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ON THE NUMBER OF ZEROS OF LINEAR COMBINATIONS OF INDEPENDENT CHARACTERISTIC POLYNOMIALS OF RANDOM UNITARY MATRICES

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Abstract. We show that for any linear combination of characteristic polynomials of independent random unitary matrices with the same determinant, the expected proportion of zeros lying on the unit circle tends to 1 as the dimension of the matrices tends to infinity. This result is the random matrix analogue of an earlier result by Bombieri and Hejhal on the distribution of zeros of linear combinations of L-functions, and thus is consistent with the conjectured links between the value distribution of the characteristic polynomial of random unitary matrices and the value distribution of L-functions on the critical line.

1. Introduction

Over the past two decades, there have been many new results at the interface of random matrix theory and analytic number theory that can be considered as evidence for the zeros of the Riemann zeta function being statistically distributed as eigenvalues of large random matrices (GUE matrices or Haar distributed unitary matrices). The interested reader can refer to [19], [13] and [21] for a detailed account with many references, and to [12] for the function field framework. Since the seminal papers by Keating and Snaith [14, 15], it is believed that the characteristic polynomial of a random unitary matrix on the unit circle provides a very accurate model for the value distribution of the Riemann zeta function (or, more generally, L-functions) on the critical line. This analogy was used by Keating and Snaith to produce the moments conjecture and since then the characteristic polynomial has been the topic of many research papers, and the moments of the characteristic polynomial have now been derived with many different methods: representation theoretic approach (see [6, 20]); super-symmetry (see [19]); analytic methods (such as Toeplitz determinants [19] and orthogonal polynomials on the unit circle [16]); and probabilistic techniques ([4]). Each method brings a new insight to the problem. Many more fine-grained properties of the characteristic polynomial have been established (e.g. the large deviations principle [9], local limit theorems [17], the analogue of the moments conjecture for finite field zeta functions [1, 11], etc.). Moreover, thanks to this analogy, one has been able to perform calculations in the random matrix world (whose analogue in the number theory world appears to be currently out of reach) to produce conjectures for the analogous arithmetic objects (see [23] for a recent account).

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There are nonetheless certain results that can be proved in both worlds, such as Selberg's central limit theorem for the Riemann zeta function and the Keating-Snaith central limit theorem for the characteristic polynomial of random unitary matrices (see [14]). In fact Selberg's central limit theorem can be proved more generally for a wide class of L-functions (see [22] and [3]).

Roughly speaking, an L-function must be defined by a Dirichlet series for $\mathrm{Re}(s) > 1$, have an Euler product (with some growth condition on the coefficients of this product), an analytic continuation (except for finitely many poles all located on the line $\mathrm{Re}(s) = 1$), and must satisfy a functional equation. Such L-functions are expected to satisfy the general Riemann hypothesis (GRH), which says that all the non-trivial zeros are located on the critical line, the line $\mathrm{Re}(s) = 1/2$.

Now if one considers a finite number of such L-functions, all satisfying the same functional equation, then one can wonder if the zeros of a linear combination of these L-functions are still on the critical line. In general, the answer is that GRH does not hold anymore for such a linear combination even though it still has a functional equation (this can be thought of coming from the fact that such a linear combination does not have an Euler product anymore). But Bombieri and Hejhal proved in [3] that nonetheless 100% of the zeros of such linear combinations are still on the critical line (under an extra assumption of "near orthogonality" which ensures that the logarithm of the L-functions are statistically asymptotically independent and that 100% of the zeros of each L-function do not occur in clumps). In this paper we will show that a similar result holds for linear combinations of independent characteristic polynomials of random unitary matrices. The result on the random matrix side is technical and difficult. In addition to being an extra piece of evidence that the characteristic polynomial is a good model for the value distribution of L-functions, the result is also remarkable when viewed in the general setting of random polynomials. The main goal of this article is to show that on average, any linear combination of characteristic polynomials of independent random unitary matrices having the same determinant has a proportion of zeros on the unit circle which tends to 1 when the dimension goes to infinity.

More precisely, if U is a unitary matrix of order $N \geqslant 1$, for $z \in \mathbb{C}$ let

$$\Phi_U(z) = \det\left(I_N - zU\right)$$

be the characteristic polynomial of U. From the fact that U is unitary, we get the functional equation

$$\Phi_U(z) = (-z)^N \det(U)\overline{\Phi_U}(1/z).$$

For z on the unit circle, this equation implies that

$$\Phi_U(z) = R(z)\sqrt{(-z)^N \det(U)},$$

where $R(z)$ is real-valued (with any convention taken for the square root). Now, let $(U_j)_{1\leqslant j\leqslant n}$ be a collection of n unitary matrices of order N, and let $(b_j)_{1\leqslant j\leqslant n}$ be real numbers.

We wish to study the number of zeros on the unit circle of the linear combination

$$F_N(z) = \sum_{j=1}^{n} b_j \Phi_{U_j}(z).$$

If we want F to have most of its zeros on the unit circle, it is reasonable to expect that we need a "unidimensional condition" for the equation $F(z) = 0$ and $|z| = 1$, i.e. a functional equation similar to the equation satisfied by U. This equation obviously exists if all the characteristic polynomials Φ_{U_j} satisfy the *same* functional equation, which happens when the matrices U_j have the same determinant. By symmetry of the unitary group, it is natural to assume that the unitary matrices have determinant 1. More precisely, the main result of the article is the following:

Theorem 1.1. *Let $(b_j)_{1\leqslant j\leqslant n}$ be non-zero real numbers. For $N \geqslant 1$, let*

$$F_N(z) := \sum_{j=1}^{n} b_j \Phi_{U_j}(z),$$

where $(U_j)_{1\leqslant j\leqslant n}$ is a family of independent matrices whose distribution is the Haar measure on the special unitary group $SU(N)$. The expected proportion of zeros of F_N on the unit circle tends to 1 as N goes to infinity, i.e.

$$\mathbb{E}\left(|\{z \in \mathbb{U} \,:\, F_N(z) = 0\}|\right) = N - o(N),$$

where $|\{z \in \mathbb{U} \,:\, F_N(z) = 0\}|$ is the number of z on the unit circle which satisfy $F_N(z) = 0$.

The whole paper is devoted to the proof of this result. Before explaining the strategy of the proof, we make a few remarks.

Remark 1.2. Theorem 1.1 can be stated as

$$\lim_{N\to\infty} \mathbb{E}\left(\frac{1}{N}|\{z \in \mathbb{U} \,:\, F_N(z) = 0\}|\right) = 1.$$

Since the random variable $\frac{1}{N}|\{z \in \mathbb{U} \,:\, F_N(z) = 0\}|$ is bounded by 1, the convergence holds in all L^p spaces for $p \geqslant 1$. It therefore also holds in probability since convergence in L^1 implies convergence in probability.

Remark 1.3. The fact that we impose the condition that the matrices have the same determinant is similar to the condition in [3] that the L-functions have the same functional equation. Moreover, in our framework, the analogue of the Riemann hypothesis is automatically satisfied since all the zeros of each characteristic polynomial are on the unit circle.

Remark 1.4. The fact that the proportion of zeros on the unit circle tends to 1 is a remarkable fact when considered as a result about random polynomials. It is well known

that the characteristic polynomial of a unitary matrix is self-inversive[1]. Self-inversive random polynomials are of interest in the context of semiclassical approximations in quantum mechanics, and determining the proportion of zeros on the unit circle is an important problem in that context. Bogomolny, Bohigas and Leboeuf [2] showed that if the first half of the coefficients are chosen as independent complex Gaussian random variables (the second half being fixed by the self-inverse symmetry), then asymptotically $\frac{1}{\sqrt{3}}$ of all the zeros are exactly on the unit circle. Hence we can say that our result is not typical of what is expected for classical random polynomials built from independent Gaussian random variables. In our framework, we do not know the distribution of the coefficients, but we do know that they are not independent. Consequently the classical methods which use the independence of the coefficients (or use the fact that they are Gaussian if one wishes to add some dependence) do not work here. Using general results on random polynomials whose coefficients are not necessarily independent or equidistributed [10], one can deduce that the zeros cluster uniformly around the unit circle. But showing that they are almost all precisely on the unit circle is a much more refined statement.

We now say a few words about our strategy of proof of Theorem 1.1. In fact we use the same general method as in [3], called the "carrier waves" method, but the ingredients of our proof are different, in the sense that they are probabilistic. For instance we use the coupling method, concentration inequalities and the recent probabilistic representation of the characteristic polynomial obtained in [4]. More precisely, for $U \in U(N)$ and $\theta \in \mathbb{R}$, we denote by $Z_U(\theta)$ the characteristic polynomial of U evaluated at $e^{-i\theta}$, i.e. $Z_U(\theta) = \Phi_U(e^{-i\theta})$. Then we make a simple transformation of the linear combination F_N in order that it is real-valued when restricted as a function on the unit circle:

$$i^N e^{iN\theta/2} F_N(e^{-i\theta}) = i^N e^{iN\theta/2} \sum_{j=1}^{n} b_j \Phi_{U_j}(e^{-i\theta}) = \sum_{j=1}^{n} b_j i^N e^{iN\theta/2} Z_{U_j}(\theta). \tag{1}$$

Using the fact that $U_j \in SU(N)$, one checks that $i^N e^{iN\theta/2} Z_{U_j}(\theta)$ is real, and thus the number of zeros of F_N on the unit circle is bounded from below by the number of sign changes, when θ increases from θ_0 to $\theta_0 + 2\pi$ (with θ_0 to be chosen carefully), of the real quantity given by the right-hand side of the equation above. The notion of carrier waves is explained in detail in [3], p. 824–827 and we do not explain it again but we would rather give a general outline. The main idea is that informally, with "high" probability and for "most" of the values of θ, one of the characteristic polynomials Z_{U_j} dominates all the others (it is the "carrier wave"). More precisely, Lemma 3.8 implies the following: If δ depends only on N and tends to zero when N goes to infinity, then there exists, with probability $1 - o(1)$, a subset of $[\theta_0, \theta_0 + 2\pi)$ with Lebesgue measure $o(1)$ such that for any θ outside this set, one can find j_0 between 1 and n such that $\left|\log|Z_{U_{j_0}}(\theta)| - \log|Z_{U_j}(\theta)|\right| > \delta\sqrt{\log N}$ for all $j \neq j_0$. In other words, one of the terms in the sum of the right-hand side of (1) should dominate all the others. Moreover, Lemma 3.13 informally gives the following: With high probability, the order of magnitude of each of the characteristic polynomials

[1]A self-inversive polynomial is one where $a_{N-k} = \exp(i\theta)\bar{a}_k$ for some $\theta \in \mathbb{R}$, where $(a_k)_{0 \leqslant k \leqslant N}$ are the coefficients of the polynomial.

does not change too quickly, so if the interval $[\theta_0, \theta_0 + 2\pi)$ is divided into sufficiently many equal subintervals, the index of the carrier wave remains the same in a "large" part of each subinterval. The zeros of $Z_{U_{j_0}}$ correspond to sign changes of $i^N e^{iN\theta/2} Z_{U_{j_0}}(\theta)$, and since this is the dominant term of (1), one gets sign changes of $i^N e^{iN\theta/2} F_N(e^{-i\theta})$. By counting all these sign changes, one deduces a lower bound for the number of zeros of F_N on the unit circle.

The main issue of the present paper is to make rigorous this informal construction, in such a way that one gets a lower bound $N - o(N)$. One of the reasons why the proof becomes technical and involved is that we have to take into account two different kinds of sets, and show that they have almost "full measure": subsets of the interval $[\theta_0, \theta_0 + 2\pi)$ and subsets of $SU(N)$.

Our proof is structured as follows: We first give two standard results (Propositions 2.1 and 2.5), one on the disintegration of the Haar measure on $U(N)$[2] and the other one which establishes a relationship between the number of eigenvalues in a given fixed arc to the variation of the imaginary part of the logarithm of the characteristic polynomial. Then we provide some estimates on the real and imaginary parts of the logarithm of the characteristic polynomial (Lemmas 3.1 and 3.2) as well as a bound on the concentration of the law of the logarithm of the characteristic polynomial (Lemma 3.3). Then we provide bounds on the oscillations of the real and imaginary parts of the logarithm of the characteristic polynomial (Lemma 3.7). These estimates, and some further intermediary ones which we establish, are also useful in their own right and complement the existing results in the literature on characteristic polynomials. We then introduce our subdivisions of the interval $[\theta_0, \theta_0 + 2\pi)$ and the corresponding relevant random sets to implement the carrier waves technique. Finally we combine all these estimates together to show that the average number of sign changes of (1) is at least $N\left(1 - O\left((\log N)^{-1/22}\right)\right)$. The exponent $-1/22$ does not play any major role in our analysis.

Notation

We gather here some notation used throughout the paper.

We write $U(N)$ for the unitary group of order N, while $SU(N)$ stands for the subgroup of elements $U(N)$ whose determinant is equal to 1. We let $\mathbb{P}_{U(N)}$ and $\mathbb{P}_{SU(N)}$ denote the probability Haar measure on $U(N)$ and $SU(N)$ respectively. Similarly we denote by $\mathbb{E}_{U(N)}$ and $\mathbb{E}_{SU(N)}$ the corresponding expectations. For n a positive integer, we let $\mathbb{P}^{(n)}_{SU(N)}$ be the n-fold product of the Haar measure on $SU(N)$, and $\mathbb{E}^{(n)}_{SU(N)}$ the corresponding expectation.

If U is a unitary matrix, we write its characteristic polynomial as $\Phi_U(z) = \det(I_N - zU)$, with $z \in \mathbb{C}$. For $\theta \in \mathbb{R}$, we denote by $Z_U(\theta)$ the characteristic polynomial of U evaluated at $e^{-i\theta}$, i.e. $Z_U(\theta) = \Phi_U(e^{-i\theta})$.

We shall denote the Lebesgue measure on $\mathbb{R}$ by λ. If $\alpha > 0$ is a constant and if I is an interval of length α, then λ_α will denote the normalized measure $\frac{1}{\alpha}\lambda$ on the interval I.

[2]Indeed, most of our results on random matrices are established for $U(N)$ and we show how to go from the results for $U(N)$ to those for $SU(N)$.

If n is an integer, we denote $[\![1, n]\!]$ the set of integers $\{1, \cdots, n\}$, and if $\mathcal{E}$ is a finite set, we write $|\mathcal{E}|$ for the number of its elements.

We shall introduce several positive quantities during the proof: $K > 0$, $M > 0$ and $\delta > 0$. The reader should have in mind that these quantities will eventually depend on N. Unless stated otherwise, $N \geqslant 4$ and K is an integer such that $2 \leqslant K \leqslant N/2$, and $M = N/K$. In the end we will use $K \sim N/(\log N)^{3/64}$ and $\delta \sim (\log N)^{-3/32}$.

2. Some general facts

In this section, we state some general facts in random matrix theory, which will be used later on.

2.1. **Disintegration of the Haar measure on unitary matrices.**

Proposition 2.1. *Let $\mathbb{P}_{U(N)}$ be the Haar measure on $U(N)$, $\mathbb{P}_{SU(N)}$ the Haar measure on $SU(N)$, and for $\theta \in \mathbb{R}$, let $\mathbb{P}_{SU(N),\theta}$ be the image of $\mathbb{P}_{SU(N)}$ by the application $U \mapsto e^{i\theta}U$ from $U(N)$ to $U(N)$. Then, we have the following equality:*

$$\int_0^{2\pi} \mathbb{P}_{SU(N),\theta} \frac{d\theta}{2\pi} = \mathbb{P}_{U(N)}, \tag{2}$$

i.e. for any continuous function F from $U(N)$ to $\mathbb{R}_+$, the expectation $\mathbb{E}_{SU(N),\theta}(F)$ of F with respect to $\mathbb{P}_{SU(N),\theta}$ is measurable with respect to θ and

$$\int_0^{2\pi} \mathbb{E}_{SU(N),\theta}(F) \frac{d\theta}{2\pi} = \mathbb{E}_{U(N)}(F).$$

Proof. One has

$$\mathbb{E}_{SU(N),\theta}(F) = \int F(Xe^{i\theta}) d\mathbb{P}_{SU(N)}(X), \tag{3}$$

which, by dominated convergence, is continuous, and a fortiori measurable with respect to θ. By integrating (3) with respect to θ, one sees that the proposition is equivalent to the following: if U is a uniform matrix on $SU(N)$, and if Z is independent, uniform on the unit circle, then ZU is uniform on $U(N)$. Now, let A be a deterministic matrix in $U(N)$. For any $d \in \mathbb{C}$ such that $d^{-N} = \det(A)$, one has $Ad \in SU(N)$, and then $ZUA = (Z/d)(UAd)$, where:

1. UAd follows the Haar measure on $SU(N)$ (since this measure is invariant by multiplication by $Ad \in SU(N)$).
2. Z/d is uniform on the unit circle (since d, as $\det(A)$, has modulus 1).
3. These two variables, which depend deterministically on the independent variables A and Z, are independent.

Hence ZUA has the same law as ZU, i.e. this law is invariant by right-multiplication by any unitary matrix. Hence, ZU follows the Haar measure on $U(N)$. □

Remark 2.2. This result also appears in the proof of Lemma 2.5 of [18].

Remark 2.3. This disintegration will allow us to deduce the estimates on $SU(N)$ we will need from the corresponding estimates on $U(N)$. Another way to proceed would be to make use of Lemma AD.7.1 from [12] which we recall here:

Lemma 2.4 ([12], p. 390). *Let $f : U(N) \to \mathbb{C}$ be a $\mathcal{C}^\infty$ bounded central function, that is $f(UV) = f(VU)$ for all $U, V \in U(N)$. Then, one has the equality with an absolutely convergent series*

$$\mathbb{E}_{SU(N)}(f(X)) = \sum_{\ell \in \mathbb{Z}} \mathbb{E}_{U(N)}\big((\det X)^\ell f(X)\big)$$

If one wished to proceed using this lemma, one would then need to estimate the terms of the series in order to show that the term corresponding to $\ell = 0$ is dominant, i.e. that the expectation $\mathbb{E}_{SU(N)}(f(X))$ is close to $\mathbb{E}_{U(N)}(f(X))$. For f being a function of the characteristic polynomial, this can be done using the probabilistic splitting defined in [4] and used in the proof of Lemma 3.2.

2.2. **Number of eigenvalues in an arc.** The result we state here relates the number of eigenvalues of a unitary matrix on a given arc to the logarithm of its characteristic polynomial. Recall that for $U \in U(N)$ and $\theta \in \mathbb{R}$, we denote by $Z_U(\theta)$ the characteristic polynomial of U evaluated at $e^{-i\theta}$. Moreover, if $e^{i\theta}$ is not an eigenvalue of U, we define the logarithm of $Z_U(\theta)$, as

$$\log Z_U(\theta) := \sum_{j=1}^{N} \log(1 - e^{i(\theta_j - \theta)}), \tag{4}$$

where $\theta_1, \dots, \theta_N$ are the zeros of Z_U in $[0, 2\pi)$, taken with multiplicity (notice that the eigenvalues of U are $e^{i\theta_1}, \dots, e^{i\theta_N}$), and where the principal branch of the logarithm is taken in the right-hand side. Note that $e^{i\theta}$ is not an eigenvalue of U almost surely under Haar measure on $U(N)$, and also almost surely under the Haar measure on $SU(N)$, except for the case when $e^{i\theta} = 1$ and $N = 1$.

We then have the following result, already stated, for example, in [9]:

Proposition 2.5. *Let $0 \leqslant s < t < 2\pi$, and let us assume that s and t are not zeros of Z_U. Then, the number of zeros of Z_U in the interval (s, t) is given as follows:*

$$\sum_{k=1}^{N} \mathbb{1}_{\{\theta_k \in (s,t)\}} = \frac{N}{2\pi}(t - s) + \frac{1}{\pi}\left(\operatorname{Im}\log Z_U(t) - \operatorname{Im}\log Z_U(s)\right). \tag{5}$$

Proof. It is sufficient to check that for all $\vartheta \in [0, 2\pi)\backslash\{s, t\}$,

$$\pi \mathbb{1}_{\{\vartheta \in (s,t)\}} = \frac{t - s}{2} + \operatorname{Im}\log\left(1 - e^{i(\vartheta - t)}\right) - \operatorname{Im}\log\left(1 - e^{i(\vartheta - s)}\right).$$

Now, for $v \in (0, 2\pi)$,

$$1 - e^{iv} = e^{iv/2}(e^{-iv/2} - e^{iv/2}) = -2i\sin(v/2)\, e^{iv/2} = 2\sin(v/2)\, e^{i(v-\pi)/2}.$$

Now, $\sin(v/2) > 0$ and $(v - \pi)/2 \in (-\pi/2, \pi/2)$ and hence

$$\operatorname{Im}\log\left(1 - e^{iv}\right) = \frac{v - \pi}{2},$$

since we take the principal branch of the logarithm. Now, for $\vartheta \in [0, 2\pi)\backslash\{s, t\}$, $\vartheta - s + 2\pi \mathbb{1}_{\{\vartheta<s\}}$ and $\vartheta - t + 2\pi \mathbb{1}_{\{\vartheta<t\}}$ are in $(0, 2\pi)$, which implies

$$\begin{aligned}\operatorname{Im}\log\left(1-e^{i(\vartheta-t)}\right) - \operatorname{Im}\log\left(1-e^{i(\vartheta-s)}\right) &= \frac{\vartheta - t - \pi + 2\pi\mathbb{1}_{\{\vartheta<t\}}}{2} - \frac{\vartheta - s - \pi + 2\pi\mathbb{1}_{\{\vartheta<s\}}}{2} \\ &= \frac{s-t}{2} + \pi\left(\mathbb{1}_{\{\vartheta<t\}} - \mathbb{1}_{\{\vartheta<s\}}\right),\end{aligned}$$

from which Proposition 2.5 follows by summing over the N zeros of Z_U in $[0, 2\pi)$. □

3. Proof of Theorem 1.1

3.1. **Conventions.** All the random matrices we will consider are defined, for some $N \geqslant 1$, on the measurable space $(\mathcal{M}_N(\mathbb{C}), \mathcal{F})$, where $\mathcal{F}$ denotes the Borel σ-algebra of the space of $N \times N$ complex matrices $\mathcal{M}_N(\mathbb{C})$. The canonical random variable with values in $\mathcal{M}_N(\mathbb{C})$, i.e. the identity function from $(\mathcal{M}_N(\mathbb{C}), \mathcal{F})$ to $\mathcal{M}_N(\mathbb{C})$ is denoted by X. For example, if F is a bounded, Borel function from $\mathcal{M}_N(\mathbb{C})$ to $\mathbb{R}$,

$$\mathbb{E}_{SU(N)}[F(X)] = \int_{\mathcal{M}_N(\mathbb{C})} F(M) d\mathbb{P}_{SU(N)}(M).$$

3.2. **An estimate on the average of the logarithm of the characteristic polynomial.**

Lemma 3.1. *There exists a universal constant $c_1 > 0$ such that for all $N \geqslant 2$, and $A \geqslant 0$,*

$$\int_0^{2\pi} \mathbb{P}_{SU(N)}\left(\left|\log Z_X(\theta)\right| \geqslant A\sqrt{\log N}\right)\frac{d\theta}{2\pi} \leqslant c_1 e^{-\frac{A}{2}\left(A \wedge \frac{\sqrt{\log N}}{2}\right)}$$

where $A \wedge \frac{\sqrt{\log N}}{2}$ denotes the minimum of A and $\frac{\sqrt{\log N}}{2}$.

Proof. For all $\lambda \geqslant 0$,

$$\begin{aligned}\int_0^{2\pi} \mathbb{P}_{SU(N)}\left(\left|\log Z_X(\theta)\right| \geqslant A\sqrt{\log N}\right)\frac{d\theta}{2\pi} &\leqslant e^{-\lambda A\sqrt{\log N}} \int_0^{2\pi} \mathbb{E}_{SU(N)}\left(e^{\lambda|\log Z_X(\theta)|}\right)\frac{d\theta}{2\pi} \\ &\leqslant e^{-\lambda A\sqrt{\log N}}\mathbb{E}_{U(N)}\left(e^{\lambda|\log Z_X(0)|}\right) \text{ (by (2))} \\ &\leqslant e^{-\lambda A\sqrt{\log N}}\mathbb{E}_{U(N)}\left(e^{\lambda(|\operatorname{Re}\log Z_X(0)|+|\operatorname{Im}\log Z_X(0)|)}\right)\end{aligned}$$

Using the inequality $e^{|a|+|b|} \leqslant e^{a+b} + e^{a-b} + e^{-a+b} + e^{-a-b}$, valid for all $a, b \in \mathbb{R}$, and writing the right-hand side of this inequality as $4\mathbb{E}\left(e^{Ba+B'b}\right)$ for B and B' two independent Bernoulli random variables, independent of U, such that $\mathbb{P}(B = 1) = 1 - \mathbb{P}(B = -1) = 1/2$, we have:

$$\begin{aligned}&\int_0^{2\pi} \mathbb{P}_{SU(N)}\left(\left|\log Z_X(\theta)\right| \geqslant A\sqrt{\log N}\right)\frac{d\theta}{2\pi} \\ &\qquad\leqslant 4e^{-\lambda A\sqrt{\log N}}\mathbb{E}_{U(N)}\left(e^{\lambda(B\ \operatorname{Re}\log Z_X(0)+B'\operatorname{Im}\log Z_X(0))}\right).\end{aligned}$$

Keating and Snaith [14] showed that for $s,t\in\mathbb{C}$ such that $\mathrm{Re}(s+it)$ and $\mathrm{Re}(s-it)$ are strictly larger than -1,

$$\mathbb{E}_{U(N)}\big(e^{s\ \mathrm{Re}\log Z_X(0)+t\ \mathrm{Im}\log Z_X(0)}\big)=\frac{G\left(1+\frac{s+it}{2}\right)G\left(1+\frac{s-it}{2}\right)G\left(1+N\right)G\left(1+N+s\right)}{G\left(1+N+\frac{s+it}{2}\right)G\left(1+N+\frac{s-it}{2}\right)G\left(1+s\right)} \quad (6)$$

where G is the Barnes G-function, defined for all $z\in\mathbb{C}$, by

$$G(z+1):=(2\pi)^{z/2}e^{-[(1+\gamma)z^2+z]/2}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^n e^{-z+(z^2/2n)},$$

with γ being the Euler constant.

In other words, one has

$$\mathbb{E}_{U(N)}\big(e^{s\ \mathrm{Re}\log Z_X(0)+t\ \mathrm{Im}\log Z_X(0)}\big)=\frac{G\left(1+\frac{s+it}{2}\right)G\left(1+\frac{s-it}{2}\right)}{G\left(1+s\right)}N^{\left(s^2+t^2\right)/4}G_{N,s,t},$$

where, by the classical estimates of the Barnes function,

$$G_{N,s,t}:=N^{-\left(s^2+t^2\right)/4}\frac{G\left(1+N\right)G\left(1+N+s\right)}{G\left(1+N+\frac{s+it}{2}\right)G\left(1+N+\frac{s-it}{2}\right)}$$

tends to 1 when N goes to infinity, uniformly on s and t if these parameters are bounded.

For any sequence $(\lambda_N)_{N\geqslant 1}$ such that $\lambda_N\in[0,1/2]$, one has (taking $s=\lambda_N B$ and $t=\lambda_N B'$):

$$\mathbb{E}_{U(N)}\Big(e^{\lambda_N(B\ \mathrm{Re}\log Z_X(0)+B'\ \mathrm{Im}\log Z_X(0))}\Big)=M(\lambda_N)N^{\lambda_N^2/2},$$

where

$$M(\lambda_N):=\mathbb{E}\left(\frac{G\left(1+\lambda_N\frac{B+iB'}{2}\right)G\left(1+\lambda_N\frac{B-iB'}{2}\right)}{G(1+\lambda_N B)}G_{N,\lambda_N B,\lambda_N B'}\right).$$

Since the function G is holomorphic, with no zero on the half-plane $\{z\in\mathbb{C}:\mathrm{Re}(z)>0\}$, and since $G_{N,\lambda B,\lambda B'}$ tends to 1 when N goes to infinity, uniformly on $\lambda\in[0,1/2]$, the quantity $M(\lambda)$ is uniformly bounded by some universal constant $c'>0$, for $\lambda\in[0,1/2]$. Hence for N going to infinity,

$$\mathbb{E}_{U(N)}\Big(e^{\lambda_N(B\ \mathrm{Re}\log Z_X(0)+B'\ \mathrm{Im}\log Z_X(0))}\Big)\leqslant c'N^{\lambda_N^2/2},$$

which implies that

$$\int_0^{2\pi}\mathbb{P}_{SU(N)}\left(\Big|\log Z_X(\theta)\Big|\geqslant A\sqrt{\log N}\right)\frac{d\theta}{2\pi}\leqslant 4c'e^{-\lambda_N A\sqrt{\log N}+(\lambda_N^2\log N)/2}.$$

Now, taking $\lambda_N = (1/2) \wedge (A/\sqrt{\log N})$ (which means λ_N is the smaller of $1/2$ and $A/\sqrt{\log N}$), we have

$$\begin{aligned}\int_0^{2\pi} \mathbb{P}_{SU(N)}\left(\left|\log Z_X(\theta)\right| \geqslant A\sqrt{\log N}\right)\frac{d\theta}{2\pi} &\leqslant 4c'e^{-\lambda_N\sqrt{\log N}[A-(\lambda_N\sqrt{\log N})/2]}\\ &\leqslant 4c'e^{-\lambda_N\sqrt{\log N}[A-(A/\sqrt{\log N})(\sqrt{\log N})/2]}\\ &\leqslant 4c'e^{-\lambda_N\sqrt{\log N}(A/2)}\\ &\leqslant 4c'e^{-[(\sqrt{\log N}/2)\wedge A](A/2)}. \qquad \square\end{aligned}$$

3.3. **An estimate on the imaginary part of the logarithm of the characteristic polynomial.** From the previous result, we obtain the following estimate for the imaginary part of the logarithm of the characteristic polynomial:

Lemma 3.2. *There exists a universal constant $c_1' > 0$ such that for all $N \geqslant 2$, $A \geqslant 0$, and $\theta \in \mathbb{R}$,*

$$\mathbb{P}_{SU(N)}\left(\left|\operatorname{Im}\log Z_X(\theta)\right| \geqslant A\sqrt{\log N}\right) \leqslant c_1' e^{-\frac{A}{2}\left(A\wedge\frac{\sqrt{\log N}}{2}\right)}.$$

Proof. We use here the probabilistic splitting established in [4] which shows that for any $U \in U(N)$, there exists, for $1 \leqslant j \leqslant N$, x_j on the unit sphere of $\mathbb{C}^j$, uniquely determined, such that

$$U = R(x_N)\begin{pmatrix} R(x_{N-1}) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} R(x_{N-2}) & 0\\ 0 & I_2\end{pmatrix}\cdots\begin{pmatrix} R(x_1) & 0\\ 0 & I_{N-1}\end{pmatrix}, \tag{7}$$

where $R(x_j)$ denotes the unique unitary matrix in $U(j)$ sending the last basis vector e_j of $\mathbb{C}^j$ to x_j, and such that the image of $I_j - R(x_j)$ is the vector space generated by $e_j - x_j$. (See also [5] for an infinite-dimensional point of view).

Moreover, the characteristic polynomial of $U \in U(N)$ is given by

$$Z_U(0) = \prod_{j=1}^{N}(1 - \langle x_j, e_j\rangle),$$

and its logarithm is

$$\log Z_U(0) = \sum_{j=1}^{N}\log(1 - \langle x_j, e_j\rangle), \tag{8}$$

when 1 is not an eigenvalue of U, taking the principal branch of the logarithm on the right-hand side. Notice that the determination of the logarithm given by this formula fits with the definition involving the eigenangles (4). Indeed, the two formulas depend continuously on the matrix U, on the connected set $\{U \in U(N), 1 \notin \operatorname{Spec}(U)\}$, and their exponentials are equal, hence, it is sufficient to check that they coincide for one matrix U. For example if $U = -I_N$ then $x_j = -e_j$ for all j and the two formulas both give $N\log 2$.

If U follows the uniform distribution on $U(N)$, then the vectors $(x_j)_{1\leqslant j\leqslant N}$ are independent and x_j is uniform on the sphere of $\mathbb{C}^j$. The determinant of U is equal to the product

of the determinants of $R(x_j)$ for $1 \leqslant j \leqslant N$, and since $R(x_1)$ is the multiplication by x_1 on $\mathbb{C}$, one has

$$\det(U) = x_1 \prod_{j=2}^{N} \Gamma_j(x_j),$$

where Γ_j is the function from $\mathbb{C}^j$ to the unit circle $\mathbb{U}$ that maps $x_j \mapsto \det(R(x_j))$. From this, we deduce that under the measure $\mathbb{P}_{SU(N),\theta}$

(1) The vectors $(x_j)_{2\leqslant j\leqslant N}$ are independent, x_j being uniform on the unit sphere of $\mathbb{C}^j$.
(2) The value of $x_1 \in \mathbb{U}$ is uniquely determined by the determinant $\det(U) = e^{iN\theta}$,

$$x_1 = e^{iN\theta} \prod_{j=2}^{N} [\Gamma_j(x_j)]^{-1}.$$

Indeed, let $\mathbb{P}'_{SU(N),\theta}$ be the probability measure on the image of $SU(N)$ by the multiplication by $e^{i\theta}$, under which the law of $(x_j)_{1\leqslant j\leqslant N}$ is given by the two bullet points above. This probability measure can be constructed as the law of the random matrix U given by the formula (7), where $(x_j)_{1\leqslant j\leqslant N}$ are random vectors whose joint distribution is given by the bullet points (1) and (2) just above. We now have to prove that $\mathbb{P}_{SU(N),\theta} = \mathbb{P}'_{SU(N),\theta}$. Let us first notice that the joint law of $(x_j)_{2\leqslant j\leqslant N}$, under the probability measure $\mathbb{P}'_{SU(N),\theta}$, does not depend on θ. Hence, under the averaged measure

$$\int_0^{2\pi} \mathbb{P}'_{SU(N),\theta} \frac{d\theta}{2\pi},$$

the vectors $(x_j)_{2\leqslant j\leqslant N}$ still have the same law, i.e. they are independent and x_j is uniform on the unit sphere of $\mathbb{C}^j$. Moreover, conditionally on $(x_j)_{2\leqslant j\leqslant N}$, $x_1 = e^{iN\theta} \prod_{j=2}^{N} [\Gamma_j(x_j)]^{-1}$, where θ is uniform on $[0, 2\pi)$. Hence $(x_j)_{1\leqslant j\leqslant N}$ are independent, x_1 is uniform on $\mathbb{U}$, and thus x_j in uniform on the unit sphere of $\mathbb{C}^j$ for all $j \in \{1, \dots, N\}$, which implies

$$\int_0^{2\pi} \mathbb{P}'_{SU(N),\theta} \frac{d\theta}{2\pi} = \mathbb{P}_{U(N)} = \int_0^{2\pi} \mathbb{P}_{SU(N),\theta} \frac{d\theta}{2\pi}.$$

Now, $\mathbb{P}_{SU(N),2\pi/N}$ is the image of $\mathbb{P}_{SU(N)}$ by multiplication by $e^{i2\pi/N} I_N$, which is a matrix in $SU(N)$. The invariance property defining the Haar measure $\mathbb{P}_{SU(N)}$ implies that $\mathbb{P}_{SU(N),2\pi/N} = \mathbb{P}_{SU(N)}$, and so $\theta \mapsto \mathbb{P}_{SU(N),\theta}$ is $(2\pi/N)$-periodic. It is the same for $\theta \mapsto \mathbb{P}'_{SU(N),\theta}$, since the values of $x_1, \dots x_N$ involved in the definition of $\mathbb{P}'_{SU(N),\theta}$ do not change if we add a multiple of $2\pi/N$ to θ. Hence,

$$\int_0^{2\pi/N} \mathbb{P}'_{SU(N),\theta} \frac{N d\theta}{2\pi} = \int_0^{2\pi/N} \mathbb{P}_{SU(N),\theta} \frac{N d\theta}{2\pi}.$$

Now, let F be a continuous, bounded function from $U(N)$ to $\mathbb{R}$. By applying the equality above to the function $U \mapsto F(U) \mathbb{1}_{\left\{\det U \in \{e^{iN\theta}, \theta \in I\}\right\}}$, for an interval $I \subset [0, 2\pi/N)$, one

deduces with obvious notation that

$$\int_I \mathbb{E}'_{SU(N),\theta}(F)\frac{d\theta}{|I|} = \int_I \mathbb{E}_{SU(N),\theta}(F)\frac{d\theta}{|I|},$$

where $|I|$ is the length of I. By the definition of $\mathbb{P}_{SU(N),\theta}$ and $\mathbb{P}'_{SU(N),\theta}$, the first measure is the image of $\mathbb{P}_{SU(N)}$ by multiplication by $e^{i\theta}$, and the second measure is the image of $\mathbb{P}'_{SU(N),0}$ by right multiplication by the matrix $\begin{pmatrix} e^{iN\theta} & 0 \\ 0 & I_{N-1} \end{pmatrix}$. Hence by continuity and boundedness of F, and by dominated convergence, $\mathbb{E}_{SU(N),\theta}(F)$ and $\mathbb{E}'_{SU(N),\theta}(F)$ are continuous with respect to θ. By considering a sequence $(I_r)_{r\geqslant 1}$ of intervals containing a given value of θ and whose length tends to zero, one deduces, by letting $r \to \infty$,

$$\mathbb{E}'_{SU(N),\theta}(F) = \mathbb{E}_{SU(N),\theta}(F).$$

We now get the equality $\mathbb{P}_{SU(N),\theta} = \mathbb{P}'_{SU(N),\theta}$, and thus the law of $(x_j)_{1\leqslant j\leqslant N}$ under $\mathbb{P}_{SU(N),\theta}$ described above.

Hence, the sequence $(x_j)_{2\leqslant j\leqslant N}$ has the same law under $\mathbb{P}_{SU(N),\theta}$ and $\mathbb{P}_{U(N)}$. We now use this fact to construct a coupling between these two probability measures on the unitary group.

The general principle of coupling is the following: When we want to show that two probability distributions $\mathbb{P}_1$ and $\mathbb{P}_2$ on a metric space have a similar behavior, a possible strategy is to construct a couple (U, U') of random variables defined on the same probability space endowed with a probability $\mathbb{P}$, such that the law of U under $\mathbb{P}$ is $\mathbb{P}_1$, the law of U' under $\mathbb{P}$ is $\mathbb{P}_2$, and the distance between U and U' is small with high probability. In the present situation, we take $(x'_j)_{1\leqslant j\leqslant N}$ to be independent with x'_j uniform on the unit sphere of $\mathbb{C}^j$ for all $j \in \{1, \ldots, N\}$. We construct, using (7), a random matrix U' following $\mathbb{P}_{U(N)}$. Then, we do the coupling by taking $x_j := x'_j$ for $2 \leqslant j \leqslant N$ and

$$x_1 := e^{iN\theta}\prod_{j=2}^{N}[\Gamma_j(x_j)]^{-1},$$

which gives a random matrix U following $\mathbb{P}_{SU(N),\theta}$. From the fact that $x_j = x'_j$ for $j \geqslant 2$ and the equation (8), we get the following:

$$\log Z_U(0) - \log Z_{U'}(0) = \log(1 - x_1) - \log(1 - x'_1),$$

and in particular,

$$|\mathrm{Im}\log Z_U(0) - \mathrm{Im}\log Z_{U'}(0)| \leqslant \pi.$$

Now, for $B := \left(A - \frac{\pi}{\sqrt{\log N}}\right)_+$, one gets:

$$\begin{aligned}
\mathbb{P}_{SU(N)}\left(\left|\operatorname{Im}\log Z_X(-\theta)\right| \geqslant A\sqrt{\log N}\right) &= \mathbb{P}_{SU(N),\theta}\left(\left|\operatorname{Im}\log Z_X(0)\right| \geqslant A\sqrt{\log N}\right)\\
&= \mathbb{P}\left(\left|\operatorname{Im}\log Z_U(0)\right| \geqslant A\sqrt{\log N}\right)\\
&\leqslant \mathbb{P}\left(\left|\operatorname{Im}\log Z_{U'}(0)\right| \geqslant A\sqrt{\log N} - \pi\right)\\
&= \mathbb{P}_{U(N)}\left(\left|\operatorname{Im}\log Z_X(0)\right| \geqslant B\sqrt{\log N}\right)\\
&= \int_0^{2\pi} \mathbb{P}_{SU(N)}\left(\left|\operatorname{Im}\log Z_X(\theta)\right| \geqslant B\sqrt{\log N}\right)\frac{d\theta}{2\pi}\\
&\leqslant \int_0^{2\pi} \mathbb{P}_{SU(N)}\left(\left|\log Z_X(\theta)\right| \geqslant B\sqrt{\log N}\right)\frac{d\theta}{2\pi}\\
&\leqslant c_1 e^{-\frac{B}{2}\left(B\wedge\frac{\sqrt{\log N}}{2}\right)}
\end{aligned}$$

Now, if $B \leqslant \frac{\sqrt{\log N}}{2}$, then

$$\begin{aligned}
\frac{A}{2}\left(A\wedge\frac{\sqrt{\log N}}{2}\right) \leqslant \frac{A^2}{2} &\leqslant \frac{1}{2}\left(B + \frac{\pi}{\sqrt{\log N}}\right)^2 = \frac{B^2}{2} + \frac{B\pi}{\sqrt{\log N}} + \frac{\pi^2}{2\log N}\\
&\leqslant \frac{B^2}{2} + \frac{\pi}{2} + \frac{\pi^2}{2\log 2}\\
&= \frac{B}{2}\left(B\wedge\frac{\sqrt{\log N}}{2}\right) + \frac{\pi}{2} + \frac{\pi^2}{2\log 2}.
\end{aligned}$$

If $B \geqslant \frac{\sqrt{\log N}}{2}$, then

$$\begin{aligned}
\frac{A}{2}\left(A\wedge\frac{\sqrt{\log N}}{2}\right) \leqslant \frac{A\sqrt{\log N}}{4} &\leqslant \frac{\sqrt{\log N}}{4}\left(B + \frac{\pi}{\sqrt{\log N}}\right) = \frac{B\sqrt{\log N}}{4} + \frac{\pi}{4}\\
&= \frac{B}{2}\left(B\wedge\frac{\sqrt{\log N}}{2}\right) + \frac{\pi}{4}.
\end{aligned}$$

Hence, we get Lemma 3.2, with

$$c_1' = c_1 e^{\frac{\pi}{2} + \frac{\pi^2}{2\log 2}}.$$ □

3.4. **Bound on the concentration of the law of the logarithm of the characteristic polynomial.**

Lemma 3.3. *For $N \geqslant 4$, $\theta \in [0, 2\pi)$ and $\delta \in (0, 1/2)$, one has, uniformly in $x_0 \in \mathbb{R}$*

$$\mathbb{P}_{SU(N)}[|\log|Z_X(\theta)| - x_0| \leqslant \delta\sqrt{\log N}] \leqslant C\delta\log(1/\delta),$$

where $C > 0$ is a universal constant.

Proof. The proof of Lemma 3.3 needs several steps.

Sublemma 3.4. *For $j \geqslant 1$ integer, $s, t \in \mathbb{R}$, let us define*

$$Q(j,s,t) := \frac{\left(j + \frac{it-s}{2}\right)\left(j + \frac{it+s}{2}\right)}{j(j+it)}.$$

Then,

1. *For $s^2+t^2 \geqslant 8j^2$, $|Q(j,s,t)| \geqslant \max\left(1, \frac{\sqrt{s^2+t^2}}{8j}\right)$.*
2. *For $j^2 \leqslant s^2+t^2 \leqslant 8j^2$, $|Q(j,s,t)| \leqslant 1$.*
3. *For $s^2+t^2 \leqslant j^2$, $|Q(j,s,t)| \leqslant e^{-(s^2+t^2)/10j^2}$.*

Proof. One has

$$Q(j,s,t) = \frac{1 - \frac{s^2+t^2}{4j^2} + it/j}{1 + it/j}. \tag{9}$$

If $s^2+t^2 \leqslant 8j^2$, it is immediate that the numerator has a smaller absolute value than the denominator, i.e. $|Q(j,s,t)| \leqslant 1$. Moreover,

$$|Q(j,s,t)|^2 = \frac{1 - \frac{s^2+t^2}{2j^2} + \frac{(s^2+t^2)^2}{16j^4} + \frac{t^2}{j^2}}{1 + \frac{t^2}{j^2}} = 1 - \frac{\left(\frac{s^2+t^2}{2j^2}\right)\left(1 - \frac{s^2+t^2}{8j^2}\right)}{1 + \frac{t^2}{j^2}}$$

and in the case where $s^2+t^2 \leqslant j^2$, one deduces

$$|Q(j,s,t)|^2 \leqslant 1 - \frac{7(s^2+t^2)}{32j^2}$$

and then

$$|Q(j,s,t)| \leqslant e^{-7(s^2+t^2)/64j^2} \leqslant e^{-(s^2+t^2)/10j^2}.$$

Now, if $s^2+t^2 \geqslant 8j^2$, the numerator in (9) has a larger absolute value than the denominator, and then $|Q(j,s,t)| \geqslant 1$. Moreover, since $(s^2+t^2)/8j^2 \geqslant 1$,

$$|Q(j,s,t)|^2 = \frac{\left(\frac{s^2+t^2}{4j^2} - 1\right)^2 + \frac{t^2}{j^2}}{1 + \frac{t^2}{j^2}} \geqslant \frac{\left(\frac{s^2+t^2}{8j^2}\right)^2 + \frac{t^2}{j^2}}{1 + \frac{t^2}{j^2}} \geqslant \frac{\left(\frac{s^2+t^2}{8j^2}\right)^2 + \frac{s^2+t^2}{j^2}}{1 + \frac{s^2+t^2}{j^2}}$$

$$\geqslant \frac{1}{64} \cdot \frac{\left(\frac{s^2+t^2}{j^2}\right)^2 + \frac{s^2+t^2}{j^2}}{1 + \frac{s^2+t^2}{j^2}} = \frac{s^2+t^2}{64j^2},$$

which finishes the proof of the sublemma. □

Sublemma 3.5. *Let $j \geqslant 1$ be an integer, let ρ_j and σ_j be the real and imaginary parts of $\log(1 - \sqrt{\beta_{1,j-1}}e^{i\theta})$, where $\beta_{1,j-1}$ is a beta random variable with $\beta(1, j-1)$ distribution and θ is independent of $\beta_{1,j-1}$, uniform on $[0, 2\pi]$. Then, for $s, t \in \mathbb{R}$,*

$$|\mathbb{E}[e^{i(t\rho_j + s\sigma_j)}]| \leqslant e^{-(s^2+t^2)/30j}$$

if $s^2+t^2 \leqslant 8j^2$, and

$$|\mathbb{E}[e^{i(t\rho_j + s\sigma_j)}]| \leqslant \frac{8}{\sqrt{s^2+t^2}}$$

if $s^2+t^2 \geqslant 8j^2$ *and* $j \geqslant 2$.

Proof. For $t \in \mathbb{R}$ and $s \in \mathbb{C}$ with real part strictly between -1 and 1,

$$\mathbb{E}[e^{i(t\rho_j+s\sigma_j)}] = \frac{\Gamma(j)\Gamma(j+it)}{\Gamma\left(j+\frac{it-s}{2}\right)\Gamma\left(j+\frac{it+s}{2}\right)} \tag{10}$$

(see [4]). Now, if t is fixed, the function

$$s \mapsto \mathbb{E}[e^{i(t\rho_j+s\sigma_j)}]$$

is holomorphic, since the imaginary part is uniformly bounded (by $\pi/2$). This implies that (10) holds for all $t \in \mathbb{R}$, $s \in \mathbb{C}$, and in particular for all $s, t \in \mathbb{R}$. Moreover,

$$\frac{\Gamma(k)\Gamma(k+it)}{\Gamma\left(k+\frac{it-s}{2}\right)\Gamma\left(k+\frac{it+s}{2}\right)} \underset{k\to\infty}{\longrightarrow} 1,$$

since $\Gamma(k+z)/\Gamma(k)$ is asymptotic to k^z for large k for all $z \in \mathbb{C}$. Hence, by using the equation $\Gamma(z+1) = z\Gamma(z)$, one deduces that

$$\mathbb{E}[e^{i(t\rho_j+s\sigma_j)}] = \prod_{k=j}^{\infty} \frac{\left(k+\frac{it-s}{2}\right)\left(k+\frac{it+s}{2}\right)}{k(k+it)} = \prod_{k=j}^{\infty} Q(k,s,t).$$

If $s^2+t^2 \leqslant 8j^2$, then $|Q(k,s,t)| \leqslant 1$ for all $k \geqslant j$ and $|Q(k,s,t)| \leqslant e^{-(s^2+t^2)/10k^2}$ for all $k \geqslant 3j$. Hence

$$|\mathbb{E}[e^{i(t\rho_j+s\sigma_j)}]| \leqslant \prod_{k=3j}^{\infty} e^{-(s^2+t^2)/10k^2} \leqslant \prod_{k=3j}^{\infty} e^{-(s^2+t^2)/10k(k+1)} = e^{-(s^2+t^2)/30j}.$$

Now let us assume $s^2+t^2 \geqslant 8j^2$. One has:

$$\mathbb{E}[e^{i(t\rho_j+s\sigma_j)}] = \frac{\Gamma(1)\Gamma(1+it)}{\Gamma\left(1+\frac{it-s}{2}\right)\Gamma\left(1+\frac{it+s}{2}\right)} \prod_{k=1}^{j-1} \frac{1}{Q(k,s,t)}$$

where all the factors $\frac{1}{Q(k,s,t)}$ have absolute value bounded by one. By considering the case where $j=1$, one deduces

$$\left|\frac{\Gamma(1)\Gamma(1+it)}{\Gamma\left(1+\frac{it-s}{2}\right)\Gamma\left(1+\frac{it+s}{2}\right)}\right| \leqslant 1,$$

and then, for $j \geqslant 2$,

$$|\mathbb{E}[e^{i(t\rho_j+s\sigma_j)}]| \leqslant \frac{1}{|Q(1,s,t)|} \leqslant \frac{8}{\sqrt{s^2+t^2}}.$$ □

Sublemma 3.6. *For* $N \geqslant 4$ *and* $\theta \in [0, 2\pi)$*, the distribution of* $\log(Z_X(\theta))$ *under Haar measure on* $U(N)$ *has a density with respect to Lebesgue measure on* $\mathbb{C}$*, which is continuous and bounded by* $C_0/\log(N)$*, where* $C_0 > 0$ *is a universal constant.*

Proof. By the results in [4] and the previous sublemma, one checks that the characteristic function Φ of $\log(Z_X(\theta)) \in \mathbb{C} \sim \mathbb{R}^2$ is given by

$$\Phi(s,t) = \prod_{j=1}^{N} \mathbb{E}[e^{i(t\rho_j + s\sigma_j)}].$$

If $s^2+t^2 \geqslant 32N$, one has $s^2+t^2 \geqslant 128 \geqslant 8j^2$ for $j \in \{2,3,4\}$. Hence,

$$|\Phi(s,t)| \leqslant |\mathbb{E}[e^{i(t\rho_2+s\sigma_2)}]||\mathbb{E}[e^{i(t\rho_3+s\sigma_3)}]||\mathbb{E}[e^{i(t\rho_4+s\sigma_4)}]| \leqslant \frac{512}{(s^2+t^2)^{3/2}}.$$

If $s^2+t^2 \leqslant 32N$, then $s^2+t^2 \leqslant 8j^2$ for all $j \geqslant 2\sqrt{N}$. Hence,

$$|\Phi(s,t)| \leqslant \prod_{2\sqrt{N}\leqslant j\leqslant N} \mathbb{E}[e^{i(t\rho_j+s\sigma_j)}] \leqslant \exp\left(-(s^2+t^2)\sum_{2\sqrt{N}\leqslant j\leqslant N}\frac{1}{30j}\right).$$

Since $e^{1/j} \geqslant \frac{j+1}{j}$, one deduces

$$\begin{aligned}|\Phi(s,t)| &\leqslant \prod_{2\sqrt{N}\leqslant j\leqslant N}\left(\frac{j}{j+1}\right)^{(s^2+t^2)/30} \leqslant \left(\frac{2\sqrt{N}+1}{N+1}\right)^{(s^2+t^2)/30} \\ &\leqslant \left(\frac{3\sqrt{N}}{N}\right)^{(s^2+t^2)/30} = e^{-\log(N/9)(s^2+t^2)/60}.\end{aligned}$$

Now, for $N \geqslant 10$,

$$\begin{aligned}\int_{\mathbb{R}^2}|\Phi(s,t)|dsdt &\leqslant \int_{\mathbb{R}^2}\frac{512}{(s^2+t^2)^{3/2}}\mathbb{1}_{\{s^2+t^2\geqslant 32N\}}dsdt + \int_{\mathbb{R}^2} e^{-\log(N/9)(s^2+t^2)/60}\mathbb{1}_{\{s^2+t^2\leqslant 32N\}}dsdt \\ &= \pi\left(\int_0^{32N} e^{-u\log(N/9)/60}du + \int_{32N}^{\infty}\frac{512}{u^{-3/2}}du\right) \\ &\leqslant \frac{60\pi}{\log(N/9)} + 1024\pi(32N)^{-1/2} \leqslant \frac{10000}{\log N},\end{aligned}$$

and for $N \in \{4,5,6,7,8,9\}$,

$$\begin{aligned}\int_{\mathbb{R}^2}|\Phi(s,t)|dsdt &\leqslant \int_{\mathbb{R}^2}\frac{512}{(s^2+t^2)^{3/2}}\mathbb{1}_{\{s^2+t^2\geqslant 32N\}}dsdt + \int_{\mathbb{R}^2}\mathbb{1}_{\{s^2+t^2\leqslant 32N\}}dsdt \\ &= \pi\left(\int_0^{32N} du + \int_{32N}^{\infty}\frac{512}{u^{-3/2}}du\right) \\ &\leqslant 32\pi N + 1024\pi(32N)^{-1/2} \leqslant 288\pi + 1024\pi(128)^{-1/2} \leqslant \frac{10000}{\log 9}.\end{aligned}$$

By applying Fourier inversion, we obtain Sublemma 3.6. □

Let us now go back to the proof of Lemma 3.3. For any $X \in U(N)$ with eigenvalues $(e^{i\theta_j})_{1\leqslant j\leqslant N}$, one has, in the case where $e^{i\theta} \neq e^{i\theta_j}$ for all $j \in \{1,\dots,N\}$,

$$\begin{aligned}\mathcal{I} := \operatorname{Im}(\log(Z_X(\theta)) &= \sum_{1\leqslant j\leqslant N} \operatorname{Im}(\log(1-e^{i(\theta_j-\theta)}))\\ &= \frac{1}{2}\sum_{1\leqslant j\leqslant N}(\theta_j-\theta) + \sum_{1\leqslant j\leqslant N}\operatorname{Im}(\log(e^{-i(\theta_j-\theta)/2}-e^{i(\theta_j-\theta)/2}))\\ &= \frac{1}{2}\operatorname{Im}(\log\,\det(X)) - \frac{N\theta}{2} + \sum_{1\leqslant j\leqslant N}\operatorname{Im}\left(\log\left(-2i\sin\left(\theta_j-\theta\right)/2\right)\right)\\ &= \frac{\mathcal{J}}{2} - \frac{N(\theta+\pi)}{2},\end{aligned}$$

where $\mathcal{J}$ denotes the version of $\operatorname{Im}(\log\,\det(X))$ lying on the interval $(-\pi,\pi]$. Hence, for any $\epsilon \in (0,\pi)$, $|\mathcal{J}| \leqslant \epsilon$ if and only if $\mathcal{I}$ is on an interval of the form $\left[\frac{2k\pi-\epsilon-N(\theta+\pi)}{2}, \frac{2k\pi+\epsilon-N(\theta+\pi)}{2}\right]$ for some $k \in \mathbb{Z}$. Now, for some $A > 0$ chosen later as a function of δ, let Φ be a continuous function from $\mathbb{C}$ to $[0,1]$ such that $\Phi(z) = 1$ if $|\operatorname{Re} z - x_0| \leqslant \delta\sqrt{\log N}$ and $|\operatorname{Im} z| \leqslant A\sqrt{\log N}$, and such that $\Phi(z) = 0$ for $|\operatorname{Re} z - x_0| \geqslant 2\delta\sqrt{\log N}$ or $|\operatorname{Im} z| \geqslant 2A\sqrt{\log N}$. For $\epsilon \in (0,\pi)$, and under the Haar measure $\mathbb{P}_{U(N)}$ on $U(N)$,

$$\begin{aligned}&\frac{\pi}{\epsilon}\mathbb{E}_{U(N)}\left[\Phi(\log(Z_X(\theta)))\mathbb{1}_{\{|\mathcal{J}|\leqslant\epsilon\}}\right]\\ &\qquad= \frac{\pi}{\epsilon}\sum_{k\in\mathbb{Z}}\mathbb{E}_{U(N)}\left[\Phi(\log(Z_X(\theta)))\mathbb{1}_{\left\{\frac{2k\pi-\epsilon-N(\theta+\pi)}{2}\leqslant\mathcal{I}\leqslant\frac{2k\pi+\epsilon-N(\theta+\pi)}{2}\right\}}\right]\\ &\qquad= \frac{\pi}{\epsilon}\sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty}dx\int_{(2k\pi-\epsilon-N(\theta+\pi))/2}^{(2k\pi+\epsilon-N(\theta+\pi))/2}dy\,D(x+iy)\Phi(x+iy)\\ &= \pi\sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty}dx\int_{-1/2}^{1/2}du\,D(x+i[k\pi-N(\theta+\pi))/2+u\epsilon])\Phi(x+i[k\pi-N(\theta+\pi))/2+u\epsilon]),\end{aligned}$$

where D denotes the density of the law of $\log(Z_X(\theta))$, with respect to the Lebesgue measure. Now,

$$D(x+i[k\pi-N(\theta+\pi))/2+u\epsilon])\Phi(x+i[k\pi-N(\theta+\pi))/2+u\epsilon])$$

is uniformly bounded by the overall maximum of D and vanishes as soon as $|x-x_0| \geqslant 2\delta\sqrt{\log N}$ or $|k|\pi \geqslant N(|\theta|+\pi)/2 + \pi/2 + 2A\sqrt{\log N}$. Since D and Φ are continuous functions, one can apply dominated convergence and deduce that

$$\frac{\pi}{\epsilon}\mathbb{E}_{U(N)}\left[\Phi(\log(Z_X(\theta)))\mathbb{1}_{\{|\mathcal{J}|\leqslant\epsilon\}}\right]$$

converges to

$$\pi\sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty}D(x+i[k\pi-N(\theta+\pi))/2])\Phi(x+i[k\pi-N(\theta+\pi))/2])dx$$

when ϵ goes to zero. On the other hand, if the matrix X follows $\mathbb{P}_{SU(N)}$ and if T is an independent uniform variable on $(-\pi, \pi]$, then $Xe^{iT/N}$ follows $\mathbb{P}_{U(N)}$ and its determinant is e^{iT}. One deduces

$$\begin{aligned}\frac{\pi}{\epsilon}\mathbb{E}_{U(N)}\left[\Phi(\log(Z_X(\theta)))\mathbb{1}_{\{|\mathcal{J}|\leqslant\epsilon\}}\right] &= \frac{\pi}{\epsilon}\mathbb{E}_{SU(N)}\left[\Phi(\log(Z_{Xe^{iT/N}}(\theta)))\mathbb{1}_{\{|T|\leqslant\epsilon\}}\right] \\ &= \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\mathbb{E}_{SU(N)}\left[\Phi(\log(Z_{Xe^{it/N}}(\theta)))\right]\,dt \\ &= \int_{-1/2}^{1/2}\mathbb{E}_{SU(N)}\left[\Phi(\log(Z_{Xe^{2iv\epsilon/N}}(\theta)))\right]\,dv\end{aligned}$$

Now, the function $X \mapsto \Phi(\log(Z_X(\theta)))$ is continuous from $U(N)$ to $[0,1]$, since Φ is continuous with compact support and $X \mapsto \log(Z_X(\theta))$ has discontinuities only at points where its real part goes to $-\infty$. One can then apply dominated convergence and obtain

$$\frac{\pi}{\epsilon}\mathbb{E}_{U(N)}\left[\Phi(\log(Z_X(\theta)))\mathbb{1}_{\{|\mathcal{J}|\leqslant\epsilon\}}\right] \underset{\epsilon\to 0}{\longrightarrow} \mathbb{E}_{SU(N)}\left[\Phi(\log(Z_X(\theta)))\right].$$

By comparing with the convergence obtained just above, one deduces that

$$\mathbb{E}_{SU(N)}\left[\Phi(\log(Z_X(\theta)))\right] = \pi\sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty} D(x+i[k\pi-N(\theta+\pi))/2])\Phi(x+i[k\pi-N(\theta+\pi))/2])dx.$$

Since $D(z) \leqslant C_0/\log N$ and

$$\mathbb{1}_{\left\{|x-x_0|\leqslant\delta\sqrt{\log N}, |y|\leqslant A\sqrt{\log N}\right\}} \leqslant \Phi(x+iy) \leqslant \mathbb{1}_{\left\{|x-x_0|\leqslant 2\delta\sqrt{\log N}, |y|\leqslant 2A\sqrt{\log N}\right\}}$$

for all $x, y \in \mathbb{R}$, one deduces

$$\mathbb{P}_{SU(N)}[|\log|Z_X(\theta)| - x_0| \leqslant \delta\sqrt{\log N}, |\operatorname{Im}\log Z_X(\theta)| \leqslant A\sqrt{\log N}] \leqslant \frac{\pi d L C_0}{\log N},$$

where $d = 4\delta\sqrt{\log N}$ is the length of the interval $[x_0 - 2\delta\sqrt{\log N}, x_0 + 2\delta\sqrt{\log N}]$ and L is the number of integers k such that $|k\pi - N(\theta+\pi))/2| \leqslant 2A\sqrt{\log N}$. Now, it is easy to check that $L \leqslant 1 + \frac{4A\sqrt{\log N}}{\pi}$, and so

$$\mathbb{P}_{SU(N)}\left[|\log|Z_X(\theta)| - x_0| \leqslant \delta\sqrt{\log N}, |\operatorname{Im}\log Z_X(\theta)| \leqslant A\sqrt{\log N}\right] \leqslant 16\, C_0\, A\delta + \frac{4\pi\delta\, C_0}{\sqrt{\log N}}.$$

Using Lemma 3.2, one obtains

$$\mathbb{P}_{SU(N)}\left[|\log|Z_X(\theta)| - x_0| \leqslant \delta\sqrt{\log N}\right] \leqslant 16\, C_0\, A\delta + \frac{4\pi\delta\, C_0}{\sqrt{\log N}} + c_1' e^{-\frac{A}{2}\left(A\wedge\frac{\sqrt{\log N}}{2}\right)}.$$

Let us now choose $A := 1 + 5\log(1/\delta)$. One finds

$$A \wedge \frac{\sqrt{\log N}}{2} = [1 + 5\log(1/\delta)] \wedge \frac{\sqrt{\log N}}{2} \geqslant \frac{\sqrt{\log 2}}{2}$$

and thus

$$\frac{A}{2}\left(A \wedge \frac{\sqrt{\log N}}{2}\right) \geqslant \frac{5\sqrt{\log 2}\log(1/\delta)}{4} \geqslant \log(1/\delta).$$

Therefore,

$$\mathbb{P}_{SU(N)}\left[|\log|Z_X(\theta)|-x_0|\leqslant\delta\sqrt{\log N}\right]\leqslant 16\,C_0\,\delta+80\,C_0\,\delta\log(1/\delta)+\frac{4\pi\delta\,C_0}{\sqrt{\log N}}+c_1'\delta.$$

Since $\delta<1/2$, one has $\delta\leqslant\delta\log(1/\delta)/\log(2)$, which implies Lemma 3.3, for

$$C=\frac{16\,C_0}{\log 2}+80\,C_0+\frac{4\pi\,C_0}{(\log 2)^{3/2}}+\frac{c_1'}{\log 2}.$$ □

3.5. **Behaviour of the oscillation in short intervals of the logarithm of the characteristic polynomial.**

Lemma 3.7. *There exists $c_2>0$ such that for $\mu\in\mathbb{R}$ and $A\geqslant 0$ and uniformly in $N\geqslant M\geqslant 2\vee\frac{|\mu|}{2\pi}$,*

$$\mathbb{P}_{SU(N)}\left(\int_0^{2\pi}\left|\operatorname{Re}\log Z_X\left(\theta+\frac{\mu}{N}\right)-\operatorname{Re}\log Z_X(\theta)\right|\frac{d\theta}{2\pi}\geqslant A\sqrt{\log M}\right)\leqslant\frac{c_2}{A^2},$$

and

$$\mathbb{P}_{SU(N)}\left(\int_0^{2\pi}\left|\operatorname{Im}\log Z_X\left(\theta+\frac{\mu}{N}\right)-\operatorname{Im}\log Z_X(\theta)\right|\frac{d\theta}{2\pi}\geqslant A\sqrt{\log M}\right)\leqslant\frac{c_2}{A^2}.$$

Proof. By symmetry of the problem, we can assume $\mu>0$. Setting

$$R_\theta:=\operatorname{Re}\log Z_X\left(\theta+\frac{\mu}{N}\right)-\operatorname{Re}\log Z_X(\theta)$$

for fixed μ (or the same expression with the imaginary part), we get:

$$\begin{aligned}\mathbb{P}_{SU(N)}\left(\int_0^{2\pi}|R_\theta|\,\frac{d\theta}{2\pi}\geqslant A\sqrt{\log M}\right)&\leqslant\frac{1}{A^2\log M}\mathbb{E}_{SU(N)}\left(\left(\int_0^{2\pi}|R_\theta|\,\frac{d\theta}{2\pi}\right)^2\right)\\&\leqslant\frac{1}{A^2\log M}\mathbb{E}_{SU(N)}\left(\int_0^{2\pi}R_\theta^2\frac{d\theta}{2\pi}\right)\\&=\frac{1}{A^2\log M}\int_0^{2\pi}\mathbb{E}_{SU(N)}\big(R_\theta^2\big)\,\frac{d\theta}{2\pi}\\&=\frac{1}{A^2\log M}\mathbb{E}_{U(N)}\big(R_0^2\big)\qquad\text{(by (2))}\end{aligned}$$

Now, under $U(N)$, the canonical matrix X is almost surely unitary. Let $\theta_1,\dots,\theta_N$ be its eigenangles in $[0,2\pi)$. For $j\in\{1,\dots,N\}$ and $t\in[0,2\pi)\backslash\{\theta_j\}$, we can expand the logarithm as a conditionally convergent series

$$\log(1-e^{i(\theta_j-t)})=-\sum_{k\geqslant 1}\frac{e^{ik(\theta_j-t)}}{k}.$$

Hence, for t such that $Z_X(t) \neq 0$,

$$\log Z_X(t) = -\sum_{j=1}^{N} \sum_{k \geqslant 1} \frac{e^{ik(\theta_j - t)}}{k} = -\sum_{k \geqslant 1} \frac{e^{-ikt}}{k} \operatorname{tr}\left(X^k\right).$$

Therefore

$$\operatorname{Re} \log Z_X(t) = -\frac{1}{2}\left(\sum_{k \geqslant 1} \frac{1}{k} e^{-ikt} \operatorname{tr}\left(X^k\right) + \sum_{k \geqslant 1} \frac{1}{k} e^{ikt} \operatorname{tr}\left(X^{-k}\right)\right) = -\frac{1}{2} \sum_{k \in \mathbb{Z}^*} \frac{1}{|k|} e^{-ikt} \operatorname{tr}\left(X^k\right),$$

and

$$\operatorname{Im} \log Z_X(t) = -\frac{1}{2i}\left(\sum_{k \geqslant 1} \frac{1}{k} e^{-ikt} \operatorname{tr}\left(X^k\right) - \sum_{k \geqslant 1} \frac{1}{k} e^{ikt} \operatorname{tr}\left(X^{-k}\right)\right) = -\frac{1}{2i} \sum_{k \in \mathbb{Z}^*} \frac{1}{k} e^{-ikt} \operatorname{tr}\left(X^k\right).$$

Here, the series in $k \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ are conditionally convergent. More precisely, for $K \geqslant 1$ set

$$S_t^{(K)} := -\frac{1}{2} \sum_{k \in \mathbb{Z}^*, |k| \leqslant K} \frac{1}{|k|} e^{-ikt} \operatorname{tr}\left(X^k\right),$$

and

$$S_t := \operatorname{Re} \log Z_X(t),$$

then $S_t^{(K)}$ tends almost surely to S_t as K goes to infinity.

Moreover, one has the following classical result, [7]: For all $p, q \in \mathbb{Z}$,

$$\mathbb{E}_{U(N)}\left(\operatorname{tr}\left(X^p\right)\overline{\operatorname{tr}\left(X^q\right)}\right) = \mathbb{1}_{\{p=q\}} |p| \wedge N. \tag{11}$$

Hence, for $K, L \geqslant 1$, $t, u \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}_{U(N)}\left(S_t^{(K)} S_u^{(L)}\right) &= \mathbb{E}_{U(N)}\left(\frac{1}{4} \sum_{p,q \in \mathbb{Z}^*, |p| \leqslant K, |q| \leqslant L} \frac{e^{-i(pt+qu)}}{|pq|} \operatorname{tr}\left(X^p\right) \operatorname{tr}\left(X^q\right)\right) \\
&= \frac{1}{4} \sum_{p,q \in \mathbb{Z}^*, |p| \leqslant K, |q| \leqslant L} \frac{e^{-i(pt+qu)}}{|pq|} \mathbb{E}_{U(N)}(\operatorname{tr}\left(X^p\right) \operatorname{tr}\left(X^q\right)) \\
&= \frac{1}{4} \sum_{p,q \in \mathbb{Z}^*, |p| \leqslant K, |q| \leqslant L} \frac{e^{-i(pt+qu)}}{|pq|} \mathbb{1}_{\{p=-q\}} |q| \wedge N \qquad \text{(from (11))} \\
&= \frac{1}{4} \sum_{k \in \mathbb{Z}^*, |k| \leqslant K \wedge L} \frac{e^{ik(u-t)}}{k^2} |k| \wedge N \\
&= \frac{1}{2} \sum_{1 \leqslant k \leqslant K \wedge L} \frac{k \wedge N}{k^2} \left(\frac{e^{ik(u-t)} + e^{-ik(u-t)}}{2}\right) \\
&= \frac{1}{2} \sum_{1 \leqslant k \leqslant K \wedge L} \frac{k \wedge N}{k^2} \cos\left(k(u-t)\right).
\end{aligned}$$

One deduces that

$$\begin{aligned}\mathbb{E}_{U(N)}\Big((S_t^{(K)}-S_t^{(L)})^2\Big) &= \mathbb{E}_{U(N)}\Big((S_t^{(K)})^2\Big)+\mathbb{E}_{U(N)}\Big((S_t^{(L)})^2\Big)-2\,\mathbb{E}_{U(N)}\Big(S_t^{(K)}S_t^{(L)}\Big)\\ &= \frac{1}{2}\sum_{k\geqslant 1}\frac{k\wedge N}{k^2}\cos\left(k(u-t)\right)\left(\mathbb{1}_{\{k\leqslant K\}}+\mathbb{1}_{\{k\leqslant L\}}-2\mathbb{1}_{\{k\leqslant K\wedge L\}}\right)\\ &= \frac{1}{2}\sum_{k\geqslant 1}\frac{k\wedge N}{k^2}\cos\left(k(u-t)\right)\;\mathbb{1}_{\{K\wedge L<k\leqslant K\vee L\}}\\ &\leqslant \frac{1}{2}\sum_{k\geqslant K\wedge L}\frac{k\wedge N}{k^2},\end{aligned}$$

which tends to zero when $K\wedge L$ goes to infinity. Hence, $S_t^{(K)}$ converges in L^2 when K goes to infinity, and the limit is necessarily S_t. Therefore,

$$\mathbb{E}_{U(N)}(S_tS_u)=\lim_{K\to\infty}\frac{1}{2}\sum_{1\leqslant k\leqslant K\wedge L}\frac{k\wedge N}{k^2}\cos\left(k(u-t)\right)=\frac{1}{2}\sum_{k\geqslant 1}\frac{k\wedge N}{k^2}\cos\left(k(u-t)\right).$$

The same computation with $\widetilde{S}_t:=\operatorname{Im}\log Z_X(t)$ gives exactly the same equality, namely

$$\mathbb{E}_{U(N)}\Big(\widetilde{S}_t\widetilde{S}_u\Big)=\frac{1}{2}\sum_{k\geqslant 1}\frac{k\wedge N}{k^2}\cos\left(k(u-t)\right)$$

It is therefore sufficient to only show the calculations for S_t, as those for $\widetilde{S}_t$ are identical. Setting $\alpha=\frac{\mu}{N}$ and using this last formula, we can write

$$\begin{aligned}\mathbb{E}_{U(N)}\big(R_0^2\big)=\mathbb{E}_{U(N)}\big((S_\alpha-S_0)^2\big) &= 2\mathbb{E}_U\big(S_0^2-S_\alpha S_0\big)\\ &= \sum_{k\geqslant 1}\frac{k\wedge N}{k^2}\left(1-\cos\left(k\alpha\right)\right)\\ &= \sum_{k\geqslant 1}\frac{k\wedge N}{k^2}-\sum_{k\geqslant 1}\frac{k\wedge N}{k^2}\cos\left(\frac{k\mu}{N}\right).\end{aligned}$$

We have

$$\sum_{k\geqslant 1}\frac{k\wedge N}{k^2}=\sum_{k=1}^{N}\frac{1}{k}+N\sum_{k>N}\frac{1}{k^2}=\log N+\gamma+O\left(\frac{1}{N}\right)+N\left(\frac{1}{N}+O\left(\frac{1}{N^2}\right)\right)$$

and using Euler-Maclaurin summation, we have the following result (see [8], p.37 for the details), that uniformly on $\alpha\in[-\pi,\pi]$

$$\sum_{k\geqslant 1}\frac{k\wedge N}{k^2}\cos(k\alpha)=-\log\left|2\sin\left(\frac{\alpha}{2}\right)\right|+\operatorname{Ci}\left(N\left|\alpha\right|\right)+\cos\left(N\alpha\right)-\frac{\pi}{2}N\left|\alpha\right|+N\alpha\operatorname{Si}\left(N\alpha\right)+O\left(\frac{1}{N}\right),$$

where

$$\mathrm{Si}\,(z) := \int_0^z \frac{\sin x}{x} dx = \frac{\pi}{2} - \frac{\cos z}{z} + \int_z^\infty \frac{\cos x}{x^2} dx$$

and

$$\mathrm{Ci}\,(z) = -\int_z^\infty \frac{\cos x}{x} dx = \gamma - \log z + \int_0^z \frac{\cos x - 1}{x} dx.$$

Let us denote $f\,(\mu) := \log \mu + \frac{\pi}{2}\mu - \cos \mu - \mathrm{Ci}\,(\mu) - \mu \mathrm{Si}\,(\mu)$. We therefore have, for N going to infinity,

$$\begin{aligned}
\mathbb{E}_{U(N)}\big(R_0^2\big) &= \log N + 1 + \gamma + f(\mu) - \log \mu + \log \left| 2 \sin\left(\frac{\mu}{2N}\right) \right| + O\left(\frac{1}{N}\right) \\
&= \log N + 1 + \gamma + f(\mu) - \log \mu + \log \left(2 \frac{\mu}{2N} \left(1 + O\left(\left(\frac{\mu}{N}\right)^2 \right) \right) \right) + O\left(\frac{1}{N}\right) \\
&= 1 + \gamma + f(\mu) + O\left(\left(\frac{\mu}{N}\right)^2 \right) + O\left(\frac{1}{N}\right). \qquad (12)
\end{aligned}$$

Let us now study the behavior of the function f.

$$\begin{aligned}
f(\mu) &= \log \mu - \cos \mu + \mu \left(\frac{\pi}{2} - \mathrm{Si}\,(\mu) \right) - \mathrm{Ci}\,(\mu) \\
&= \log \mu - \cos \mu + \mu \left(\frac{\cos \mu}{\mu} - \int_\mu^\infty \frac{\cos x}{x^2} dx \right) - \left(\gamma - \log \mu + \int_0^\mu \frac{\cos x - 1}{x} dx \right) \\
&= -\gamma - \mu \int_\mu^\infty \frac{\cos x}{x^2} dx + \int_0^\mu \frac{\cos x - 1}{x} dx.
\end{aligned}$$

Bounding the integrals uniformly in $\mu > 0$ one has that

$$\begin{aligned}
f(\mu) &= -\gamma - \mu O\left(\int_\mu^\infty \frac{1}{x^2} dx \right) + O\left(\int_0^\mu \left(\frac{1}{x} \wedge 1 \right) dx \right) \\
&= -\gamma + O\,(1) + O\,(1 + \log(\mu \vee 1)) = O\left(\log\left(\frac{\mu}{2\pi} \vee 2 \right) \right),
\end{aligned}$$

where we have used the standard notation $\mu \vee 1$ to mean the maximum of μ and 1.

Substituting this into (12) yields

$$\mathbb{E}_{U(N)}\big(R_0^2\big) = O\left(\log\left(\frac{\mu}{2\pi} \vee 2 \right) \right) = O\,(\log M)\,. \qquad (13)$$

□

3.6. **Control in probability of the mean oscillation of the logarithm of the characteristic polynomials.**

Lemma 3.8. *Given $n \in \mathbb{N}$, consider an i.i.d. sequence $(U_j)_{1 \leqslant j \leqslant n}$ of random matrices following the Haar measure on $SU(N)$, and let*

$$L_j(\theta) := \frac{\log \left| Z_{U_j}(\theta) \right|}{\sqrt{\frac{1}{2} \log N}}.$$

For $\delta \in (0,1/2)$, consider the random set

$$\mathscr{E}_\delta := \bigcup_{i=1}^{n} \left(\left\{ \theta \in [0,2\pi] \ / \ |L_i(\theta)| \geqslant \delta^{-1} \right\} \cup \bigcup_{j \neq i} \left\{ \theta \in [0,2\pi] \ / \ |L_j(\theta) - L_i(\theta)| \leqslant \delta \right\} \right).$$

There exists $c_3 > 0$, depending only on n, such that for all $N \geqslant 4$

$$\mathbb{E}(\lambda_{2\pi}(\mathscr{E}_\delta)) \leqslant c_3 \delta \log(1/\delta),$$

where $\lambda_{2\pi}$ denotes the normalised Lebesgue measure on $[0,2\pi]$.

Proof. One has

$$\begin{aligned}\mathbb{E}_{SU(N)}(\lambda_{2\pi}(\mathscr{E}_\delta)) &= \int_0^{2\pi} \frac{d\theta}{2\pi} \mathbb{P}_{SU(N)} \left(\theta \in \mathscr{E}_\delta \right) \\ &\leqslant \sum_{i=1}^{n} \int_0^{2\pi} \frac{d\theta}{2\pi} \mathbb{P}_{SU(N)} \left(|L_i(\theta)| \geqslant \delta^{-1} \right) + \sum_{1 \leqslant i \neq j \leqslant n} \int_0^{2\pi} \frac{d\theta}{2\pi} \mathbb{P}_{SU(N)} \left(|L_j(\theta) - L_i(\theta)| \leqslant \delta \right).\end{aligned}$$

In the last second sum, since $i \neq j$, $L_i(\theta)$ is independent of $L_j(\theta)$, and hence, conditioning on $L_i(\theta)$, one gets

$$\mathbb{P}_{SU(N)} \left(|L_j(\theta) - L_i(\theta)| \leqslant \delta \right) = \mathbb{E}_{SU(N)}(H(L_i(\theta)))$$

where

$$H(x) = \mathbb{P}_{SU(N)} \left(|L_j(\theta) - x| \leqslant \delta \right)$$

One deduces that for all $i \neq j$

$$\mathbb{P}_{SU(N)} \left(|L_j(\theta) - L_i(\theta)| \leqslant \delta \right) \leqslant \sup_{\theta \in [0,2\pi], x \in \mathbb{R}} \mathbb{P}_{SU(N)} \left(\left| \log |Z_X(\theta)| - x \right| \leqslant \delta \sqrt{\frac{1}{2} \log N} \right)$$

Using Markov's inequality, i.e. $\mathbb{P}(X \geqslant t) \leqslant \mathbb{E}(X)/t$ for all $t > 0$ and all random variables $X \geqslant 0$, one thus obtains

$$\begin{aligned}\mathbb{E}_{SU(N)}(\lambda_{2\pi}(\mathscr{E}_\delta)) &\leqslant n \int_0^{2\pi} \frac{d\theta}{2\pi} \mathbb{P}_{SU(N)} \left(\left| \log |Z_X(\theta)| \right| \geqslant \delta^{-1} \sqrt{\frac{1}{2} \log N} \right) \\ &\quad + n(n-1) \sup_{\theta \in [0,2\pi], x \in \mathbb{R}} \mathbb{P}_{SU(N)} \left(\left| \log |Z_X(\theta)| - x \right| \leqslant \delta \sqrt{\frac{1}{2} \log N} \right) \\ &\leqslant n c_1 e^{-\frac{\delta^{-1}}{\sqrt{2}} \left(\frac{\delta^{-1}}{\sqrt{2}} \wedge \frac{\sqrt{\log N}}{2} \right)} + n(n-1) C(\delta/\sqrt{2}) \log(\sqrt{2}/\delta).\end{aligned}$$

Now,

$$e^{-\frac{\delta^{-1}}{\sqrt{2}} \left(\frac{\delta^{-1}}{\sqrt{2}} \wedge \frac{\sqrt{\log N}}{2} \right)} \leqslant e^{-\frac{\delta^{-1}}{\sqrt{2}} \left(\frac{2}{\sqrt{2}} \wedge \frac{\sqrt{\log 2}}{2} \right)} \leqslant e^{-\frac{\delta^{-1}}{5}} = O(\delta \log(1/\delta))$$

and

$$\delta \log(\sqrt{2}/\delta) \leqslant \delta \log(\sqrt{\delta^{-1}}/\delta) = \frac{3\delta}{2} \log(1/\delta),$$

which gives Lemma 3.8. □

3.7. **Control in expectation of the oscillation of the logarithm of the characteristic polynomials on a small period.** In what follows, assume the dimension $N \geqslant 4$. Let K be an integer such that $2 \leqslant K \leqslant N/2$, defined as a function of N which is asymptotically equivalent to $N/(\log N)^{3/64}$ when N goes to infinity. We denote

$$M := N/K \geqslant 2,$$

which is asymptotic to $(\log N)^{3/64}$, and we also define a parameter $\delta \in (0, 1/4)$ as a function of N, asymptotic to $(\log N)^{-3/32}$ when N goes to infinity. For $\theta_0 \in [0, 2\pi]$, we denote, for $0 \leqslant k \leqslant K$.

$$\theta_k := \theta_0 + \frac{2\pi k}{K} = \theta_0 + \frac{2\pi k M}{N},$$

and for $0 \leqslant k \leqslant K-1$,

$$\Delta := \theta_{k+1} - \theta_k = \frac{2\pi}{K} = \frac{2\pi M}{N}.$$

The angle θ_0 is chosen in such a way that the following technical condition is satisfied:

$$\sum_{k=0}^{K-1} \mathbb{E}_{SU(N)} \left(\left| \operatorname{Im} \log Z_X(\theta_k + (1-\sqrt{\delta})\Delta) - \operatorname{Im} \log Z_X(\theta_k + \sqrt{\delta}\Delta) \right| \right)$$
$$\leqslant K \mathbb{E}_{SU(N)} \left(\int_0^{2\pi} \frac{d\theta}{2\pi} \left| \operatorname{Im} \log Z_X(\theta + (1-\sqrt{\delta})\Delta) - \operatorname{Im} \log Z_X(\theta + \sqrt{\delta}\Delta) \right| \right).$$

This choice is always possible. Indeed, if the converse (strict) inequality were true for all θ_0, then one would get a contradiction by integrating with respect to $\theta_0 \in [0, 2\pi/K)$. We then define the interval $J := [\theta_0, \theta_0 + 2\pi) = [\theta_0, \theta_K)$. Note that all the objects introduced here can be defined only as a function of N. Moreover, by applying Lemma 3.7 to $\theta + \sqrt{\delta}\Delta$ and $\mu = N(1 - 2\sqrt{\delta})\Delta \leqslant 2\pi M$, we deduce that the assumption made on θ_0 implies that

$$\sum_{k=0}^{K-1} \mathbb{E}_{SU(N)} \left(\left| \operatorname{Im} \log Z_X(\theta_k + (1-\sqrt{\delta})\Delta) - \operatorname{Im} \log Z_X(\theta_k + \sqrt{\delta}\Delta) \right| \right) = O\left(K\sqrt{\log M} \right). \tag{14}$$

We now introduce the **2-oscillation** of the real and imaginary parts of the logarithm of the characteristic polynomial.

Definition 3.9. For $\theta \in J$ and $\mu \in [0, 2\pi M]$, and for the canonical matrix $X \in U(N)$, the 2-oscillations of $\operatorname{Re} \log Z_X$ and $\operatorname{Im} \log Z_X$ are defined by

$$\Delta_\mu R_\theta := \frac{1}{\sqrt{\log(M)}} \left| \operatorname{Re} \log Z_X\left(\theta + \frac{\mu}{N}\right) - \operatorname{Re} \log Z_X(\theta) \right|$$
$$\Delta_\mu I_\theta := \frac{1}{\sqrt{\log(M)}} \left| \operatorname{Im} \log Z_X\left(\theta + \frac{\mu}{N}\right) - \operatorname{Im} \log Z_X(\theta) \right|$$

In case of several matrices $(X_j)_{1 \leqslant j \leqslant n}$, we denote the corresponding 2-oscillations by $\Delta_\mu R_\theta^{(j)}$ and $\Delta_\mu I_\theta^{(j)}$.

Now, we need to introduce several random sets. The most important ones can be informally described as follows:

(1) A set $\mathscr{N}_1$ of indices k such that the average of the 2-oscillations $\Delta_\mu R_\theta$ and $\Delta_\mu I_\theta$ of the logarithm of the characteristic polynomials for $\theta \in [\theta_k, \theta_{k+1}]$ and $\mu \in [0, 2\pi M]$ is sufficiently small.
(2) For $k \in \mathscr{N}_1$, a subset $\mathscr{G}_k$ of $[\theta_k, \theta_{k+1}]$ for which the average of the 2-oscillations with respect to $\mu \in [0, 2\pi M]$ is small enough.
(3) A subset $\mathscr{N}_2$ of $\mathscr{N}_1$ of "good" indices, such that there exists $\theta \in [\theta_k, \theta_k + \sqrt{\delta}\Delta]$, both in $\mathscr{G}_k$ and $\mathscr{E}_\delta^c$. This last set, introduced in Lemma 3.8, corresponds to the fact that the logarithms of the absolute values of the characterize polynomials are not too large and not too close from each other. It is from this last condition that we can define the "carrier wave".
(4) For $k \in \mathscr{N}_2$, and for some $\theta_k^* \in [\theta_k, \theta_k + \sqrt{\delta}\Delta] \cap \mathscr{G}_k \cap \mathscr{E}_\delta^c$, a subset $\mathscr{Y}_k$ of $[0, 2\pi M]$ such that the 2-oscillations $\Delta_\mu R_{\theta_k^*}$ and $\Delta_\mu I_{\theta_k^*}$ are sufficiently small. This condition will ensure that the carrier wave index corresponding to $\theta = \theta_k^* + \mu/N$ does not depend on $\mu \in \mathscr{Y}_k$.
(5) From this property, we deduce that, for each pair of consecutive gaps between zeros of the carrier wave, which are sufficiently large to contain an angle of the form $\theta_k^* + \mu/N$ for $k \in \mathscr{N}_2$ and $\mu \in \mathscr{Y}_k$ ("roomy gaps"), one can find, with the notation of the introduction, a sign change of $i^N e^{iN\theta/2} F_N(e^{-i\theta})$, and hence a zero of F_N.

All these sets will be precisely defined later in this paper. They are constructed in a way such that their measure is "large" with "high" probability (again, in a manner to be made precise). The corresponding estimates will then be used to prove our main result, Theorem 1.1.

Lemma 3.10. *Let $\mathbb{P}_{SU(N)}^{(n)}$ be the n-fold product of the Haar measure on $SU(N)$, $\mathbb{E}_{SU(N)}^{(n)}$ the corresponding expectation, and $(X_j)_{1\leqslant j\leqslant n}$ the canonical sequence of n matrices in $SU(N)$. Then:*

(1) *There exists a random set $\mathscr{N}_1 \subset [\![0, K-1]\!]$ such that $\mathbb{E}_{SU(N)}^{(n)}(|\mathscr{N}_1|) \geqslant (1-\delta)K$ and $\mathbb{P}_{SU(N)}^{(n)}$-a.s., $\forall\, (j,k) \in [\![1,n]\!] \times \mathscr{N}_1$,*

$$\int_{\theta_k}^{\theta_{k+1}} \int_0^{2\pi M} \Delta_\mu R_\theta^{(j)} \frac{d\mu}{2\pi M} \frac{d\theta}{2\pi} = O\left(\frac{1}{\delta K}\right)$$

and

$$\int_{\theta_k}^{\theta_{k+1}} \int_0^{2\pi M} \Delta_\mu I_\theta^{(j)} \frac{d\mu}{2\pi M} \frac{d\theta}{2\pi} = O\left(\frac{1}{\delta K}\right)$$

(2) *$\mathbb{P}_{SU(N)}^{(n)}$-a.s., $\forall\, k \in \mathscr{N}_1$, $\exists \mathscr{G}_k \subset [\theta_k, \theta_{k+1})$ such that $\lambda_{2\pi}(\mathscr{G}_k) \geqslant (1-\delta)/K$ and, $\forall\, \theta \in \mathscr{G}_k, j \in [\![1,n]\!]$,*

$$\int_0^{2\pi M} \Delta_\mu R_\theta^{(j)} \frac{d\mu}{2\pi M} = O\left(\frac{1}{\delta^2}\right) \text{ and } \int_0^{2\pi M} \Delta_\mu I_\theta^{(j)} \frac{d\mu}{2\pi M} = O\left(\frac{1}{\delta^2}\right) \tag{15}$$

Here, the implied constant in the $O(\cdot)$ symbols depends only on n.

Proof. By (13) and the similar estimate for the imaginary part, we have uniformly (with a universal implied constant),

$$\mathbb{E}_{U(N)}\big((\Delta_\mu R_0)^2\big) + \mathbb{E}_{U(N)}\big((\Delta_\mu I_0)^2\big) = O(1).$$

The Cauchy-Schwarz inequality ensures that

$$\mathbb{E}_{U(N)}(\Delta_\mu R_0) + \mathbb{E}_{U(N)}(\Delta_\mu I_0) = O(1),$$

i.e.

$$\int_J \mathbb{E}_{SU(N)}(\Delta_\mu R_\theta + \Delta_\mu I_\theta)\,\frac{d\theta}{2\pi} = O(1),$$

which implies

$$\int_J \int_0^{2\pi M} \mathbb{E}_{SU(N)}(\Delta_\mu R_\theta + \Delta_\mu I_\theta)\,\frac{d\mu}{2\pi M}\frac{d\theta}{2\pi} = O(1).$$

Splitting the interval J into K equal pieces and applying this estimate to n independent matrices $(X_j)_{1\leqslant j\leqslant n}$ following the Haar measure on $SU(N)$, one gets

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{K-1}\mathbb{E}^{(n)}_{SU(N)}\left(\int_{\theta_k}^{\theta_{k+1}}\int_0^{2\pi M}(\Delta_\mu R_\theta^{(j)} + \Delta_\mu I_\theta^{(j)})\frac{d\mu}{2\pi M}\frac{d\theta}{2\pi}\right) = O(1). \tag{16}$$

Applying Markov's inequality, we deduce that there exists a universal constant $\kappa > 0$, such that

$$\mathbb{E}^{(n)}_{SU(N)}\left(\mathrm{card}\left\{(j,k)\in[\![1,n]\!]\times[\![0,K-1]\!]\,:\,\int_{\theta_k}^{\theta_{k+1}}\int_0^{2\pi M}\Delta_\mu R_\theta^{(j)}\frac{d\mu}{2\pi M}\frac{d\theta}{2\pi}\geqslant\frac{\kappa n}{K\delta}\right\}\right)\leqslant\frac{\delta K}{2}$$

and

$$\mathbb{E}^{(n)}_{SU(N)}\left(\mathrm{card}\left\{(j,k)\in[\![1,n]\!]\times[\![0,K-1]\!]\,:\,\int_{\theta_k}^{\theta_{k+1}}\int_0^{2\pi M}\Delta_\mu I_\theta^{(j)}\frac{d\mu}{2\pi M}\frac{d\theta}{2\pi}\geqslant\frac{\kappa n}{K\delta}\right\}\right)\leqslant\frac{\delta K}{2}.$$

Setting

$$\mathscr{N}_1 := \bigcap_{j=1}^{n}\left\{k\in[\![0,K-1]\!]\;:\;\int_{\theta_k}^{\theta_{k+1}}\int_0^{2\pi M}\Delta_\mu R_\theta^{(j)}\frac{d\mu}{2\pi M}\frac{d\theta}{2\pi}\leqslant\frac{\kappa n}{K\delta},\;\int_{\theta_k}^{\theta_{k+1}}\int_0^{2\pi M}\Delta_\mu I_\theta^{(j)}\frac{d\mu}{2\pi M}\frac{d\theta}{2\pi}\leqslant\frac{\kappa n}{K\delta}\right\}$$

we thus get

$$\mathbb{E}^{(n)}_{SU(N)}(|\mathscr{N}_1|)\geqslant(1-\delta)K. \tag{17}$$

Now, for $k\in\mathscr{N}_1$, let us set

$$\mathscr{G}_k := [\theta_k,\theta_{k+1})\cap\bigcap_{j=1}^{n}\left\{\int_0^{2\pi M}\Delta_\mu R_\cdot^{(j)}\frac{d\mu}{2\pi M}\leqslant\frac{2\kappa n^2}{\delta^2},\int_0^{2\pi M}\Delta_\mu I_\cdot^{(j)}\frac{d\mu}{2\pi M}\leqslant\frac{2\kappa n^2}{\delta^2}\right\}.$$

Applying Markov's inequality again, we get that $\mathbb{P}_{SU(N)}^{(n)}$-a.s. we have

$$\lambda_{2\pi}\left(\mathscr{G}_k\right) \geqslant (1-\delta)/K. \tag{18}$$

□

Definition 3.11 (Good indices)**.** An index $k \in [\![0, K-1]\!]$ is said to be **good** if:

(1) $k \in \mathscr{N}_1$,
(2) $\mathscr{E}_\delta^c \cap \mathscr{G}_k \cap [\theta_k, \theta_k + \sqrt{\delta}\Delta) \neq \emptyset$.

We denote by $\mathscr{N}_2$ the set of good indices, that is

$$\mathscr{N}_2 := \left\{ k \in \mathscr{N}_1 \Big/ \mathscr{E}_\delta^c \cap \mathscr{G}_k \cap [\theta_k, \theta_k + \sqrt{\delta}\Delta) \neq \emptyset \right\}. \tag{19}$$

An index is said to be **bad** if it is not good.

Lemma 3.12. *With the notation above, the set of good indices satisfies*

$$\mathbb{E}_{SU(N)}^{(n)}(|\mathscr{N}_2|) = K\left(1 - O\left(\sqrt{\delta}\log(1/\delta)\right)\right),$$

where the implied constant in the $O(\cdot)$ symbol depends only on n.

Proof. If $k \in \mathscr{N}_2^c$, either $k \in \mathscr{N}_1^c$, or $k \in \mathscr{N}_1$ and $\mathscr{E}_\delta^c \cap \mathscr{G}_k \cap [\theta_k, \theta_k + \sqrt{\delta}\Delta) = \emptyset$. This is, $\mathscr{N}_2^c = \mathscr{N}_1^c \cup \widetilde{\mathscr{N}_1}$ where

$$\widetilde{\mathscr{N}_1} := \left\{ k \in \mathscr{N}_1 \,:\, \mathscr{G}_k \cap [\theta_k, \theta_k + \sqrt{\delta}\Delta) \subset \mathscr{E}_\delta \right\}.$$

By (17), we have $\mathbb{E}_{SU(N)}^{(n)}(|\mathscr{N}_1^c|) \leqslant \delta K$.

For all $k \in \widetilde{\mathscr{N}_1}$, we have $\mathscr{E}_\delta \supset \mathscr{G}_k \cap [\theta_k, \theta_k + \sqrt{\delta}\Delta)$, i.e. $\mathscr{E}_\delta \supset \bigcup_{k\in\widetilde{\mathscr{N}_1}} \mathscr{G}_k \cap [\theta_k, \theta_k + \sqrt{\delta}\Delta)$, where the union is disjoint, and thus, $\lambda_{2\pi}\left(\mathscr{E}_\delta\right) \geqslant \left|\widetilde{\mathscr{N}_1}\right| \min_k \lambda_{2\pi}\left(\mathscr{G}_k \cap [\theta_k, \theta_k + \sqrt{\delta}\Delta)\right)$.

By (18) we have, $\mathbb{P}_{SU(N)}^{(n)}$-a.s.,

$$\begin{aligned}\lambda_{2\pi}\left(\mathscr{G}_k \cap [\theta_k, \theta_k + \sqrt{\delta}\Delta)\right) &\geqslant \lambda_{2\pi}\left(\mathscr{G}_k\right) + \lambda_{2\pi}\left([\theta_k, \theta_k + \sqrt{\delta}\Delta)\right) - \lambda_{2\pi}\left([\theta_k, \theta_k + \Delta)\right) \\ &\geqslant \frac{1}{K}\left((1-\delta) + \sqrt{\delta} - 1\right).\end{aligned}$$

Now, since $\delta < 1/4$, we obtain $\mathbb{P}_{SU(N)}^{(n)}$-a.s.

$$\lambda_{2\pi}\left(\mathscr{G}_k \cap [\theta_k, \theta_k + \sqrt{\delta}\Delta)\right) \geqslant \frac{\sqrt{\delta}}{2K}. \tag{20}$$

This implies that $\mathbb{P}_{SU(N)}^{(n)}$-a.s.

$$\left|\widetilde{\mathscr{N}_1}\right| \leqslant \frac{2K}{\sqrt{\delta}} \lambda_{2\pi}\left(\mathscr{E}_\delta\right).$$

Now, by Lemma (3.8), $\mathbb{E}_{SU(N)}^{(n)}\left(\left|\widetilde{\mathscr{N}_1}\right|\right) = O\left(K\sqrt{\delta}\log(1/\delta)\right)$ and so

$$\mathbb{E}_{SU(N)}^{(n)}(|\mathscr{N}_2^c|) \leqslant \mathbb{E}_{SU(N)}^{(n)}(|\mathscr{N}_1^c|) + \mathbb{E}_{SU(N)}^{(n)}\left(\left|\widetilde{\mathscr{N}_1}\right|\right) \leqslant \delta K + O\left(K\sqrt{\delta}\log(1/\delta)\right). \qquad \square$$

3.8. **Speed of the good oscillation of the logarithm of the characteristic polynomials.**

Lemma 3.13. *With the notation above, and $\mathbb{P}^{(n)}_{SU(N)}$-a.s., $\forall\, k \in \mathscr{N}_2$, there exists a random set $\mathscr{Y}_k \subset [0, 2\pi M]$, and $\theta_k^* \in \mathscr{E}_\delta^c \cap \mathscr{G}_k \cap [\theta_k, \theta_k + \sqrt{\delta}\Delta)$, such that*

$$\lambda_{2\pi M}\left(\mathscr{Y}_k\right) = 1 - O\left(\delta^{-2}(\log N)^{-1/4}(\log M)^{1/2}\right),$$

where $\lambda_{2\pi M}$ is $1/(2\pi M)$ times the Lebesgue measure, and for all $j \in [\![1, n]\!]$, $\mu \in \mathscr{Y}_k$,

$$\Delta_\mu R^{(j)}_{\theta_k^*} = O\left(\frac{(\log N)^{1/4}}{(\log M)^{1/2}}\right) \text{ and } \Delta_\mu I^{(j)}_{\theta_k^*} = O\left(\frac{(\log N)^{1/4}}{(\log M)^{1/2}}\right). \tag{21}$$

Again, the implied constant in the $O(\cdot)$ symbol depends only on n.

Proof. Let $k \in \mathscr{N}_2$ and $\theta_k^* \in \mathscr{E}_\delta^c \cap \mathscr{G}_k \cap [\theta_k, \theta_k + \sqrt{\delta}\Delta)$. Set

$$\mathscr{Y}_k := \bigcap_{j=1}^{n} \left\{ \Delta_\cdot R^{(j)}_{\theta_k^*} \leqslant \varepsilon,\ \Delta_\cdot I^{(j)}_{\theta_k^*} \leqslant \varepsilon \right\}$$

where

$$\varepsilon := \frac{(\log N)^{1/4}}{(\log M)^{1/2}}.$$

Applying Markov's inequality, we get

$$\begin{aligned}
\lambda_{2\pi M}\left(\mathscr{Y}_k^c\right) &\leqslant \lambda_{2\pi M}\left(\bigcup_{j=1}^{n} \left\{ \Delta_\cdot R^{(j)}_{\theta_k^*} \geqslant \varepsilon \right\}\right) + \lambda_{2\pi M}\left(\bigcup_{j=1}^{n} \left\{ \Delta_\cdot I^{(j)}_{\theta_k^*} \geqslant \varepsilon \right\}\right) \\
&\leqslant \frac{2n}{\varepsilon} \max_{1 \leqslant j \leqslant n} \left(\int_0^{2\pi M} \Delta_\mu R^{(j)}_{\theta_k^*} \frac{d\mu}{2\pi M} \vee \int_0^{2\pi M} \Delta_\mu I^{(j)}_{\theta_k^*} \frac{d\mu}{2\pi M} \right) = \frac{1}{\varepsilon} O\left(\delta^{-2}\right),
\end{aligned}$$

by (15), which gives the announced result. □

3.9. **The number of sign changes.**

Let us go back to Theorem 1.1. We need to estimate the number of zeros of F_N on the unit circle, or equivalently, the number of values of $\theta \in J$ such that the following quantity vanishes:

$$i^N e^{iN\theta/2} F_N(e^{-i\theta}) = i^N e^{iN\theta/2} \sum_{j=1}^{n} b_j \Phi_{U_{N,j}}(e^{-i\theta}) = \sum_{j=1}^{n} b_j i^N e^{iN\theta/2} Z_{U_{N,j}}(\theta). \tag{22}$$

Using the fact that $U_{N,j} \in SU(N)$, one checks that $i^N e^{iN\theta/2} Z_{U_{N,j}}(\theta)$ is real, and then the number of zeros of F_N on the unit circle is bounded from below by the number of sign changes, when θ increases from θ_0 to $\theta_0 + 2\pi$, of the real quantity given by the right-hand side of (22). Now, the order of magnitude of $\log|Z_{U_{N,j}}(\theta)|$ is $\sqrt{\log N}$ and more precisely, Lemma 3.8 informally means that for most values of θ, the values of $\log|Z_{U_{N,j}}(\theta)|$ for $1 \leqslant j \leqslant n$ are pairwise separated by an interval of length of order $\sqrt{\log N}$. Hence, one of the terms in the sum at the right-hand side of (22) should dominate all the others. If j_0 is the corresponding index, one can expect that the sign changes of (22) can, at least locally,

be related to the corresponding sign changes of $i^N e^{iN\theta/2} Z_{U_{N,j_0}}(\theta)$, which are associated to the zeros of the characteristic polynomial $Z_{U_{N,j_0}}$. This should give a lower bound on the number of sign changes of (22).

This informal discussion motivates the following definition.

Definition 3.14. With the notation of the previous subsections, for all $k \in \mathscr{N}_2$, we define the **carrier wave index** by:

$$j_k := \operatorname{Arg}\max_j \left\{ \operatorname{Re}\log Z_{X_j}(\theta_k^*) \right\},$$

where θ_k^* is the random angle introduced in Lemma 3.13. Moreover, to each $k \in \mathscr{N}_2$ we associate the interval

$$J_k := \left[\theta_k^*, \theta_k^* + (1-\sqrt{\delta})\Delta \right].$$

As $\theta_k^* \in \mathscr{O}_\delta^c$, we have that for all $j \neq j_k$, $\operatorname{Re}\log Z_{X_j}(\theta_k^*) \leqslant \operatorname{Re}\log Z_{X_{j_k}}(\theta_k^*) - \frac{\delta}{\sqrt{2}}\sqrt{\log N}$. From (21), we deduce that for all $j \neq j_k$ and for all $\mu \in \mathscr{Y}_k$ we have

$$\operatorname{Re}\log Z_{X_j}\left(\theta_k^* + \frac{\mu}{N}\right) \leqslant \operatorname{Re}\log Z_{X_{j_k}}\left(\theta_k^* + \frac{\mu}{N}\right) - \frac{\delta}{\sqrt{2}}(\log N)^{1/2} + O\left((\log N)^{1/4}\right). \quad (23)$$

Now, since

$$1/\delta = O((\log N)^{1/10}),$$

with a universal implied constant, we then get, for a universal $c > 0$,

$$\frac{\left|Z_{X_j}\left(\theta_k^* + \frac{\mu}{N}\right)\right|}{\left|Z_{X_{j_k}}\left(\theta_k^* + \frac{\mu}{N}\right)\right|} \leqslant \exp\left(-2c(\log N)^{4/10} + O\left((\log N)^{1/4}\right)\right) \leqslant \exp\left(-c(\log N)^{4/10}\right),$$

for N large enough, depending only on n. This implies

$$\begin{aligned}\left|\sum_{j\neq j_k} b_j Z_{X_j}\left(\theta_k^* + \frac{\mu}{N}\right)\right| &\leqslant \frac{\sum_j |b_j|}{\min_j |b_j|}\left|b_{j_k} Z_{X_{j_k}}\left(\theta_k^* + \frac{\mu}{N}\right)\right| \exp\left(-c(\log N)^{4/10}\right) \\ &\leqslant \frac{1}{2}\left|b_{j_k} Z_{X_{j_k}}\left(\theta_k^* + \frac{\mu}{N}\right)\right|\end{aligned}$$

for $N \geqslant N_0$, where N_0 depends only on $n, b_1, \ldots, b_n$. Hence, for $k \in \mathscr{N}_2$, $\mu \in \mathscr{Y}_k$ and $\theta = \theta_k^* + \mu/N$, the quantity

$$G(\theta) := \sum_{j=1}^n b_j i^N e^{iN\theta/2} Z_{X_j}(\theta),$$

which is $\mathbb{P}_{SU(N)}^{(n)}$-a.s. real, has the same sign as its term of index j_k.

Theorem 1.1 is proven if we show that the expectation of number of sign changes of $G(\theta)$ for $\theta \in J$, under $\mathbb{P}_{SU(N)}^{(n)}$, is bounded from below by $N - o(N)$. Hence, it is sufficient to get

$$\mathbb{E}_{SU(N)}^{(n)}\left(\sum_{k\in\mathscr{N}_2} \mathcal{S}_k\right) \geqslant N - o(N),$$

where $\mathcal{S}_k$ is the number of sign changes of $b_{j_k} i^N e^{iN\theta/2} Z_{X_{j_k}}(\theta)$, for $\theta \in J_k \cap \{\theta_k^* + \frac{\mu}{N}, \mu \in \mathscr{Y}_k\}$.

Now, for $k \in \mathscr{N}_2$, let $\alpha_{k,1} \leqslant \alpha_{k,2} \leqslant \cdots \leqslant \alpha_{k,\nu_k}$ be the eigenangles, counted with multiplicity, of X_{j_k} in the interval J_k. The sign of $b_j i^N e^{iN\theta/2} Z_{X_{j_k}}$ alternates between the different intervals $(\alpha_{k,1}, \alpha_{k,2}), (\alpha_{k,2}, \alpha_{k,3}), \ldots, (\alpha_{k,\nu_k - 1}, \alpha_{k,\nu_k})$. Hence, for each pair of consecutive intervals containing an angle $\theta = \theta_k^* + \frac{\mu}{N}, \mu \in \mathscr{Y}_k$, we get a contribution of at least 1 for the quantity $\mathcal{S}_k$.

Every element of J_k can be written as $\theta_k^* + \frac{\mu}{N}$, for

$$0 \leqslant \mu \leqslant (1 - \sqrt{\delta}) N\Delta \leqslant N\Delta = 2\pi M.$$

The Lebesgue measure of the elements of J_k for which $\mu \notin \mathscr{Y}_k$ is then bounded by

$$\frac{1}{N} \lambda(\mathscr{Y}_k^c) = \frac{2\pi M}{N} \lambda_{2\pi M}(\mathscr{Y}_k^c),$$

where λ denotes the standard Lebesgue measure. Hence, if an interval $(\alpha_{k,\nu}, \alpha_{k,\nu+1})$ has a length strictly greater than this bound, it necessarily contains some $\theta = \theta_k^* + \frac{\mu}{N}$ for which $\mu \in \mathscr{Y}_k$. For some $c' > 0$ depending only on n, this condition is implied by

$$\alpha_{k,\nu+1} - \alpha_{k,\nu} > c' \frac{M}{N} \delta^{-2} (\log N)^{-1/4} (\log M)^{1/2}.$$

We will say that $(\alpha_{k,\nu}, \alpha_{k,\nu+1})$ is a **roomy gap** if this inequality is satisfied, and a **narrow gap** if

$$\alpha_{k,\nu+1} - \alpha_{k,\nu} \leqslant c' \frac{M}{N} \delta^{-2} (\log N)^{-1/4} (\log M)^{1/2}.$$

By the previous discussion, $\mathcal{S}_k$ is at least the number of pairs of consecutive roomy gaps among the intervals $(\alpha_{k,1}, \alpha_{k,2}), (\alpha_{k,2}, \alpha_{k,3}), \ldots, (\alpha_{k,\nu_k - 1}, \alpha_{k,\nu_k})$. If there is no narrow gap, the number of such pairs is $(\nu_k - 2)_+ \geqslant \nu_k - 2$. Moreover, if among the intervals, we replace a roomy gap by a narrow gap, this removes at most two pairs of consecutive roomy gaps. Hence, we deduce, for all $k \in \mathscr{N}_2$, that

$$\mathcal{S}_k \geqslant \nu_k - 2 - 2\psi_k,$$

where ν_k is the number of zeros of $Z_{X_{j_k}}$ in the interval J_k and ψ_k the number of narrow gaps among these zeros. Hence, we get the lower bound:

$$\mathbb{E}_{SU(N)}^{(n)} \left(\sum_{k \in \mathscr{N}_2} \mathcal{S}_k \right) \geqslant \mathbb{E}_{SU(N)}^{(n)} \left(\sum_{k \in \mathscr{N}_2} \nu_k - 2K - 2\psi \right), \tag{24}$$

where ψ is the total number of narrow gaps among the zeros in $[0, 2\pi)$ of all the functions $(Z_j)_{1 \leqslant j \leqslant n}$.

Now, $\mathbb{P}_{SU(N)}^{(n)}$-a.s., for all $k \in \mathscr{N}_2$, we have

$$\begin{aligned}
\nu_k &= \left|\{\theta \in \left[\theta_k^*, \theta_k^* + (1-\sqrt{\delta})\Delta\right], Z_{j_k}(\theta) = 0\}\right| \\
&\geqslant \left|\{\theta \in \left[\theta_k + \sqrt{\delta}\Delta, \theta_k + (1-\sqrt{\delta})\Delta\right], Z_{j_k}(\theta) = 0\}\right| \\
&= \frac{N(1-2\sqrt{\delta})\Delta}{2\pi} + \frac{1}{\pi}\left(\operatorname{Im}\log Z_{X_{j_k}}(\theta_k + (1-\sqrt{\delta})\Delta) - \operatorname{Im}\log Z_{X_{j_k}}(\theta_k + \sqrt{\delta}\Delta)\right) \\
&\geqslant \frac{N}{K}(1-2\sqrt{\delta}) - \frac{1}{\pi}\sum_{j=1}^{n}\left|\left(\operatorname{Im}\log Z_{X_j}(\theta_k + (1-\sqrt{\delta})\Delta) - \operatorname{Im}\log Z_{X_j}(\theta_k + \sqrt{\delta}\Delta)\right)\right|,
\end{aligned}$$

the second equality coming from Proposition 2.5.

Adding this inequality for all $k \in \mathscr{N}_2$, and taking the expectation yields the estimate

$$\begin{aligned}
\mathbb{E}_{SU(N)}^{(n)}\left(\sum_{k\in\mathscr{N}_2}\nu_k\right) &\geqslant \frac{N}{K}(1-2\sqrt{\delta})\mathbb{E}_{SU(N)}^{(n)}(|\mathscr{N}_2|) \\
&\quad - \sum_{j=1}^{n}\sum_{k=0}^{K-1}\mathbb{E}_{SU(N)}^{(n)}\left[\left|\left(\operatorname{Im}\log Z_{X_j}(\theta_k + (1-\sqrt{\delta})\Delta) - \operatorname{Im}\log Z_{X_j}(\theta_k + \sqrt{\delta}\Delta)\right)\right|\right]
\end{aligned}$$

and using (14) to estimate the terms on the right-hand side we therefore conclude that

$$\begin{aligned}
\mathbb{E}_{SU(N)}^{(n)}\left(\sum_{k\in\mathscr{N}_2}\nu_k\right) &\geqslant \frac{N}{K}(1-2\sqrt{\delta})K(1 - O(\sqrt{\delta}\log(1/\delta)) + O(K\sqrt{\log M}) \\
&\geqslant N(1 - O(\sqrt{\delta}\log(1/\delta)) + O\left(\frac{N\sqrt{\log M}}{M}\right). \qquad (25)
\end{aligned}$$

It remains to estimate

$$\mathbb{E}_{SU(N)}^{(n)}[2\psi] = 2n\mathbb{E}_{SU(N)}[\chi] = 2n\mathbb{E}_{U(N)}[\chi],$$

where χ denotes the number of narrow gaps between the eigenvalues of the canonical unitary matrix X. The replacement of $SU(N)$ by $U(N)$ is possible since the notion of narrow gap is invariant by rotation of the eigenvalues. Note that this estimate of the number of narrow gaps is the analog of the assumption made by Bombieri and Hejhal on the proportion of small gaps between zeros of L-functions. Now, the last expectation can be estimated by the following result:

Lemma 3.15. *For $N \geqslant 1$ and $\epsilon > 0$, let U be a Haar-distributed matrix on $U(N)$ and let χ_ε be the number of pairs of eigenvalues of U whose argument differ by at most ε/N. Then, $\mathbb{E}[\chi_\varepsilon] = O(N\varepsilon^3)$.*

Proof. For $\theta_1, \theta_2 \in \mathbb{R}$, the two-point correlation density of the eigenvalues of U at $e^{i\theta_1}$ and $e^{i\theta_2}$, with respect to the Haar probability measure on the unitary group, is given by

$$\rho(e^{i\theta_1}, e^{i\theta_2}) = N^2 \left[1 - \left(\frac{\sin[N(\theta_2-\theta_1)/2]}{N\sin[(\theta_2-\theta_1)/2]}\right)^2\right].$$

Now,

$$N|\sin[(\theta_2-\theta_1)/2]| \leqslant N|\theta_2-\theta_1|/2$$

and then

$$\left(\frac{\sin[N(\theta_2-\theta_1)/2]}{N\sin[(\theta_2-\theta_1)/2]}\right)^2 \geqslant \left(\frac{\sin x}{x}\right)^2$$

for $x = N(\theta_2-\theta_1)/2$. Now, for all $x \in \mathbb{R}$, $|\sin x| \geqslant \sin|x| \geqslant |x| - |x|^3/6$, which implies

$$\left(\frac{\sin x}{x}\right)^2 \geqslant \left(1 - \frac{x^2}{6}\right)^2 \geqslant 1 - \frac{x^2}{3}$$

and

$$\rho(e^{i\theta_1}, e^{i\theta_2}) \leqslant N^2\left[1 - \left(\frac{\sin x}{x}\right)^2\right] \leqslant \frac{N^2x^2}{3} = \frac{N^4(\theta_2-\theta_1)^2}{6}.$$

Integrating the correlation function for $\theta_1 \in [0, 2\pi)$ and $\theta' := \theta_2 - \theta_1 \in [-\varepsilon/N, \varepsilon/N]$ gives

$$\mathbb{E}[\chi_\varepsilon] \leqslant \int_0^{2\pi} \frac{d\theta}{2\pi} \int_{-\varepsilon/N}^{\varepsilon/N} \frac{d\theta'}{2\pi} \frac{N^4(\theta')^2}{6} \leqslant N^4 \int_{-\varepsilon/N}^{\varepsilon/N} (\theta')^2 d\theta' = O\left(N^4(\varepsilon/N)^3\right). \qquad \square$$

From this result, applied for

$$\varepsilon = c' M\delta^{-2}(\log N)^{-1/4}(\log M)^{1/2}$$

we get the estimate

$$\mathbb{E}^{(n)}_{SU(N)}[2\psi] = O(N\varepsilon^3) = O\left(NM^3\delta^{-6}(\log N)^{-3/4}(\log M)^{3/2}\right). \tag{26}$$

Substituting the estimates (25) and (26) into (24), and recalling that $M := N/K$ so $2K = O(N/M)$, we find that

$$\mathbb{E}^{(n)}_{SU(N)}\left(\sum_{k\in\mathscr{N}_2} \mathcal{S}_k\right) \geqslant N\left[1 - O\left(\sqrt{\delta}\log(1/\delta) + \frac{\sqrt{\log M}}{M} + M^3\delta^{-6}(\log N)^{-3/4}(\log M)^{3/2}\right)\right].$$

From the values taken for δ and M, we get

$$\sqrt{\delta}\log(1/\delta) = O\left((\log N)^{-3/64}\log\log N\right),$$

$$\frac{\sqrt{\log M}}{M} = O\left(\sqrt{\log\log N}(\log N)^{-3/64}\right)$$

and

$$M^3\delta^{-6}(\log N)^{-3/4}(\log M)^{3/2} = O\left((\log N)^{9/64}(\log N)^{18/32}(\log N)^{-3/4}(\log\log N)^{3/2}\right)$$
$$= O\left((\log N)^{-3/64}(\log\log N)^{3/2}\right).$$

Finally, we get

$$\mathbb{E}_{SU(N)}^{(n)}\left(\sum_{k\in\mathscr{N}_2}\mathcal{S}_k\right) = N\left(1 - O\left((\log N)^{-1/22}\right)\right),$$

which completes the proof of Theorem 1.1. □

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