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Random Decentralized Market Processes for Stable Job Matchings with Competitive Salaries*

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Abstract

We analyze a decentralized process in a basic labor market where finitely many heterogeneous firms and workers meet directly and randomly in pursuit of higher payoffs over time and agents may behave myopically. We find a general random decentralized market process that almost surely converges in finite time to a competitive equilibrium of the market. A key proposition en route to this result exhibits a finite sequence of successive bilateral trades from an arbitrary initial market state to a stable matching between firms and workers with a scheme of competitive salary offers.

Keywords: Decentralized market, job matching, random path, competitive salary, stability. *JEL classification*: C62, D72.

1 Introduction

Adam Smith's Invisible Hand captures the self-regulating nature of a decentralized market where self-interested market participants, making independent decisions freely, can settle the market on a competitive equilibrium outcome. Traditionally a fictitious Walrasian auctioneer has been used to match the supply and demand of each commodity (service) at its competitive price (wage). However, many competitive markets, labor markets being a leading example, involve mainly uncoordinated

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bilateral trades and are typically decentralized. The purpose of this paper is to analyze the long-run behavior of a general random market process in a basic labor market where transactions take the form of bilateral trades so as to mimic the decentralized behavior of the labor market.

We consider a labor market where finite heterogeneous firms and workers meet directly and randomly to search for higher payoffs over time. In the market, all agents make their own decisions independently and can behave myopically, perhaps because information is dispersed and agents may not have a complete picture of the entire market. When a worker and a firm match as partners, they generate a joint surplus which is then split within the matched pair. Each agent can dissolve her current partnership unilaterally if standing alone becomes a better option. A worker and a firm, currently not matched, can form a new partnership as long as doing so makes none of the two worse off and at least one strictly better off — in this case the firm fires its previous worker and the worker abandons her previous firm, if any, and the deserted parties can be worse off. We call such transactions bilateral trades or pair improvements. In such a market process, quits and layoffs routinely arise as a result of agents seeking better matches and it is also possible that workers eventually return to their previous employers but with different contracts. The random process proceeds spontaneously and is decentralized, in that every agent acts only according to her own interests without any centralized coordination, and unforeseen and unexpected market outcomes can emerge from the agents' actions under imperfect information about the market.

The basic question we consider is whether the above random, chaotic, and dynamic decentralized process eventually leads the market to efficient assignments of workers to firms and in particular to a competitive equilibrium.[1] We establish that this market process converges with probability one to a competitive equilibrium of the market in finite time, so long as each possible bilateral trade conditional on the current market state arises with an arbitrary but positive probability in the process (Theorem 1). An interpretation of this positive probability is that although information is imperfect and dispersed among all market participants, it flows sufficiently freely so that the agents are informed about and can therefore respond to newly arrived opportunities. A crucial step for establishing Theorem 1 is to show that the random process is not trapped in trading cycles indefinitely. To this end, we demonstrate via a novel algorithm the existence of a finite sequence of successive bilateral trades from an *arbitrary* initial market state to a competitive equilibrium (Proposition 1).

[1]There are many different types of market processes. See, for example, Gale and Shapley (1962) for marriage matching problems, Shapley and Scarf (1974) for housing markets, Crawford and Knoer (1981) and Kelso and Crawford (1982) for job matching problems, Demange et al. (1986) for auction markets, and Abdulkadiroğlu and Sönmez (2003) for school choice problems. By a centralized process, we mean that individuals make decisions independently but a "clearing house" or a central planner coordinates all activities. For instance, in auctions, an auctioneer collects the demands of all bidders and then adjusts prices. In a decentralized process, individuals make decisions independently and locally without any coordination from a central planner or organization.

Our study is closely related to the seminal work by Crawford and Knoer (1981), who consider a similar labor market and propose a deterministic salary adjustment process for the market in which firms make offers and workers then accept or reject the offers. The salary adjustment process, a generalization of the deferred acceptance algorithm of Gale and Shapley (1962), converges to a competitive equilibrium of the market.[2] Our market process is random and uncoordinated, beginning with any market state and having no central planner to guide transactions. Crawford and Knoer's process is however deterministic and does not involve any uncertainty. In addition, their process starts with a specific market state and requires firms to use retention to maintain payoff monotonicity of every worker.[3] Such monotonicity cannot hold in our processes, where a bilateral trade, while improving the welfare of the involved pair, typically makes the abandoned firm and worker worse off. *The overall market welfare is hence not necessarily monotone after a sequence of bilateral trades, making a design relying on monotonicity arguments difficult if not impossible.*

Our study is also related to the seminal work by Roth and Vande Vate (1990), who reexamine the Gale-Shapley marriage matching model and develop a decentralized process that finds almost surely a stable matching between men and women.[4] A key difference here is that our model admits monetary transfers and has a competitive equilibrium supported by competitive prices, while it is known that the marriage matching model generally does not have competitive prices to support any stable matching. Another major difference is that the *stability* solution in the marriage matching literature is strictly weaker than ours which coincides with competitive equilibrium. Furthermore, a key step of the algorithm in Roth and Vande Vate (1990) maintains strict payoff monotonicity of one side of the market, which fails in our setting because we work with a more general notion of blocking involving both a pair of agents and surplus division.[5] Hence, while Roth and Vande Vate (1990) provide a decentralized foundation for stability in the marriage matching, our study offers a decentralized framework for competitive equilibrium in the assignment market.

Finally, it is worth mentioning another strand of literature concerning Feldman (1974) and Green (1974). While their processes are deterministic for certain classes of NTU games, the current process is random and also involves significant indivisibility.

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[2]Koopmans and Beckmann (1957) prove the existence of a competitive equilibrium in such a market. Shapley and Shubik (1971) demonstrate that the core of this market is a lattice and coincides with the set of competitive equilibrium price vectors.

[3]While we also construct an algorithm leading any initial market state to a competitive equilibrium, the role of our algorithm is to ensure that the random market process eventually converges. We discuss the detailed difference between our algorithm and the salary adjustment process in Section 3.1.

[4]Gale and Shapley (1962) show the existence of a stable marriage matching via a deferred acceptance process. For models closely related to Roth and Vande Vate (1990), see Diamantoudi et al. (2006), Kojima and Ünver (2006), and Klaus and Klijn (2007).

[5]We thank a referee for pointing this out.

2 The Model

Consider a labor market with heterogeneous firms and workers. Denote the (finite) set of firms as F and the (finite) set of workers as W such that $F \cap W = \varnothing$. Each firm hires at most one worker and each worker accepts at most one job.[6] A matching or assignment μ is a one-to-one mapping from $F \cup W$ to itself such that for all $x \in F \cup W$, x is either self-matched ($\mu(x) = x$), or is matched to a member of the other side of the market. Agent $\mu(x)$ is called x's partner under μ.

A worker w's productivity at firm f is denoted by $V(f, w)$, which is also interpreted as the surplus generated by the pair (f, w). For all $x \in F \cup W$, value $V(x, x)$ is agent x's stand-alone value, interpreted as x's outside option and we allow for heterogeneous outside options. We assume that all values $V(f, w)$, $V(f, f)$ and $V(w, w)$ are measured in terms of an *indivisible* commodity (money), i.e., $V(f, w)$, $V(f, f)$, $V(w, w) \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. Such a discrete modeling assumption is fairly natural and standard, as for instance one cannot specify a monetary payoff more closely than to its nearest penny.[7] We denote the labor market by (F, W, V).

A state or outcome of the market (F, W, V) consists of a matching μ and a payoff vector $u \in \mathbb{R}^{F \cup W}$ such that $u(x) = V(x, x)$ for any $x \in I(\mu)$ and $u(x) + u(\mu(x)) = V(x, \mu(x))$ for $x \notin I(\mu)$, where $\mathbb{R}$ is the set of real numbers and $I(\mu) = \{h \in F \cup W \mid \mu(h) = h\}$ is the set of self-matched agents at μ. A market state (μ, u) is *individually rational* if $u(x) \geq V(x, x)$ for all $x \in F \cup W$. Define the *market value* associated with μ as $\sum_{f \in F \setminus I(\mu)} V(f, \mu(f)) + \sum_{i \in I(\mu)} V(i, i)$. A matching μ is *efficient* if no other matching generates a higher market value than μ does.

A *blocking pair* of a state (μ, u) is a pair (f, w) of firm f and worker w that are not matched under μ but both can improve their well-being by abandoning their partners at μ and matching with each other, i.e., there are $r_f, r_w \in \mathbb{R}$ with $r_f + r_w = V(f, w)$ such that $r_w \geq u(w)$ and $r_f \geq u(f)$ with at least one strict inequality. A state (μ, u) can also be blocked by a single agent x if x is not self-matched at μ, but prefers to be single, or $r_x = V(x, x) > u(x)$, i.e., (x, x) is also called a blocking pair of (μ, u). A market state (μ, u) is *stable* or equivalently *a competitive equilibrium* if $u(f) + u(w) \geq V(f, w)$ and $u(x) \geq V(x, x)$ for all $f \in F$, $w \in W$, $x \in F \cup W$. Namely, a state is stable if it is not blocked by any single agent or any pair of firm and worker. It is known that if (μ, u) is stable, then μ is efficient. Observe that the domain we use to define a market outcome, equilibrium, blocking pair and others is *the reals* $\mathbb{R}$.

As a blocking pair may result in multiple states, arising from different specifications of surplus division, we define a concept of *pair improvement* to fully describe

[6] This is the *unit-demand* assumption in the literature. See, e.g., Shapley and Shubik (1971), Crawford and Knoer (1981), Demange et al. (1986).

[7] See, e.g., Ausubel (2006), Demange et al. (1986), and Roth and Sotomayor (1990). Technically speaking, the values of $V(f, w)$, $V(f, f)$ and $V(w, w)$ being integral guarantees the existence of a stable market state with integral payoffs. This also allows us to work exclusively on blocking payoffs with integral payoffs which reflect real life transactions. See Lemma 1 below.

the process from a blocking pair.

Definition 1 *Given a blocking pair* (f,w) *of a state* (μ,u)*, a new state* (μ',u') *is said to be a* ***pair improvement*** *of* (μ,u) *through* (f,w) *if (1)* $\mu'(x)=\mu(x)$ *and* $u'(x)=u(x)$ *for any* $x\in(F\cup W)\setminus\{f,w,\mu(f),\mu(w)\}$*, (2) under* μ'*,* f *and* w *are matched, while* $\mu(f)$ *and* $\mu(w)$ *are self-matched, and (3)* $u'(f)=r_f$ *and* $u'(w)=r_w$ *such that* $r_f+r_w=V(f,w)$*, while* $u'(\mu(f))=V(\mu(f),\mu(f))$ *and* $u'(\mu(w))=V(\mu(w),\mu(w))$*.*

A pair improvement mimics a real life transaction between a firm and a worker and is intuitively interpreted as a specific form of bilateral trade. We hence use *pair improvement* and *bilateral trade* interchangeably. Notice that when a firm hires a new employee or a worker joins a new firm, both parties are better off but the abandoned parties are usually worse off than before. A pair improvement hence has opposing effects on the involved agents. Consequently, the market value along a path of successive pair improvements need not be monotone.

Our labor market can be regarded as a general assignment market. It is well known that an assignment market admits at least one competitive equilibrium and that the set of stable outcomes (i.e., strict core) coincides with that of competitive equilibria (Shapley and Shubik 1971). In addition, as all valuations are integers and the market structure is totally unimodular,[8] our labor market must have at least one stable outcome with an integral payoff vector $u\in\mathbb{Z}^{F\cup W}$. See Ausubel (2006) and Sun and Yang (2009) for more general results. Finally, while a blocking pair is defined through *real* payoffs $r_f,r_w\in\mathbb{R}$, our next lemma shows that it is sufficient to focus only on blocking pairs with *integer payoffs* when searching for a stable outcome in the market.

Lemma 1 *Let* $V(f,w)$*,* $V(f,f)$ *and* $V(w,w)$ *be integral for all* $f\in F$ *and* $w\in W$*. If a state* (μ,u) *with integral* $u\in\mathbb{Z}^{F\cup W}$ *is not blocked by any pair* (f,w) *with integral* $(r_f,r_w)\in\mathbb{Z}\times\mathbb{Z}$*, then it cannot be blocked by any pair* (f',w') *with real* $(r_{f'},r_{w'})\in\mathbb{R}\times\mathbb{R}$*. Consequently,* (μ,u) *must be a competitive equilibrium.*

Proof. Suppose to the contrary that the statement is not true. Then there would exist a state (μ,u) such that $u\in\mathbb{Z}^{F\cup W}$ is not blocked by any pair (f,w) with $(r_f,r_w)\in\mathbb{Z}\times\mathbb{Z}$, but is blocked by a pair (f',w') with $(r'_f,r'_w)\in\mathbb{R}\times\mathbb{R}$. Because (f',w') blocks (μ,u), $r_{f'}+r_{w'}=V(f',w')$, $r_{f'}\geq u(f')$ and $r_{w'}\geq u(w')$ with at least one strict inequality. Since $V(f',w')$ and $u\in\mathbb{Z}^{F\cup W}$ are integral, we must have that either both $r_{f'}$ and $r_{w'}$ are integral or neither $r_{f'}$ nor $r_{w'}$ is integral. The former case cannot happen by hypothesis. In the latter case, we must have $r_{f'}>u(f')$ and $r_{w'}>u(w')$. Now let $f=f'$ and $w=w'$. We can round up r_f to its next higher integer s_f and round down r_w to its next lower integer s_w. Because u and $V(f,w)$ are

[8]The market structure can be expressed as a matrix with all its entries being 1 or 0. The matrix is *totally unimodular* because all its subdeterminants equal 1, -1, or 0.

integral, clearly we have $s_f + s_w = V(f, w)$, $s_f > u(f)$ and $s_w \geq u(w)$. By definition, (μ, u) is blocked by (f, w) with $(s_f, s_w) \in \mathbb{Z} \times \mathbb{Z}$, contradicting the hypothesis. ■

Lemma 1 indicates that although competitive equilibrium is defined on the real domain $\mathbb{R}$, we can actually ignore all fractional numbers and focus only on integer payoffs or salaries which correspond exactly to real life transactions. It provides a rationale behind the market process and the algorithm to be presented below which make use of only the integer domain $\mathbb{Z}$ instead of the real domain $\mathbb{R}$.

3 Main Results

In this section we present our central result Theorem 1, which demonstrates that starting with an arbitrary initial market state, any random and decentralized process in which every bilateral trade conditional on the current market state occurs with a positive probability will converge with probability one to a competitive equilibrium in finite time. To achieve this goal, we first present a crucial mathematical result Proposition 1 establishing the existence of a finite sequence of successive bilateral trades from any initial market state to a competitive equilibrium. For this purpose we will prove Proposition 1 in Section 3.1 and then Theorem 1 in Section 3.2.

3.1 A Key Proposition

To prove our key Proposition 1 we will construct a desired path from an arbitrary initial market state to a stable state. The crucial feature of the desired path is that it only employs successive "local adjustments" or pair improvements to reach a stable state. This constraint gives rise to two complications in our setting: First, both matchings and payoffs along the path have to be chosen carefully so as to satisfy the pair improvement requirement. Second, because an agent's payoff can remain the same after each pair improvement, trading cycles naturally arise in the process.[9] This leads to a challenging complication of designing a systematic procedure to lead the path out of trading cycles and also to prevent the same cycles from recurring.

We proceed in two main steps: We first design a path of successive pair improvements toward stability for an "almost stable" outcome. Using this as a building block, we then construct a required path for an arbitrary initial market outcome.[10]

An outcome (μ, u) is *almost stable* if there exists a self-matched worker w^0 and (μ, u) is stable if w^0 is excluded from the economy, i.e., (μ, u) restricted to $F \cup (W \setminus$

[9] A trading cycle occurs when starting from a state (μ, u), a successive path of pair improvements results in the same market state (μ, u).

[10] Following an older version of the current study (Chen, Fujishige and Yang 2010), Biró et al. (2012) recently obtain similar results in a more general setting of stable roommates problems with transfers. While our algorithm is implementable in practice, they have to assume the existence of an explicitly given competitive equilibrium and hence their algorithm shall be regarded as a thought experiment and cannot be used in reality.

$\{w^0\}$) is stable. For each $w \in W$, the set of w's *best firms* under (μ, u) is $F_w(u) = \{f \in F \mid V(f,w) - u(f) = \max\{V(f',w) - u(f') \mid f' \in F\}\}$. A *list* L_w of w's all best firms in $F_w(u)$ is a permutation (i.e., linear ordering) of all elements of $F_w(u)$, which is fixed whenever $F_w(u)$ remains the same. An *alternating path* for (μ, u) starting from w^0 is an alternating sequence of unmatched and matched firm-worker pairs "(f^1, w^0), $(f^1, w^1 = \mu(w^1))$, (f^2, w^1), $(f^2, w^2 = \mu(f^2))$, ..., $(f^{l-1}, w^{l-1} = \mu(f^{l-1}))$, (f^l, w^{l-1})" such that (i) all firms and workers are distinct in the path, (ii) no firm in the path is self-matched, and (iii) for $i = 1, \ldots, l$, firm f^i is a best firm for worker w^{i-1} when w^{i-1} becomes self-matched, i.e., $f^i \in F_{w^{i-1}}(u) \setminus \{\mu(w^{i-1})\}$.[11]

These definitions will mainly serve as key tools to systematically tackle trading cycles. Roughly speaking, the set $F_w(u)$ is a "depository" of firms with which worker w wants to form a blocking pair. The list L_w is an index, specifying the order w should follow in applying to the firms in $F_w(u)$. Each list L_w is used cyclically so that the first element of L_w becomes the next firm when w reaches the end of L_w. Since L_w is merely a book-keeping device, the permutation can be arbitrary. However, L_w should be fixed for fixed $F_w(u)$ so that when a trading cycle arises, every worker w involved in the cycle has been matched to all firms in $F_w(u)$ during the cycle. An alternating path is a device to lead the path out of a cycle so that each adjustment is a legitimate pair improvement. For an unmatched pair (f^k, w^{k-1}) in an alternating path ($k \geq 1$), after breaking up with her current firm (if any), worker w^{k-1} then applies to firm f^k, the next most-preferred firm she would like to be matched with.

Denote the initial almost stable outcome as (μ^0, u^0) which is not stable but becomes stable without a self-matched worker w^0. For reasons of bookkeeping, we reset the list L_w cyclically for each $w \in W$ so that the first firm in L_w is matched with w in μ^0. The following algorithm constructs a path of successive pair improvements leading (μ^0, u^0) to stability.

The Algorithm

Step 1. Let w^0 apply to f^0, the first firm of list L_{w^0}, and (w^0, f^0) is a blocking pair of (μ^0, u^0) as (μ^0, u^0) without worker w^0 is stable. Match f^0 and w^0 together and let w^0 receive the entire blocking surplus to obtain (μ^1, u^1) so that $\mu^1(f^0) = w^0$ and $\mu^1(x) = \mu^0(x)$ for every other agent $x \in F \cup W$ except possibly f^0's partner $\mu^0(f^0)$ under μ^0, and $u^1(w^0) = V(f^0, w^0) - u^0(f^0)$, $u^1(x) = u^0(x)$ for every other agent $x \in F \cup W$ except possibly $\mu^0(f^0)$. If firm f^0 is self-matched under μ^0, then we are done as f^0 is currently a best firm for w^0. If not, let w^1 be $\mu^0(f^0)$ and go to **Step 2**.

Step 2. By construction, there is no blocking pair for (μ^1, u^1) restricted to $F \cup (W \setminus \{w^1\})$. If (μ^1, u^1) is stable, then we are done. If not, proceed similarly as in **Step 1** to obtain (μ^2, u^2) so that w^1 is matched with f^1 according to L_{w^1}, receiving the entire blocking surplus and $w^2 = \mu^1(f^1)$ if f^1 is not self-matched in μ^1.[12] Continue

[11]For $l \geq 2$, we also call such a sequence without the last pair (f^l, w^{l-1}) an alternating path.

[12]Here f^1 will be the second element of L_{w^1} as the first element is f^0, w^1's previous partner.

in this fashion until we reach an outcome $\left(\mu^k, u^k\right)$, which is either a stable outcome or $(\mu^k, u^k) = (\mu^{k'}, u^{k'})$ for some integer k' with $0 \leq k' < k$, that is, we have found a trading cycle, in which case go to **Step 3**.

Step 3. Collect the firms involved in the cycle in F_Q which is the set of firms whose matched workers change at least once during the cycle. As the firms in F_Q are over-demanded by the workers in the cycle, we adjust the firms' payoffs upwards so as to lead the process out of the cycle and such adjustments can only be done via pair improvements. We use the following **Augment** procedure to accomplish this.

Augment: Rename w^k and μ^k in the outcome $\left(\mu^k, u^k\right)$ as w^* and μ respectively. Let F^* be a bookkeeping set that we temporarily store firms in F_Q that have already been treated with payoff increase (hence initially $F^* := \varnothing$). The **Augment** procedure ends when $F_Q = F^*$. This is done by repeatedly carrying out the following steps:

A1. Construct an alternating path for $\left(\mu, u^k\right)$ from w^* to a firm f^* whose payoff has not been increased, i.e., $f^* \in F_Q \setminus F^*$ so that all the firms in the alternating path except the last firm f^* are in F^*:[13] $\underline{(f_1, w_0 = w^*)}$, $(f_1, w_1 = \mu(f_1))$, $\underline{(f_2, w_1)}$, $(f_2, w_2 = \mu(f_2))$, …, $(f_{l-1}, w_{l-1} = \mu(f_{l-1}))$, $\underline{(f_l = f^*, w_{l-1})}$ where $f_1, \ldots, f_{l-1} \in F^*$ and pairs with underscores are currently not matched together under μ.

A2. Starting from the end of the alternating path and proceeding in the reverse order, for each unmatched pair (f, w), we match f with w (so that $\mu(f)$ and $\mu(w)$ for $w \neq w^*$ become self-matched) and let $u(w) = V(f, w) - u^k(f) - 1$ and $u(f) = u^k(f) + 1$ with any newly self-matched agent receiving her outside option. Proceed like this starting from pair $(f_l = f^*, w_{l-1}))$ until we match (and specify the payoffs for) the first pair $(f_1, w_0 = w^*)$.

A3. As $f_1, \ldots, f_{l-1} \in F^*$, only the last firm f^*'s payoff has increased by exactly 1 after **A2**. Call the new matching after **A2** as μ' and update F^* so that $F^* := F^* \cup \{f^*\}$. If the updated set $F^* \neq F_Q$, then rename f^*'s previous partner $\mu(f^*)$ to be the next w^*, and μ' to be the next μ and go to step **A1.**

Step 4. Denote the outcome after step **Step 3** as (μ, u). If (μ, u) is stable, then we are done. If not, then let w^0 be the self-matched worker that appeared at the most recent updating of μ and rename (μ, u) as (μ^0, u^0). Update the list L_w according to $F_w\left(u^0\right)$ for all $w \in W$ so that the first firm in L_w is matched with w in μ^0. Go to **Step 1.**

We have considered blocking pairs with integer payoffs in the algorithm, which is sufficient for our purpose due to Lemma 1. Notice that trading cycles can arise in Step 1 and Step 2, as a result of a set of workers competing for their best firms (collected in F_Q) and these workers receiving the entire blocking surpluses in the process. Step 3 is the key step of the algorithm, where we search for a sequence of pair improvements so as to lead the path out of a trading cycle, and to ensure that the same cycle will not be encountered again. Getting out of the cycle requires

[13] By construction, the first alternating path hence has to have length 1.

that the over-demanded firms in F_Q receive higher payoffs. The Augment procedure increases the payoff of each over-demanded firm in F_Q by the smallest increment of 1 with the constraint that each such adjustment has to be a pair improvement. We adjust the payoffs of the over-demanded firms one at a time using an alternating path. Since such payoff adjustment for an over-demanded firm can potentially affect various firms, we construct the alternating path so that (1) only the firms whose payoffs have already been adjusted are involved when we treat a newly added over-demanded firm and (2) these firms (whose payoffs have been adjusted) are contained in the path so that their payoffs do not decrease when we adjust upwards the payoff of the new over-demanded firm.

Lemma 2 *The Algorithm finds a stable outcome after a finite sequence of successive pair improvements for any almost stable market state.*

The path of successive pair improvements toward stability from an *arbitrary* initial market state can then be constructed by applying Lemma 2.[14] The main idea is to inductively construct an increasing "internally stable" set, i.e., for a given market state, if we restrict our attention to the agents in the "internally stable" set, there are no blocking pairs or feasible pair improvements. In each inductive step of adding an agent to enlarge the "internally stable" set, we can apply Lemma 2 as the process is isomorphic to the procedure of finding a path of pair improvements toward stability for an almost stable outcome.[15] Such an inductive construction, together with Lemma 2, enables us to establish the following:

Proposition 1 *Consider a labor market (F, W, V) with an arbitrary initial market state (μ^0, u^0). There exists a finite sequence of successive pair improvements which leads (μ^0, u^0) to a competitive equilibrium.*

As a by-product, Proposition 1 provides an alternative deterministic adjustment process toward competitive equilibrium, as well as a constructive proof of the existence of stable outcomes for the assignment market. Importantly, it differs from the salary adjustment process in Crawford and Knoer (1981) on two accounts: First, our deterministic process can start with an arbitrary market state, while the salary adjustment process has to start with a specific market state. Second, the transactions in our process only take the form of bilateral trades. The salary adjustment process, like the deferred acceptance algorithm, employs adjustments that violate the property of pair improvements. In particular, the salary adjustment process uses retention where workers temporarily hold job offers from the firms without accepting any offer until toward the end of the process, and as a result, the payoffs of every agent on one side

[14]The proof of Lemma 2 is relegated to an Appendix.

[15]The detail of the construction is omitted and can be found in our working paper Chen, Fujishige and Yang (2010).

of the market are non-decreasing in the process. Convergence is hence monotonic in the salary adjustment process, rather than cyclical as in our process.[16]

3.2 A Random Decentralized Market Process

We now address our central result on probabilistic convergence of a general random decentralized market process. Consider an arbitrary initial market state of the labor market. Each agent, though knowing some information about the market (e.g., her current payoff, her stand-alone payoff, and perhaps some other existing better firms/workers), may not have a complete picture of the entire market and hence may act myopically in future transactions. To be specific, suppose that agents at each point of time randomly receive opportunities, a real-world analogy being that firms and workers constantly obtain various job-related information from advertisements, labor market intermediaries, or friends. Suppose further that once an agent or a pair of worker and firm finds an opportunity to improve the status quo, they will do so by abandoning their current partners and forming new partnerships. This process continues until a stable state (a competitive equilibrium) is reached.

Our question is whether such a general random and decentralized market process converges to a competitive equilibrium eventually. The following theorem answers this question in the affirmative. Formally, we prove that starting from an arbitrary market state, this market process converges probabilistically to a competitive equilibrium, provided that at any point in time, each pair improvement of the current market state arises with a positive probability. Notice that the salient feature of finite successive pair improvements in Proposition 1 is essential to capture the decentralized nature of the random market process. Other convergent paths where this feature is absent are unable to achieve our goal in Theorem 1.

Theorem 1 *Start with an arbitrary initial market state of a labor market (F, W, V). If every pair improvement occurs with a positive probability bounded away from zero in a random decentralized rematching process, then almost surely, convergence to a competitive equilibrium obtains in finite time.*

We now discuss how to establish the Theorem via Proposition 1. Observe that it suffices to consider all individually rational market states (μ, u) with integral payoff vector $u \in \mathbb{Z}^{F \cup W}$, because any individual who gets less than her stand-alone value can immediately abandon her partner to achieve at least her stand-alone value. Let $\mathcal{A}(F, W, V)$ denote the set of all such market states. Notice that $\mathcal{A}(F, W, V)$ is finite, as the number of workers and firms is finite, the number of matchings is finite, any value of $V(x)$ or $V(f, w)$ is finite, u is integral, and every (μ, u) is individually rational.

[16]Demange et al. (1986) also provide a similar process (an ascending price mechanism) as the salary adjustment process that finds a stable market state in finite time. A Supplementary Material of this article provides examples, extra comparison of our algorithm with the construction in Demange et al. (1986) and Crawford and Knoer (1980), and other auxiliary results.

Suppose now that the market opens with an arbitrary state $(\mu^0, u^0) \in \mathcal{A}(F, W, V)$. Consider a random process with the finite set of market states $\mathcal{A}(F, W, V)$. The transition probabilities of the states in $\mathcal{A}(F, W, V)$ are defined in a way such that for every unstable state $(\mu, u) \in \mathcal{A}(F, W, V)$, each pair improvement of (μ, u) is chosen with a positive probability bounded away from zero.[17] Proposition 1 implies that the constructed random process does not oscillate among unstable states indefinitely and hence the associated random sequence converges to a stable state in $\mathcal{A}(F, W, V)$ with probability one.

From an economic point of view, however, it is more natural for a decentralized random process to feature time-dependent transition probabilities associated with every market state in the process. Consider now a similar but more general decentralized random process with discrete time, finite set $\mathcal{A}(F, W, V)$ of states, and possibly *time-dependent* transition probabilities among all states. The market again opens with an arbitrary initial market state $(\mu^0, u^0) \in \mathcal{A}(F, W, V)$ at time $t = 0$. Assume that every transition probability from an unstable state to another state is no less than a fixed number $\varepsilon \in (0, 1)$ at any time, namely, every possible pair improvement occurs with a positive probability. With only two classes of states (stable and unstable), it follows that starting from any state (μ, u) in $\mathcal{A}(F, W, V)$, the process either finds a stable state in $\mathcal{A}(F, W, V)$ and remains stable afterwards, or continues to move from one unstable state to another unstable state in $\mathcal{A}(F, W, V)$, as the random process by construction always arrives at a state in $\mathcal{A}(F, W, V)$. Suppose that the random process does not converge to a stable state with probability one in the limit. This necessarily implies that at some point, after reaching an unstable (μ, u) in $\mathcal{A}(F, W, V)$, the random process oscillates among a (finite) set of unstable states in $\mathcal{A}(F, W, V)$ indefinitely. Since each possible pair improvement is chosen with a probability no less than ε at each point of time, there is then some (μ', u') in $\mathcal{A}(F, W, V)$ such that no finite path of pair improvements toward stability exists, no matter how one chooses the associated pair improvements, which contradicts Proposition 1, completing the proof of the Theorem.

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[17] Strictly speaking, we can define the entire set of states as the product $\mathcal{A}(F, W, V) \times \mathbb{Z}_+$ of market states $\mathcal{A}(F, W, V)$ and the discrete time set $\mathbb{Z}_+$ starting from 0. Then the transition probability from a state to another state is a probability from an element in $\mathcal{A}(F, W, V) \times \{t\}$ to another element in $\mathcal{A}(F, W, V) \times \{t+1\}$. A transition probability may reflect, for example, how likely the agents involved in the corresponding pair improvement would meet and would split the surplus once they meet, as well as the current market structure.

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Appendix

Proof of Lemma 2. The key issue is to verify the validity of (trading cycle) **Step 3**. We hence divide the proof into two parts: We first show the existence of an alternating path in **A1** of **Step 3**, followed by a proof showing that every adjustment in **Step 3** is a pair improvement.

Let F_Q (resp., W_Q) be the set of firms (resp., workers) whose matched partners change at least once during the cycle. We claim that whenever $F^* \neq F_Q$, there exists an alternating path from w^* to a firm $f^l \in F_Q \backslash F^*$ and that all other firms involved in the alternating path are in F^*. Suppose there does not exist any such alternating path for (μ, u^k) from w^* to $F_Q \setminus F^*$. Let $\hat{F}$ and $\hat{W}$, respectively, be the set of all firms and that of all workers in $F_Q \cup W_Q$ *reachable* from w^* by alternating paths for (μ, u^k), where an agent is *reachable* from w^* if the agent belongs to an alternating path originating from $w^* \in \hat{W}$. Notice that all agents reachable from w^* by alternating paths for (μ, u^k) have to be in $F_Q \cup W_Q$.

Given $\hat{F}$, $\hat{W}$, and the non-existence of alternating paths, we have

$$\begin{aligned} |\hat{F}| + 1 &= |\hat{W}|,\ |F_Q \setminus \hat{F}| = |W_Q \setminus \hat{W}| > 0 \text{ and} \quad (1) \\ \nexists \text{ matched } (f, w) \text{ s.t } w &\in \hat{W},\ f \in (F_Q \setminus \hat{F}) \cap F_w\left(u^k\right), \quad (2) \end{aligned}$$

where observe that F_w is defined using u^k. Condition (1) is derived from the definition of $\hat{F}$ and $\hat{W}$ and the fact that in a restricted situation, every matching arising in the cycle has only one self-matched worker. According to (2), no worker in $\hat{W}$ can be matched with firms in $(F_Q \setminus \hat{F})$. In particular, (1) and (2) jointly imply that no agent in $\hat{F} \cup \hat{W}$ is matched with an agent in $(F_Q \backslash \hat{F}) \cup (W_Q \backslash \hat{W})$.

Since we have a matching that matches all $f \in F_Q$ to workers in the cycle, (2) then implies the existence of a matched pair (f', w') such that $f' \in \hat{F}$ and $w' \in W_Q \setminus \hat{W}$. However, f' cannot be matched to w' since $F_Q \setminus \hat{F}$ must be matched to $W_Q \setminus \hat{W}$ due to (1) and (2).[18] This contradiction establishes the existence of a desired alternating path in the execution of **A1**.

We next show that every adjustment executed in **Augment** is a pair improvement. Let the alternating path found in **A1** in **Step 3** be "$(f^1, w^0{=}\ w^*)$, $(f^1, w^1{=}\mu(f^1))$, (f^2, w^1), $(f^2, w^2{=}\mu(f^2))$, …, $(f^{l-1}, w^{l-1}{=}\mu(f^{l-1}))$, (f^l, w^{l-1}))" for some $l \in \mathbb{N}$. (Notice that $\mu(f^l)$ becomes the unique self-matched worker after **A2**, which is the w^* for the next round of **A1**.)

[18] Alternatively, if $f' \in \hat{F}$ is matched with $w' \in W_Q \setminus \hat{W}$, we then cannot have that $|F_Q \setminus \hat{F}| = |W_Q \setminus \hat{W}| > 0$ and that firms in $F_Q \setminus \hat{F}$ are exactly matched with workers in $W_Q \setminus \hat{W}$.

If $l = 1$, then $(f^1, w^0 = w^*)$ is a blocking pair (recall that since $f^1 \in F_{w^0}\left(u^k\right)$, f^1 and w^0 were matched during the cycle).

If $l = 2$, we have $u(f^l) = u^k(f^l)$, $u(w^{l-1}) = u^k(w^{l-1}) - 1$, and $u^k(f^l) + u^k(w^{l-1}) = V(f^l, w^{l-1})$. Hence $V(f^l, w^{l-1}) - u(f^l) - u(w^{l-1}) = 1$ (recall that by construction, for any $f \in F_Q$ the value of $u^i(f)$ remains the same for $i = k', \ldots, k$, and that for each $w \in W_Q$ all the values of $V(f, w) - u^i(f)$ for $f \in F_w$ and $i = k', \ldots, k$ are the same. Here k' and k are the parameters in **Step 2**). Hence, (f^l, w^{l-1}) is a blocking pair and this validates the operations in **A2**. We make f^l matched to w^{l-1} and update u as in **A2** of **Augment**. (In effect, $u(f^l)$ is increased by one and $u(w^{l-1})$ remains the same.) Then f^{l-1} becomes self-matched. If $l = 2$, then $(f^1, w^0 = w^*)$ is a blocking pair (recall again that f^1 and w^0 were matched during the cycle, so that they prefer being matched to being self-matched.)

If $l \geq 3$, we have $u(f^{l-1}) = V(f^{l-1}, f^{l-1}) < u^k(f^{l-1}) + 1 = V(f^{l-1}, w^{l-2}) - u(w^{l-2})$, so that pair (f^{l-1}, w^{l-2}) becomes a blocking pair. We then perform **A2**. Repeat this process until we match f^1 with $w^0 = w^*$, which completes an execution of **A2**.

Upon finishing the **Augment** procedure, $u(f)$ of each $f \in F_Q$ is increased by one.

When proceeding from **Step 3** to **Step 4**, w^k being the current self-matched worker, no blocking pair exists in $(F \cup W) \setminus \{w^k\}$. Recall that $F_Q = \cup_{w \in W_Q} F_w\left(u^k\right)$, implying that for any (f, w) such that $f \in F \setminus F_Q$ and $w \in W_Q$, we have $V(f, w) - u(f) \leq V(\mu(w), w) - u(\mu(w))$, where (μ, u) is the final state after **Step 3** is completed.

Each time we execute **Step 3**, at least one value of $u(f)$ increases and $u(f)$ for each $f \in F$ is non-decreasing throughout the algorithm (where we neglect the temporary steps in which $u(f)$ becomes $V(f, f)$ during the execution of Augment). Moreover, the set of possible integer values of $u(f)$ $(f \in F)$ is finite. Since the number of all matchings for fixed $F_w\left(u^k\right)$ $(w \in W)$ is bounded by $|W|!$, after at most $|W|!$ updatings of μ, we either get into a cycle, where some $u(f)$ is increased after **Step 3**, or the algorithm terminates. We therefore conclude that the algorithm terminates after a finite number of steps. ■