

This is a repository copy of *Probability distribution for vacuum energy flux fluctuations in two spacetime dimensions*.

White Rose Research Online URL for this paper:

<https://eprints.whiterose.ac.uk/224826/>

Version: Published Version

---

**Article:**

Fewster, Christopher [orcid.org/0000-0001-8915-5321](https://orcid.org/0000-0001-8915-5321) and Ford, L. H. (2025) Probability distribution for vacuum energy flux fluctuations in two spacetime dimensions. *Physical Review D*. 085015. ISSN 2470-0029

<https://doi.org/10.1103/PhysRevD.111.085015>

---

**Reuse**

This article is distributed under the terms of the Creative Commons Attribution (CC BY) licence. This licence allows you to distribute, remix, tweak, and build upon the work, even commercially, as long as you credit the authors for the original work. More information and the full terms of the licence here:

<https://creativecommons.org/licenses/>

**Takedown**

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing [eprints@whiterose.ac.uk](mailto:eprints@whiterose.ac.uk) including the URL of the record and the reason for the withdrawal request.

# Probability distribution for vacuum energy flux fluctuations in two spacetime dimensions

Christopher J. Fewster<sup>\*</sup>

*Department of Mathematics, University of York, Heslington, York YO10 5DD, United Kingdom  
and York Centre for Quantum Technologies, University of York,  
Heslington, York YO10 5DD, United Kingdom*

L. H. Ford<sup>†</sup>

*Institute of Cosmology, Department of Physics and Astronomy, Tufts University,  
Medford, Massachusetts 02155, USA*



(Received 4 December 2024; accepted 20 March 2025; published 9 April 2025)

The probability distribution for vacuum fluctuations of the energy flux in two dimensions is constructed, along with the joint distribution of energy flux and energy density. Our approach is based on previous work on probability distributions for the energy density in two-dimensional conformal field theory. In both cases, the relevant stress tensor component must be averaged in time, and the results are sensitive to the form of the averaging function. Here we present results for two classes of such functions, which include the Gaussian and Lorentzian functions. The distribution for the energy flux is symmetric, unlike that for the energy density. In both cases, the distribution may possess an integrable singularity. The functional form of the flux distribution function involves a modified Bessel function and is distinct from the shifted Gamma form for the energy density. By considering the joint distribution of energy flux and energy density, we show that the distribution of energy flux tends to be more centrally concentrated than that of the energy density. We also determine the distribution of energy fluxes, conditioned on the energy density being negative. Some applications of the results are also discussed.

DOI: [10.1103/PhysRevD.111.085015](https://doi.org/10.1103/PhysRevD.111.085015)

## I. INTRODUCTION

This paper deals with some exact solutions for the vacuum state probability distribution for the fluctuations of the energy flux in two-dimensional conformal quantum field theories. The vacuum expectation value of the flux vanishes, but an individual measurement can return a nonzero result, which is equally likely to be positive or negative. Furthermore, the probability of finding a given magnitude for the flux is independent of its sign, so the distribution will be symmetric. This work extends previous exact results for the energy density in two dimensions [1–4], and is related to approximate results in four dimensions [5–10]. The flux operator must be averaged in time to be well defined, and the averaging function describes the details of a physical measurement. Here we consider two classes of averaging

functions which have previously been used for the energy density; these include the Gaussian and Lorentzian functions as special cases. We also consider the joint distribution of energy flux and energy density and, by computing the probability that the absolute value of the flux is less than the absolute value of energy density, show that the energy flux is the typically more centrally concentrated of the two. Furthermore, we show that the distribution of fluxes changes markedly when conditioned on the energy density being negative, despite the fact that the probability for obtaining a negative energy density is typically greater than 1/2. Some applications of the results to four-dimensional models, numerical simulations of quantum fluctuations, and analog models in condensed matter systems are also discussed.

## II. VACUUM ENERGY DENSITY

Here we review results from Refs. [1–3] for the probability distributions for the energy density in (unitary, positive energy) conformal field theories (CFTs) including the massless free scalar field as a special case. The stress tensor for such a theory is determined by right-moving and left-moving components,  $T_R(u)$  and  $T_L(v)$ , which have the same central charge assuming parity invariance. Here  $u = t - x$  and  $v = t + x$  are null coordinates on Minkowski

<sup>\*</sup>Contact author: [chris.fewster@york.ac.uk](mailto:chris.fewster@york.ac.uk)

<sup>†</sup>Contact author: [ford@cosmos.phy.tufts.edu](mailto:ford@cosmos.phy.tufts.edu)

*Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP<sup>3</sup>.*

spacetime. These components have a well-defined probability distribution only if they have been averaged by a suitable sampling function,  $f(u)$  or  $f(v)$ , conveniently normalized so that

$$\int_{-\infty}^{\infty} f(u) du = 1. \quad (2.1)$$

Let  $\omega_R$  denote the dimensionless averaged operator

$$\omega_R = \tau^2 \bar{T}_R = \tau^2 \int_{-\infty}^{\infty} f(u) T_R(u) du, \quad (2.2)$$

and define  $\omega_L$  and  $\bar{T}_L$  similarly in terms of  $T_L(v)$ . Here  $\tau$  is a timescale, for example a characteristic width of the sampling function. We are interested in the probability distribution  $P_{L,R}(\omega)$  for the outcomes of individual measurements of  $\omega_{L,R}$  in the vacuum state of the theory. Explicit results [1–3] have been found for specific choices of  $f$ , including generalizations of the Gaussian and Lorentzian functions. In all these cases, the probability distributions are of the form of a shifted Gamma distribution,

$$P_{L,R}(\omega) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta(\omega + \omega_0) e^{-\beta(\omega + \omega_0)} (\omega + \omega_0)^{\alpha-1}, \quad (2.3)$$

with dimensionless positive parameters  $\omega_0$ ,  $\alpha$ , and  $\beta$  which depend upon the choice of  $f(u)$  and the central charge of the CFT. Note that our notation differs slightly from that of Refs. [1–3], where  $\omega$  and  $\omega_0$  have dimensions of length<sup>-2</sup> and  $\beta$  has dimensions of length<sup>2</sup>. Features of note are that  $P_{L,R}$  vanishes for  $\omega < -\omega_0$ , which is also the quantum inequality bound for a null component of energy density [1,4], and displays an integrable singularity as  $\omega \rightarrow -\omega_0 +$  if  $\alpha < 1$ . The probability of obtaining a negative result,  $\int_{-\omega_0}^0 P_{L,R}(\omega) d\omega$  is (often substantially) greater than the probability  $\int_0^\infty P_{L,R}(\omega) d\omega$  of a positive result, but the mean of the distribution is zero, in agreement with the expectation of  $\omega_{L,R}$  in the vacuum state.

The energy density is the sum of the right-moving and left-moving components:

$$\rho = T_R + T_L. \quad (2.4)$$

Because these components are decoupled from one another (see Sec. IV) and have the same central charge, their vacuum fluctuations are identical and independent, and the energy density probability distribution is a convolution of  $P_R$  and  $P_L$ :

$$P_\rho(x) = \int_{-\infty}^{\infty} d\omega P_L(\omega) P_R(x - \omega). \quad (2.5)$$

Here  $x = \rho\tau^2$  is the dimensionless averaged energy density. Explicit evaluation of the above integral using Eq. (2.3) leads to

$$P_\rho(x) = \frac{\beta^{2\alpha}}{\Gamma(2\alpha)} \theta(x + 2\omega_0) e^{-\beta(x + 2\omega_0)} (x + 2\omega_0)^{2\alpha-1}. \quad (2.6)$$

In effect, the parameters  $\omega_0$  and  $\alpha$  have been doubled, reflecting the fact that these are proportional to the central charge of the conformal field theory.

Note that  $P_\rho(x) = 0$  for  $x < -2\omega_0$ , the lower bound on the allowed averaged energy density in the vacuum state [4]. Also note that  $P_\rho(x)$  decays exponentially for large  $x$ , and displays an integrable singularity at the lower bound  $-2\omega_0$  if  $0 < \alpha < 1/2$ . Although this result is specific to two-dimensional spacetime, some of its features, such as the existence of a lower bound on the energy density probability distribution, also apply in four dimensions. More generally, the explicit two-dimensional results can be a guide to possible effects in four dimensions. For example, they were used by Carlip *et al.* [11,12] to explore effects near a cosmological singularity.

### III. VACUUM ENERGY FLUX

The dimensionless averaged energy flux operator is determined by an off-diagonal component of the averaged stress tensor

$$F = \tau^2 \bar{T}^{tx} = \tau^2 (\bar{T}_R - \bar{T}_L). \quad (3.1)$$

The probability distribution for vacuum energy flux fluctuation may be constructed as a convolution, in analogy with that for the energy density:

$$\begin{aligned} P_F(F) &= \int_{-\infty}^{\infty} d\omega P_L(\omega) P_R(-F + \omega) \\ &= \int_{-\infty}^{\infty} d\omega P_L(\omega + F/2) P_R(\omega - F/2). \end{aligned} \quad (3.2)$$

This is an even function of  $F$ , as follows from physical grounds and the fact that  $P_R = P_L$ . We now proceed to use Eq. (2.3) to find it explicitly in cases where  $P_L$  and  $P_R$  are shifted Gamma distributions:

$$\begin{aligned}
P_F(F) &= \frac{\beta^{2\alpha} e^{-2\beta\omega_0}}{\Gamma(\alpha)^2} \int_{-\infty}^{\infty} d\omega \theta(\omega + F/2 + \omega_0) \theta(\omega - F/2 + \omega_0) e^{-2\beta\omega} [(\omega + \omega_0)^2 - F^2/4]^{\alpha-1} \\
&= \frac{\beta^{2\alpha} e^{-2\beta\omega_0}}{\Gamma(\alpha)^2} \int_{|F|/2 - \omega_0}^{\infty} d\omega e^{-2\beta\omega} [(\omega + \omega_0)^2 - F^2/4]^{\alpha-1} \\
&= \frac{\beta^{2\alpha}}{\Gamma(\alpha)^2} \int_{|F|/2}^{\infty} d\omega e^{-2\beta\omega} (\omega^2 - F^2/4)^{\alpha-1} \\
&= \frac{\beta^{2\alpha}}{\Gamma(\alpha)^2} (|F|/2)^{2\alpha-1} \int_1^{\infty} d\eta e^{-\beta|F|\eta} (\eta^2 - 1)^{\alpha-1}.
\end{aligned} \tag{3.3}$$

Note that  $P_F(F)$  is independent of the parameter  $\omega_0$ . Using standard formulas (see 3.387.3 in [13] or 10.32.8 in [14]) we may evaluate the integral as a modified Bessel function to find

$$P_F(F) = \frac{\beta(\beta|F|/2)^{\alpha-1/2}}{\sqrt{\pi}\Gamma(\alpha)} K_{\alpha-1/2}(\beta|F|). \tag{3.4}$$

Here the constants  $\alpha$  and  $\beta$  depend upon the specific choice of sampling function and the assumption that  $P_L = P_R$  are shifted Gamma distributions. The distribution  $P_F$  is a centered symmetric variance-gamma distribution in which  $\alpha$  is the shape parameter and  $\beta$  is an inverse width scale; in fact, it has been known for a long time that the difference of two identically distributed Gamma distributions is distributed in this way [15], and the same applies to the shifted Gamma case.

The asymptotic probability distribution for large argument is

$$\begin{aligned}
P_F(F) &\sim \frac{\beta}{2\Gamma(\alpha)} \left(\frac{\beta|F|}{2}\right)^{\alpha-1} e^{-\beta|F|} [1 + O((\beta|F|)^{-1})], \\
\beta|F| &\gg 1.
\end{aligned} \tag{3.5}$$

Thus, the probability of a large fluctuation decreases exponentially with the same decay constant as for  $P_\rho$ .

Near the origin, we can use the asymptotic formula

$$(z/2)^\nu K_\nu(z) \sim \begin{cases} \frac{1}{2}\Gamma(\nu) & \nu > 0 \\ -\log(z/2) & \nu = 0 \\ \frac{1}{2}\Gamma(-\nu)(z/2)^{2\nu} & \nu < 0, \end{cases} \tag{3.6}$$

to see that the distribution satisfies

$$P_F(F) \sim \frac{\beta(\beta|F|/2)^{2\alpha-1}\Gamma(1/2-\alpha)}{2\sqrt{\pi}\Gamma(\alpha)} \tag{3.7}$$

for  $\alpha < 1/2$ ,

$$P_F(F) \sim \frac{\beta}{\pi} \log(\beta|F|/2) \tag{3.8}$$

for  $\alpha = 1/2$ , and

$$P_F(F) \sim \frac{\beta\Gamma(\alpha-1/2)}{2\sqrt{\pi}\Gamma(\alpha)} \tag{3.9}$$

for  $\alpha > 1/2$ . In particular,  $P_F$  has an integrable singularity at  $F = 0$  for  $\alpha \leq 1/2$ , and is continuous for  $\alpha > 1/2$ .

The variance of the flux fluctuations is

$$\langle F^2 \rangle = \int_{-\infty}^{\infty} dF F^2 P_F(F) = \frac{2\alpha}{\beta^2}, \tag{3.10}$$

as can be seen using the integral identity 6.561.16 in [13] or 10.43.19 in [14]. It is also of interest to examine the cumulative distribution function

$$P_{<}(F) = \int_{-\infty}^F dy P_F(y), \tag{3.11}$$

which is the probability to find a value less than  $F$  in a measurement. It may be calculated in terms of modified Bessel and Struve functions as

$$\begin{aligned}
P_{<}(F) &= \frac{1}{2} \{ 1 + \beta F [K_{\alpha-1/2}(\beta|F|) L_{\alpha-3/2}(\beta|F|) \\
&\quad + K_{\alpha-3/2}(\beta|F|) L_{\alpha-1/2}(\beta|F|)] \}.
\end{aligned} \tag{3.12}$$

Here we have again used the identities 6.561.16, as well as 6.561.4, in Ref. [13] (see also 10.43.2 in Ref. [14]). We may use either the above result, or numerical integration of Eq. (3.4), to obtain equivalent numerical results for  $P_{<}(F)$ .

#### IV. THE JOINT DISTRIBUTION OF AVERAGED FLUX AND ENERGY DENSITY

The results of Secs. II and III can be understood from a more general perspective that leads to further results. As the two averaged null components  $\omega_R, \omega_L$  of the stress tensor commute, they have a joint probability distribution, whose joint moment generating function is the product of the two independent generating functions

$$\langle e^{s\omega_R + t\omega_L} \rangle = \langle e^{s\omega_R} \rangle \langle e^{t\omega_L} \rangle, \tag{4.1}$$

because the vacuum expectation values factorize (see the Appendix)

$$\langle \omega_R^m \omega_L^n \rangle = \langle \omega_R^m \rangle \langle \omega_L^n \rangle. \quad (4.2)$$

Consequently, the joint probability distribution function is the product of the two individual ones, giving

$$\text{Prob}((\omega_R, \omega_L) \in \Delta) = \int_{\Delta} P_R(\lambda) P_L(\mu) d\lambda d\mu \quad (4.3)$$

for any Borel subset  $\Delta \subset \mathbb{R}^2$ . In the above expressions, the generating functions may be understood as formal series in  $s$  and  $t$ . Petersen's theorem (see [16], Theorem 14.6) states that the joint moments uniquely determine the joint probability distribution provided that the marginal distributions are uniquely determined by their moments. This is certainly satisfied if  $P_{L,R}$  are given by shifted Gamma distributions, because their moments satisfy the Hamburger moment condition [1].

The averaged flux  $F = \omega_R - \omega_L$  and energy density  $\rho = \omega_R + \omega_L$  also commute and have a joint probability distribution given in terms of that of  $\omega_L$  and  $\omega_R$ . Specifically,

$$\begin{aligned} \text{Prob}((\rho, F) \in \Delta) &= \int_{\tilde{\Delta}} P_R(\lambda) P_L(\mu) d\lambda d\mu \\ &= \int_{\tilde{\Delta}} P(\rho, F) d\rho dF, \end{aligned} \quad (4.4)$$

where  $\tilde{\Delta} = \{(\lambda, \mu) \in \mathbb{R}^2 : (\lambda + \mu, \lambda - \mu) \in \Delta\}$ , and the joint probability density function is

$$P(\rho, F) = \frac{1}{2} P_R\left(\frac{1}{2}(\rho + F)\right) P_L\left(\frac{1}{2}(\rho - F)\right). \quad (4.5)$$

The factor of  $\frac{1}{2}$  arises from the change of variables. Integrating out  $F$  or  $\rho$  in (4.7), one obtains the formulas (2.5) and (3.4) for the marginal distributions of  $\rho$  and  $F$  as above. To understand the significance of the joint probability density function, consider the simple example  $\Delta = \{\rho_0 \leq \rho \leq \rho_0 + \Delta\rho, F_0 \leq F \leq F_0 + \Delta F\}$ , where  $\Delta\rho \ll |\rho_0|$  and  $\Delta F \ll |F_0|$ , in which limit we have

$$\text{Prob}((\rho, F) \in \Delta) \approx P(\rho_0, F_0) \Delta\rho \Delta F. \quad (4.6)$$

Due to  $P_L$  and  $P_R$  being supported in  $[-\omega_0, \infty)$ ,  $P$  effectively has a factor of  $\theta(\rho + F + 2\omega_0)\theta(\rho - F + 2\omega_0)$  and one easily sees that  $P$  is supported in the set of  $(\rho, F) \in \mathbb{R}^2$  such that  $|F| \leq \rho + 2\omega_0$ . As  $\rho \rightarrow -2\omega_0$ , the quantum inequality lower bound, the flux must vanish in the following sense: for any  $\rho_0 > -2\omega_0$ , the conditional random variable  $F|(\rho < \rho_0)$  (i.e.,  $F$  conditioned on  $\rho$  being less than  $\rho_0$ ) takes values in the interval  $[-2\omega_0 - \rho_0, 2\omega_0 + \rho_0]$  and therefore the expectation value of its absolute value

is bounded above by  $2\omega_0 + \rho_0$ , which vanishes in the limit  $\rho_0 \rightarrow -2\omega_0$ . Thus, in this limit,  $F|(\rho < \rho_0)$  converges in mean to the random variable taking the constant value 0. Note that this conclusion does not depend upon the specific functional forms of  $P_R$  and  $P_L$ .

In the case where  $P_R$  and  $P_L$  are identical shifted Gamma distributions, a similar calculation to one given in Eq. (3.3) shows that

$$\begin{aligned} \text{Prob}((\rho, F) \in \Delta) &= \frac{2(\beta/2)^{2\alpha}}{\Gamma(\alpha)^2} \int_0^\infty d\rho' \int_{-\rho'}^{\rho'} dF \chi_\Delta(\rho' - 2\omega_0, F) e^{-\beta\rho'} \\ &\times ((\rho')^2 - F^2)^{\alpha-1}, \end{aligned} \quad (4.7)$$

where  $\chi_\Delta$  is the characteristic function of  $\Delta$ , i.e.,  $\chi_\Delta(\rho, F) = 1$  for  $(\rho, F) \in \Delta$  and  $\chi_\Delta(\rho, F) = 0$  otherwise.

Using the joint distribution, we can determine, for example, the probability that  $|F| < |\rho|$ , corresponding to  $\Delta = \{(\rho, F) : |F| < |\rho|\}$ , for which  $\tilde{\Delta}$  is the union of the first and third open quadrants of  $\mathbb{R}^2$ . With  $P_L = P_R$  (but not necessarily of the shifted Gamma form) this gives

$$\text{Prob}(|F| < |\rho|) = p^2 + (1 - p)^2 = 1 - 2p + 2p^2 \quad (4.8)$$

where  $p = \text{Prob}(\omega_{L,R} < 0)$ . One typically has  $p > \frac{1}{2}$ , because the expectation is zero and the distribution has a long positive tail but cannot take negative values below the quantum inequality bound. It follows that  $\text{Prob}(|F| < |\rho|) > \frac{1}{2}$ , as well, indicating the flux tends to be more centrally concentrated than the energy density. This effect can be quite marked: for instance, with  $c = 1$  and Gaussian smearing,  $p = 0.89$  [1] and  $\text{Prob}(|F| < |\rho|) = 0.81$  (quoting values to 2d.p.).

As another example, we can compute the probability density function  $P_{F|\rho < 0}(F)$  of the flux, conditioned on the energy density being negative (thus lying in  $[-2\omega_0, 0]$ ), which is given in general by

$$P_{F|\rho < 0}(F) = \frac{1}{2q} \int_{-2\omega_0}^0 d\rho P_R\left(\frac{1}{2}(\rho + F)\right) P_L\left(\frac{1}{2}(\rho - F)\right), \quad (4.9)$$

where  $q = \text{Prob}(\rho < 0)$ . This formula is obtained by setting  $\Delta = (-2\omega_0, 0) \times (-\infty, F)$  in (4.5) and differentiating with respect to  $F$ , then dividing by  $q$  to normalize the distribution. Given that  $P_L$  and  $P_R$  both have support  $[-\omega_0, \infty)$ , we see that  $P_{F|\rho < 0}(F) = 0$  for  $|F| > 2\omega_0$ , so this distribution has compact support  $[-2\omega_0, 2\omega_0]$ . In the case where  $P_{L,R}$  are identical shifted Gamma distributions, this gives



$$\begin{aligned}
P_{F|\rho < 0}(F) &= \frac{\beta^{2\alpha} e^{-2\beta\omega_0}}{q\Gamma(\alpha)^2 2^{2\alpha-1}} \int_{-2\omega_0}^0 d\rho \theta(\rho + F + 2\omega_0) \\
&\quad \times \theta(\rho - F + 2\omega_0) e^{-\beta\rho} ((\rho + 2\omega_0)^2 - F^2)^{\alpha-1} \\
&= \frac{\beta^{2\alpha} \theta(2\omega_0 - |F|)}{q\Gamma(\alpha)^2 2^{2\alpha-1}} \int_{|F|}^{2\omega_0} d\rho' e^{-\beta\rho'} ((\rho')^2 - F^2)^{\alpha-1} \\
&= \frac{\beta(\beta|F|/2)^{2\alpha-1}}{q\Gamma(\alpha)^2} Q(\alpha, \beta|F|; |F|/(2\omega_0)) \theta(2\omega_0 - |F|),
\end{aligned} \tag{4.10}$$

where

$$Q(\alpha, \eta; z) = \int_1^{1/z} ds e^{-\eta s} (s^2 - 1)^{\alpha-1}. \tag{4.11}$$

Consider the limit in which  $|F| \rightarrow 2\omega_0^-$ , so  $1/z = 1 + \epsilon$ , with  $0 < \epsilon = 2\omega_0/|F| - 1 \ll 1$ , and  $\eta = 2\omega_0\beta/(1 + \epsilon) \rightarrow 2\omega_0\beta$ . Now  $Q/\epsilon^\alpha$  becomes

$$\epsilon^{-\alpha} Q = \frac{1}{\epsilon^\alpha} \int_0^\epsilon dx e^{-\eta(1+x)} [(1+x)^2 - 1]^{\alpha-1} \rightarrow e^{-2\omega_0\beta} \frac{2^{\alpha-1}}{\alpha}, \tag{4.12}$$

as can be confirmed by a dominated convergence argument, leading to

$$P_{F|\rho < 0}(F) \approx \frac{\beta^{2\alpha} \omega_0^{\alpha-1} e^{-2\beta\omega_0}}{2q\alpha\Gamma(\alpha)^2} (2\omega_0 - |F|)^\alpha \tag{4.13}$$

as  $|F|$  approaches  $2\omega_0$  from below. An example of this behavior will appear in Fig. 3 below.

We can also investigate the limit in which  $|F| \ll \omega_0$ . Here, it is convenient to use the penultimate expression in (4.10), also noting that the exponential factor lies in the interval  $[e^{-2\omega_0\beta}, 1]$  for all  $\rho' \in [0, 2\omega_0]$ . Thus

$$P_{F|\rho < 0}(F) \asymp I(F) = \int_{|F|}^{2\omega_0} d\rho' ((\rho')^2 - F^2)^{\alpha-1} \tag{4.14}$$

as  $|F| \rightarrow 0$ , where the  $\asymp$  symbol means that the left-hand side is bounded above and below by constant multiples of the right-hand side. There are three cases. For  $0 < \alpha < \frac{1}{2}$ , we have

$$P_{F|\rho < 0}(F) \asymp F^{2\alpha-1} \int_1^{2\omega_0/|F|} ds (s^2 - 1)^{\alpha-1} \tag{4.15}$$

and because the integral is finite and nonzero in the limit  $|F| \rightarrow 0$ , it follows that  $P_{F|\rho < 0}(F) \asymp F^{2\alpha-1}$ , which is an integrable singularity (see Fig. 3 for an example). On the other hand, if  $\alpha = \frac{1}{2}$ , the above integral can be evaluated exactly and grows logarithmically in  $2\omega_0/|F|$ , so

$P_{F|\rho < 0}(F) \asymp \log(2\omega_0/|F|)$ . Finally, if  $\alpha > \frac{1}{2}$ , an integration by parts argument based on multiplying and dividing the integrand of Eq. (4.14) by  $2\rho'$  gives

$$I(F) = \frac{((2\omega_0)^2 - F^2)^\alpha}{4\omega_0\alpha} + \frac{1}{2\alpha} \int_{|F|}^{2\omega_0} d\rho' \frac{1}{(\rho')^2} ((\rho')^2 - F^2)^\alpha, \tag{4.16}$$

where  $I(F)$  is the integral on the right-hand side of Eq. (4.14). As  $(\rho')^{-2} < ((\rho')^2 - F^2)^{-1}$ , the second term on the right-hand side is bounded above by  $I(F)/(2\alpha)$ . On rearranging,

$$I(F) < \frac{2\alpha}{2\alpha - 1} \frac{((2\omega_0)^2 - F^2)^\alpha}{4\omega_0\alpha}, \tag{4.17}$$

which shows that  $P_{F|\rho < 0}(F)$  remains bounded as  $|F| \rightarrow 0$  for  $\alpha > \frac{1}{2}$ . Note that the behavior we have given here results in the same singularity structure exhibited by  $P_F(F)$  as  $|F| \rightarrow 0$ , described in Eqs. (3.7)–(3.9).

## V. SPECIFIC SAMPLING FUNCTIONS

Here we discuss some choices of sampling function for which the probability  $P_{L,R}$  distributions take the shifted Gamma form given in Eq. (2.3).

### A. Gaussian-like functions

The class of functions of the form

$$f_{a,b}(u) = \gamma u^{2a} e^{-bu^2}, \tag{5.1}$$

where  $a$  is a non-negative integer,  $b > 0$  and  $\gamma = b^{a+1/2}/\Gamma(a+1/2)$ , was discussed in Ref. [3]. Here we may take

$$\tau = \frac{1}{\sqrt{b}}. \tag{5.2}$$

The results of Ref. [3] may be expressed in our present notation as

$$\alpha = \frac{c(4a-1)}{24(2a-1)} \tag{5.3}$$

and

$$\beta = \pi. \tag{5.4}$$

Here  $c$  is the central charge of the conformal field theory, and  $c = 1$  for a massless scalar field. The usual Gaussian, which was treated in Ref. [1], is the case  $a = 0$ , so

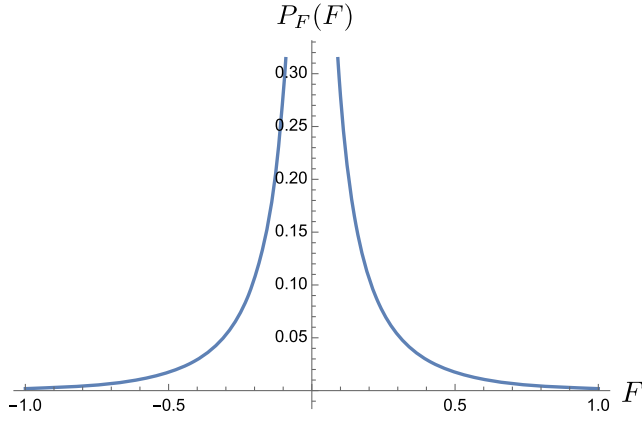


FIG. 1. The probability distribution,  $P_F(F)$ , with Gaussian averaging is plotted as a function of the dimensionless flux,  $F$ .

$$\alpha = \frac{c}{24}, \quad \beta = \pi. \quad (5.5)$$

The vacuum probability distribution for the Gaussian averaged flux of a massless scalar field,  $c = 1$ , is plotted in Fig. 1. Here  $P_F \propto |F|^{-11/12}$  near the origin.

The cumulative distribution function  $P_<(F)$  is plotted for the Gaussian averaged scalar field case with  $c = 1$  in Fig. 2. Note that the integrable singularity in  $P_F(F)$  leads to  $P_<(F)$  approximating a step function. Numerical evaluation gives  $P_<(10^{-10}) = 0.585$  and  $P_<(-10^{-10}) = 0.415$ , so the average slope in the interval  $|F| < 10^{-10}$  is

$$\frac{\Delta P_<}{\Delta F} \approx 8.5 \times 10^8, \quad (5.6)$$

which is consistent with the lower panel in Fig. 2.

The distribution  $P_{F|\rho<0}(F)$  given in Eq. (4.10) is plotted in Fig. 3 for values  $\alpha = c/24$ ,  $\beta = \pi$ , with central charge  $c = 8$  to indicate better the shape of the plot near the end of the support.

### B. Lorentzian-like functions

Another class of functions considered in Ref. [3] takes the form

$$g_{n,a,b}(u) = \frac{Cu^{2a}}{(b^2 + u^2)^n}, \quad (5.7)$$

where  $0 \leq a < n$  are integers and  $b > 0$ , and we may now take

$$\tau = b. \quad (5.8)$$

Now

$$\alpha = \frac{cn(4a^2 - 4an - 4a + n)}{12(2a - 2n - 1)(n + 1)(2a - 1)}, \quad (5.9)$$

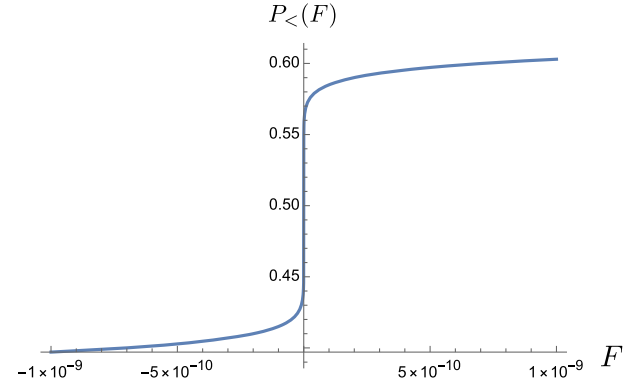
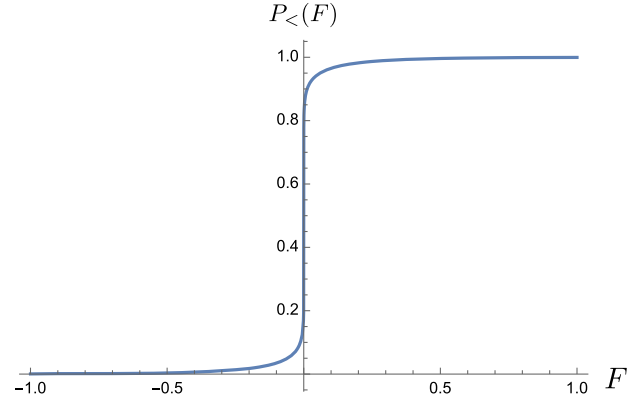


FIG. 2. The cumulative probability distribution,  $P_<(F)$ , with Gaussian averaging is plotted as a function of the dimensionless flux,  $F$ , on two different scales. These plots may be obtained by numerical integration of Eq. (3.4) or numerical evaluation of the exact expression, Eq. (3.12).

and

$$\beta = \frac{4n\pi}{4(n-a)^2 - 1}. \quad (5.10)$$

The usual Lorentzian is

$$g_{1,0,b}(u) = \frac{b}{\pi(u^2 + b^2)}, \quad (5.11)$$

for which

$$\alpha = \frac{c}{72}, \quad \beta = \frac{4\pi}{3}, \quad (5.12)$$

In the massless scalar field case,  $c = 1$ ,  $P_F(\omega)$  is qualitatively similar to the Gaussian averaged case plotted in Fig. 1. However, here  $P_F \propto |\omega|^{-17/18}$  near the origin, and  $P_<(\omega)$  is plotted in Fig. 4.

We find in this case that  $P_<(10^{-10}) = 0.775$  and  $P_<(-10^{-10}) = 0.2255$ , so the average slope in the interval  $|F| < 10^{-10}$  is of order

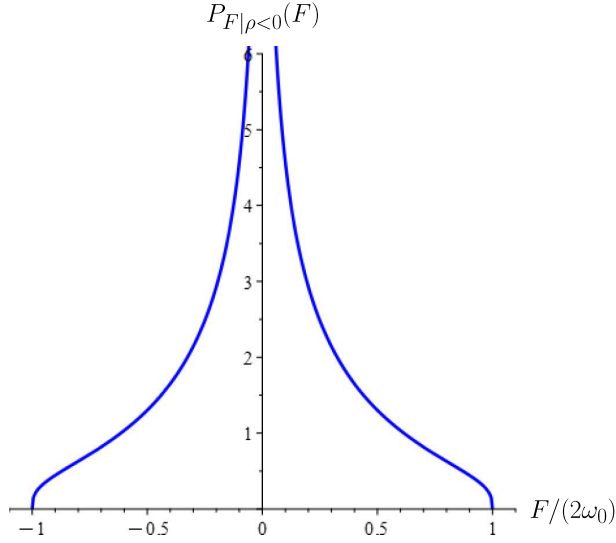


FIG. 3. The probability distribution of flux, conditioned on the energy density being negative.

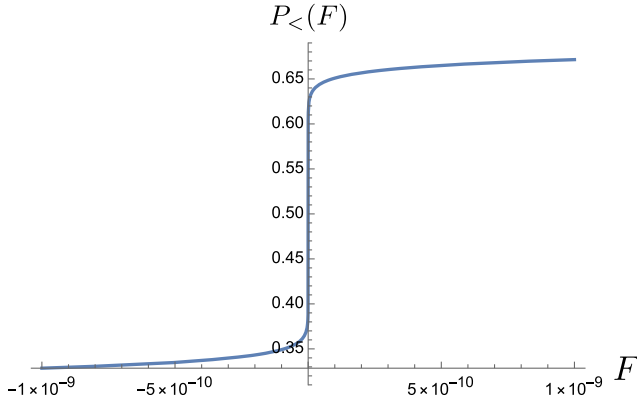


FIG. 4. The cumulative probability distribution,  $P_<(F)$ , with Lorentzian averaging is plotted as a function of the dimensionless flux,  $F$ .

$$\frac{\Delta P_<}{\Delta F} \approx 2.7 \times 10^9. \quad (5.13)$$

This is about a factor of 3 larger than the Gaussian case, and is presumably due to the more singular behavior of  $P_F$  for the Lorentzian case.

Note that for the massless scalar field case,  $c = 1$ , both the Gaussian and Lorentzian averaging functions lead to  $\alpha < \frac{1}{2}$ , and hence an integrable singularity in  $P_F$ . This need not be the case for some of the generalized averaging functions discussed above. For example, if we set  $a = \frac{23}{44}$  in Eqs. (5.1) and (5.3), we obtain a generalized Gaussian for which  $\alpha = 1$ , leading a probability distribution which is finite everywhere. This is a good illustration of how sensitive stress tensor probability distributions are to the details of the measurement process.

### C. Compactly supported functions

Compactly supported functions, those which vanish outside of finite intervals, are the appropriate descriptions for physical measures of finite duration. All of the smearing functions discussed above have tails, and hence are not compactly supported. A class of infinitely differentiable compactly supported functions was described in Ref. [5], and used in Refs. [8–10]. They have Fourier transforms which decay as an exponential of a fractional power of frequency:

$$\hat{f}(\omega) \sim e^{-a|\omega|^{\alpha_p}}, \quad 0 < \alpha_p < 1. \quad (5.14)$$

Although the Lorentzian function is not compactly supported, its Fourier transform corresponds to the  $\alpha_p = 1$  limit of this class.

The focus of Ref. [5] was on the rate of growth of moments and the asymptotic probability distributions for stress tensor components in four-dimensional spacetime. The explicit example used was that of  $:\dot{\phi}^2:$ , the normal ordered square of the time derivative of a massless scalar field. This operator appears as part of the energy density and other stress tensor components, and its asymptotic probability distribution was assumed to model those of a general component. If we set  $p = 1$  in the discussion in Sec. IV of Ref. [5], we can infer the asymptotic probability distribution for  $:\dot{\phi}^2:$  in two dimensions sampled by a compactly supported function. If we assume that asymptotic distribution also holds for the flux in two dimensions, we have

$$P(F) \sim e^{-\beta|F|^{\alpha_p}} \quad (5.15)$$

for some constant  $\beta$ . The  $\alpha_p = 1$  limit of this expression agrees with the form found in Eq. (3.5).

## VI. SUMMARY

We have treated the probability distributions for vacuum fluctuations of the energy flux in two-dimensional spacetime. These distributions depend upon the details of the sampling function used to find spacetime averages of the flux operator. In many, but not all cases, the distribution has an integrable singularity at the origin. In particular, if  $\alpha > \frac{1}{2}$ , the distribution  $P(F)$  is finite at  $F = 0$ . In all cases,  $P(F)$  decays as an exponential for large arguments, an illustration of the non-Gaussian character of vacuum stress tensor fluctuations.

In addition, in Sec. IV, we construct a joint probability distribution for the energy flux and energy density operators which have been averaged with the same sampling function. The joint distribution may be used, for example, to compute a modified probability distribution for the energy flux under some condition on the energy density, such as being negative.



Both of these sets of results have potential physical applications to four-dimensional models and to condensed matter systems with effectively one spatial dimension. Two-dimensional models have often been used to infer insights into possible behavior in four dimensions [11,12]. Electromagnetic vacuum flux fluctuations could have observable effects on the motion of electrons [17], and the two dimensional flux fluctuation models may be useful in further studies of this effect.

Zero point density fluctuations in a fluid [18] are an analog model for quantum stress tensor fluctuations. Thus, phonons in a one-space-dimensional system form an analog model for two-spacetime-dimensional quantum field theories, and one which may be experimentally accessible.

The results of this paper are expected to be useful for future numerical simulations of stress tensor fluctuations. Simulations of energy density fluctuations without correlations were performed in Refs. [11,12], and simulations of Gaussian field fluctuations including correlations between different times were treated in Ref. [19]. The results of Sec. III for  $P(F)$  will be useful for simulations of energy flux fluctuations which include correlations between different times.

The first step in a numerical simulation is an algorithm to generate a set of outcomes which obey a given probability distribution. This is usually done using a cumulative probability distribution, such as  $P_<(F)$ . The numerical challenges are greater when  $P(F)$  has an integrable singularity, and hence  $P_<(F)$  is close to a step function. The nonsingular  $\alpha > \frac{1}{2}$  cases seem likely to lead to more stable simulations, but all values of  $\alpha$  are of interest. A first step will be a simulation of flux fluctuations at different times using both our results for  $P(F)$  and the flux-flux correlation function.

Future simulations of both the energy flux and density fluctuations could employ either a correlation function or a joint probability distribution. A preliminary study using a flux-density correlation function was done in Sec. III of Ref. [20], where the correlation between energy density and flux in adjacent space time regions was discussed. An extension of this work which uses the detailed probability distributions treated in the present paper is now possible. The joint distribution also provides information about correlations about flux and density fluctuations, but when averaged over the same space time region. The possibility of a joint distribution for flux and energy density averaged over different regions is an open question.

## ACKNOWLEDGMENTS

C. J. F. thanks Henning Bostelmann and Gandalf Lechner for a useful conversation concerning the Appendix. This work was supported in part by the National Science Foundation under Grant No. PHY-2207903 and EPSRC Grant No. EP/Y000099/1.

## DATA AVAILABILITY

No data were created or analyzed in this study.

## APPENDIX: FACTORIZATION OF THE JOINT MOMENT GENERATING FUNCTION

The decoupling of the left- and right-moving expectation values is well known in CFT, but we did not find a simple direct argument in the literature. However an argument is easily given, assuming that the CFT obeys standard properties of a quantum field theory, specifically the translational invariance of the vacuum and the cluster property [21] (more directly, but less simply, one could argue from the spectral properties of the translation operator, cf. [22,23]). Suppose  $Z$  is any product of  $m$  factors of  $\omega_R$  and  $n$  factors of  $\omega_L$ . Because they commute, one has

$$\langle Z \rangle = \langle \omega_R^m \omega_L^n \rangle = \langle U(\lambda, \lambda) \omega_R^m \omega_L^n U(-\lambda, \lambda) \rangle \quad (\text{A1})$$

for any  $\lambda \in \mathbb{R}$ , where  $U(t, x)$  is the unitary operator implementing translation through  $(t, x)$ , and we have used the translation invariance of the vacuum. But as translation through  $(\lambda, \lambda)$  [respectively,  $(-\lambda, \lambda)$ ] leaves  $u$  (respectively,  $v$ ) unchanged,  $[U(\lambda, \lambda), \omega_R] = 0 = [U(-\lambda, \lambda), \omega_L]$ . Consequently,

$$\langle Z \rangle = \langle \omega_R^m U(0, 2\lambda) \omega_L^n \rangle = \langle \omega_R^m U(0, 2\lambda) \omega_L^n U(0, 2\lambda)^{-1} \rangle \quad (\text{A2})$$

using translation invariance of the vacuum again. As  $\lambda \rightarrow \infty$ , the left-hand side is constant, but the right-hand side tends to  $\langle \omega_R^m \rangle \langle \omega_L^n \rangle$  by the cluster property for large spacelike separations, giving  $\langle Z \rangle = \langle \omega_R^m \omega_L^n \rangle$ . Thus, for any integer  $N \geq 0$ ,

$$\frac{1}{N!} \langle (s\omega_R + t\omega_L)^N \rangle = \sum_{\substack{m,n \geq 0 \\ m+n=N}} \frac{s^m t^n}{m!n!} \langle \omega_R^m \rangle \langle \omega_L^n \rangle, \quad (\text{A3})$$

and Eq. (4.1) for the joint moment generating functions follows, understood as an equality of formal power series in  $s$  and  $t$ .

- [1] C. J. Fewster, L. H. Ford, and T. A. Roman, Probability distributions of smeared quantum stress tensors, *Phys. Rev. D* **81**, 121901(R) (2010).
- [2] C. J. Fewster and S. Hollands, Probability distributions for the stress tensor in conformal field theories, *Lett. Math. Phys.* **109**, 747 (2019).
- [3] M. C. Antony and C. J. Fewster, Explicit examples of probability distributions for the energy density in two-dimensional conformal field theory, *Phys. Rev. D* **101**, 025010 (2020).
- [4] C. J. Fewster and S. Hollands, Quantum energy inequalities in two-dimensional conformal field theory, *Rev. Math. Phys.* **17**, 577 (2005).
- [5] C. J. Fewster and L. H. Ford, Probability distributions for quantum stress tensors measured in a finite time interval, *Phys. Rev. D* **92**, 105008 (2015).
- [6] C. J. Fewster, L. H. Ford, and T. A. Roman, Probability distributions for quantum stress tensors in four dimensions, *Phys. Rev. D* **85**, 125038 (2012).
- [7] C. J. Fewster and L. H. Ford, Probability distributions for space and time averaged quantum stress tensors, *Phys. Rev. D* **101**, 025006 (2020).
- [8] E. D. Schiappacasse, C. J. Fewster, and L. H. Ford, Vacuum quantum stress tensor fluctuations: A diagonalization approach, *Phys. Rev. D* **97**, 025013 (2018).
- [9] P. Wu, E. D. Schiappacasse, and L. H. Ford, Space and time averaged quantum stress tensor fluctuations, *Phys. Rev. D* **103**, 125014 (2021).
- [10] P. Wu, E. D. Schiappacasse, and L. H. Ford, Frequency spectra analysis of space- and time-averaged quantum stress tensor fluctuations, *Phys. Rev. D* **107**, 036013 (2023).
- [11] S. Carlip, R. A. Mosna, and J. P. M. Pitelli, Vacuum fluctuations and the small scale structure of spacetime, *Phys. Rev. Lett.* **107**, 021303 (2011).
- [12] S. Carlip, Ricardo A. Mosna, and J. P. M. Pitelli, Quantum fields, geometric fluctuations, and the structure of space-time, *Phys. Rev. D* **102**, 126018 (2020).
- [13] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. (Academic Press, New York, 2007).
- [14] NIST Digital library of mathematical functions, Release 1.2.2 of 2024-09-15, edited by F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, <https://dlmf.nist.gov/>.
- [15] S. Kullback, The distribution laws of the difference and quotient of variables independently distributed in Pearson type III laws, *Ann. Math. Stat.* **7**, 51 (1936).
- [16] K. Schmüdgen, The moment problem, in, *Graduate Texts in Mathematics* (Springer, Cham, 2017), Vol. 277.
- [17] L. H. Ford, Vacuum radiation pressure fluctuations on electrons, *Phys. Rev. D* **110**, 096002 (2024).
- [18] P. Wu and L. H. Ford, Large zero point density fluctuations in fluids, *Phys. Rev. Res.* **2**, 032028 (2020).
- [19] E. R. Taylor, S. Yenko, and L. H. Ford, Numerical simulation of quantum field fluctuations, *Phys. Rev. D* **109**, 116010 (2024).
- [20] L. H. Ford and T. A. Roman, Energy density-flux correlations in an unusual quantum state and in the vacuum, *Phys. Rev. D* **76**, 064012 (2007).
- [21] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That*, Princeton Landmarks in Physics (Princeton University Press, Princeton, NJ, 2000), corrected third printing of the 1978 edition.
- [22] D. Maison, Eine Bemerkung zu Clustereigenschaften, *Commun. Math. Phys.* **10**, 48 (1968).
- [23] A. Petrov, Asymptotic behaviour of the unitary representations of the Poincaré group, *Rep. Math. Phys.* **4**, 183 (1973).