\Big] < \infty. \tag{3.5}$$

If all the above conditions are satisfied, then the system

$$\big(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{F}},\hat{\mathbb{P}},\hat{W},u\big)$$

will be called a martingale solution to problem (3.2) on the interval $[0,T]$ with the initial distribution μ_0.

A system $\big(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{F}},\hat{\mathbb{P}},\hat{W},u\big)$ will be called a **martingale solution** to problem (3.2) with the initial distribution μ_0 iff all the above conditions are defined with the interval $[0,T]$ being replaced by $[0,\infty)$ and the condition (3.3) is replaced by

$$u(\cdot,\omega) \in \mathcal{C}\big([0,\infty),\mathrm{H}_w\big) \cap L^2_{\mathrm{loc}}([0,\infty);\mathrm{V}), \tag{3.6}$$

and inequality (3.5) holds for every $T > 0$.

Here, H_w denotes the Hilbert space H endowed with the weak topology and $\mathcal{C}([0,T],\mathrm{H}_w)$ and $\mathcal{C}([0,\infty),\mathrm{H}_w)$ denote the spaces of H valued weakly continuous functions defined on $[0,T]$ and $[0,\infty)$, respectively.

In the case when μ_0 is equal to the law on H of a given random variable $u_0 : \Omega \to \mathrm{H}$ then, somehow incorrectly, a martingale solution to problem (3.2) will also be called a martingale solution to problem (3.2) with the initial data u_0. Fully correctly, it should be called a martingale solution to problem (3.2) with the initial data having the same law as u_0. In particular, in this case we require that the laws on H of u_0 and $u(0)$ are equal.

If no confusion seems likely, a system $\big(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{F}},\hat{\mathbb{P}},\hat{W},u\big)$ *from Definition 3.2 will be called a* **martingale solutions.**

Remark 3.3. *Let us recall the following observation from [16]. Since* $\|u\| := |\nabla u|_{L^2}$ *and* $\langle \mathcal{A}u,u\rangle = \big((u,u)\big) := \big(\nabla u,\nabla u\big)_{L^2}$, *we have*

$$(2-\eta)\|u\|^2 = 2\langle \mathcal{A}u,u\rangle - \eta\|u\|^2, \quad u\in \mathrm{V}.$$

Hence inequality (G2) *can be written equivalently in the following form*

$$\text{(G2')} \qquad 2\langle \mathcal{A}u,u\rangle - \|G(u)\|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})} \geq \eta\|u\|^2 - \lambda_0|u|^2_{\mathrm{H}} - \rho, \qquad u\in \mathrm{V},$$

Inequality (G2') is the same as considered by Flandoli and Gątarek in [31] for Stochastic NSEs in bounded domains. The assumption $\eta = 2$ corresponds to the case when the noise term does not depend on ∇u. We will prove that the set of measures induced on appropriate space by the solutions of the Galerkin equations is tight provided that the map G from part **(H.2)** of Assumption 3.1 satisfies inequalities (G3) and (G2). Inequality (G3) and

condition (G4) from part **(H.2)** of Assumption 3.1 will be important in passing to the limit as $n \to \infty$ in the Galerkin approximation. Condition (G4) is essential in the case of unbounded domain $\mathcal{O}$. It is wort mentioning that the following example of the noise term, analyzed in details in [16, Section 6], is covered by part **(H.2)** of Assumption 3.1.

EXAMPLE 3.4. Let us consider the noise term written classically as

$$\big[G(u)\big](t,x)\,dW(t) := \sum_{i=1}^{\infty}\big[\big(b_i(x)\cdot\nabla\big)u(t,x) + c_i(x)u(t,x)\big]d\beta_i(t), \tag{3.7}$$

where

$$\begin{aligned}
&\beta_i, \quad i\in\mathbb{N}, \text{ are i.i.d. standard } \mathbb{R}\text{-valued Brownian Motions},\\
&b_i : \overline{\mathcal{O}} \to \mathbb{R}^d, \quad i\in\mathbb{N}, \text{ are functions of class } \mathcal{C}^\infty\text{class},\\
&c_i : \overline{\mathcal{O}} \to \mathbb{R}, \quad i\in\mathbb{N}, \text{ are functions of } \mathcal{C}^\infty \text{ - of class },
\end{aligned}$$

are given. Assume that

$$C_1 := \sum_{i=1}^{\infty}\big(\|b_i\|^2_{L^\infty} + \|\operatorname{div} b_i\|^2_{L^\infty} + \|c_i\|^2_{L^\infty}\big) < \infty \tag{3.8}$$

and there exists $a \in (0,2]$ such that for all $\zeta = (\zeta_1,...,\zeta_d) \in \mathbb{R}^d$ and all $x \in \mathcal{O}$,

$$\sum_{i=1}^{\infty}\sum_{j,k=1}^{d} b_i^j(x)b_i^k(x)\big)\zeta_j\zeta_k \le 2\sum_{j,k=1}^{d}\delta_{jk}\zeta_j\zeta_k - a|\zeta|^2 = (2-a)|\zeta|^2. \tag{3.9}$$

This noise term can be reformulated in the following manner. Let $\mathrm{K} := l^2(\mathbb{N})$, where $l^2(\mathbb{N})$ denotes the space of all sequences $(h_i)_{i\in\mathbb{N}} \subset \mathbb{R}$ such that $\sum_{i=1}^{\infty} h_i^2 < \infty$. It is a Hilbert space with the scalar product given by $\big(h,k\big)_{l^2} := \sum_{i=1}^{\infty} h_i k_i$, where $h = (h_i)$ and $k = (k_i)$ belong to $l^2(\mathbb{N})$. Putting

$$G(u)h = \sum_{i=1}^{\infty}\big[\big(b_i\cdot\nabla\big)u + c_i u\big]\,h_i, \qquad u\in\mathrm{V}, \quad h=(h_i)\in l^2(\mathbb{N}), \tag{3.10}$$

we infer that the mapping G fulfils all conditions stated in assumption **(H.2)**, see [16, Section 6] for details.

REMARK 3.5. Let us explain that via the isomorphism between the space V and its dual V′, condition **(H.2)** (iii) in Assumption 3.1 is understood in the usual sense, i.e. for every orthonormal basis $(e_k) \subset \mathrm{K}$

$$\sum_k |G(u)(e_k)|^2_{\mathrm{V}'} \leq C(1 + |u|^2_{\mathrm{H}}), \quad u \in \mathrm{H}.$$

In fact, conditions **(H.2)** (iii) and (iv) in Assumption 3.1 can be replaced by the following more general:

(iii') The map $G : \mathrm{V} \to \mathcal{T}_2(\mathrm{K}, \mathrm{H})$ extends to a measurable map $g : \mathrm{H} \to \mathcal{L}(\mathrm{K}, \mathrm{V}')$ such that for some $C > 0$ for every $u \in \mathrm{H}$

(G3') $$\sup_{v \in \mathrm{V}, \|v\|_{\mathrm{V}} \leq 1} \sup_{k \in \mathrm{K}, \|k\|_{\mathrm{K}} \leq 1} |{}_{\mathrm{V}'}\langle g(u)(k), v\rangle_{\mathrm{V}}|^2 \leq C(1 + |u|^2_{\mathrm{H}}).$$

(iv') and, for every $\psi \in \mathcal{V}$ the function

(G4') $$\psi^{**} g : \mathrm{H}_{\mathrm{loc}} \ni u \mapsto \left\{ \mathrm{K} \ni y \mapsto {}_{\mathrm{V}'}\langle g(u)y, \psi\rangle_{\mathrm{V}} \in \mathbb{R} \right\} \in \mathrm{K}'$$

is continuous.

REMARK 3.6. Note that by Definition 3.2 every solution to problem (3.2) satisfies equality (3.4) for all $\mathrm{v} \in \mathcal{V}$. However, equality (3.4) holds not only for $\mathrm{v} \in \mathcal{V}$ but also for all $\mathrm{v} \in \mathrm{V}$. Indeed, this follows from the density of $\mathcal{V}$ in the space V and the fact that each term in (3.4) is well defined and continuous with respect to $\mathrm{v} \in \mathrm{V}$. This remark is important while using the Itô formula in the proof of Lemma 5.8.

REMARK 3.7. *Let assumptions **(H.1)-(H.5)** be satisfied. If the system* $(\hat\Omega, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, \hat W, u)$ *is a martingale solution of problem* (3.2) *on the interval* $[0,\infty)$*, then* $\hat{\mathbb{P}}$*-a.e. paths of the process* $u(t)$*,* $t\in[0,\infty)$*, are* V'*-valued continuous functions, i.e. for* $\hat{\mathbb{P}}$*-a.e.* $\omega\in\hat\Omega$

$$u(\cdot,\omega)\in\mathcal{C}\big([0,\infty),\mathrm{V}'\big), \tag{3.11}$$

and equality (3.4) *can be rewritten as the following one, understood in the space* V'*,*

$$\begin{aligned} u(t)+\int_0^t \mathcal{A}u(s)\,ds+\int_0^t B(u(s))\,ds \\ = u(0)+\int_0^t f(s)\,ds+\int_0^t G(u(s))\,d\hat W(s),\quad t\in[0,\infty). \end{aligned} \tag{3.12}$$

PROOF. Let us fix any $T>0$. Let us notice that since the map G satisfies inequality (G3) in Assumption 3.1, by inequality (3.5) we infer that

$$\hat{\mathbb{E}}\Big[\int_0^T |G(u(s))|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{V}')}\,ds\Big]\le C\,\hat{\mathbb{E}}\Big[\int_0^T (1+|u(s)|^2_{\mathrm{H}})\,ds\Big]<\infty.$$

Thus the process μ defined by

$$\mu(t):=\int_0^t G(u(s))\,d\hat W(s),\quad t\in[0,T],$$

is a V'-valued square integrable continuous martingale.

Remark. *The process* μ *is an* H*-valued square integrable continuous martingale, as well.*

PROOF. Since the map G satisfies inequality (G2) in Assumption 3.1, using inequality (3.5) we deduce that

$$\hat{\mathbb{E}}\Big[\int_0^T |G(u(s))|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}\,ds\Big]\le\hat{\mathbb{E}}\Big[\int_0^T [(2-\eta)\|u(s)\|^2+\lambda_0|u(s)|^2_{\mathrm{H}}+\rho]\,ds\Big]<\infty.$$

Thus $\mu(t)$, $t \in [0,T]$, is an H-valued square integrable continuous martingale.

□

In the framework of Remark 3.7, by the regularity assumption (3.3), we infer that for $\hat{\mathbb{P}}$-a.e. $\omega \in \hat{\Omega}$

$$\mathcal{A}u(\cdot,\omega) \in L^2(0,T;\mathrm{V}'), \quad B(u(\cdot,\omega),u(\cdot,\omega)) \in L^{4/3}(0,T;\mathrm{V}').$$

By assumption (H.3), in particular, $f \in L^p(0,T;\mathrm{V}')$. Hence for $\hat{\mathbb{P}}$-a.e. $\omega \in \hat{\Omega}$ the functions

$$\begin{aligned}
&[0,T] \ni t \mapsto \int_0^t \mathcal{A}u(s,\omega)\,ds \in \mathrm{V}',\\
&[0,T] \ni t \mapsto \int_0^t B(u(s,\omega),(u(s,\omega))\,ds \in \mathrm{V}',\\
&[0,T] \ni t \mapsto \int_0^t f(s)\,ds \in \mathrm{V}'
\end{aligned}$$

are well defined and continuous. Using (3.4) we infer that for $\hat{\mathbb{P}}$-a.e. $\omega \in \hat{\Omega}$

$$u(\cdot,\omega) \in \mathcal{C}([0,T],\mathrm{V}')$$

and for every $t \in [0,T]$ equality (3.12) holds. Since $T > 0$ has been chosen in an arbitrary way, regularity condition (3.11) and equality (3.12) hold. The proof of the claim is thus complete. □

4. The continuous dependence of the solutions on the initial state and the external forces in 2D and 3D domains. In this section we will concentrate on martingale solutions to problem (3.2) on a fixed interval $[0,T]$. The main result is Theorem 4.11. We will also need some modification of Theorem 5.1 in [16], contained in Theorem 4.8.

As in [16] in the proofs we will use the following structure. Let us fix $s > \frac{d}{2} + 1$ and notice that the space V_s is dense in V and the natural embedding $\mathrm{V}_s \hookrightarrow \mathrm{V}$ is continuous. By [32, Lemma 2.5], see also [16, Lemma C.1], there exists a separable Hilbert space U such that U is a dense subset of V_s and

$$\text{(4.1)} \qquad \textit{the natural embedding } \iota_s : U \hookrightarrow \mathrm{V}_s \textit{ is compact .}$$

Then we also have

$$\text{(4.2)} \qquad U \hookrightarrow \mathrm{V}_s \hookrightarrow \mathrm{H} \cong \mathrm{H}' \hookrightarrow \mathrm{V}_s' \hookrightarrow U',$$

where H' and U' are the dual spaces of H and U, respectively, H' being identified with H and the dual embedding $\mathrm{H}' \hookrightarrow U'$ is compact as well.

In the next definition we will recall definition of a topological space $\mathcal{Z}_T$ that plays an important rôle in our approach, see page 1629 and Section 3 in [16].

To define the space $\mathcal{Z}_T$ we will need the following four spaces.

$$\begin{aligned}
\mathcal{C}([0,T],U') &:= \text{the space of continuous functions } u : [0,T] \to U' \\
&\qquad \text{with the topology induced by the norm} \\
&\qquad |u|_{\mathcal{C}([0,T],U')} := \sup_{t\in[0,T]} |u(t)|_{U'} \\
L^2_w(0,T;\mathrm{V}) &:= \text{the space } L^2(0,T;\mathrm{V}) \text{ with the weak topology,} \\
L^2(0,T;\mathrm{H}_{\mathrm{loc}}) &:= \text{the space of all measurable functions } u : [0,T] \to \mathrm{H}
\end{aligned}$$

such that for all $R \in \mathbb{N}$

$$p_{T,R}(u) := \left(\int_0^T \int_{\mathcal{O}_R} |u(t,x)|^2 \, dxdt \right)^{\frac{1}{2}} < \infty$$

with the topology generated by the seminorms

$(p_{T,R})_{R\in\mathbb{N}}$.

Let H_w denote the Hilbert space H endowed with the weak topology and let us put

$\mathcal{C}([0,T];\mathrm{H}_w) :=$ the space of weakly continuous functions $u : [0,T] \to \mathrm{H}$

endowed with the weakest topology such that for all

for all $h \in \mathrm{H}$ the mappings

$\mathcal{C}([0,T];\mathrm{H}_w) \ni u \mapsto \big(u(\cdot), h\big)_{\mathrm{H}} \in \mathcal{C}([0,T];\mathbb{R})$ are continuous.

Definition 4.1. For $T > 0$ let us put

$$\mathcal{Z}_T := \mathcal{C}([0,T];U') \cap L^2_w(0,T;\mathrm{V}) \cap L^2(0,T;\mathrm{H}_{\mathrm{loc}}) \cap \mathcal{C}([0,T];\mathrm{H}_w) \tag{4.3}$$

and let $\mathcal{T}_T$ be the supremum of the corresponding four topologies, i.e. the smallest topology on $\mathcal{Z}_T$ such that the four natural embeddings from $\mathcal{Z}_T$ are continuous. The space $\mathcal{Z}_T$ will also considered with the Borel σ-algebra, i.e. the smallest σ-algebra containing the family $\mathcal{T}_T$.

The following auxiliary result which is needed in the proof of Theorem 4.11, cannot be deduced directly from the Kuratowski Theorem, see Counterexample C.4 in the C.

Lemma 4.2. *Assume that $T > 0$. Then the following fours sets $\mathcal{C}([0,T];\mathrm{H}) \cap \mathcal{Z}_T$, $\mathcal{C}([0,T];\mathrm{V}) \cap \mathcal{Z}_T$, $L^2(0,T;V) \cap \mathcal{Z}_T$ and $\mathcal{C}([0,T];\mathrm{V}') \cap \mathcal{Z}_T$ are Borel subsets of $\mathcal{Z}_T$ and the corresponding embedding tranforms Borel*

sets into Borel subsets of $\mathcal{Z}_T$. Moreover, the following $\mathbb{R}_+ \cup \{+\infty\}$-valued functions

$$\mathcal{Z}_T \ni u \mapsto \begin{cases} \sup_{s\in[0,T]} |u(s)|^2_{\mathrm{H}}, & \text{if } u \in \mathcal{C}([0,T];\mathrm{H}) \cap \mathcal{Z}_T \\ \infty, & \text{otherwise,} \end{cases}$$

$$\mathcal{Z}_T \ni u \mapsto \begin{cases} \int_0^T \|u(s)\|^2 \, ds, & \text{if } u \in L^2(0,T;\mathrm{V}) \cap \mathcal{Z}_T, \\ \infty & \text{otherwise,} \end{cases}$$

are Borel.

Proof. Because $\mathcal{C}([0,T];U') \cap L^2(0,T;\mathrm{H}_{\mathrm{loc}})$ is a Polish space, by the Kuratowski Theorem $\mathcal{C}([0,T];\mathrm{H})$ is Borel subset of $\mathcal{C}([0,T];U') \cap L^2(0,T;\mathrm{H}_{loc})$. Hence the intersection $\mathcal{C}([0,T];\mathrm{H}) \cap \mathcal{Z}_T$ is a Borel subset of the intersection $\mathcal{C}([0,T];U') \cap L^2(0,T;\mathrm{H}_{loc}) \cap \mathcal{Z}_T$ which happens to be equal to $\mathcal{Z}_T$.

We can argue in the same way in the case of the spaces $\mathcal{C}([0,T];\mathrm{V}) \cap \mathcal{Z}_T$ and $\mathcal{C}([0,T];\mathrm{V}') \cap \mathcal{Z}_T$.

The proof in case the space $L^2(0,T;\mathrm{V})$ is analogous, one needs to begin with an observation that by the Kuratowski Theorem the set $L^2(0,T;\mathrm{V})$ is Borel subset of $L^2(0,T;\mathrm{H}_{loc})$. We have used a fact that a product of Borel set in $\mathcal{C}([0,T];U') \cap L^2(0,T;\mathrm{H}_{loc})$ and the set $\mathcal{Z}_T$ is a Borel subset of the latter.

The same argument applies to the proof that i_T and j_T map Borel subsets of their corresponding domains to Borel sets in $\mathcal{Z}_T$. The last part of Lemma is a consequence Proposition C.2. □

4.1. *Tightness criterion and Jakubowski's version of the Skorokhod theorem.* One of the main tools in this section is the tightness criterion in the space $\mathcal{Z}_T$ defined in identity (4.3). We will use a slight generalization of of

the criterion stated in Corollary 3.9 from [16], compare with the proof of Lemma 5.4 therein. Namely, we will consider the sequence of stochastic processes defined on their own probability spaces. Let $(\Omega_n, \mathcal{F}_n, \mathbb{F}_n, \mathbb{P}_n)$, $n \in \mathbb{N}$, be a sequence of probability spaces with the filtration $\mathbb{F}_n = (\mathcal{F}_{n,t})_{t\geq 0}$.

COROLLARY 4.3. **(tightness criterion)** *Assume that $(X_n)_{n\in\mathbb{N}}$ is a sequence of continuous $\mathbb{F}_n$-adapted U'-valued processes defined on Ω_n and such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}_n \Big[\sup_{s\in[0,T]} |X_n(s)|_{\mathrm{H}}^2 \Big] < \infty, \tag{4.4}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}_n \left[\int_0^T \|X_n(s)\|^2 \, ds \right] < \infty, \tag{4.5}$$

(a) *and for every $\varepsilon > 0$ and for every $\eta > 0$ there exists $\delta > 0$ such that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $[0,T]$-valued $\mathbb{F}_n$-stopping times one has*

$$\sup_{n\in\mathbb{N}} \sup_{0\leq\theta\leq\delta} \mathbb{P}_n \big\{ |X_n(\tau_n+\theta) - X_n(\tau_n)|_{U'} \geq \eta \big\} \leq \varepsilon. \tag{4.6}$$

Let $\tilde{\mathbb{P}}_n$ be the law of X_n on the Borel σ-field $\mathcal{B}(\mathcal{Z}_T)$. Then for every $\varepsilon > 0$ there exists a compact subset K_ε of $\mathcal{Z}_T$ such that

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{P}}_n(K_\varepsilon) \geq 1 - \varepsilon.$$

The proof of Corollary 4.3 is essentially the same as the proof of [16, Corollary 3.9].

If the sequence $(X_n)_{n\in\mathbb{N}}$ satisfies condition **(a)** then we say that it satisfies the Aldous condition **[A]** in U' on [0,T]. If it satisfies condition **(a)** for each $T > 0$, we say that it satisfies the Aldous condition **[A]** in U'.

Obviously, the class of U'-valued processes satisfying the Aldous condition is a real vector space. Below we will formulate a sufficient condition for the Aldous condition. This idea has been used in the proof of Lemma 5.4 in [16] but it has not been formulated in such a way.

LEMMA 4.4. *Assume that Y is a separable Banach space, $\sigma \in (0,1]$ and that $(u_n)_{n\in\mathbb{N}}$ is a sequence of continuous $\mathbb{F}_n$-adapted Y-valued processes indexed by $[0,T]$ for some $T>0$, such that*

(a') *there exists $C>0$ such that for every $\theta>0$ and for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $[0,T]$-valued $\mathbb{F}_n$-stopping times with one has*

$$\mathbb{E}_n\left[|u_n(\tau_n+\theta)-u_n(\tau_n)|_Y\right] \le C\theta^\sigma. \tag{4.7}$$

*Then the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition **[A]** on $[0,T]$.*

PROOF. Let us fix $\eta>0$ and $\varepsilon>0$. By the Chebyshev inequality and the estimate (4.7) we obtain

$$\mathbb{P}_n\big(\big\{|u_n(\tau_n+\theta)-u_n(\tau_n)|_Y \ge \eta\big\}\big) \le \frac{1}{\eta}\mathbb{E}_n\left[|u_n(\tau_n+\theta)-u_n(\tau_n)|_Y\right]$$
$$\le \frac{C\cdot\theta^\sigma}{\eta}, \qquad n\in\mathbb{N}.$$

Let us $\delta := \left[\frac{\eta\cdot\varepsilon}{C}\right]^{\frac{1}{\sigma}}$. Then we have

$$\sup_{n\in\mathbb{N}}\sup_{1\le\theta\le\delta}\mathbb{P}_n\big\{|u_n(\tau_n+\theta)-u_n(\tau_n)|_Y\ge\eta\big\}\le\varepsilon,$$

This completes the proof of Lemma 4.4. □

REMARK 4.5. As can be seen in (4.3), the space $\mathcal{Z}_T$ is defined as an intersection of four spaces, one of them being the space $\mathcal{C}([0,T];U')$. The latter space plays, in fact, only an auxiliary rôle. Let us recall that the space

U, see (4.1) and [16, Section 2.3], is important in the construction of the solutions to stochastic Navier-Stokes equations via the Galerkin method in the case of an unbounded domain, i.e. when the embedding $\mathrm{V} \subset \mathrm{H}$ is not compact. (In the case of a bounded domain we can take, e.g. $U := \mathrm{V}_s$ for sufficiently large s.) In particular, the orthonormal basis of the space H, which we use in the Galerkin method is contained in U, so the Galerkin solutions "live in" the space U.

With the space U in hand, in [16] we prove an appropriate compactness and tightness criteria in the space $\mathcal{Z}_T$, see [16, Lemma 3.3 and Corollary 3.9]. Let us emphasize that in order to prove the relative compactness of an appropriate set in the Fréchet space $L^2(0,T;\mathrm{H}_{loc})$ first we need to prove a certain generalization of the classical Dubinsky Theorem, see [16, Lemma 3.1], where the space $\mathcal{C}([0,T];U')$ is used. This result is related to the Aldous condition in the space U' in the tightness criterion, (4.6) in Corollary 4.3 and [16, Corollary 3.9(c)].

We will use Corollary 4.3 to prove Theorems 4.9 and 4.11, below. Even though, the presence of the space $\mathcal{C}([0,T];U')$ in the definition of the space $\mathcal{Z}_T$ is natural in the context of the Galerkin approximation solutions, it's presence in the context of Theorems 4.9 and 4.11 where we consider sequences of the solutions of the Navier-Stokes equations seems to be unnecessary. However, again because of the lack of the compactness of the embedding $\mathrm{V} \subset \mathrm{H}$ to prove tightness in Theorem 4.9 we still use Corollary 4.3 in its original form.

In the proofs of the theorems on the existence of a martingale solution and on the continuous dependence of the data we use a version of the Skorokhod

theorem for nonmetric spaces. For convenience of the reader let us recall the following Jakubowski's [34] version of the Skorokhod Theorem, see also Brzeźniak and Ondreját [18].

THEOREM 4.6. (Theorem 2 in [34]). *Let $(\mathcal{X},\tau)$ be a topological space such that there exists a sequence (f_m) of continuous functions $f_m : \mathcal{X} \to \mathbb{R}$ that separates points of $\mathcal{X}$. Let (X_n) be a sequence of $\mathcal{X}$-valued Borel random variables. Suppose that for every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset \mathcal{X}$ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{P}(\{X_n \in K_\varepsilon\}) > 1-\varepsilon.$$

Then there exists a subsequence $(X_{n_k})_{k\in\mathbb{N}}$, a sequence $(Y_k)_{k\in\mathbb{N}}$ of $\mathcal{X}$-valued Borel random variables and an $\mathcal{X}$-valued Borel random variable Y defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that

$$\mathcal{L}(X_{n_k}) = \mathcal{L}(Y_k), \qquad k = 1,2,...$$

and for all $\omega \in \Omega$:

$$Y_k(\omega) \xrightarrow{\tau} Y(\omega) \qquad \text{as } k \to \infty.$$

Note that the sequence (f_m) defines another, weaker topology on $\mathcal{X}$. However, this topology restricted to σ-compact subsets of $\mathcal{X}$ is equivalent to the original topology τ. Let us emphasize that thanks to the assumption on the tightness of the set of laws $\{\mathcal{L}(X_n), n \in \mathbb{N}\}$ on the space $\mathcal{X}$ the maps Y and Y_k, $k \in \mathbb{N}$, in Theorem 4.6 are measurable with respect to the Borel σ-field in the space $\mathcal{X}$.

The following result has been proved in the proof of [16, Corollary 3.12] for the spaces $\mathcal{Z}_T$.

LEMMA 4.7. *The topological space $\mathcal{Z}_T$ satisfies the assumptions of Theorem 4.6.*

4.2. *The existence and properties of martingale solutions on $[0,T]$.* We will concentrate on martingale solutions to problem (3.2) on a fixed interval $[0,T]$. The following result is a slight generalisation of Theorem 5.1 in [16]. In comparison to [16] the deterministic initial state has been replaced by the random one satisfying assumption (H.3). However, our attention will be focused on the estimates satisfied by the solutions of the Navier-Stokes equations. We claim that there exists a solution u satisfying estimate $\hat{\mathbb{E}}\big[\sup_{t\in[0,T]}|u(t)|_{\mathrm{H}}^q\big]\leq C_1(p,q)$ for every $q\in[2,p]$, (and not only for $q=2$ as stated in inequality (5.1) in [16]). Moreover, we analyse what is the relation between the constant $C_1(p,q)$ and the initial state u_0 and the external forces f. The same concerns the estimate on $\hat{\mathbb{E}}[\int_0^T\|u(t)\|^2\,dt]$. These results generalise [16, Theorem 5.1]. In the second part of Theorem 4.8 we will prove another estimate on u in the case when $\mathcal{O}$ is a 2D or 3D Poincaré domain, see (4.11) below. This estimate will be of crucial importance in the proof of existence of an invariant measure in 2D case. The proof of Theorem 4.8 is based on the Galerkin method. The analysis of the Galerkin equations is postponed to A. Recall also that in assumption (H.3) we have put $\frac{\eta}{2-\eta}=\infty$ when $\eta=2$.

THEOREM 4.8. *Let assumptions **(H.1)-(H.5)** be satisfied. In particular, we assume that p satisfies (3.1), i.e.*

$$p\in\big[2,2+\frac{\eta}{2-\eta}\big),$$

*where $\eta\in(0,2]$ is given in assumption **(H.2)**.*

(1) For every $T > 0$ and $R_1, R_2 > 0$ if μ_0 is a Borel probability measure on H, *$f \in L^p([0,\infty);\mathrm{V}')$ satisfy $\int_{\mathrm{H}} |x|^p \mu_0(dx) \leq R_1$ and $|f|_{L^p(0,T;\mathrm{V}')} \leq R_2$, then there exists a* ***martingale solution*** *$\big(\hat\Omega, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, \hat W, u\big)$ to problem (3.2) with the initial law μ_0 which satisfies the following estimates: for every $q \in [1,p]$ there exist constants $C_1(p,q)$ and $C_2(p)$, depending also on T, R_1 and R_2, such that*

$$\hat{\mathbb{E}}\big(\sup_{s\in[0,T]} |u(s)|_{\mathrm{H}}^q\big) \leq C_1(p,q), \tag{4.8}$$

putting $C_1(p) := C_1(p,p)$, in particular,

$$\hat{\mathbb{E}}\big(\sup_{s\in[0,T]} |u(s)|_{\mathrm{H}}^p\big) \leq C_1(p), \tag{4.9}$$

and

$$\hat{\mathbb{E}}\big[\int_0^T |\nabla u(s)|_{L^2}^2\, ds\big] \leq C_2(p). \tag{4.10}$$

(2) Moreover, if $\mathcal{O}$ is a Poincaré domain and the map G satisfies inequality (G2) in Assumption 3.1 with $\lambda_0 = 0$, then there exists a martingale solution $\big(\hat\Omega, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, u\big)$ of problem (3.2) satisfying additionally the following inequality for every $T > 0$

$$\frac{\eta}{2}\hat{\mathbb{E}}\bigg[\int_0^T |\nabla u(s)|_{L^2}^2\, ds\bigg] \leq \hat{\mathbb{E}}[\,|u(0)|_{\mathrm{H}}^2\,] + \frac{2}{\eta}\int_0^T |f(s)|_{\mathrm{V}'}^2\, ds + \rho T. \tag{4.11}$$

Proof of Theorem 4.8 is postponed to B.

4.3. *The continuous dependence.* We prove the following results related to the continuous dependence on the deterministic initial condition and deterministic external forces. Roughly speaking, we will show that if $(u_{0,n}) \subset \mathrm{H}$ and $(f_n) \subset L^p(0,T;\mathrm{V}')$ are sequences of initial conditions and external forces

approaching $u_0 \in \mathrm{H}$ and $f \in L^p(0,T,\mathrm{V}')$, respectively, then a sequence (u_n) of martingale solutions of the Navier-Stokes equations with the data $(u_{0,n}, f_n)$, satisfying inequalities (4.8)-(4.10), contains a subsequence of solutions, on a changed probability basis, convergent to a martingale solution with the initial condition u_0 and the external force f. Note that existence of such solutions u_n, $n \in \mathbb{N}$, is guaranteed by Theorem 4.8. This result holds both in 2D and 3D possibly unbounded domains with smooth boundaries. Moreover, in the case of $2D$ domains, because of the existence and uniqueness of the strong solutions, stronger result holds. Namely, the solutions u_n, $n \in \mathbb{N}$, satisfy inequalities (4.8)-(4.10) and not only a subsequence but the whole sequence of solutions (u_n) is convergent to the solution of the Navier-Stokes equation with the data u_0 and f. Their proofs are de facto, modifications of the proofs of corresponding parts of Theorem 5.1 from [16], where Galerkin approximations are substituted by solutions u_n, $n \in \mathbb{N}$. However, the last part of the proof is different. Namely, contrary to the case of the Galerkin aproximations, the martingale $\tilde{M}_n$ defined by (5.16) in [16] is, in general, not square integrable. It would be square integrable, for example, if inequality (4.8) held with some $q > 4$. This holds in the case, when the noise term does not depend on ∇u or if we impose such restriction on η that $\frac{\eta}{2-\eta} > 4$. However, to cover the general case, this part of the proof is different.

In what follows we do not assume that $\mathcal{O}$ is a Poincaré domain.

THEOREM 4.9. *Let assumptions* ***(H.1)-(H.3)*** *and* ***(H.5)*** *be satisfied and let $T > 0$. Assume that $\left(u_{0,n}\right)_{n=1}^{\infty}$ is a bounded* H*-valued sequence and $(f_n)_{n=1}^{\infty}$ is a bounded $L^p(0,T;\mathrm{V}')$-valued sequence. Let $R_1 > 0$ and $R_2 > 0$*

be such that $\sup_{n\in\mathbb{N}} |u_{0,n}|_{\mathrm{H}} \leq R_1$ *and* $\sup_{n\in\mathbb{N}} \|f_n\|_{L^p(0,T;\mathrm{V}')} \leq R_2$. *Let*

$$\big(\hat{\Omega}_n, \hat{\mathcal{F}}_n, \hat{\mathbb{F}}_n, \hat{\mathbb{P}}_n, \hat{W}_n, u_n\big)$$

be a martingale solution of problem (3.2) *with the initial data* $u_{0,n}$ *and the external force* f_n *and satisfying inequalities* (4.8)-(4.10). *Then, the set of Borel measures* $\big\{\mathcal{L}(u_n), n \in \mathbb{N}\big\}$ *is tight on the space* $(\mathcal{Z}_T, \mathcal{T}_T)$.

Proof. Let us fix $T > 0$ and p satisfying condition (3.1). Let $\big(u_{0,n}\big)_{n=1}$ and $\big(f_n\big)_{n=1}$ be bounded H-valued, resp. $L^p(0,T;\mathrm{V}')$-valued, sequences. Let

$$\big(\hat{\Omega}_n, \hat{\mathcal{F}}_n, \hat{\mathbb{F}}_n, \hat{\mathbb{P}}_n, \hat{W}_n, u_n\big)$$

be a corresponding martingale solution of problem (3.2) with the initial data u_0^n and the external force f_n, and satisfying inequalities (4.8)-(4.10). Such a solution exists by Theorem 4.8.

To show that the set of measures $\big\{\mathcal{L}(u_n), n \in \mathbb{N}\big\}$ are tight on the space $(\mathcal{Z}_T, \mathcal{T}_T)$, where $\mathcal{Z}_T$ is defined in (4.3), we argue as in the proof of Lemma 5.4 in [16] and apply Corollary 4.3. We first observe that due to estimates (4.8) (with $q = 2$) and (4.10), conditions (4.4) and (4.5) of Corollary 4.3 are satisfied. Thus, it is sufficient to prove condition (**a**), i.e. that the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition [**A**]. By Lemma 4.4 it is sufficient to proof the condition (**a'**).

We have now to choose our steps very carefully as we no longer treat strong solutions to an SDE in a finite dimensional Hilbert space but instead a weak solution to an SPDE in an infinite dimensional Hilbert space.

Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of stopping times taking values in $[0,T]$. Since

each process satisfies equation (3.4), by Remark 3.7 we have

$$\begin{aligned} u_n(t) &= u_{0,n} - \int_0^t \mathcal{A}u_n(s)\,ds - \int_0^t B\big(u_n(s)\big)\,ds + \int_0^t f_n(s)\,ds \\ &\quad + \int_0^t G(u_n(s))\,dW(s) \\ &=: J_1^n + J_2^n(t) + J_3^n(t) + J_4^n(t) + J_5^n(t), \qquad t \in [0,T], \end{aligned}$$

where the above equality is understood in the space V'. Let us choose and $\theta > 0$. It is sufficient to show that each sequence J_i^n of processes, $i = 1, \cdots, 5$ satisfies the sufficient condition (**a'**) from Lemma 4.4.

Obviously the term J_1^n which is constant in time, satisfies whatever we want. We will only deal with the other terms. In fact, we will check that the terms J_2^n, J_4^n, J_5^n satisfy condition (**a'**) from Lemma 4.4 in the space $Y = \mathrm{V}'$ and the term J_3^n satifies this condition in $Y = \mathrm{V}_s'$ with $s > \frac{d}{2} + 1$. Since the embeddings $\mathrm{V}_s' \subset U'$ and $\mathrm{V}' \subset U'$ are continuous, we infer that (**a'**) from Lemma 4.4 holds in the space $Y = U'$, as well.

Ad J_2^n. Since the linear operator $\mathcal{A} : \mathrm{V} \to \mathrm{V}'$ is bounded, by the Hölder inequality and (4.10), we have

$$\begin{aligned} \mathbb{E}_n\left[|J_2^n(\tau_n+\theta) - J_2^n(\tau_n)|_{\mathrm{V}'}\right] &\le \mathbb{E}_n\left[\int_{\tau_n}^{\tau_n+\theta} \left|\mathcal{A}u_n(s)\right|_{\mathrm{V}'} ds\right] \\ &\le \theta^{\frac{1}{2}}\left(\mathbb{E}_n\left[\int_0^T \|u_n(s)\|^2\,ds\right]\right)^{\frac{1}{2}} \le C_2(p)\cdot\theta^{\frac{1}{2}}. \end{aligned} \tag{4.12}$$

Ad J_3^n. Let $s > \frac{d}{2} + 1$ Similarly, since $B : \mathrm{H} \times \mathrm{H} \to \mathrm{V}_s'$ is bilinear and continuous (and hence bounded so that the norm $\|B\|$ of $B : \mathrm{H} \times \mathrm{H} \to \mathrm{V}_s'$ is

finite), then by (4.8) we have the following estimates

$$
\begin{aligned}
(4.13)\quad & \mathbb{E}_n\left[|J_3^n(\tau_n+\theta)-J_3^n(\tau_n)|_{\mathrm{V}_s'}\right]=\mathbb{E}_n\left[\Big|\int_{\tau_n}^{\tau_n+\theta}B\big(u_n(r)\big)\,dr\Big|_{\mathrm{V}_s'}\right]\\
&\le c\mathbb{E}_n\left[\int_{\tau_n}^{\tau_n+\theta}|B\big(u_n(r)\big)|_{\mathrm{V}_s'}\,dr\right]\le c\|B\|\,\mathbb{E}_n\left[\int_{\tau_n}^{\tau_n+\theta}|u_n(r)|_{\mathrm{H}}^2\,dr\right]\\
&\le c\|B\|\cdot\mathbb{E}_n\Big[\sup_{r\in[0,T]}|u_n(r)|_{\mathrm{H}}^2\Big]\cdot\theta\le c\|B\|\,C_1(p,2)\cdot\theta.
\end{aligned}
$$

Remark. The above argument works as well for $d=3$. However for $d=2$ we have the following different proof which exploits inequality (2.12) (which is valid only the the two dimensional case).

$$
\begin{aligned}
(4.14)\quad & \mathbb{E}_n\left[|J_3^n(\tau_n+\theta)-J_3^n(\tau_n)|_{\mathrm{V}'}\right]\le\mathbb{E}_n\left[\int_{\tau_n}^{\tau_n+\theta}\big|B\big(u_n(r)\big)\big|_{\mathrm{V}'}\,dr\right]\\
&\le c_2\mathbb{E}_n\int_{\tau_n}^{\tau_n+\theta}|u_n(r)|_{L^2}\,|\nabla u_n(r)|_{L^2}\,dr\\
&\le c_2\left[\mathbb{E}_n\sup_{r\in[\tau_n,\tau_n+\theta]}|u_n(r)|_H^2\right]^{\frac12}\left[\mathbb{E}_n\int_{\tau_n}^{\tau_n+\theta}|\nabla u_n(r)|_{L^2}^2\,dr\right]^{\frac12}\theta^{\frac12}\\
&\le c_2\left[\mathbb{E}_n\sup_{r\in[0,T]}|u_n(r)|_{\mathrm{H}}^2\right]^{\frac12}\left[\mathbb{E}_n\int_0^T|\nabla u_n(r)|_{L^2}^2\,dr\right]^{\frac12}\theta^{\frac12}\\
&\le c_2[C_1(p,2)]^{\frac12}[C_2(p)]^{\frac12}\theta^{\frac12}.
\end{aligned}
$$

Ad J_4^n. Since the sequence (f_n) is weakly convergent in $L^p(0,T;\mathrm{V}')$, it is, in particular, bounded in $L^p(0,T;\mathrm{V}')$. Using the Hölder inequality, we have

$$
\begin{aligned}
(4.15)\quad & \mathbb{E}_n\left[|J_4^n(\tau_n+\theta)-J_4^n(\tau_n)|_{\mathrm{V}'}\right]=\mathbb{E}_n\left[\Big|\int_{\tau_n}^{\tau_n+\theta}f_n(s)\,ds\Big|_{\mathrm{V}'}\right]\\
&\le\theta^{\frac{p-1}{p}}\left(\mathbb{E}_n\left[\int_0^T|f_n(s)|_{\mathrm{V}'}^p\,ds\right]\right)^{\frac1p}=\theta^{\frac{p-1}{p}}|f_n|_{L^p(0,T;\mathrm{V}')}=c_4\cdot\theta^{\frac{p-1}{p}},
\end{aligned}
$$

where $c_4:=\sup_{n\in\mathbb{N}}|f_n|_{L^p(0,T;\mathrm{V}')}$.

Ad J_5^n. By assumption (G3) and inequality (4.8), we obtain the following

inequalities

$$
\begin{aligned}
(4.16)\quad \mathbb{E}_n\left[|J_5^n(\tau_n+\theta)-J_5^n(\tau_n)|_{\mathrm{V}'}\right] &\le \left\{\mathbb{E}_n\left[|J_5^n(\tau_n+\theta)-J_5^n(\tau_n)|_{\mathrm{V}'}^2\right]\right\}^{\frac{1}{2}}\\
&= \left[\mathbb{E}_n\int_{\tau_n}^{\tau_n+\theta}\|G(u_n(s))\|_{\mathcal{T}_2(Y,\mathrm{V}')}^2\,ds\right]^{\frac{1}{2}}\\
&\le \left[C\cdot\mathbb{E}_n\int_{\tau_n}^{\tau_n+\theta}(1+|u_n(s)|_{\mathrm{H}}^2)\,ds\right]^{\frac{1}{2}}\\
&\le \left[C\big(1+\left[\mathbb{E}_n\big[\sup_{s\in[0,T]}|u_n(s)|_{\mathrm{H}}^2\big]\right)\theta\right]^{\frac{1}{2}}\\
&\le \left[C(1+C_1(2))\theta\right]^{\frac{1}{2}} =: c_5\cdot\theta^{\frac{1}{2}}.
\end{aligned}
$$

Thus the proof of Theorem 4.9 is complete. □

Remark 4.10. *It is easy to be convinced that u_n take values in $\mathcal{Z}_T$ but it's not so obvious to see that in fact u_n are Borel measurable functions. This is so because our construction of the martingale solution is based on Jakubowski's version of the Skorokhod Theorem, see Theorem 4.6 for details.*

The main result about the continuous dependence of the solutions of the Navier-Stokes equations on the initial state and deterministic external forces, which covers both cases of 2D and 3D domains, is expressed in the following theorem 4.11. Stronger version for 2D domains will be formulated in the next section, see Theorem 5.9.

Theorem 4.11. *Let conditions **(H.1)-(H.3)** and **(H.5)** of Assumption 3.1 be satisfied and let $T>0$. Assume that $\left(u_{0,n}\right)_{n=1}^{\infty}$ is an H-valued sequence that is convergent weakly in H to $u_0\in\mathrm{H}$ and $(f_n)_{n=1}^{\infty}$ is an $L^p(0,T;\mathrm{V}')$-valued sequence that is weakly convergent in $L^p(0,T;\mathrm{V}')$ to $f\in L^p(0,T;\mathrm{V}')$. Let $R_1>0$ and $R_2>0$ be such that $\sup_{n\in\mathbb{N}}|u_{0,n}|_{\mathrm{H}}\le R_1$ and*

$\sup_{n\in\mathbb{N}} \|f_n\|_{L^p(0,T;\mathrm{V}')} \le R_2$. *Let*

$$\big(\hat\Omega_n, \hat{\mathcal{F}}_n, \hat{\mathbb{F}}_n, \hat{\mathbb{P}}_n \hat W_n, u_n\big)$$

be a martingale solution of problem (3.2) *with the initial data* u_0^n *and the external force* f_n *and satisfying inequalities* (4.8)-(4.10). *Then there exist*

- *a subsequence* $(n_k)_k$,
- *a stochastic basis* $\big(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}\big)$, *where* $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t\}_{t\ge 0}$,
- *a cylindrical Wiener process* $\tilde W = \tilde W(t)$, $t \in [0,\infty)$ *defined on this basis,*
- *and progressively measurable processes* $\tilde u$, $\big(\tilde u_{n_k}\big)_{k\ge 1}$ *(defined on this basis) with laws supported in* $\mathcal{Z}_T$ *such that*

(4.17)

$\tilde u_{n_k}$ *has the same law as* u_{n_k} *on* $\mathcal{Z}_T$ *and* $\tilde u_{n_k} \to \tilde u$ *in* $\mathcal{Z}_T$, $\quad \tilde{\mathbb{P}}$ *- a.s.,*

for every $q \in [1,p]$

$$\tilde{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\tilde u(s)|_{\mathrm{H}}^q\Big] < \infty, \tag{4.18}$$

and the system

$$\big(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde W, \tilde u\big)$$

is a solution to problem (3.2).

In particular, for all $t \in [0,T]$ *and all* $\mathrm{v} \in \mathrm{V}$

$$\begin{aligned}\big(\tilde u(t),\mathrm{v}\big)_{\mathrm{H}} - \big(\tilde u(0),\mathrm{v}\big)_{\mathrm{H}} + \int_0^t \langle \mathcal{A}\tilde u(s),\mathrm{v}\rangle\,ds + \int_0^t \langle B\big(\tilde u(s)\big),\mathrm{v}\rangle\,ds \\ = \int_0^t \langle f(s),\mathrm{v}\rangle\,ds + \Big\langle \int_0^t G\big(\tilde u(s)\big)\,d\tilde W(s),\mathrm{v}\Big\rangle\end{aligned}$$

and

$$\tilde{\mathbb{E}}\Big[\int_0^T \|\tilde u(s)\|^2\,ds\Big] < \infty. \tag{4.19}$$

PROOF. Since the product topological space $\mathcal{Z}_T \times \mathcal{C}([0,T],\mathrm{K})$ satisfies the assumptions of Theorem 4.6, by applying it together with Theorem 4.9, there exists a subsequence (n_k), a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $\mathcal{Z}_T \times \mathcal{C}([0,T],\mathrm{K})$-valued Borel random variables $\big(\tilde{u}, \tilde{W}\big), \big(\tilde{u}_k, \tilde{W}_k\big)$, $k \in \mathbb{N}$ such that each $\tilde{W}$ and $\tilde{W}_k$, $k \in \mathbb{N}$ is an K-valued Wiener process and such that

(4.20)

the laws on $\mathcal{B}(\mathcal{Z}_T \times \mathcal{C}([0,T],\mathrm{K}))$ of (u_{n_k}, W) and $(\tilde{u}_k, \tilde{W}_k)$ are equal.

where $\mathcal{B}(\mathcal{Z}_T \times \mathcal{C}([0,T],\mathrm{K}))$ is the Borel σ-algebra on $\mathcal{Z}_T \times \mathcal{C}([0,T],\mathrm{K})$, and, with $\hat{\mathrm{K}}$ being an auxiliary Hilbert space such that $\mathrm{K} \subset \hat{\mathrm{K}}$ and the natural embedding $\mathrm{K} \hookrightarrow \hat{\mathrm{K}}$ is Hilbert-Schmidt,

(4.21)

$\big(\tilde{u}_k, \tilde{W}_k\big)$ converges to $\big(\tilde{u}, \tilde{W}\big)$ in $\mathcal{Z}_T \times \mathcal{C}([0,T],\hat{\mathrm{K}})$ $\quad \tilde{\mathbb{P}}$-almost surely on $\tilde{\Omega}$.

Note that since $\mathcal{B}(\mathcal{Z}_T \times \mathcal{C}([0,T],\mathrm{K})) \subset \mathcal{B}(\mathcal{Z}_T) \times \mathcal{B}(\mathcal{C}([0,T],\mathrm{K}))$, the function u is $\mathcal{Z}_T$ Borel random variable.

Define a corresponding sequence of filtrations by

(4.22)

$$\tilde{\mathbb{F}}_k = (\tilde{\mathcal{F}}_k(t))_{t\geq 0}, \text{ where } \tilde{\mathcal{F}}_k(t) = \sigma\big(\{\big(\tilde{u}_k(s), \tilde{W}_k(s)\big),\ s \leq t\}\big),\ \ t \in [0,T].$$

To conclude the proof, we need to show that the random variable $\tilde{u}$ gives rise to a martingale solution. The proof of this claim is very similar to the proof of Theorem 2.3 in [43]. Let us denote the subsequence $(\tilde{u}_{n_k})_k$ again by $(\tilde{u}_n)_n$.

The few differences are:

(i) The finite dimensional space H_n is replaced by the whole space H. But now, by Lemma 4.2 the space $\mathcal{C}([0,T];\mathrm{V}') \cap \mathcal{Z}_T$ is a Borel subset of $\mathcal{Z}_T$ and

since by Remark 3.7 $u_n \in \mathcal{C}([0,T];\mathrm{V}')$, $\mathbb{P}$-a.s. and $\tilde{u}_n$ and u_n have the same laws on $\mathcal{Z}_T$, we infer that

$$\tilde{u}_n \in \mathcal{C}([0,T];\mathrm{V}') \qquad n \geq 1, \quad \tilde{\mathbb{P}}\text{-a.s.}$$

(ii) The operator P_n has to be replaced by the identity. But this is rather a simplification as for instance we do not need Lemmas 2.3 and 2.4 from [16].

In addition to point (i) above, we have that for every $q \in [1,p]$, we have

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{E}} \left(\sup_{0\leq s\leq T} |\tilde{u}_n(s)|_{\mathrm{H}}^q \right) \leq C_1(p,q), \tag{4.23}$$

Similarly,

$$\tilde{u}_n \in L^2(0,T;\mathrm{V}) \qquad n \geq 1, \quad \mathbb{P}\text{-a.s.}$$

and

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{E}} \left[\int_0^T \|\tilde{u}_n(s)\|_{\mathrm{V}}^2 \, ds \right] \leq C_2(p). \tag{4.24}$$

By inequality (4.24) we infer that the sequence $(\tilde{u}_n)$ contains a subsequence, still denoted by $(\tilde{u}_n)$, convergent weakly in the space $L^2([0,T]\times\tilde{\Omega};\mathrm{V})$. Since by (4.21) $\tilde{\mathbb{P}}$-a.s. $\tilde{u}_n \to \tilde{u}$ in $\mathcal{Z}_T$, we conclude that $\tilde{u} \in L^2([0,T]\times\tilde{\Omega};\mathrm{V})$, i.e.

$$\tilde{\mathbb{E}} \left[\int_0^T |\tilde{u}(s)|^2 \, ds \right] < \infty. \tag{4.25}$$

Similarly, by inequality (4.23) with $q = p$ we can choose a subsequence of $(\tilde{u}_n)$ convergent weak star in the space $L^p(\tilde{\Omega}; L^\infty(0,T;\mathrm{H}))$ and, using (4.21), infer that

$$\tilde{\mathbb{E}} \left[\sup_{0\leq s\leq T} |\tilde{u}(s)|_{\mathrm{H}}^p \right] < \infty. \tag{4.26}$$

Then, of course, for every $q \in [1,p]$,

$$\tilde{\mathbb{E}}\Big[\sup_{0\le s\le T}|\tilde{u}(s)|_{\mathrm{H}}^{q}\Big]<\infty. \tag{4.27}$$

The remaining proof will be done in two steps.

Step 1. Let us choose and fix $s>\frac{d}{2}+1$. We will first prove the following Lemma.

LEMMA 4.12. *For all* $\varphi\in \mathrm{V}_s$

(a) $\lim_{n\to\infty}\tilde{\mathbb{E}}\big[\int_0^T\big|\big(\tilde{u}_n(t)-\tilde{u}(t),\varphi\big)_{\mathrm{H}}\big|^2\,dt\big]=0$,

(b) $\lim_{n\to\infty}\tilde{\mathbb{E}}\big[\big|\big(\tilde{u}_n(0)-\tilde{u}(0),\varphi\big)_{\mathrm{H}}\big|^2\big]=0$,

(c) $\lim_{n\to\infty}\tilde{\mathbb{E}}\big[\int_0^T\big|\int_0^t\langle\mathcal{A}\tilde{u}_n(s)-\mathcal{A}\tilde{u}(s),\varphi\rangle\,ds\big|\,dt\big]=0$,

(d) $\lim_{n\to\infty}\tilde{\mathbb{E}}\big[\int_0^T\big|\int_0^t\langle B(\tilde{u}_n(s))-B(\tilde{u}(s)),\varphi\rangle\,ds\big|\,dt\big]=0$,

(e) $\lim_{n\to\infty}\tilde{\mathbb{E}}\big[\int_0^T\big|\int_0^t\langle f_n(s)-f(s),\varphi\rangle\,ds\big|\,dt\big]=0$,

(f) $\lim_{n\to\infty}\tilde{\mathbb{E}}\big[\int_0^T\big|\langle\int_0^t[G(\tilde{u}_n(s))-G(\tilde{u}(s))]\,d\tilde{W}(s),\varphi\rangle\big|^2\,dt\big]=0$.

PROOF OF LEMMA 4.12. Let us fix $\varphi\in\mathrm{V}_s$. **Ad (a).** Since by (4.21) $\tilde{u}_n\to\tilde{u}$ in $\mathcal{C}([0,T];\mathrm{H}_w)$ $\tilde{\mathbb{P}}$-a.s., $\big(\tilde{u}_n(\cdot),\varphi\big)_{\mathrm{H}}\to\big(\tilde{u}(\cdot),\varphi\big)_{\mathrm{H}}$ in $\mathcal{C}([0,T];\mathbb{R})$, $\tilde{\mathbb{P}}$-a.s. Hence, in particular, for all $t\in[0,T]$

$$\lim_{n\to\infty}\big(\tilde{u}_n(t),\varphi\big)_{\mathrm{H}}=\big(\tilde{u}(t),\varphi\big)_{\mathrm{H}},\qquad \tilde{\mathbb{P}}\text{-a.s.}$$

Since by (4.23), $\sup_{t\in[0,T]}|\tilde{u}_n(t)|_{\mathrm{H}}^2<\infty$, $\tilde{\mathbb{P}}$-a.s., using the dominated convergence theorem we infer that

$$\lim_{n\to\infty}\int_0^T\big|\big(\tilde{u}_n(t)-\tilde{u}(t),\varphi\big)_{\mathrm{H}}\big|^2\,dt=0\qquad\tilde{\mathbb{P}}\text{-a.s. .} \tag{4.28}$$

By the Hölder inequality and (4.23) for every $n\in\mathbb{N}$ and every $r\in\big(1,1+\frac{p}{2}\big]$

$$\begin{aligned}\tilde{\mathbb{E}}\Big[\Big|\int_0^T|\tilde{u}_n(t)-\tilde{u}(t)|_{\mathrm{H}}^2\,dt\Big|^r\Big]&\le c\tilde{\mathbb{E}}\Big[\int_0^T\big(|\tilde{u}_n(t)|_{\mathrm{H}}^{2r}+|\tilde{u}(t)|_{\mathrm{H}}^{2r}\big)\,dt\Big]\\&\le\tilde{c}C_1(p,2r),\end{aligned} \tag{4.29}$$

where $c, \tilde{c}$ are some positive constants. To conclude the proof of assertion (a) it is sufficient to use (4.28), (4.29) and the Vitali Theorem.

Ad (b). Since by (4.21) $\tilde{u}_n \to \tilde{u}$ in $\mathcal{C}(0,T;\mathrm{H}_w)$ $\tilde{\mathbb{P}}$-a.s. and $\tilde{u}$ is continuous at $t=0$, we infer that $\big(\tilde{u}_n(0),\varphi\big)_{\mathrm{H}} \to \big(\tilde{u}(0),\varphi\big)_{\mathrm{H}}$, $\tilde{\mathbb{P}}$-a.s. Now, assertion (b) follows from (4.23) and the Vitali Theorem.

Ad (c). Since by (4.21) $\tilde{u}_n \to \tilde{u}$ in $L^2_w(0,T;\mathrm{V})$, $\tilde{\mathbb{P}}$-a.s., by (2.6) we infer that $\tilde{\mathbb{P}}$-a.s.

$$\begin{aligned} \lim_{n\to\infty} \int_0^t \langle \mathcal{A}\tilde{u}_n(s),\varphi\rangle \, ds &= \lim_{n\to\infty} \int_0^t \big(\big(\tilde{u}_n(s),\varphi\big)\big) \, ds \\ &= \int_0^t \big(\big(\tilde{u}(s),\varphi\big)\big) \, ds = \int_0^t \langle \mathcal{A}\tilde{u}(s),\varphi\rangle \, ds. \end{aligned} \tag{4.30}$$

By (2.6), the Hölder inequality and estimate (4.24) we infer that for all $t\in[0,T]$ and $n\in\mathbb{N}$

$$\begin{aligned} \tilde{\mathbb{E}}\Big[\Big|\int_0^t \langle \mathcal{A}\tilde{u}_n(s),\varphi\rangle \, ds\Big|^2\Big] &= \tilde{\mathbb{E}}\Big[\Big|\int_0^t \big(\big(\tilde{u}_n(s),\varphi\big)\big) \, ds\Big|^2\Big] \\ &\le \; c\,\|\varphi\|^2_{\mathrm{V}_s}\,\tilde{\mathbb{E}}\Big[\int_0^T \|\tilde{u}_n(s)\|^2_{\mathrm{V}} \, ds\Big] \le \tilde{c}C_2(p), \end{aligned} \tag{4.31}$$

where $c,\tilde{c}>0$ are some constants. By (4.30), (4.31) and the Vitali Theorem we conclude that for all $t\in[0,T]$

$$\lim_{n\to\infty} \tilde{\mathbb{E}}\Big[\Big|\int_0^t \langle \mathcal{A}\tilde{u}_n(s) - \mathcal{A}\tilde{u}(s),\varphi\rangle \, ds\Big|\Big] = 0.$$

Assertion (c) follows now from (4.24) and the dominated convergence theorem.

Ad (d). Since by (4.24) and (2.3) the sequence $(\tilde{u}_n)$ is bounded in $L^2(0,T;\mathrm{H})$ and by (4.21) $\tilde{u}_n \to \tilde{u}$ in $L^2(0,T;\mathrm{H}_{\mathrm{loc}})$, $\tilde{\mathbb{P}}$-a.s., by Lemma B.1 in [16] we infer

that $\tilde{\mathbb{P}}$-a.s. for all $t \in [0,T]$ and $\varphi \in \mathrm{V}_s$

$$\lim_{n\to\infty} \int_0^t \langle B(\tilde{u}_n(s)) - B(\tilde{u}(s)), \varphi\rangle \, ds = 0. \tag{4.32}$$

Using the Hölder inequality, Lemma 2.2 and (4.23) we infer that for all $t \in [0,T]$, $r \in \left(0, \frac{p}{2}\right]$ and $n \in \mathbb{N}$ the following inequalities hold

$$\begin{aligned} \tilde{\mathbb{E}}\Big[\Big|\int_0^t \langle B(\tilde{u}_n(s)), \varphi\rangle \, ds\Big|^{1+r}\Big] &\le \tilde{\mathbb{E}}\Big[\Big(\int_0^t |B(\tilde{u}_n(s))|_{\mathrm{V}_s'} |\varphi|_{\mathrm{V}_s} \, ds\Big)^{1+r}\Big] \\ &\le (c_2|\varphi|_{\mathrm{V}_s})^{1+r} \, t^r \, \mathbb{E}\Big[\int_0^t |\tilde{u}_n(s)|_{\mathrm{H}}^{2+2r} \, ds\Big] \le \tilde{C}\tilde{\mathbb{E}}\big[\sup_{s\in[0,T]} |\tilde{u}_n(s)|_{\mathrm{H}}^{2+2r}\big] \\ &\le \tilde{C} C_1(p, 2+2r). \end{aligned} \tag{4.33}$$

By (4.32), (4.33) and the Vitali Theorem we obtain for all $t \in [0,T]$

$$\lim_{n\to\infty} \tilde{\mathbb{E}}\Big[\Big|\int_0^t \langle B(\tilde{u}_n(s)) - B(\tilde{u}(s)), \varphi\rangle \, ds\Big|\Big] = 0. \tag{4.34}$$

Using again Lemma 2.2 and estimate (4.23), we obtain for all $t \in [0,T]$ and $n \in \mathbb{N}$

$$\tilde{\mathbb{E}}\Big[\Big|\int_0^t \langle B(\tilde{u}_n(s)), \varphi\rangle \, ds\Big|\Big] \le c\tilde{\mathbb{E}}\big[\sup_{s\in[0,T]} |\tilde{u}_n(s))|_{\mathrm{H}}^2\big] \le cC_1(p,2),$$

where $c > 0$ is a constant. Hence by (4.34) and the dominated convergence theorem, we infer that assertion (d) holds.

Ad (e). Assertion (e) follows because the sequence (f_n) converges weakly in $L^p(0,T;\mathrm{V}')$ to f and $\mathrm{V}_s \subset \mathrm{V}$.

Ad (f). Let us notice that for all $\varphi \in \mathrm{V}$ we have

$$\begin{aligned} &\int_0^t \|\langle G(\tilde{u}_n(s)) - G(\tilde{u}(s)), \varphi\rangle\|_{\mathcal{T}_2(\hat{\mathrm{K}};\mathbb{R})}^2 \, ds \\ &= \int_0^t \|\varphi^{**} G(\tilde{u}_n)(s) - \varphi^{**} G(\tilde{u})(s)\|_{\mathcal{T}_2(\hat{\mathrm{K}};\mathbb{R})}^2 \, ds \\ &\le \|\varphi^{**} G(\tilde{u}_n) - \varphi^{**} G(\tilde{u})\|_{L^2([0,T];\mathcal{T}_2(\hat{\mathrm{K}};\mathbb{R}))}^2, \end{aligned}$$

where $\varphi^{**}G$ is the map defined by (G4) in assumption **(H.2)**. Since by (4.21) $\tilde{u}_n \to \tilde{u}$ in $L^2(0,T;\mathrm{H}_{\mathrm{loc}})$, $\tilde{\mathbb{P}}$-a.s., by (G4) we infer that for all $t \in [0,T]$ and $\varphi \in \mathrm{V}$

$$\lim_{n\to\infty} \int_0^t \|\langle G(\tilde{u}_n(s)) - G(\tilde{u}(s)), \varphi\rangle\|^2_{\mathcal{T}_2(\hat{\mathrm{K}};\mathbb{R})}\, ds = 0. \tag{4.35}$$

By (G3) and (4.23) we obtain the following inequalities for every $t \in [0,T]$, $r \in \big(1, 1+\frac{p}{2}\big]$ and $n \in \mathbb{N}$

$$\begin{aligned}
&\tilde{\mathbb{E}}\Big[\Big|\int_0^t \|\langle G(\tilde{u}_n(s)) - G(\tilde{u}(s)), \varphi\rangle\|^2_{\mathcal{T}_2(\hat{\mathrm{K}};\mathbb{R})}\, ds\Big|^r\Big] \\
&\quad\le c\,\tilde{\mathbb{E}}\Big[|\varphi|^{2r}_{\mathrm{V}} \cdot \int_0^t \big\{|G(\tilde{u}_n(s))|^{2r}_{\mathcal{T}_2(\hat{\mathrm{K}};\mathrm{V}')} + |G(\tilde{u}(s))|^{2r}_{\mathcal{T}_2(\hat{\mathrm{K}};\mathrm{V}')}\big\}\, ds\Big] \\
&\quad\le c_1\,\tilde{\mathbb{E}}\Big[\int_0^T (1 + |\tilde{u}_n(s)|^{2r}_{\mathrm{H}} + |\tilde{u}(s)|^{2r}_{\mathrm{H}})\, ds\Big] \\
&\quad\le \tilde{c}\Big\{1 + \tilde{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\tilde{u}_n(s)|^{2r}_{\mathrm{H}} + \sup_{s\in[0,T]} |\tilde{u}(s)|^{2r}_{\mathrm{H}})\Big]\Big\} \le \tilde{c}(1 + 2C_1(p,2r)),
\end{aligned} \tag{4.36}$$

where $c, c_1, \tilde{c}$ are some positive constants. Using the Vitali theorem, by (4.35), (4.36) we infer that for all $\varphi \in \mathrm{V}$

$$\lim_{n\to\infty} \tilde{\mathbb{E}}\Big[\int_0^t \|\langle G(\tilde{u}_n(s)) - G(\tilde{u}(s)), \varphi\rangle\|^2_{\mathcal{T}_2(\hat{\mathrm{K}};\mathbb{R})}\, ds\Big] = 0. \tag{4.37}$$

Hence by the properties of the Itô integral we infer that for all $t \in [0,T]$ and $\varphi \in \mathrm{V}$

$$\lim_{n\to\infty} \tilde{\mathbb{E}}\Big[\Big|\Big\langle \int_0^t \big[G(\tilde{u}_n(s)) - G(\tilde{u}(s))\big]\, d\tilde{W}(s), \varphi\Big\rangle\Big|^2\Big] = 0. \tag{4.38}$$

By the Itô isometry, since the map G satisfies inequality (G3) in part **(H.2)** of Assumption 3.1, and estimate (4.23) we have for all $\varphi \in \mathrm{V}$, $t \in [0,T]$ and

$n \in \mathbb{N}$

$$
\begin{aligned}
(4.39)\quad & \tilde{\mathbb{E}}\Big[\Big|\Big\langle \int_0^t \big[G(\tilde{u}_n(s)) - G(\tilde{u}(s))\big]\, d\tilde{W}(s), \varphi\Big\rangle\Big|^2\Big] \\
&= \tilde{\mathbb{E}}\Big[\int_0^t \|\langle G(\tilde{u}_n(s)) - G(\tilde{u}(s)), \varphi\rangle\|^2_{\mathcal{T}_2(\hat{\mathrm{K}};\mathbb{R})}\, ds\Big] \\
&\leq c\Big\{1 + \tilde{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\tilde{u}_n(s)|^2_{\mathrm{H}} + \sup_{s\in[0,T]} |\tilde{u}(s)|^2_{\mathrm{H}})\Big]\Big\} \leq c(1 + 2C_1(p,2)),
\end{aligned}
$$

where $c > 0$ is some constant. Thus by (4.38), (4.39) and the Lebesgue Dominated Convergence Theorem we infer that for all $\varphi \in \mathrm{V}$

$$
(4.40)\quad \lim_{n\to\infty} \int_0^T \tilde{\mathbb{E}}\Big[\Big|\Big\langle \int_0^t \big[G(\tilde{u}_n(s)) - G(\tilde{u}(s))\big]\, d\tilde{W}(s), \varphi\Big\rangle\Big|^2\Big] = 0.
$$

To conclude the proof of assertion (f), it is sufficient to notice that since $s > \frac{d}{2} + 1$, $\mathrm{V}_s \subset \mathrm{V}$ and thus (4.40) holds for all $\varphi \in \mathrm{V}_s$. The proof of Lemma 4.12 is thus complete. □

As a direct consequence of Lemma 4.12 we get the following corollary which we precede by introducing some auxiliary notation. Analogously to [10] and [43], let us denote

(4.41)

$$
\begin{aligned}
\Lambda_n(\tilde{u}_n, \tilde{W}_n, \varphi)(t) &:= \big(\tilde{u}_n(0), \varphi\big)_{\mathrm{H}} - \int_0^t \langle \mathcal{A}\tilde{u}_n(s), \varphi\rangle ds - \int_0^t \langle B(\tilde{u}_n(s)), \varphi\rangle ds \\
&\quad + \int_0^t \langle f_n(s), \varphi\rangle\, ds + \Big\langle \int_0^t G(\tilde{u}_n(s))\, d\tilde{W}_n(s), \varphi\Big\rangle, \quad t \in [0,T],
\end{aligned}
$$

and

$$
\begin{aligned}
(4.42)\quad \Lambda(\tilde{u}, \tilde{W}, \varphi)(t) &:= \big(\tilde{u}(0), \varphi\big)_{\mathrm{H}} - \int_0^t \langle \mathcal{A}\tilde{u}(s), \varphi\rangle ds - \int_0^t \langle B(\tilde{u}(s)), \varphi\rangle ds \\
&\quad + \int_0^t \langle f(s), \varphi\rangle\, ds + \Big\langle \int_0^t G(\tilde{u}(s))\, d\tilde{W}(s), \varphi\Big\rangle, \quad t \in [0,T].
\end{aligned}
$$

COROLLARY 4.13. *For every* $\varphi \in \mathrm{V}_s$,

$$\lim_{n\to\infty} |\big(\tilde{u}_n(\cdot),\varphi\big)_{\mathrm{H}} - \big(\tilde{u}(\cdot),\varphi\big)_{\mathrm{H}}|_{L^2([0,T]\times\tilde{\Omega})} = 0 \tag{4.43}$$

and

$$\lim_{n\to\infty} |\Lambda_n(\tilde{u}_n,\tilde{W}_n,\varphi) - \Lambda(\tilde{u},\tilde{W},\varphi)|_{L^1([0,T]\times\tilde{\Omega})} = 0. \tag{4.44}$$

PROOF OF COROLLARY 4.13. Assertion (4.43) follows from the equality

$$|\big(\tilde{u}_n(\cdot),\varphi\big)_{\mathrm{H}} - \big(\tilde{u}(\cdot),\varphi\big)_{\mathrm{H}}|^2_{L^2([0,T]\times\tilde{\Omega})} = \tilde{\mathbb{E}}\Big[\int_0^T |\big(\tilde{u}_n(t)-\tilde{u}(t),\varphi\big)_{\mathrm{H}}|^2\,dt\Big]$$

and Lemma 4.12 (a). Let us move to the proof of assertion (4.44). Note that by the Fubini theorem, we have

$$\begin{aligned}&|\Lambda_n(\tilde{u}_n,\tilde{W}_n,\varphi) - \Lambda(\tilde{u},\tilde{W},\varphi)|_{L^1([0,T]\times\tilde{\Omega})}\\ &= \int_0^T \tilde{\mathbb{E}}\big[|\Lambda_n(\tilde{u}_n,\tilde{W}_n,\varphi)(t) - \Lambda(\tilde{u},\tilde{W},\varphi)(t)|\,\big]dt.\end{aligned}$$

To conclude the proof of Corollary 4.13 it is sufficient to note that by Lemma 4.12 (b)-(f), each term on the right hand side of (4.41) tends at least in $L^1([0,T]\times\tilde{\Omega})$ to the corresponding term in (4.42). □

Step 2. Since u_n is a solution of the Navier-Stokes equation, for all $t\in[0,T]$ and $\varphi\in\mathcal{V}$

$$\big(u_n(t),\varphi\big)_{\mathrm{H}} = \Lambda_n(u_n,W,\varphi)(t), \qquad \mathbb{P}\text{-a.s.}$$

In particular,

$$\int_0^T \mathbb{E}\big[|\big(u_n(t),\varphi\big)_{\mathrm{H}} - \Lambda_n(u_n,W,\varphi)(t)|\,\big]\,dt = 0.$$

Since $\mathcal{L}(u_n, W) = \mathcal{L}(\tilde{u}_n, \tilde{W}_n)$,

$$\int_0^T \tilde{\mathbb{E}}\big[\,\big|\big(\tilde{u}_n(t), \varphi\big)_{\mathrm{H}} - \Lambda_n(\tilde{u}_n, \tilde{W}_n, \varphi)(t)\big|\,\big]\, dt = 0.$$

Moreover, by (4.43) and (4.44)

$$\int_0^T \tilde{\mathbb{E}}\big[\,\big|\big(\tilde{u}(t), \varphi\big)_{\mathrm{H}} - \Lambda(\tilde{u}, \tilde{W}, \varphi)(t)\big|\,\big]\, dt = 0.$$

Hence for l-almost all $t \in [0,T]$ and $\tilde{\mathbb{P}}$-almost all $\omega \in \tilde{\Omega}$

$$\big(\tilde{u}(t), \varphi\big)_{\mathrm{H}} - \Lambda(\tilde{u}, \tilde{W}, \varphi)(t) = 0,$$

i.e. for l-almost all $t \in [0,T]$ and $\tilde{\mathbb{P}}$-almost all $\omega \in \tilde{\Omega}$

$$\begin{aligned} &\big(\tilde{u}(t), \varphi\big)_{\mathrm{H}} + \int_0^t \langle \mathcal{A}\tilde{u}(s), \varphi\rangle\, ds + \int_0^t \langle B(\tilde{u}(s)), \varphi\rangle\, ds \\ &\quad = \big(\tilde{u}(0), \varphi\big)_{\mathrm{H}} + \int_0^t \langle f(s), \varphi\rangle\, ds + \Big\langle \int_0^t G(\tilde{u}(s))\, d\tilde{W}(s), \varphi \Big\rangle. \end{aligned} \tag{4.45}$$

Since a Borel $\tilde{u}$ is $\mathcal{Z}_T$-valued random variable, in particular $\tilde{u} \in \mathcal{C}([0,T]; \mathrm{H}_w)$, i.e. $\tilde{u}$ is weakly continuous, we infer that equality (4.45) holds for all $t \in [0,T]$ and all $\varphi \in \mathcal{V}$. Since $\mathcal{V}$ is dense in V, equality (4.45) holds for all $\varphi \in \mathrm{V}$, as well. Putting $\tilde{\mathfrak{A}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathbb{F}})$, we infer that the system $(\tilde{\mathfrak{A}}, \tilde{W}, \tilde{u})$ is a martingale solution of equation (3.2). By (4.25) and (4.27) the process $\tilde{u}$ satisfies inequalities (4.19) and (4.18). The proof of Theorem 4.11 is thus complete. □

Remark 4.14. *It seems to us that the same argument works if the space $\mathcal{Z}_T$ defined in (4.3) is replaced by a bigger space $\hat{\mathcal{Z}}_T$ defined by*

$$\hat{\mathcal{Z}}_T := L^2_w(0,T;\mathrm{V}) \cap L^2(0,T;\mathrm{H}_{\mathrm{loc}}) \cap \mathcal{C}([0,T];\mathrm{H}_w). \tag{4.46}$$

In particular, to prove that the sequence $(\tilde{u}_n)$ *given in* (4.20)*, whose existence follows from the Skorokhod Theorem, converges to a solution of the Navier-Stokes equation, it is sufficient to use the convergence of* $(\tilde{u}_n)$ *in the space* $\hat{\mathcal{Z}}_T$*.*

5. The case of 2D domains. A special result proved recently in [16] is about the existence and uniqueness of strong solutions for 2-D stochastic Navier Stokes equations in unbounded domains with a general noise.

Let us present the framework and the results. Let us recall Lemma 7.2 from [16].

LEMMA 5.1. *Let* $d = 2$ *and assume that all conditions in parts* ***(H.1)-(H.3)*** *and* ***(H.5)*** *of Assumption 3.1 are satisfied. Assume that* $\mu_0 = \delta_{u_0}$ *for some deterministic* $u_0 \in \mathrm{H}$*. Let* $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{W}, \hat{\mathbb{P}}, u)$ *be a martingale solution of problem* (3.2)*, in particular,*

$$\hat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |u(t)|_{\mathrm{H}}^2 + \int_0^T |\nabla u(t)|^2\, dt\Big] < \infty. \tag{5.1}$$

Then for $\hat{\mathbb{P}}$*-almost all* $\omega \in \hat{\Omega}$ *the trajectory* $u(\cdot,\omega)$ *is equal almost everywhere to a continuous* H*-valued function defined on* $[0,T]$*.* $\hat{\mathbb{P}}$*-a.s. and*

$$\begin{aligned} u(t) \;=\; u_0 \;-\; \int_0^t \big[\mathcal{A}u(s) + B(u(s))\big]\, ds + \int_0^t f(s)ds \\ + \int_0^t G(u(s))d\hat{W}(s), \quad t \in [0,T]. \end{aligned} \tag{5.2}$$

Let us emphasize that equality (5.2) is understood as the one in the space V', see Remark 3.7.

The next result is [16, Lemma 7.3].

LEMMA 5.2. *Assume that all conditions in parts **(H.1)**-**(H.3)** and **(H.5)** of Assumption 3.1 are satisfied. In addition we assume that the Lipschitz constant of G is smaller than $\sqrt{2}$, i.e. the map G satisfies condition (G1) in part **(H.2)** of Assumption 3.1 with $L < \sqrt{2}$. Assume that $u_0 \in \mathrm{H}$. If u_1 and u_2 are two solutions of problem (3.2) defined on the same filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ and the same Wiener process $\hat{W}$, then $\hat{\mathbb{P}}$-a.s. for all $t \in \mathbb{R}_+$, $u_1(t) = u_2(t)$.*

Because from now we will be dealing with the pathwise uniqueness of solutions let us formulate the following assumption on the stochastic basis.

ASSUMPTION 5.3. *Assume that $\big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\big)$ is a stochastic basis with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ and $W = \big(W(t)\big)_{t\geq 0}$ is a cylindrical Wiener process in a separable Hilbert space K defined on this stochastic basis.*

We will often consider problem (3.2) with the initial data $\mu_0 = \delta_{u_0}$ for some deterministic $u_0 \in \mathrm{H}$, and hence we explicitly rewrite that problem in the following way:

(5.3)
$$\begin{cases} du(t) + \mathcal{A}u(t)\,dt + B\big(u(t), u(t)\big)\,dt = f(t)\,dt + G\big(u(t)\big)\,dW(t), \quad t \geq 0, \\ u(0) = u_0, \end{cases}$$

To avoid any confusion, a martingale solution to problem (5.3) with initial data $u_0 \in \mathrm{H}$, is a martingale solution to problem (3.2) with $\mu_0 = \delta_{u_0}$.

For the completeness of the exposition let us also recall a notion of a strong solution.

DEFINITION 5.4. Assume that $u_0 \in \mathrm{H}$ and $f : [0,\infty) \to \mathrm{V}'$. Assume Assumption 5.3. We say that an $\mathbb{F}$-progressively measurable process $u : [0,\infty) \times \Omega \to \mathrm{H}$ with $\mathbb{P}$ - a.e. paths

$$u(\cdot,\omega) \in \mathcal{C}\big([0,\infty),\mathrm{H}_w\big) \cap L^2_{\mathrm{loc}}([0,\infty);\mathrm{V})$$

is a **strong solution** to problem (5.3), i.e.,

$$\begin{cases} du(t) + \mathcal{A}u(t)\,dt + B\big(u(t),u(t)\big)\,dt = f(t)\,dt + G\big(u(t)\big)\,dW(t), \quad t \geq 0, \\ u(0) = u_0, \end{cases}$$

if and only if for all $t \in [0,\infty)$ and all $\mathrm{v} \in \mathcal{V}$ the following identity holds $\mathbb{P}$ - a.s.

$$\begin{aligned} \big(u(t),\mathrm{v}\big)_{\mathrm{H}} \;&+\; \int_0^t \langle \mathcal{A}u(s),\mathrm{v}\rangle\,ds + \int_0^t \langle B(u(s),u(s)),\mathrm{v}\rangle\,ds \\ &=\; \big(u_0,\mathrm{v}\big)_{\mathrm{H}} + \int_0^t \langle f(s),\mathrm{v}\rangle\,ds + \Big\langle \int_0^t G(u(s))\,dW(s),\mathrm{v}\Big\rangle \end{aligned}$$

and for all $T > 0$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |u(t)|^2_{\mathrm{H}} + \int_0^T |\nabla u(t)|^2\,dt\Big] < \infty. \tag{5.4}$$

Let us recall two basic concepts of uniqueness of the solution, i.e. pathwise uniqueness and uniqueness in law, see [33], [45]. Please note the following difference between problems (3.2) and (5.3). In the former, a law of the initial data is prescribed, while in the latter a initial data is given.

DEFINITION 5.5. We say that solutions of problem (5.3) has **pathwise uniqueness property** if and only if for all $u_0 \in \mathrm{H}$ and $f : [0,\infty) \to \mathrm{V}'$

the following condition holds

(5.5) *if* $u^i,\ i=1,2,$ *are strong solutions of problem* (5.3) on $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W)$ *satisfying Assumption 5.3, then* $\mathbb{P}$*-a.s. for all* $t\in[0,\infty),\ u^1(t)=u^2(t)$.

Assume that $u_0\in \mathrm{H}$ and $f:[0,\infty)\to \mathrm{V}'$. A solution u to problem (5.3) on $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W)$ satisfying Assumption 5.3, is said to be pathwise unique iff for every solution $\tilde{u}$ to problem (5.3) on the same $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W)$, one has

$$\mathbb{P}\text{-a.s. for all } t\in[0,\infty),\ \ u(t)=\tilde{u}(t).$$

Definition 5.6. We say that problem (3.2) has **uniqueness in law property** iff for every Borel measure μ on H and every $f:[0,\infty)\to \mathrm{V}'$ the following condition holds

(5.6) *if* $(\Omega^i,\mathcal{F}^i,\mathbb{F}^i,\mathbb{P}^i,W^i,u^i),\ i=1,2,$ *are solutions of problem* (3.2) *then* $Law_{\mathbb{P}^1}(u^1)=Law_{\mathbb{P}^2}(u^2)$ on $\mathcal{C}\big([0,\infty),\mathrm{H}_w\big)\cap L^2_{\mathrm{loc}}([0,\infty);\mathrm{V})$,

where $Law_{\mathbb{P}^i}(u^i)$, $i=1,2$, are by definition probability measures on $\mathcal{C}\big([0,\infty),\mathrm{H}_w\big)\cap L^2_{\mathrm{loc}}([0,\infty);\mathrm{V})$.

Corollary 5.7. *Assume that conditions* ***(H.1)-(H.3)*** *and* ***(H.5)*** *of Assumption 3.1 are satisfied and that the map* G *satisfies inequality* (G1) *in part* ***(H.2)*** *of Assumption 3.1 with a constant* L *smaller than* $\sqrt{2}$. *Assume also that* $\big(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W\big)$ *satisfies Assumption 5.3. Then for every* $u_0\in \mathrm{H}$.

1) *There exists a pathwise unique strong solution* u *on* $\big(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W\big)$ *of problem* (5.3).

2) Moreover, if u is a strong solution of problem (5.3) on $\big(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W\big)$, then for $\mathbb{P}$-almost all $\omega\in\Omega$ the trajectory $u(\cdot,\omega)$ is equal almost everywhere to a continuous H*-valued function defined on $[0,\infty)$.*

3) The martingale solution of problem (3.2) with $\mu_0=\delta_{u_0}$ is unique in law. In particular, if $\big(\Omega^i,\mathcal{F}^i,\mathbb{F}^i,\mathbb{P}^i,W^i,u^i\big)$, $i=1,2$ t are such solutions to problem (3.2), then for all $t\geq 0$, the laws on H *of* H*-valued random variables $u^1(t)$ and $u^2(t)$ coincide.*

Proof. The proof of part (3) given in [16] yields the uniqueness in law in the trajectory the space $\mathcal{C}\big([0,\infty),\mathrm{H}_w\big)\cap L^2_{\mathrm{loc}}([0,\infty);\mathrm{V})$, hence in $\mathcal{C}\big([0,T],\mathrm{H}_w\big)\cap L^2(0,T;\mathrm{V})$ for every $T>0$. □

Let us emphasize that, by definition, we require a martingale solution of the Navier-Stokes equation to satisfy inequality (3.5), i.e.

$$\hat{\mathbb{E}}\Big[\sup_{t\in[0,T]}|u(t)|^2_{\mathrm{H}}+\int_0^T|\nabla u(t)|^2\,dt\Big]<\infty.$$

In Theorem 4.8, covering both 2D and 3D domains, we proved that there exists a martingale solution satisfying stronger estimates, i.e. (4.8)-(4.11). However, in the case when $\mathcal{O}$ is a 2D domain, we can prove that every martingale solution satisfies these inequalities.

Lemma 5.8. *Assume that $d=2$ and that conditions **(H.1)-(H.3)** and **(H.5)** from Assumption 3.1 are satisfied. Then the following holds.*

(1) For every $T>0$, $R_1>0$ and $R_2>0$ there exist constants $C_1(p)$ and $C_2(p)$ depending also on T, R_1 and R_2 such that if μ_0 is a Borel probability measure on H, $f\in L^p(0,T;\mathrm{V}')$ satisfy $\int_H|x|^p\mu_0(dx)\leq R_1$ and $|f|_{L^p(0,T;\mathrm{V}')}\leq R_2$, then every martingale solution of problem (3.2)

with the initial data μ_0 and the external force f, satisfies the following estimates

$$\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|u(s)|_{\mathrm{H}}^{p}\Big)\leq C_1(p) \tag{5.7}$$

and

$$\hat{\mathbb{E}}\Big[\int_0^T|u(s)|_{\mathrm{H}}^{p-2}|\nabla u(s)|^2\,ds\Big]\leq C_2(p). \tag{5.8}$$

In particular,

$$\hat{\mathbb{E}}\Big[\int_0^T|\nabla u(s)|^2\,ds\Big]\leq C_2:=C_2(2). \tag{5.9}$$

(2) Moreover, if $\mathcal{O}$ is a Poincaré domain and the map G satisfies inequality (G2) *in part* ***(H.2)*** *of Assumption 3.1 with $\lambda_0=0$ (and with $\rho\in[0,\infty)$ and $\eta\in(0,2]$), then the process u satisfies additionally the following inequality for every $t\geq 0$*

(5.10)
$$\hat{\mathbb{E}}[\,|u(t)|_{\mathrm{H}}^2\,]+\frac{\eta}{2}\hat{\mathbb{E}}\bigg[\int_0^t|\nabla u(s)|^2\,ds\bigg]\leq\hat{\mathbb{E}}[\,|u(0)|_{\mathrm{H}}^2\,]+\frac{2}{\eta}\int_0^t|f(s)|_{\mathrm{V}'}^2\,ds+\rho t.$$

The proof of Lemma 5.8 is similar to the proof of estimates (5.4), (5.5) and (5.6) from Appendix in [16]. The difference is that the solution process u to which the Itô formula (in a classical form, see for instance [33]) was applied was taking values in a finite dimensional Hilbert space H_n and u was a solution in the most classical way. Now, u_n is martingale solution to problem (3.2), see Definition 3.2.

If we assume that $d=2$, by Lemma III.3.4 p. 198 in [54], we infer that the regularity assumption (3.3) implies that

$$B\big(u(\cdot,\omega),u(\cdot,\omega)\big)\in L^2_{\mathrm{loc}}([0,\infty);\mathrm{V}')\ \text{for}\ \hat{\mathbb{P}}\text{-a.a. }\omega\in\Omega.$$

This however does not imply that

$$\hat{\mathbb{E}} \int_0^T |B(u(t),u(t))|^2_{\mathrm{V}'}\,dt < \infty$$

what is necessary in order to apply the infinite dimensional Itô Lemma from [47].

Fortunately, we can proceed as in the proof of the uniqueness result, i.e. Lemma 7.3 from [16], i.e. introduce a family τ_N, $N \in \mathbb{N}$ of the stopping times defined by

$$\tau_N := \inf\{t \in [0,\infty) : |u(t)|_{\mathrm{H}} \geq N\}, \qquad N \in \mathbb{N}. \tag{5.11}$$

and then consider a stopped process $u(t \wedge \tau_N)$, $t \geq 0$. Note that with this definition of the stopping time τ_N, we have

$$\hat{\mathbb{E}} \int_0^{T\wedge\tau_N} |B(u(t),u(t))|^2_{\mathrm{V}'}\,dt \leq CN^2\hat{\mathbb{E}} \int_0^T \|u(t)\|^2\,dt < \infty.$$

Remark. If $d = 3$, then

$$B\big(u(\cdot,\omega),u(\cdot,\omega)\big) \in L^{4/3}(0,T;\mathrm{V}') \quad \text{ for } \hat{\mathbb{P}}\text{-a.a. } \omega \in \Omega.$$

Thus, in this case the above procedure with the stopping time τ_N does not help.

Proof of Lemma 5.8. Let us fix p satisfying condition (3.1). As in the proof of Lemma A.1, we apply the Itô formula from [47] to the function F defined by

$$F : \mathrm{H} \ni x \mapsto |x|^p_{\mathrm{H}} \in \mathbb{R}.$$

With the above comments in mind and using Remark 3.6, we have, for $t \in [0,\infty)$,

(5.12)

$$\begin{aligned}|u(t\wedge\tau_N)|^p - |u(0)|^p &= \int_0^{t\wedge\tau_N} \Big[p\,|u(s)|^{p-2}\langle u(s), -\mathcal{A}u(s) - B\big(u(s)\big) + f(s)\rangle \\ &\qquad + \frac{1}{2}\mathrm{Tr}\left[F''(u(s))\big(G(u(s)), G(u(s))\big)\right]\Big]\,ds \\ &\quad + p\int_0^{t\wedge\tau_N} |u(s)|^{p-2}\langle u(s), G(u(s))\,d\hat{W}(s)\rangle \\ &= \int_0^{t\wedge\tau_N} \Big[-p\,|u(s)|^{p-2}\|u(s)\|^2 + p\,|u(s)|^{p-2}\langle u(s), f(s)\rangle \\ &\qquad + \frac{1}{2}\mathrm{Tr}\big[F''(u(s))\big(G(u(s)), G(u(s))\big)\big]\Big]\,ds \\ &\quad + p\int_0^{t\wedge\tau_N} |u(s)|^{p-2}\langle u(s), G(u(s))\,d\hat{W}(s)\rangle.\end{aligned}$$

Proceeding as in the proof of Lemma A.1, we obtain

$$\begin{aligned}(5.13)\quad &|u(t\wedge\tau_N)|^p + \delta\int_0^{t\wedge\tau_N} |u(s)|^{p-2}|\nabla u(s)|^2\,ds \\ &\le |u(0)|^p + K_p(\lambda_0,\rho)\int_0^{t\wedge\tau_N} |u(s)|^p ds + \frac{2\rho}{p}t + \varepsilon^{-p/2}\int_0^{t\wedge\tau_N} |f(t)|^p_{\mathrm{V}'}\,ds \\ &+p\int_0^t |u(s)|^{p-2}\langle u(s), G(u(s))\,d\hat{W}(s)\rangle, \quad t\in[0,\infty),\end{aligned}$$

where $K_p(\lambda_0,\rho) = \frac{p-1}{2}[\lambda_0 p + 2 + \rho(p-2)]$.

By the definition of the stopping time τ_N we infer that the process

$$\mu_N(t) := \int_0^{t\wedge\tau_N} |u(s)|^{p-2}\langle u(s), G(u(s))\,d\hat{W}(s)\rangle, \qquad t\in[0,\infty)$$

is a martingale. Indeed, if we define a map

$$g : \mathrm{V} \ni u \mapsto \{\mathrm{K}\ni k \mapsto \langle u, G(u)k\rangle \in \mathrm{H}\} \in \mathcal{T}_2(\mathrm{K},\mathbb{R})$$

then $\mu_N(t) = \int_0^{t\wedge\tau_N} |u(s)|^{p-2} g(u(s))dW(s)$ and, since the map G satisfies inequality (G2) in part **(H.2)** of Assumption 3.1, we infer that for every

$t \geq 0$,

$$
\begin{aligned}
&\int_0^{t\wedge\tau_N} \| \, |u(s)|^{p-2} g(u(s))\|^2_{\mathcal{T}_2(\mathrm{K},\mathbb{R})} \, ds \\
&= \int_0^{t\wedge\tau_N} |u(s)|^{p-2} \| \, g(u(s))\|^2_{\mathcal{T}_2(\mathrm{K},\mathbb{R})} \, ds \\
&\leq \int_0^{t\wedge\tau_N} |u(s)|^{p-2} |u(s)|^2 \|G(u(s))\|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})} \, ds \\
&\leq \int_0^{t\wedge\tau_N} |u(s)|^{p} \big[(2-\eta)\, |\nabla u(t)|^2 + \lambda_0 |u(t)|^2 + \rho\big] \, ds \\
&\leq (2-\eta) N^p \int_0^{t\wedge\tau_N} |\nabla u(t)|^2 \, dt + t N^p (\lambda_0 N^2 + \rho).
\end{aligned} \tag{5.14}
$$

Hence by inequality (3.5) we infer that

$$
\hat{\mathbb{E}} \int_0^{t\wedge\tau_N} \| \, |u(s)|^{p-2} g(u(s)) \, \|^2_{\mathcal{T}_2(\mathrm{K},\mathbb{R})} \, ds < \infty, \quad t \geq 0.
$$

and thus we infer, as claimed, that the process μ_N is a martingale. Hence, $\mathbb{E}[\mu_N(t)] = 0$. Let us now fix $T > 0$. By taking expectation in inequality (5.13) we infer that

$$
\begin{aligned}
&\hat{\mathbb{E}} \left[|u(t\wedge\tau_N)|^p \right] \leq \hat{\mathbb{E}}[|u(0)|^p] \\
&+ K_p(\lambda_0,\rho) \int_0^{t\wedge\tau_N} \hat{\mathbb{E}} \left[|u(s)|^p \right] ds + \frac{2\rho}{p} (t\wedge\tau_N) + \varepsilon^{-p/2} (t\wedge\tau_N) |f|^p_{\mathrm{V}'} \\
&\leq \hat{\mathbb{E}}[|u(0)|^p] \\
&+ K_p(\lambda_0,\rho) \int_0^{t\wedge\tau_N} \hat{\mathbb{E}} \left[|u(s\wedge\tau_N)|^p \right] ds + T\big(\frac{2\rho}{p} + \varepsilon^{-p/2} |f|^p_{\mathrm{V}'}\big), \ t \in [0,T].
\end{aligned}
$$

Hence by the Gronwall Lemma there exists a constant $C = C_p(T,\eta,\lambda_0,\rho,\hat{\mathbb{E}}[|u(0)|^p], |f|_{L^p(0,T;\mathrm{V}')}) > 0$ such that

$$
\hat{\mathbb{E}} \left[|u(t\wedge\tau_N)|^p \right] \leq C, \qquad t \in [0,T]. \tag{5.15}
$$

Using this bound in (5.13) we also obtain

$$
\hat{\mathbb{E}} \left[\int_0^{T\wedge\tau_N} |u(s)|^{p-2} |\nabla u(s)|^2 \, ds \right] \leq C \tag{5.16}
$$

for a new constant $C = \tilde{C}_p(\eta, \hat{\mathbb{E}}\,|u(0)|^p, \hat{\mathbb{E}}\int_0^T |f(s)|^p_{\mathrm{V}'}\,ds) > 0$. Finally, taking the limit $N \to \infty$ and observing that $T \wedge \tau_N \to T$, by the Lebesgue dominated convergence Theorem we infer that for the same constant C we have

$$\sup_{t\in[0,T]} \hat{\mathbb{E}}\left[|u(t)|^p\right] \leq C, \tag{5.17}$$

$$\hat{\mathbb{E}}\left[\int_0^T |u(s)|^{p-2}|\nabla u(s)|^2\,ds\right] \leq C. \tag{5.18}$$

This completes the proof of estimates (5.8) and (5.9). The proof of inequality (5.7) is the same as the proof of inequality (A.2) and thus omitted.

To prove inequality (5.10) in the case $\mathcal{O}$ is a Poincaré domain we use the same arguments as the proof of inequality (A.5). This time however, the solution to the Galerkin approximating equation is replaced by the stopped process $u(t \wedge \tau_N)$, $t \geq 0$. Let us recall that in the space V we consider the inner product $\big((\cdot,\cdot)\big)$ given by (2.2).

By identity (5.12) with $p = 2$, we have

$$\begin{aligned} |u(t\wedge\tau_N)|^2 \;-\; |u(0)|^2 \;&=\; \int_0^{t\wedge\tau_N} \Big\{-2\,\|u(s)\|^2 + 2\,\langle u(s), f\rangle \\ &+\; \frac{1}{2}\mathrm{Tr}\big[F''(u(s))\big(G(u(s)), G(u(s))\big)\big]\Big\}\,ds \\ &+\; 2\int_0^{t\wedge\tau_N} \langle u(s), G(u(s))\,d\hat{W}(s)\rangle, \quad t \geq 0. \end{aligned}$$

Since $\hat{\mathbb{E}}\big(\int_0^{t\wedge\tau_N} \langle G(u(s)), u(s)\,d\hat{W}(s)\rangle\big) = 0$, we infer that

$$\begin{aligned} \hat{\mathbb{E}}|u(t\wedge\tau_N)|^2_{\mathrm{H}} \;\leq\; &\hat{\mathbb{E}}[\,|u(0)|^2_{\mathrm{H}}\,] + \hat{\mathbb{E}}\int_0^{t\wedge\tau_N} \big\{-2\|u(s)\|^2 + 2\langle f(s), u(s)\rangle\big\}\,ds \\ &+ \hat{\mathbb{E}}\int_0^{t\wedge\tau_N} |G(u(s))|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}\,ds. \end{aligned}$$

Taking next the $N \to \infty$ limit, since the map G satisfies inequality (G2) in part **(H.2)** of Assumption 3.1 with $\lambda_0 = 0$, i.e.
$|G(u(s))|^2_{\mathcal{T}_2(K,\mathrm{H})} \leq (2-\eta)\|u(s)\|^2 + \varrho$, we get

(5.19)
$$\hat{\mathbb{E}}|u(t)|^2_{\mathrm{H}} \leq -\eta\mathbb{E}\int_0^t \|u(s)\|^2\,ds + \hat{\mathbb{E}}[\,|u(0)|^2_{\mathrm{H}}\,] + 2\hat{\mathbb{E}}\int_0^t \langle f(s), u(s)\rangle\,ds + \varrho t.$$

Since $2\langle f, u(s)\rangle \leq \frac{\eta}{2}|\nabla u(s)|^2 + \frac{2}{\eta}|f|^2_{\mathrm{V}'}$ we infer that

(5.20)
$$\hat{\mathbb{E}}|u(t)|^2_{\mathrm{H}} \leq -\frac{\eta}{2}\hat{\mathbb{E}}\int_0^t \|u(s)\|^2\,ds + \hat{\mathbb{E}}[|u(0)|^2_{\mathrm{H}}] + \frac{2}{\eta}\int_0^t |f(s)|^2_{\mathrm{V}'} + \varrho t, \quad t \geq 0.$$

The proof of inequality (5.10) is thus complete. This completes the proof of Lemma 5.8. □

Note that if $f : [0,\infty) \to \mathrm{V}'$ is constant, then $f \in L^p(0,T;\mathrm{V}')$ for every $T > 0$ and p satisfying condition **(H.3)** of Assumption 3.1. In this case we will write $f \in \mathrm{V}'$.

By Theorem 4.11 Corollary 5.7 and Lemma 5.8 we obtain the following result about the continuous dependence of the solutions to 2D SNSEs with respect to the initial data and the external forces.

THEOREM 5.9. *Let $d = 2$. Let parts **(H.1)**-**(H.2)**, **(H.5)** and (G1) with a constant L smaller than $\sqrt{2}$, of Assumption 3.1, be satisfied. Assume that $u_0 \in \mathrm{H}$, $f \in \mathrm{V}'$ and that an H-valued sequence $\big(u_{0,n}\big)_{n=1}^{\infty}$ is weakly convergent in H to u_0, and that an V'-valued sequence $\big(f_n\big)_{n=1}^{\infty}$ is weakly convergent in V' to f. Let*

$$\big(\Omega_n, \mathcal{F}_n, \mathbb{F}_n, \mathbb{P}_n, W_n, u_n\big)$$

be a martingale solution of problem (5.3) *on* $[0,\infty)$ *with the initial data* $u_{0,n}$ *and the external force* f_n. *Then for every* $T>0$ *there exist*

- *a subsequence* $(n_k)_k$,
- *a stochastic basis* $\big(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{F}},\tilde{\mathbb{P}}\big)$, *where* $\tilde{\mathbb{F}}=\{\tilde{\mathcal{F}}_t\}_{t\geq 0}$,
- *a cylindrical Wiener process* $\tilde W=\tilde W(t)$, $t\in[0,\infty)$ *defined on this basis,*
- *and* $\mathbb{F}$*-progressively measurable processes* $\tilde u(t)$, $\big(\tilde u_{n_k}(t)\big)_{k\geq 1}, t\in[0,T]$ *(defined on this basis) with laws supported in* $\mathcal{Z}_T$ *such that*

(5.21) $\tilde u_{n_k}$ *has the same law as* u_{n_k} *on* $\mathcal{Z}_T$ *and* $\tilde u_{n_k}\to\tilde u$ *in* $\mathcal{Z}_T$, $\quad\tilde{\mathbb{P}}$ *- a.s.*

and the system

$$\big(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{F}},\tilde{\mathbb{P}},\tilde W,\tilde u\big) \tag{5.22}$$

is a martingale solution to problem (5.3) *on the interval* $[0,T]$ *with the initial law* δ_{u_0}. *In particular, for all* $t\in[0,T]$ *and* $\mathrm{v}\in\mathrm{V}$

$$\begin{aligned}\big(\tilde u(t),\mathrm{v}\big)_{\mathrm{H}} &- \big(\tilde u(0),\mathrm{v}\big)_{\mathrm{H}}+\int_0^t\langle\mathcal{A}\tilde u(s),\mathrm{v}\rangle\,ds+\int_0^t\langle B\big(\tilde u(s)\big),\mathrm{v}\rangle\,ds\\ &=\int_0^t\langle f,\mathrm{v}\rangle\,ds+\Big\langle\int_0^t G\big(\tilde u(s)\big)\,d\tilde W(s),\mathrm{v}\Big\rangle.\end{aligned}$$

Moreover, the process $\tilde u$ *satisfies the following inequality for every* p *satisfying condition* (3.1) *and* $q\in[1,p]$

$$\tilde{\mathbb{E}}\Big[\sup_{s\in[0,T]}|\tilde u(s)|_{\mathrm{H}}^q\Big]+\tilde{\mathbb{E}}\Big[\int_0^T\|\tilde u(s)\|^2\,ds\Big]<\infty. \tag{5.23}$$

Proof. Let p be any exponent satisfying condition (3.1). Since the sequences $\big(u_{0,n}\big)_{n=1}^\infty\subset\mathrm{H}$ and $(f_n)_{n=1}^\infty\subset\mathrm{V}'$ convergent weakly in H and V′, respectively, we infer that there exist $R_1>0$ and $R_2>0$ such that

$$\sup_{n\in\mathbb{N}}|u_{0,n}|_{\mathrm{H}}\leq R_1\quad\text{and}\quad\sup_{n\in\mathbb{N}}\|f_n\|_{\mathrm{V}'}\leq R_2.$$

By Lemma 5.8 we infer that the processes u_n, $n \in \mathbb{N}$, satisfy inequalities (4.8)-(4.10). Thus the first part of the assertion follows directly from Theorem 4.11. Inequality (5.23) follows again from Lemma 5.8. The proof of theorem is thus complete. □

REMARK 5.10. *Although this has not been studied in the present paper, we believe that methods developed here can be used to study the continuous dependence of the solutions on other parameters entering our equations, for instance the linear operator A, the nonlinearity B and the diffusion operator G.*

6. Existence of an invariant measure for Stochastic NSEs on 2-dimensional domains. In this section we assume that $d = 2$. Since we are interested in the existence of invariant measures we assume that the domain $\mathcal{O}$ satisfies the Poincaré condition see (2.4). [3] However, our results are true for general domains for the stochastic damped Navier-Stokes equations, see for instance [22].

Since we assume that $\mathcal{O}$ is a Poincaré domain, by the Poincaré inequality, see (2.4), the functional given by the formula

$$\|u\| = |\nabla u|_{L^2}, \qquad u \in \mathrm{V}, \tag{6.1}$$

is a norm in the space V equivalent to the norm given by (2.3).

In the sequel, in the space V *we consider the norm given by* (6.1).

We aim in this section to prove that, under some natural assumptions, problem (3.2) has an invariant measure. Let us fix, as in Assumptions 5.3,

[3] It is well known that this condition holds if the domain $\mathcal{O}$ is bounded in some direction, i.e. there exists a vector $h \in \mathbb{R}^d$ such that $\mathcal{O} \cap (h + \mathcal{O}) = \emptyset$.

a stochastic basis $\big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\big)$ with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$; a canonical cylindrical Wiener process $W = W(t)$ in a separable Hilbert space K defined on the stochastic basis $\big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\big)$. We also fix a function $G : \mathrm{H} \to \mathcal{T}_2(\mathrm{K}, \mathrm{V}')$ satisfying condition **(H.2)** in Assumption 3.1 and, in addition, the Lipschitz condition (G1) with a constant L smaller than $\sqrt{2}$, and inequality (G2) with $\lambda_0 = 0$. The last assumption on λ_0 corresponds to the fact that in $\mathcal{O}$ we consider the norm given by (6.1). In what follows the initial data u_0 will be an element of the space H. By $u(t, u_0)$, $t \geq 0$, we denote the unique solution to the problem (5.3) (defined on the above stochastic basis satisfying Assumptions 5.3).

For any bounded Borel function $\varphi \in \mathcal{B}_b(\mathrm{H})$ and $t \geq 0$ we define

$$(P_t\varphi)(u_0) = \mathbb{E}[\varphi(u(t, u_0))], \quad u_0 \in \mathrm{H}. \tag{6.2}$$

Since by Lemma 5.1 the trajectories $u(\cdot, u_0)$ are continuous, $(P_t)_{t\geq 0}$ is a stochastically continuous semigroup on the Banach space $\mathcal{C}_b(\mathrm{H})$. This means that for every $\varphi \in \mathcal{C}_b(\mathrm{H})$ and $u_0 \in \mathrm{H}$

$$\lim_{t\to 0} P_t\varphi(u_0) = u_0.$$

As a consequence of Corollary 5.7 we have the following result.

PROPOSITION 6.1. *The family $u(t, u_0)$, $t \geq 0$, $u_0 \in \mathrm{H}$ is Markov. In particular, $P_{t+s} = P_tP_s$ for $t, s \geq 0$.*

The proof of Proposition 6.1 is standard and thus omitted, see e.g. [1], [26, Section 9.2], [50, Section 9.7].

PROPOSITION 6.2. *The semigroup P_t is bw-Feller, i.e. if $\phi : \mathrm{H} \to \mathbb{R}$ is a bounded sequentially weakly continuous function and $t > 0$ then $P_t\phi : \mathrm{H} \to \mathbb{R}$*

is also a bounded sequentially weakly continuous function. In particular, if $u_{0n} \to u_0$ *weakly in* H *then*

$$P_t\phi(u_{0n}) \to P_t\phi(u_0).$$

Proof of Proposition 6.2. Let us choose and fix $t > 0$, $u_0 \in \mathrm{H}$ and an H-valued sequence (u_{0n}) that is weakly convergent to u_0 in H. Let also $\phi : \mathrm{H} \to \mathbb{R}$ be a bounded sequentially weakly continuous function. Let us choose an auxiliary time $T \in (t, \infty)$.

Since obviously the function $P_t\phi : \mathrm{H} \to \mathbb{R}$ is bounded, we only need to prove that it is sequentially weakly continuous.

Let $u_n(\cdot) = u(\cdot, u_{0n})$, respectively $u(\cdot) = u(\cdot, u_0)$, be a strong solution of problem (5.3) on $[0, \infty)$ with the initial data u_{0n}, resp. u_0. We assume that these processes are defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$. By Theorem 5.9 there exist (depending on T)

- a subsequence $(n_k)_k$,
- a stochastic basis $\big(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}\big)$, where $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_s\}_{s\in[0,T]}$,
- a cylindrical Wiener process $\tilde W = \tilde W(s)$, $s \in [0, T]$ defined on this basis,
- and an $\mathbb{F}$-progressively measurable processes $\tilde u(s)$, $\big(\tilde u_{n_k}(s)\big)_{k\geq 1}, s \in [0, T]$ (defined on this basis) with laws supported in $\mathcal{Z}_T$ such that

(6.3) $\tilde u_{n_k}$ *has the same law as* u_{n_k} *on* $\mathcal{Z}_T$ *and* $\tilde u_{n_k} \to \tilde u$ *in* $\mathcal{Z}_T$, $\tilde{\mathbb{P}}$ - *a.s.*

and the system

$$\big(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde W, \tilde u\big) \tag{6.4}$$

is a martingale solution to problem (5.3) on the interval $[0,T]$ with the initial data u_0. In particular, by (6.3), $\tilde{\mathbb{P}}$-almost surely

$$\tilde{u}_{n_k}(t) \to \tilde{u}(t) \quad \text{weakly in H.}$$

Since the function $\phi : \mathrm{H} \to \mathbb{R}$ is sequentially weakly continuous, we infer that $\tilde{\mathbb{P}}$-a.s.,

$$\phi(\tilde{u}_{n_k}(t)) \to \phi(\tilde{u}(t)) \quad \text{in } \mathbb{R}.$$

Therefore, since the function $\phi : \mathrm{H} \to \mathbb{R}$ is also bounded, by the Lebesgue Dominated Convergence Theorem we infer that

$$\lim_{k\to\infty} \tilde{\mathbb{E}}[\phi(\tilde{u}_{n_k}(t))] = \tilde{\mathbb{E}}[\phi(\tilde{u}(t))]. \tag{6.5}$$

From the equality of laws of $\tilde{u}_{n_k}$ and u_{n_k}, $k \in \mathbb{N}$, on the space $\mathcal{Z}_T$ we infer that

$$\tilde{\mathbb{E}}[\phi(\tilde{u}_{n_k}(t))] = \mathbb{E}[\phi(u_{n_k}(t))] = P_t\phi(u_{0n_k}). \tag{6.6}$$

Since by assumptions $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, u)$ is a martingale solution of equation (5.3) with the initial data u_0 and $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{u}\right)$ is also a martingale solution with the initial of equation (5.3) with the initial data u_0 and since the solution of (5.3) is unique in law, we infer that

$$\text{the processes } u \text{ and } \tilde{u} \text{ have the same law on the space } \mathcal{Z}_t.$$

Hence

$$\tilde{\mathbb{E}}[\phi(\tilde{u}(t))] = \mathbb{E}[\phi(u(t))] = P_t\phi(u_0). \tag{6.7}$$

Thus by (6.5), (6.6) and (6.7), we infer that

$$\lim_{k\to\infty} P_t\phi(u_{0n_k}) = P_t\phi(u_0).$$

Using the sub-subsequence argument, we infer that the whole sequence $(P_t\phi(u_{0n}))_{n\in\mathbb{N}}$ is convergent and

$$\lim_{n\to\infty} P_t\phi(u_{0n}) = P_t\phi(u_0),$$

which completes the proof of Proposition 6.2. □

REMARK 6.3. From inequality (5.10) and the Poincaré inequality (2.4), it follows that the following inequality holds for the strong solution u of problem (5.3) defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$

$$\int_0^t \mathbb{E}|u(s)|_{\mathrm{H}}^2\, ds \;\le\; \frac{2}{C\eta}|u_0|_{\mathrm{H}}^2 \;+\; \frac{2}{C\eta}\Big(\frac{2}{\eta}|f|_{\mathrm{V}'}^2 + \varrho\Big)t, \quad t \ge 0. \tag{6.8}$$

PROOF OF INEQUALITY (6.8). Let us fix $t \ge 0$. By the Poincaré inequality (2.4) for almost all $s \in [0,t]$,

$$|u(s)|_{\mathrm{H}}^2 \;\le\; \frac{1}{C}|\nabla u(s)|_{L^2}^2.$$

By (5.10), in particular, we obtain

$$\frac{\eta}{2}\mathbb{E}\int_0^t |\nabla u(s)|^2\, ds \;\le\; |u_0|_{\mathrm{H}}^2 + \Big(\frac{2}{\eta}|f|_{\mathrm{V}'}^2 + \varrho\Big)t$$

Hence we infer that

$$\int_0^t \mathbb{E}|u(s)|_{\mathrm{H}}^2\, ds \le \frac{1}{C}\mathbb{E}\int_0^t |\nabla u(s)|^2\, ds \le \frac{2}{C\eta}|u_0|_{\mathrm{H}}^2 + \frac{2}{C\eta}\Big(\frac{2}{\eta}|f|_{\mathrm{V}'}^2 + \varrho\Big)t, \quad t \ge 0,$$

i.e. inequality (6.8) holds. □

Using inequality (6.8) we deduce the following result.

COROLLARY 6.4. *Let $u_0 \in \mathrm{H}$ and let $u(t)$, $t \ge 0$, be the unique solution to the problem (5.3) starting from u_0. Then there exists $T_0 \ge 0$ such that for*

every $\varepsilon > 0$ there exists $R > 0$ such that

$$\sup_{T\geq T_0} \frac{1}{T}\int_0^T (P_s^*\delta_{u_0})(\mathrm{H}\setminus \bar{\mathbb{B}}_R)\,ds \;\leq\; \varepsilon, \tag{6.9}$$

where $\bar{\mathbb{B}}_R = \{\mathrm{v}\in \mathrm{H}:\ |\mathrm{v}|_{\mathrm{H}}\leq R\}$.

Proof. Using the Chebyshev inequality and inequality (6.8) we infer that for every $T\geq 0$ and $R>0$

$$\begin{aligned}
\frac{1}{T}\int_0^T (P_s^*\delta_{u_0})(\mathrm{H}\setminus \bar{\mathbb{B}}_R)\,ds &= \frac{1}{T}\int_0^T \mathbb{P}(\{|u(s)|_{\mathrm{H}}>R\})\,ds\\
&\leq \frac{1}{TR^2}\int_0^T \mathbb{E}|u(s)|_{\mathrm{H}}^2\,ds\\
&\leq \frac{1}{TR^2}\Big[\frac{2}{C\eta}|u_0|_{\mathrm{H}}^2+\frac{2}{C\eta}\Big(\frac{2}{\eta}|f|_{\mathrm{V}'}^2+\varrho\Big)T\Big]\\
&= \frac{1}{TR^2}\frac{2}{C\eta}|u_0|_{\mathrm{H}}^2+\frac{1}{R^2}\frac{2}{C\eta}\Big(\frac{2}{\eta}|f|_{\mathrm{V}'}^2+\varrho\Big).
\end{aligned}$$

Thus the assertion follows. □

By Proposition 6.2, Corollary 6.4 and the Maslowski-Seidler Theorem [42, Proposition 3.1] we deduce the following main result of our paper.

Theorem 6.5. *Let $\mathcal{O}\subset\mathbb{R}^2$ be a Poincaré domain. Let assumptions **(H.1)-(H.2)** and **(H.5)** be satisfied. In addition we assume that the function G satisfies condition (G1) with $L<\sqrt{2}$ and inequality (G2) with $\lambda_0 = 0$. Then there exists an invariant measure of the semigroup $(P_t)_{t\geq 0}$ defined by (6.2), i.e. a probability measure μ on* H *such that*

$$P_t^*\mu \;=\; \mu.$$

Remark 6.6. *In this section we have used strong solutions. In particular, in order to show a global inequality (6.8) which was a basis for Corollary 6.4. However, we could have easily avoided this. For instance, instead*

of the global inequality (6.8) *we could prove that every martingale solution* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, u)$ *of equation* (5.3) *with the initial data* u_0 *on the time interval* $[0,T]$ *satisfies inequality* (6.8) *for only* $t \in [0,T]$ *but with constants* C, η *and* ρ *independent of* T.

APPENDIX A: UNIFORM ESTIMATES OF THE SOLUTIONS GALERKIN APPROXIMATIN EQUATIONS

Let us recall that the proof of existence of a martingale solution of the Navier-Stokes equations, given in [16], is based on the Faedo-Galerkin approximation in the space H_n, see (5.2) in the cited paper. In order to continue we need to choose and fix a stochastic basis and thus we assume that Assumption 5.3 holds. We also fix an $\mathcal{F}_0$-measurable H-valued random variable. Then the n-th equation is the following one in the space H_n.

$$\text{(A.1)} \qquad \begin{cases} du_n(t) = -\big[P_n \mathcal{A} u_n(t) + B_n\big(u_n(t)\big) - P_n f(t)\big]\, dt \\ \qquad\qquad + P_n G\big(u_n(t)\big)\, dW(t), \quad t > 0, \\ u_n(0) = P_n u_0. \end{cases}$$

Recall that H_n is a finite dimensional subspace spanned by the n first eigenvectors of the operator L given by (2.19) in [16], P_n is defined by [16, (2.25)] and B_n is defined on p. 1636 in [16]. For details see [16, Lemmas 2.3 and 2.4]. In particular, P_n restricted to H is the orthogonal projection. The existence of a solution of equation (A.1) is guaranteed by Lemma 5.2 in [16].

The following result corresponds to Lemma 5.3 from [16]. The proof of estimates (A.2), (A.3) and (A.5), is similar to the proof of estimates (5.4), (5.5) and (5.6) from Appendix A in [16]. However, we provide the details to indicate the dependence of appropriate constants on the data, which

will be important in the proof of continuous dependence of the solutions of the Navier-Stokes equations on the initial state u_0 and the external forces f. Moreover, if $\mathcal{O}$ is the Poincaré domain, we prove a new estimate, see (A.5). This estimate is of crucial importance in the proof of the existence of invariant measure. Recall that we have put $\frac{\eta}{2-\eta} = \infty$ when $\eta = 2$.

Lemma A.1. *Let Assumption 5.3 and parts **(H.2)**,**(H.3)** and **(H.5)** of Assumption 3.1 be satisfied. In particular, we assume that p satisfies (3.1), i.e.*

$$p \in \big[2, 2 + \frac{\eta}{2-\eta}\big),$$

*where $\eta \in (0, 2]$ is given in **(H.2)**.*

(1) Then for every $T > 0$, ν, R_1 and R_2 there exist constants $C_1(p)$, $\tilde{C}_2(p)$, $C_2(p)$, such that if $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathrm{H})$, $f \in L^p([0,\infty); \mathrm{V}')$ satisfy $\mathbb{E}[|u_0|_{\mathrm{H}}^p] \le R_1$ and $|f|_{L^p(0,T;\mathrm{V}')} \le R_2$, then every solution u_n of Galerkin equation (A.1) with the initial data u_0 and the external force f satisfies the following estimates

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big(\sup_{s\in[0,T]} |u_n(s)|_{\mathrm{H}}^p \Big) \le C_1(p) \tag{A.2}$$

and

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\int_0^T |u_n(s)|_{\mathrm{H}}^{p-2} |\nabla u_n(s)|^2 \, ds \Big] \le \tilde{C}_2(p), \tag{A.3}$$

and

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\int_0^T |\nabla u_n(s)|^2 \, ds \Big] \le C_2(p). \tag{A.4}$$

(2) Moreover, if $\mathcal{O}$ is a Poincaré domain and inequality (G2) holds with

$\lambda_0 = 0$, *then for every* $t > 0$

$$\sup_{n\in\mathbb{N}}\left(\mathbb{E}[\,|u_n(t)|_{\mathrm{H}}^2\,] + \frac{\eta}{2}\mathbb{E}\left[\int_0^t |\nabla u_n(s)|^2\,ds\right]\right) \le \mathbb{E}[\,|u_0|_{\mathrm{H}}^2\,] + \frac{2}{\eta}\int_0^t |f(s)|_{\mathrm{V}'}^2\,ds + \rho t. \tag{A.5}$$

Proof of Lemma A.1. Let us fix p satisfying condition (3.1). We apply the Itô formula from [47] to the function F defined by

$$F : \mathrm{H} \ni x \mapsto |x|_{\mathrm{H}}^p \in \mathbb{R}.$$

In the sequel we will omit the subscript H and write $|\cdot| := |\cdot|_{\mathrm{H}}$. Note that

$$F'(x) = d_x F = p\cdot|x|^{p-2}\cdot x, \quad \|F''(x)\| = \|d_x^2 F\| \le p(p-1)\cdot|x|^{p-2}, \quad x \in \mathrm{H}.$$

With the above comments in mind, we have, for $t \in [0,\infty)$,

$$\begin{aligned}
|u_n(t)|^p - |u_n(0)|^p &= \int_0^t \Big[p\,|u_n(s)|^{p-2}\langle u_n(s), -\mathcal{A}u_n(s) - B_n\big(u_n(s)\big) + P_n f(s)\rangle \\
&\quad + \frac{1}{2}\mathrm{Tr}\,\big[F''(u_n(s))\big(P_nG(u_n(s)), P_nG(u_n(s))\big)\big]\Big]\,ds \\
&\quad + p\int_0^t |u_n(s)|^{p-2}\langle u_n(s), P_nG(u_n(s))\,dW(s)\rangle \\
&= \int_0^t \Big[-p\,|u_n(s)|^{p-2}\|u_n(s)\|^2 + p\,|u_n(s)|^{p-2}\langle u_n(s), P_n f(s)\rangle \\
&\quad + \frac{1}{2}\mathrm{Tr}\big[F''(u_n(s))\big(P_nG(u_n(s)), P_nG(u_n(s))\big)\big]\Big]\,ds \\
&\quad + p\int_0^t |u_n(s)|^{p-2}\langle u_n(s), P_nG(u_n(s))\,dW(s)\rangle.
\end{aligned} \tag{A.6}$$

Since

$$\mathrm{Tr}\,\big[F''(u)\big(P_nG(u), P_nG(u)\big)\big] \le p(p-1)\,|u|^{p-2}\cdot|G(u)|_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}^2, \quad u \in \mathrm{V},$$

and by (G2)

$$|G(u)|_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}^2 \le (2-\eta)\,|\nabla u|^2 + \lambda_0|u|^2 + \rho, \quad u \in \mathrm{V},$$

and since by (2.3) and the Young inequality with exponents $2, \frac{2p}{p-2}$ and p, for $u \in \mathrm{V}$ and $f \in \mathrm{V}'$

$$\begin{aligned} |u|^{p-2}\langle f,u\rangle &\leq |u|^{p-2}\|u\|_{\mathrm{V}}\,|f|_{\mathrm{V}'} = |u|^{p-2}(|u|^2+|\nabla u|^2)^{\frac12}\,|f|_{\mathrm{V}'} \\ &\leq \frac{\varepsilon}{2}(|u|^2+|\nabla u|^2)\,|u|^{p-2}+(\frac12-\frac1p)|u|^p+\frac{\varepsilon^{-p/2}}{p}\,|f|^p_{\mathrm{V}'} \\ &\leq \frac{\varepsilon}{2}|\nabla u|^2\,|u|^{p-2}+(\frac{1+\varepsilon}{2}-\frac1p)|u|^p+\frac{\varepsilon^{-p/2}}{p}\,|f|^p_{\mathrm{V}'}, \end{aligned}$$

we infer that

$$\begin{aligned} |u_n(t)|^p &+ \big[p-p\frac{\varepsilon}{2}-\frac12 p(p-1)(2-\eta)\,\big]\int_0^t |u_n(s)|^{p-2}|\nabla u_n(s)|^2\,ds \\ &\leq |u_n(0)|+\int_0^t\Big[(\frac{p(1+\varepsilon)}{2}-1)|u_n(s)|^p+\varepsilon^{-p/2}\,|f(s)|^p_{\mathrm{V}'} \\ &\qquad+\frac12 p(p-1)\,|u_n(s)|^{p-2}\cdot\big(\lambda_0|u_n(s)|^2+\rho\big)\Big]\,ds \\ &\qquad+p\int_0^t |u_n(s)|^{p-2}\langle u_n(s),P_nG(u_n(s))\,dW(s)\rangle \\ &= \int_0^t\Big[\Big(\frac{\lambda_0}{2}p(p-1)+\frac{p(1+\varepsilon)}{2}-1\Big)\,|u_n(s)|^p \\ &\qquad+\frac{\rho}{2}p(p-1)\,|u_n(s)|^{p-2}+\varepsilon^{-p/2}\,|f(s)|^p_{\mathrm{V}'}\Big]\,ds \\ &\qquad+p\int_0^t |u_n(s)|^{p-2}\langle u_n(s),P_nG(u_n(s))\,dW(s)\rangle \end{aligned}$$

Let us choose $\varepsilon\in(0,1)$ such that $\delta=\delta(p,\eta):=p-p\frac{\varepsilon}{2}-\frac12 p(p-1)(2-\eta)>0$, or equivalently,

$$\varepsilon \;<\; 1\wedge[2-(p-1)(2-\eta)].$$

Notice that under condition (3.1) such ε exists. Denote also

$$K_p(\lambda_0,\rho):=\frac{\lambda_0}{2}p(p-1)+p-1+\rho p(1-\frac2p)\frac{p-1}{2}=\frac{p-1}{2}[\lambda_0 p+2+\rho(p-2)].$$

Thus, since by Young inequality $x^{p-2}\leq(1-\frac2p)x^p+\frac2p 1^{p/2}$ for $x\geq0$, we

obtain

$$
\begin{aligned}
|u_n(t)|^p \;+\; & \delta \int_0^t |u_n(s)|^{p-2} |\nabla u_n(s)|^2 \, ds \\
& \leq \; |u(0)|^p + K_p(\lambda_0, \rho) \int_0^t |u_n(s)|^p \, ds + \rho(p-1)t \\
& + \; \varepsilon^{-p/2} \int_0^t |f(s)|_{\mathrm{V}'}^p \, ds \\
& + \; p \int_0^t |u_n(s)|^{p-2} \langle u_n(s), P_n G(u_n(s)) \, dW(s) \rangle, \quad t \in [0, \infty).
\end{aligned} \tag{A.7}
$$

Since u_n is the solutions of the Galerkin equation, we infer that the process

$$
\mu_n(t) := \int_0^t |u_n(s)|^{p-2} \langle u_n(s), P_n G(u_n(s)) \, dW(s) \rangle, \qquad t \in [0, \infty)
$$

is a square integrable martingale. Indeed, if we define a map

$$
g : \mathrm{V} \ni u \mapsto \{ \mathrm{K} \ni k \mapsto \langle u, P_n G(u) k \rangle \in \mathrm{H} \} \in \mathcal{T}_2(\mathrm{K}, \mathbb{R})
$$

then $\mu_n(t) = \int_0^t |u_n(s)|^{p-2} g(u_n(s)) dW(s)$ and hence, by assumption (G2) and the fact that P_n is the orthogonal projection in H we infer that for every $t \geq 0$,

$$
\begin{aligned}
& \int_0^t \| \, |u_n(s)|^{p-2} g(u_n(s)) \|_{\mathcal{T}_2(\mathrm{K},\mathbb{R})}^2 \, ds \\
& \qquad = \; \int_0^t |u_n(s)|^{p-2} \| \, g(u_n(s)) \|_{\mathcal{T}_2(\mathrm{K},\mathbb{R})}^2 \, ds \\
& \qquad \leq \; \int_0^t |u_n(s)|^{p-2} |u_n(s)|^2 \| P_n G(u_n(s)) \|_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}^2 \, ds \\
& \qquad \leq \; \int_0^t |u_n(s)|^p \big[(2-\eta) \, |\nabla u_n(t)|^2 + \lambda_0 |u_n(t)|^2 + \rho \big] \, ds.
\end{aligned} \tag{A.8}
$$

Hence by the fact that u_n is a Galerkin solution we infer that

$$
\mathbb{E} \int_0^t \| \, |u_n(s)|^{p-2} g(u_n(s)) \, \|_{\mathcal{T}_2(\mathrm{K},\mathbb{R})}^2 \, ds < \infty, \quad t \geq 0.
$$

and thus we infer, as claimed, that the process μ_n is a square integrable martingale. Hence, $\mathbb{E}[\mu_n(t)] = 0$. Let us now fix $T > 0$. By taking expectation

in inequality (A.7) we infer that

$$\begin{aligned}\mathbb{E}\big[\,|u_n(t)|^p\,\big] &\le \mathbb{E}\big[\,|u_0|^p\,\big] + K_p(\lambda_0,\rho)\int_0^t \mathbb{E}\left[|u_n(s)|^p\right] ds \\ &+ \ \rho(p-1)t + \varepsilon^{-p/2}\mathbb{E}\int_0^t |f(s)|^p_{\mathrm{V}'}\, ds \\ &\le \mathbb{E}\big[\,|u_0|^p\,\big] + K_p(\lambda_0,\rho)\int_0^t \mathbb{E}\left[|u_n(s)|^p\right] ds + \rho(p-1)T \\ &+ \ \varepsilon^{-p/2}\mathbb{E}\int_0^T |f(s)|^p_{\mathrm{V}'}\, ds, \quad t\in[0,T].\end{aligned}$$

Hence by the Gronwall Lemma there exists a constant $\tilde{C}_p = \tilde{C}_p(T,\eta,\lambda_0,\rho,\mathbb{E}[|u_0|^p],\|f\|_{L^p(0,T;\mathrm{V}')}) = \tilde{C}_p(T,\eta,\lambda_0,\rho,R_1,R_2) > 0$ such that

$$\mathbb{E}\left[|u_n(t)|^p\right] \le \tilde{C}_p, \qquad t\in[0,T], \quad n\in\mathbb{N},$$

i.e.

$$\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\left[|u_n(t)|^p\right] \ \le \ \tilde{C}_p. \tag{A.9}$$

Using this bound in (A.7) we also obtain

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\int_0^T |u_n(s)|^{p-2}|\nabla u_n(s)|^2\, ds\right] \ \le \ \tilde{C}_2(p) \tag{A.10}$$

for a new constant $\tilde{C}_2(p) = C_2(p,T,\eta,\lambda_0,\rho,\mathbb{E}[|u_0|^p],\|f\|_{L^p(0,T;\mathrm{V}')})$ $= \tilde{C}_2(p,T,\eta,\lambda_0,\rho,R_1,R_2)$. This completes the proof of estimates (A.3). Since $\mathbb{E}[|u_0|^2] \le (\mathbb{E}[|u_0|^p])^{\frac{2}{p}} \le R_1^{2/p}$, we infer that (A.4) holds with another constant $C_2(p)$.

Let us move to the proof of estimate (A.2). By the Burkholder-Davis-Gundy inequality, see [27], the Schwarz inequality and inequality (G2), there

exists a constant c_p such that for any $t \geq 0$,

(A.11)

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq s\leq t}\left|\int_0^s p\,|u_n(\sigma)|^{p-2}\langle u_n(\sigma), P_nG(u_n(\sigma))\,dW(\sigma)\rangle\right|\right]\\
&\leq c_p\cdot\mathbb{E}\left[\left(\int_0^t |u_n(\sigma)|^{2p-2}\cdot|P_nG(u_n(\sigma))|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}\,d\sigma\right)^{\frac12}\right]\\
&\leq c_p\cdot\mathbb{E}\left[\sup_{0\leq\sigma\leq t}|u_n(\sigma)|^{\frac p2}\left(\int_0^t |u_n(\sigma)|^{p-2}\cdot|G(u_n(\sigma))|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}\,d\sigma\right)^{\frac12}\right]\\
&\leq \frac12\mathbb{E}\big[\sup_{0\leq s\leq t}|u_n(s)|^p\big]+\frac12c_p^2\int_0^t |u_n(\sigma)|^{p-2}\cdot|G(u_n(\sigma))|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}\,d\sigma\\
&\leq \frac12\mathbb{E}\big[\sup_{0\leq s\leq t}|u_n(s)|^p\big]\\
&\qquad+\frac12c_p^2\int_0^t |u_n(\sigma)|^{p-2}\cdot\big[(2-\eta)|u_n(\sigma)|^2+\lambda_0\,|u_n(\sigma)|^2+\rho\big]\,d\sigma\\
&\leq \frac12\mathbb{E}\big[\sup_{0\leq s\leq t}|u_n(s)|^p\big]+\frac12c_p^2\frac{2\rho}{p}t+\frac12c_p^2(2-\eta)\mathbb{E}\left[\int_0^t |u_n(\sigma)|^p\|u_n(\sigma)\|^2\,d\sigma\right]\\
&\qquad+\frac12c_p^2\left(\lambda_0+\rho\Big(1-\frac2p\Big)\right)\cdot\mathbb{E}\left[\int_0^t |u_n(\sigma)|^p\,d\sigma\right].
\end{aligned}
$$

Using (A.11) in (A.7), by inequalities (A.9) and (A.10) we infer that

$$
\begin{aligned}
\mathbb{E}\big[\sup_{0\leq s\leq t}|u_n(s)|^p\big] \;\leq\;& \mathbb{E}[\,|u_0|^p\,]\\
&+\left[K_p(\lambda_0,\rho)+\frac12c_p^2\left(\lambda_0+\rho\Big(1-\frac2p\Big)\right)\right]\int_0^t\mathbb{E}\left[|u_n(s)|^p\right]ds\\
&+\left(\frac{2\rho}{p}+c_p^2\frac{\rho}{p}\right)t+\varepsilon^{-p/2}\int_0^t|f(s)|^p_{\mathrm{V}'}\,ds\\
&+\frac12\mathbb{E}\big[\sup_{0\leq s\leq t}|u_n(s)|^p\big]+\frac12c_p^2(2-\eta)\mathbb{E}\left[\int_0^t|u_n(\sigma)|^p\|u_n(\sigma)\|^2\,d\sigma\right]\\
\leq\;& \mathbb{E}[\,|u_0|^p\,]+\left[K_p(\lambda_0,\rho)+\frac12c_p^2\left(\lambda_0+\rho\Big(1-\frac2p\Big)\right)\right]\tilde C_pt\\
&+\frac{\rho}{p}(2+c_p^2)\,t+\varepsilon^{-p/2}\int_0^t|f(s)|^p_{\mathrm{V}'}\,ds\\
&+\frac12\mathbb{E}\big[\sup_{0\leq s\leq t}|u_n(s)|^p\big]+\frac12c_p^2(2-\eta)C_2(p),\quad t\geq0.
\end{aligned}
$$

Thus for a fixed $T > 0$

$$\mathbb{E}\big[\sup_{0\le s\le T}|u_n(s)|^p\big] \;\le\; C_1(p),$$

where

$$\begin{aligned} C_1(p) &= C_1(p,T,\eta,\lambda_0,\rho,R_1,R_2) \\ &:= 2R_1 + 2\bigg[K_p(\lambda_0,\rho) + \frac{1}{2}c_p^2\Big(\lambda_0 + \rho\Big(1-\frac{2}{p}\Big)\Big)\bigg]\tilde{C}_pT \\ &\quad +2\bigg(\frac{2\rho}{p} + c_p^2\frac{\rho}{p}\bigg)T + 2\varepsilon^{-p/2}R_2 + c_p^2(2-\eta)C_2(p). \end{aligned}$$

This completes the proof of estimate (A.2).

To prove inequality (A.5) let us assume that $\mathcal{O}$ is a Poincaré domain and inequality (G2) holds with $\lambda_0 = 0$. Recall that now in the space V we consider the inner product $\big((\cdot,\cdot)\big)$ given by (2.2). By identity (A.6) from the previous proof with $p = 2$, we have

$$\begin{aligned} |u_n(t)|^2 - |u(0)|^2 = \int_0^t \Big\{ &-2\,\|u_n(s)\|^2 + 2\,\langle u_n(s), f(s)\rangle \\ &+ \frac{1}{2}\mathrm{Tr}\big[F''(u_n(s))\big(G(u_n(s)), G(u_n(s))\big)\big]\Big\}\,ds \\ &+ 2\int_0^t \langle u_n(s), P_nG(u_n(s))\,dW(s)\rangle, \quad t \ge 0. \end{aligned}$$

Since $\mathbb{E}\big(\int_0^t \langle P_nG(u_n(s)), u_n(s)\,dW(s)\rangle\big) = 0$, we infer that

$$\begin{aligned} \mathbb{E}|u_n(t)|_{\mathrm{H}}^2 &\le \mathbb{E}[\,|u_0|_{\mathrm{H}}^2\,] + \mathbb{E}\int_0^t \big\{-2\|u_n(s)\|^2 + 2\langle f(s), u_n(s)\rangle\big\}\,ds \\ &+ \mathbb{E}\int_0^t |P_nG(u_n(s))|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}\,ds \end{aligned}$$

Using assumption (G2) with $\lambda_0 = 0$, i.e $|G(u_n(s))|^2_{\mathcal{T}_2(K,\mathrm{H})} \le (2-\eta)\|u_n(s)\|^2 + \varrho$, we get

$$\begin{aligned} \mathbb{E}|u(t)|_{\mathrm{H}}^2 &\le -\eta\mathbb{E}\int_0^t \|u_n(s)\|^2\,ds + \mathbb{E}[\,|u_0|_{\mathrm{H}}^2\,] \\ &+ 2\,\mathbb{E}\int_0^t \langle f(s), u(s)\rangle\,ds + \varrho t. \end{aligned} \tag{A.12}$$

Since $2\langle f,u\rangle \leq \frac{\eta}{2}|\nabla u_n|^2 + \frac{2}{\eta}|f|^2_{\mathrm{V}'}$, for $u \in \mathrm{V}, f \in \mathrm{V}'$ we infer that

$$\begin{aligned} \mathbb{E}|u_n(t)|^2_{\mathrm{H}} &\leq -\frac{\eta}{2}\mathbb{E}\int_0^t \|u_n(s)\|^2\,ds + \mathbb{E}[|u_0|^2_{\mathrm{H}}] \\ &+ \frac{2}{\eta}\int_0^t |f(s)|^2_{\mathrm{V}'} + \varrho t, \quad t \geq 0. \end{aligned} \tag{A.13}$$

The proof of inequality (A.5) is thus complete. □

APPENDIX B: PROOF OF THEOREM 4.8

Similarly to the proof of Theorem 5.1 in [16] the present proof is based on the Galerkin method. We will use the fact the the laws of the Galerkin solutions form a tight set of probability measures on $\mathcal{Z}_T$. We will use the Jakubowski's version of the Skorokhod theorem 4.6, as well. However, some details are different.

Let us fix positive numbers T, R_1 and R_2. Let us assume that μ is a Borel probability measure on H, $f \in L^p([0,\infty);\mathrm{V}')$ which satisfy $\int_{\mathrm{H}} |x|^p \mu(dx) \leq R_1$ and $|f|_{L^p(0,T;\mathrm{V}')} \leq R_2$. Similarly to the previous section we choose and fix a stochastic basis and thus we assume that Assumption 5.3 holds. We also fix an $\mathcal{F}_0$-measurable H-valued random variable whose law is equal to μ.

As in the proof of [16, Theorem 5.1] let $(u_n)_{n\in\mathbb{N}}$ be a sequence of the solutions of the Galerkin equations. Then the set of laws $\{\mathcal{L}(u_n), n \in \mathbb{N}\}$ is tight on the space $(\mathcal{Z}_T, \sigma(\mathcal{T}_T))$, where $\sigma(\mathcal{T}_T)$ denotes the topological σ-field. By theorem 4.6 there exists a subsequence (n_k), a probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and, on this space $\mathcal{Z}_T$-valued random variables u, $\tilde u_{n_k}$, $k \in \mathbb{N}$, and a sequence

of K-valued Wiener processes $\tilde{W}, \tilde{W}_{n_k}$, $k \in \mathbb{N}$ such that

(B.1) the variables (u_{n_k}, W) and $(\tilde{u}_{n_k}, \tilde{W}_{n_k})$ have the same laws on the Borel σ-algebra $\mathcal{B}\big(\mathcal{Z}_T \times \mathcal{C}([0,T], \mathrm{K})\big)$

and

(B.2) $(\tilde{u}_{n_k}, \tilde{W}_{n_k})$ converges to $(u, \tilde{W})$ in $\mathcal{Z}_T \times \mathcal{C}([0,T]; \mathrm{K})$ almost surely on $\tilde{\Omega}$.

In particular,

(B.3) $\tilde{u}_{n_k}$ converges to u in $\mathcal{Z}_T$ almost surely on $\tilde{\Omega}$.

We will denote the subsequence $(\tilde{u}_{n_k}, \tilde{W}_{n_k})$ again by $(\tilde{u}_n, \tilde{W}_n)$. Define a corresponding sequence of filtrations by

$$\tilde{\mathbb{F}}_n = (\tilde{\mathcal{F}}_{n,t})_{t\geq 0}, \text{ where } \tilde{\mathcal{F}}_{n,t} = \sigma\{(\tilde{u}_n(s), \tilde{W}_n(s)),\ s \leq t\},\ \ t \in [0,T]. \tag{B.4}$$

To obtain (4.8), we modify the proof from [16] at pages 1650-51. Namely, using Lemma A.1, we infer that the processes $\tilde{u}_n$, $n \in \mathbb{N}$, satisfy the following inequalities

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{E}}\Big(\sup_{s\in[0,T]} |\tilde{u}_n(s)|_{\mathrm{H}}^p \Big) \ \leq\ C_1(p) \tag{B.5}$$

and

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{E}}\Big[\int_0^T |\nabla \tilde{u}_n(s)|_{L^2}^2 \, ds \Big] \ \leq\ C_2(p). \tag{B.6}$$

Let us emphasize that the constants $C_1(p)$ and $C_2(p)$, being the same as in Lemma A.1, depend on T, R_1 and R_2. Using inequality (B.5) we choose a subsequence, still denoted by $(\tilde{u}_n)$, convergent weak star in the space $L^p(\tilde{\Omega}; L^\infty(0,T;\mathrm{H}))$ and infer that

$$\mathbb{E}\Big[\sup_{s\in[0,T]} |u(s)|_{\mathrm{H}}^p \Big] \ \leq\ C_1(p) \tag{B.7}$$

and that the limit process u satisfies (B.7), as well. This completes the proof of inequality (4.9). To prove (4.8) let us fix $q \in [1,p)$. Notice that for every $t \in [0,T]$

$$|u(t)|^q \;=\; (|u(t)|^p)^{q/p} \;\le\; \Big(\sup_{t\in[0,T]} |u(t)|^p\Big)^{q/p}.$$

Thus, $\sup_{t\in[0,T]} |u(t)|^q \le \Big(\sup_{t\in[0,T]} |u(t)|^p\Big)^{q/p}$, and so by the Hölder inequality

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]} |u(t)|^q\Big] \;&\le\; \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} |u(t)|^p\Big)^{q/p}\Big]\\ &\le\; \Big(\mathbb{E}\Big[\sup_{t\in[0,T]} |u(t)|^p\Big]\Big)^{q/p} \;\le\; \big(C_1(p)\big)^{q/p},\end{aligned}$$

which means that inequality (4.8) holds with the constant $C_1(p,q) := \big(C_1(p)\big)^{q/p}$.

By inequality (B.6) we infer that the sequence $(\tilde u_n)$ contains further subsequence, denoted again by $(\tilde u_n)$, convergent weakly in the space $L^2([0,T]\times\tilde\Omega;\mathrm{V})$ to u. Moreover, it is clear that

$$\tilde{\mathbb{E}}\Big[\int_0^T |\nabla u(s)|^2_{L^2}\,ds\Big] \;\le\; C_2(p) \tag{B.8}$$

and the process u satisfies (4.10).

To prove the second part of the theorem we assume that $\mathcal{O}$ is a Poincaré domain and inequality (G2) holds with $\lambda_0 = 0$. In this case, by Lemma A.1, instead of inequality (B.6) we can use the following one corresponding to the uniform estimates (A.5),

$$\frac{\eta}{2}\sup_{n\in\mathbb{N}}\mathbb{E}\left[\int_0^T |\nabla\tilde u_n(s)|^2_{L^2}\,ds\right] \le \mathbb{E}[\,|u_0|^2_{\mathrm{H}}\,] + \frac{2}{\eta}\int_0^T |f(s)|^2_{\mathrm{V}'}\,ds + \rho T, \tag{B.9}$$

choose a subseqence convergent weakly in the space $L^2([0,T]\times\tilde\Omega;\mathrm{V})$ to u and infer that the limit process satisfies the same estimate, which proves

estimate (4.11). We will prove that the system $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, u)$ is a martingale solution of problem (3.2).

Step 1. Let us fix $\varphi \in U$. Analogously to [10] and [43], let us denote

$$\begin{aligned}\Lambda_n(\tilde{u}_n, \tilde{W}_n, \varphi)(t) \;:=\; & \big(\tilde{u}_n(0), \varphi\big)_{\mathrm{H}} - \int_0^t \langle P_n \mathcal{A}\tilde{u}_n(s), \varphi\rangle ds \\ & - \int_0^t \langle B_n(\tilde{u}_n(s)), \varphi\rangle ds + \int_0^t \langle f_n(s), \varphi\rangle \, ds \\ & + \Big\langle \int_0^t P_n G(\tilde{u}_n(s)) \, d\tilde{W}_n(s), \varphi \Big\rangle, \quad t \in [0,T],\end{aligned} \tag{B.10}$$

and

(B.11)

$$\begin{aligned}\Lambda(u, \tilde{W}, \varphi)(t) \;:=\; & \big(u(0), \varphi\big)_{\mathrm{H}} - \int_0^t \langle \mathcal{A}u(s), \varphi\rangle ds - \int_0^t \langle B(u(s)), \varphi\rangle ds \\ & + \int_0^t \langle f(s), \varphi\rangle \, ds + \Big\langle \int_0^t G(u(s)) \, d\tilde{W}(s), \varphi \Big\rangle, \quad t \in [0,T].\end{aligned}$$

Using Lemma 2.4(c) from [16], see also [43, Lemma 5.4], we can prove the following lemma analogous to Lemma 4.12.

Lemma B.1. *For all $\varphi \in U$*

(a) $\lim_{n\to\infty} \tilde{\mathbb{E}}\big[\int_0^T |\big(\tilde{u}_n(t) - u(t), \varphi\big)_{\mathrm{H}}|^2 \, dt\big] = 0$,

(b) $\lim_{n\to\infty} \tilde{\mathbb{E}}\big[|\big(\tilde{u}_n(0) - u(0), \varphi\big)_{\mathrm{H}}|^2\big] = 0$,

(c) $\lim_{n\to\infty} \tilde{\mathbb{E}}\big[\int_0^T \big|\int_0^t \langle P_n \mathcal{A}\tilde{u}_n(s) - \mathcal{A}u(s), \varphi\rangle \, ds\big| \, dt\big] = 0$,

(d) $\lim_{n\to\infty} \tilde{\mathbb{E}}\big[\int_0^T \big|\int_0^t \langle B_n(\tilde{u}_n(s)) - B(u(s)), \varphi\rangle \, ds\big| \, dt\big] = 0$,

(e) $\lim_{n\to\infty} \tilde{\mathbb{E}}\big[\int_0^T \big|\int_0^t \langle P_n f_n(s) - f(s), \varphi\rangle \, ds\big| \, dt\big] = 0$,

(f) $\lim_{n\to\infty} \tilde{\mathbb{E}}\big[\int_0^T \big|\langle \int_0^t [P_n G(\tilde{u}_n(s)) - G(u(s))] \, d\tilde{W}(s), \varphi\rangle\big|^2 \, dt\big] = 0$.

Directly from Lemma 1 we get the following corollary

Corollary B.2. *For every $\varphi \in U$,*

$$\lim_{n\to\infty} |\big(\tilde{u}_n(\cdot), \varphi\big)_{\mathrm{H}} - \big(u(\cdot), \varphi\big)_{\mathrm{H}}|_{L^2([0,T]\times\tilde{\Omega})} = 0 \tag{B.12}$$

and

$$\lim_{n\to\infty} |\Lambda_n(\tilde{u}_n, \tilde{W}_n, \varphi) - \Lambda(u, \tilde{W}, \varphi)|_{L^1([0,T]\times\tilde{\Omega})} = 0. \tag{B.13}$$

Proof. Assertion (B.12) follows from the equality

$$|\big(\tilde{u}_n(\cdot), \varphi\big)_{\mathrm{H}} - \big(\tilde{u}(\cdot), \varphi\big)_{\mathrm{H}}|^2_{L^2([0,T]\times\tilde{\Omega})} = \tilde{\mathbb{E}}\Big[\int_0^T |\big(\tilde{u}_n(t) - \tilde{u}(t), \varphi\big)_{\mathrm{H}}|^2 \, dt\Big]$$

and Lemma 4.12 (a). To prove (B.13) let us note that by the Fubini theorem, we have

$$\begin{aligned} &|\Lambda_n(\tilde{u}_n, \tilde{W}_n, \varphi) - \Lambda(u, \tilde{W}, \varphi)|_{L^1([0,T]\times\tilde{\Omega})} \\ &= \int_0^T \tilde{\mathbb{E}}\big[|\Lambda_n(\tilde{u}_n, \tilde{W}_n, \varphi)(t) - \Lambda(u, \tilde{W}, \varphi)(t)|\,\big] dt. \end{aligned}$$

To complete the proof of (B.13) it is sufficient to note that by Lemma 1 (b)-(f), each term on the right hand side of (B.10) tends at least in $L^1([0,T]\times\tilde{\Omega})$ to the corresponding term in (B.11). □

Step 2. Since u_n is a solution of the Galerkin equation, for all $t \in [0,T]$ and $\varphi \in U$

$$\big(u_n(t), \varphi\big)_{\mathrm{H}} = \Lambda_n(u_n, W, \varphi)(t), \qquad \mathbb{P}\text{-a.s.}$$

In particular,

$$\int_0^T \mathbb{E}\big[|\big(u_n(t), \varphi\big)_{\mathrm{H}} - \Lambda_n(u_n, W, \varphi)(t)|\,\big]\, dt = 0.$$

Since $\mathcal{L}(u_n, W) = \mathcal{L}(\tilde{u}_n, \tilde{W}_n)$, using (B.12) and (B.13) we infer that

$$\int_0^T \tilde{\mathbb{E}}\big[|\big(u(t), \varphi\big)_{\mathrm{H}} - \Lambda(u, \tilde{W}, \varphi)(t)|\,\big]\, dt = 0.$$

Hence for l-almost all $t \in [0,T]$ and $\tilde{\mathbb{P}}$-almost all $\omega \in \tilde{\Omega}$

$$\big(u(t),\varphi\big)_{\mathrm{H}} - \Lambda(u,\tilde{W},\varphi)(t) = 0, \tag{B.14}$$

Since u is $\mathcal{Z}_T$-valued random variable, in particular $u \in \mathcal{C}([0,T];\mathrm{H}_w)$, i.e. u is weakly continuous, we infer that equality (B.14) holds for all $t \in [0,T]$ and all $\varphi \in U$. Since U is dense in V, equality (B.14) holds for all $\varphi \in \mathrm{V}$, as well. Putting $\tilde{\mathfrak{A}} := (\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}},\tilde{\mathbb{F}})$, by (B.14) and (B.11) we infer that the system $(\tilde{\mathfrak{A}},\tilde{W},u)$ is a martingale solution of equation (3.2). The proof of Theorem 4.8 is thus complete.

APPENDIX C: KURATOWSKI THEOREM

The following is the classical form of the celebrated Kuratowski Theorem.

Theorem C.1. *Assume that X_1, X_2 are two Polish spaces with their Borel σ-fields denoted respectively by $\mathcal{B}(X_1), \mathcal{B}(X_2)$. If $\phi : X_1 \longrightarrow X_2$ is an injective Borel measurable map, then for any $E_1 \in \mathcal{B}(X_1)$, $E_2 := \phi(E_1) \in \mathcal{B}(X_2)$.*

Let us formulate a simple corollary to the above result.

Proposition C.2. *Suppose that X_1, X_2 are two topological spaces with their Borel σ-fields denoted respectively by $\mathcal{B}(X_1), \mathcal{B}(X_2)$. Suppose that $\phi : X_1 \longrightarrow X_2$ is an injective Borel measurable map such that for any $E_1 \in \mathcal{B}(X_1)$, $E_2 := \phi(E_1) \in \mathcal{B}(X_2)$. Then if $g : X_1 \to \mathbb{R}$ is a Borel measurable map then a function $f : X_2 \to \mathbb{R}$ defined by*

$$f(x_2) = \begin{cases} g(\phi^{-1}(x_2)), & \text{if } x_2 \in \phi(X_1), \\ \infty, & \text{if } x_2 \in X_2 \setminus \phi(X_1), \end{cases} \tag{C.1}$$

is also Borel measurable.

PROOF. Note that $g = f \circ \phi$.

$$f^{-1}(A) = \phi[g^{-1}(A)], \quad A \subset \mathbb{R}.$$

Thus, if $A \in \mathcal{B}(\mathbb{R})$, then by assumptions $g^{-1}(A) \in \mathcal{B}(X_1)$. Hence by Theorem C.1 we infer that $\phi[g^{-1}(A)] \in \mathcal{B}(X_2)$ and thus by the equality above, we infer that $f^{-1}(A) \in \mathcal{B}(X_2)$. The proof is complete. □

One may wonder if the following a generalization of the above result to non Polish spaces is valid.

Theorem C.3. *Let X_1 and X_2 be a topological spaces such that for each $i = 1, 2$ there exists a sequence $\{f_{i,m}\}$ of continuous functions $f_{i,m} : X_i \to \mathbb{R}$ that separate points of X_i. Let us denote by $\mathscr{S}_i$ the σ-algebra generated by the maps $\{f_{i,m}\}$. If $\phi : X_1 \longrightarrow X_2$ is an injective measurable map, then for any $E_1 \in \mathscr{S}_1$, $E_2 := \phi(E_1) \in \mathscr{S}_2$.*

The following Counterexample shows that the answer to the above question is No.

Counterexample C.4. *1) Define $f_k(x) = e^{2ikx\pi}$, $x \in [0, 1)$, for every integer k (trigonometric functions).*

2) Let X_1 be a non-Borel subset of $[0, 1)$ equipped with the euclidean metric.

3) Let X_2 denote $[0, 1)$ with the Euclidean metric.

4) Denote by f_k^1 the restriction of f_k to X_1.

5) Then f_k^1 are continuous and separate points in X_1.

6) Then f_k are continuous and separate points in X_2.

7) $\sigma(f_k) = Borel(X_2)$ by Stone-Weierstrass.

8) $\sigma(f_k^1) = \{A \cap X_1 : A \in \sigma(f_k)\} = \{A \cap X_1 : A \in Borel(X_2)\} = Borel(X_1)$.

9) Let $\varphi : X_1 \to X_2$ be the identity mapping.

10) φ is a continuous injection.

11) $\varphi[X_1]$ is not Borel in X_2.

REFERENCES

[1] S. Albeverio, Z. Brzeźniak and Jiang-Lun Wu, *Existence of global solutions and invariant measures for stochastic differential equations driven by Poisson type noise with non-Lipschitz coefficients*, J. Math. Anal. Appl. **371**, no. 1, 309-322 (2010)

[2] S. Albeverio and M. Röckner, *Dirichlet forms, quantum fields and stochastic quantization. Stochastic analysis, path integration and dynamics* (Warwick, 1987), 1-21, Pitman Res. Notes Math. Ser., 200, Longman Sci. Tech., Harlow (1989)

[3] A. Avez, *Propriétés ergodiques des endomorphismes dilatants des variétés compactes*, C. R. Acad. Sci. Paris Sér. A-B 266, A610-A612 (1968)

[4] J. M. Ball, *Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations*, J. Nonlinear Sci. **7**, no. 5, 475-502 (1997)

[5] V. Bogachev and M. Röckner, *Elliptic equations for measures on infinite-dimensional spaces and applications*, Probab. Theory Related Fields **120**, no. 4, 445-496 (2001)

[6] L. Breiman, PROBABILITY, Addison-Wesley Publishing Company, Reading, Mass.-London-Don Mills, Ont. 1968.

[7] H. Brezis, ANALYSE FONCTIONNELLE, Masson, 1983.

[8] Z. Brzeźniak and D. Gątarek, *Martingale solutions and invariant measures for stochastic evolution equations in Banach spaces*, Stochastic Process. Appl. **84**, no. 2, 187-225 (1999)

[9] Z. Brzeźniak, B. Goldys and T. Jegaraj, *Large deviations and transitions between equilibria for stochastic Landau-Lifshitz-Gilbert equation*, submitted

[10] Z. Brzeźniak, E. Hausenblas and P. Razafimandimby, *Martingale solutions for stochastic equations of reaction diffusion type driven by Lévy noise or Poisson random measure*, Preprint ArXiv: 1010.59 33v2

[11] Z. Brzeźniak,E. Hausenblas and E. Motyl, *Uniqueness in law of the stochastic convolution process driven by Lévy noise*, Electron. J. Probab. **18**, no. 57, 15 pp. (2013)

[12] Z. Brzeźniak, Y. Li, *Asymptotic compactness and absorbing sets for 2D stochastic*

Navier-Stokes equations on some unbounded domains, Trans. Am. Math. Soc., **358**, 5587-5629 (2006)

[13] Z. Brzeźniak, H. Long and I. Simão, *Invariant measures for stochastic evolution equations in M-type 2 Banach spaces*, J. Evol. Equ. **10**, no. 4, 785-810 (2010)

[14] Z. Brzeźniak, Y. Li, *Asymptotic compactness and absorbing sets for 2D stochastic Navier-Stokes equations on some unbounded domains*, Trans. Am. Math. Soc., **358**, 5587-5629 (2006)

[15] Z. Brzeźniak and E. Motyl, *The existence of martingale solutions to the stochastic Boussinesq equations*, Global and Stochastic Analysis Vol. 1, No. 2, 175-216 (2011)

[16] Z. Brzeźniak and E. Motyl, *Existence of a martingale solution of the stochastic Navier-Stokes equations in unbounded 2D and 3D-domains*, J. Differential Equations **254**, no. 4, 1627-1685 (2013)

[17] Z. Brzeźniak, M. Ondreját, *Weak solutions to stochastic wave equations with values in Riemannian manifolds*. Comm. Partial Differential Equations **36**, no. 9, 1624-1653 (2011)

[18] Z. Brzeźniak, M. Ondreját, *Stochastic geometric wave equations with values in compact Riemannian homogeneous spaces*, arXiv: 1101.4835v1, 2011, Ann. Probab. **41**, no. 3B, 1938-1977 (2013)

[19] Z. Brzeźniak, M. Ondrejat and J. Seidler, *Invariant measures for stochastic nonlinear beam and wave equations*, J. Differential Equations **260**, 4157-4179 (2015)

[20] Z. Brzeźniak, S. Peszat, *Stochastic two dimensional Euler equations*, Ann. of Prob., Vol. 29, No.4, 1796-1832 (2001)

[21] Chow, Pao-Liu; Khasminskii, Rafail Z. *Stationary solutions of nonlinear stochastic evolution equations*, Stochastic Anal. Appl. **15**, no. 5, 671-699 (1997)

[22] P. Constantin and F. Ramos, *Inviscid limit for damped and driven incompressible Navier-Stokes equations in* R^2, Comm. Math. Phys. **275**, no. 2, 529-551 (2007)

[23] D. Daners, *Perturbation of semi-linear evolution equations under weak assumptions at initial time*, J. Differential Equations **210**, no. 2, 352-382 (2005)

[24] G. Da Prato and A. Debussche, *Two-dimensional Navier-Stokes equations driven by a space-time white noise*, J. Funct. Anal. **196**, no. 1, 180-210 (2002)

[25] G. Da Prato and A. Debussche, *Strong solutions to the stochastic quantization equa-*

tions. Ann. Probab. **31**, no. 4, 1900-1916 (2003)

[26] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, 1992.

[27] G. Da Prato and J. Zabczyk, Ergodicity for Infinite Dimensional Systems, London Mathematical Society Lecture Notes Series 229, Cambridge University Press, 1996.

[28] Dynkin, E. B. Основания теории марковских процессов. (Russian) [Foundations of the theory of Markov processes] Teorija Verojatnosteĭi Matematičeskaja Statistika Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow 1959, 227 pp.

[29] Dynkin, E. B. Theory of Markov processes. Translated from the Russian by D. E. Brown; edited by T. Kövàry, Prentice-Hall, Inc., Englewood Cliffs, N.J.; Pergamon Press, Oxford-London-Paris 1961 ix+210 pp.

[30] F. Flandoli, *Dissipativity and invariant measures for stochastic Navier-Stokes equations*, NoDEA **1**, 403–423 (1994)

[31] F. Flandoli, D. Gątarek, *Martingale and stationary solutions for stochastic Navier-Stokes equations*, Probab. Theory Related Fields **102** no. 3 , 367–391 (1995)

[32] K. Holly and M. Wiciak, *Compactness method applied to an abstract nonlinear parabolic equation*, Selected problems of Mathematics, Cracow University of Technology, 95-160 (1995)

[33] Ikeda, N., Watanabe, S., *Stochastic Differential Equations and Diffusion Processes*, North-Holland Publishing Company, Amsterdam, 1981.

[34] A. Jakubowski, *The almost sure Skorohod representation for subsequences in nonmetric spaces*, Teor. Veroyatnost. i Primenen. **42**, no. 1, 209-216 (1997); translation in Theory Probab. Appl. **42** no.1, 167-174 (1998)

[35] J.U. Kim, *Invariant measures for the stochastic von Karman plate equation*, SIAM J. Math. Anal. **36**, no. 5, 1689-1703 (2005)

[36] N. Kryloff and N. Bogoliouboff, *La théorie générale de la mesure dans son application l'étude des systémes dynamiques de la mécanique non linéaire*, Ann. of Math. (2) **38**, no. 1, 65-113 (1937)

[37] K. Kuratowski, Topologie, vol. I. (French) 3ème ed. Monografie Matematyczne, Tom XX, Polskie Towarzystwo Matematyczne, Warszawa, 1952.

[38] A. Lasota and G. Pianigiani, *Invariant measures on topological spaces*. Boll. Un. Mat. Ital. B (5) **14**, no. 2, 592-603 (1977)

[39] G. Leha and G. Ritter, *Lyapunov functions and stationary distributions of stochastic evolution equations*, Stochastic Anal. Appl. **21**, no. 4, 763-799 (2003)

[40] Leha, Gottlieb; Ritter, Gunter; Wakolbinger, Anton *An improved Lyapunov-function approach to the behavior of diffusion processes in Hilbert spaces*, Stochastic Anal. Appl. **15**, no. 1, 59-89 (1997)

[41] Leha, Gottlieb; Ritter, Gunter *Lyapunov-type conditions for stationary distributions of diffusion processes on Hilbert spaces*, Stochastics Stochastics Rep. **48**, no. 3-4, 195-225 (1994)

[42] B. Maslowski and J. Seidler, *On sequentially weakly Feller solutions to SPDE's*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., **10**, 2, 69-78 (1999)

[43] E. Motyl, *Stochastic hydrodynamic-type evolution equations driven by Lévy noise in 3D unbounded domains - abstract framework and applications*, Stoch. Processes and their Appl. **124**, 2052-2097 (2014)

[44] M. Ondreját, *Existence of global mild and strong solutions to stochastic hyperbolic evolution equations driven by a spatially homogeneous Wiener process*, J. Evol. Equ. **4**, no. 2, 169-191 (2004)

[45] M. Ondreját, *Uniqueness for stochastic evolution equations in Banach spaces*, Dissertationes Math. 426, 1-63 (2004)

[46] J. C. Oxtoby and S. M. Ulam, *On the existence of a measure invariant under a transformation*. Ann. of Math. (2) 40 , 560-566 (1939)

[47] E. Pardoux, *Stochastic partial differential equations and filtering of diffusion processes*, Stochastics, Vol. 3, 127-167 (1979)

[48] G. Parisi and Y. S. Wu, *Perturbation theory without gauge fixing*, Sci. Sinica **24**, no. 4, 483-496 (1981)

[49] K. R. Parthasarathy, Probability measures on metric spaces, Academic Press, New York and London, 1967.

[50] S. Peszat and J. Zabczyk, Stochastic Partial Differential Equations with Lévy Noise. An Evolution Equation Approach. Encyclopedia of Mathematics and

its Applications **113**, Cambridge University Press, Cambridge, 2007.

[51] M. Röckner and Z. Sobol, *Kolmogorov equations in infinite dimensions: well-posedness and regularity of solutions, with applications to stochastic generalized Burgers equations*, Ann. Probab. **34**, no. 2, 663-727 (2006)

[52] R. Rosa, *The global attractor for the 2D Navier-Stokes flow on some unbounded domains*, Nonlinear Analysis **32**, 71-85 (1998)

[53] W.A. Strauss, *On continuity of functions with values in various Banach spaces*, Pacific J. Math., **19**, 3, 543-555, (1966)

[54] R. Temam, Navier-Stokes equations. Theory and numerical analysis, Reprint of the 1984 edition. AMS Chelsea Publishing, Providence, RI, 2001. xiv+408 pp.

[55] R. Temam, Navier-Stokes equations and nonlinear functional analysis, SIAM, Philadelphia, Pensylvania, 1995.

[56] T. Schmidt, *Strict Interior Approximation Of Sets Of Finite Perimeter And Functions Of Bounded Variation*, preprint.

[57] N. N. Vakhania, V.I. Tarieladze and S.A. Chobanyan, Probability distributions on Banach spaces. Translated from the Russian and with a preface by Wojbor A. Woyczynski. Mathematics and its Applications (Soviet Series), 14. D. Reidel Publishing Co., Dordrecht, 1987.

[58] M.J. Vishik and A.V. Fursikov, Mathematical Problems of Statistical Hydromechanics, Kluwer Academic Publishers, Dordrecht, 1988.

Department of Mathematics, University of York, Heslington, York, YO105DD, United Kingdom
E-mail: zdzislaw.brzezniak@york.ac.uk

Faculty of Mathematics and Computer Science, University of Łódź, ul. Banacha 22,
91-238 Łódź, Poland
E-mail: emotyl@math.uni.lodz.pl

The Institute of Information Theory and Automation of the Czech Academy of Sciences,
Pod Vodárenskou věží 4, CZ-182 08, Praha 8, Czech Republic
E-mail: ondrejat@utia.cas.cz; **??**