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Inhomogeneous Diophantine Approximation on M_0-sets with restricted denominators

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Abstract

Let $F \subseteq [0,1]$ be a set that supports a probability measure μ with the property that $|\widehat{\mu}(t)| \ll (\log|t|)^{-A}$ for some constant $A > 0$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers. If $\mathcal{A}$ is lacunary and $A > 2$, we establish a quantitative inhomogeneous Khintchine-type theorem in which (i) the points of interest are restricted to F and (ii) the denominators of the 'shifted' rationals are restricted to $\mathcal{A}$. The theorem can be viewed as a natural strengthening of the fact that the sequence $(q_n x \bmod 1)_{n\in\mathbb{N}}$ is uniformly distributed for μ almost all $x \in F$. Beyond lacunary, our main theorem implies the analogous quantitative result for sequences $\mathcal{A}$ for which the prime divisors are restricted to a finite set of k primes and $A > 2k$. Loosely speaking, for such sequences our result can be viewed as a quantitative refinement of the fundamental theorem of Davenport, Erdös & LeVeque (1963) in the theory of uniform distribution.

Subject classification: 11K60, 11J71, 11K06 11J83, 11K16, 11K70

1 Introduction and results

1.1 Motivation and lacunary results

We start by setting the scene. Throughout, F will be a subset of the unit interval $\mathbb{I} := [0,1]$ that supports a non-atomic probability measure μ. As usual, the Fourier transform of μ is defined by

$$\widehat{\mu}(t) := \int e^{-2\pi itx}\,\mathrm{d}\mu(x) \qquad (t \in \mathbb{R}).$$

The set F is called an M_0-*set* if $\widehat{\mu}(t)$ vanishes at infinity. It is well known that the decay rate of the Fourier transform is related to the Hausdorff dimension of the support of μ. Indeed, a classical result of Frostman states that if $|\widehat{\mu}(t)| \leq c\,|t|^{-\eta/2}$ for some constants $c, \eta > 0$, then $\dim F \geq \min\{1, \eta\}$. Further details and references of can be found in [8]. Throughout, $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ will be an increasing sequence of natural numbers. Recall, that $\mathcal{A}$ is said to be *lacunary* if there exists a constant $K > 1$ such that

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$$\frac{q_{n+1}}{q_n} \geq K \qquad (n \in \mathbb{N})\,. \tag{1}$$

The fundamental theorem of Davenport, Erdös & LeVeque [7] in the theory of uniform distribution, shows that the generic distribution properties of a sequence $(q_n x)_{n\in\mathbb{N}}$ with x restricted to the support of μ are intimately related to the decay rate of $\widehat{\mu}$.

Theorem DEL (Davenport, Erdös & LeVeque). *Let μ be a probability measure supported on a subset F of $\mathbb{I}$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers. If*

$$\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{m,n=1}^{N} \widehat{\mu}(h(q_m - q_n)) \ < \ \infty \tag{2}$$

for all integers $h \neq 0$, then the sequence $(q_n x)_{n\in\mathbb{N}}$ is uniformly distributed modulo one for μ–almost all $x \in F$.

In the case the sequence $\mathcal{A}$ is lacunary, the theorem gives rise to the following elegant statement.

Corollary DEL. *Let μ be a probability measure supported on a subset F of $\mathbb{I}$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be a lacunary sequence of natural numbers. Let $f : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function such that*

$$\sum_{n=2}^{\infty} \frac{f(n)}{n \log n} \ < \ \infty \tag{3}$$

and suppose that

$$\widehat{\mu}(t) = O\left(f(|t|)\right) \qquad as \quad |t| \to \infty\,. \tag{4}$$

Then the sequence $(q_n x)_{n\in\mathbb{N}}$ is uniformly distributed modulo one for μ–almost all $x \in F$.

The deduction of the corollary from the theorem is reasonably straightforward. However, for the sake of completeness and the reader's convenience we provide the details at the end of the paper in §7 - Appendix A.

Remark 1. Reinterpreting the corollary in terms of normal numbers, it implies that μ–almost all numbers in F are normal. Thus, it provides a useful mechanism for proving the existence of normal numbers in a given subset of real numbers. Indeed, Corollary DEL implies that if a given set F supports a probability measure μ such that for some $\epsilon > 0$

$$\widehat{\mu}(t) = O\left((\log\log|t|)^{-(1+\epsilon)} \right) \qquad \text{as} \quad |t| \to \infty\,, \tag{5}$$

then μ–almost all numbers in F are normal. This observation is key, for example, in showing that there are normal numbers which are badly approximable – see Remark 6 in §1.1.1 below. For completeness, we mention that Theorem DEL (and its corollary) is valid for non-integer sequences and that this is at the heart of addressing the long standing problem of when normality to one base, not necessarily integer, implies normality to another (see [20] and references within).

Remark 2. In the language of Kahane [15] and Lyons [17], the conclusion of the corollary is equivalent to saying that the μ-measure of every lacunary W^*-set is zero – see [17, Theorem 4]. Basically, a Borel set $F \subset \mathbb{I}$ is a lacunary W^*-set if there exists a lacunary sequence $\mathcal{A}$ such that $(q_n x)_{n\in\mathbb{N}}$ is not uniformly distributed modulo one for any $x \in F$.

Let $\gamma \in \mathbb{I}$ and let $B = B(\gamma, r) \subseteq \mathbb{I}$ denote the ball centred at γ with radius $r \leq 1/2$. By the definition of uniform distribution, Corollary DEL implies that for μ–almost all $x \in F$ the sequence $(q_n x)_{n\in\mathbb{N}}$ modulo one 'hits' the ball B the 'expected' number of times. In other words, for μ–almost all $x \in F$

$$\lim_{N\to\infty} \frac{1}{N} \#\big\{1 \leq n \leq N : \|q_n x - \gamma\| \leq r\big\} = 2r\,, \tag{6}$$

where $\|\alpha\| := \min\{|\alpha - m| : m \in \mathbb{Z}\}$ denotes the distance from $\alpha \in \mathbb{R}$ to the nearest integer. In this paper, we consider the situation in which the radius of the ball is allowed to shrink with time. With this in mind, let $\psi : \mathbb{N} \to \mathbb{I}$ be a real, positive function and consider the counting function

$$R(x, N) \;=\; R(x, N; \gamma, \psi, \mathcal{A}) := \#\big\{1 \leq n \leq N : \|q_n x - \gamma\| \leq \psi(q_n)\big\}. \tag{7}$$

As alluded to in the definition, we will often simply write $R(x, N)$ for $R(x, N; \gamma, \psi, \mathcal{A})$ since the other three dependencies will be clear from the context and are usually fixed. Our first result implies that if $\widehat{\mu}$ decays quickly enough, then for μ–almost all $x \in F$ the sequence $(q_n x)_{n\in\mathbb{N}}$ modulo one 'hits' the shrinking ball $B(\gamma, \psi(q_n))$ the 'expected' number of times.

Theorem 1. *Let μ be a probability measure supported on a subset F of $\mathbb{I}$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be a lacunary sequence of natural numbers. Let $\gamma \in \mathbb{I}$ and $\psi : \mathbb{N} \to \mathbb{I}$ be a real, positive function. Suppose there exists a constant $A > 2$, so that*

$$\widehat{\mu}(t) = O\left((\log |t|)^{-A}\right) \qquad as \quad |t| \to \infty\,. \tag{8}$$

Then, for any $\varepsilon > 0$, we have that

$$R(x, N) \;=\; 2\Psi(N) + O\Big(\Psi(N)^{2/3}\big(\log(\Psi(N) + 2)\big)^{2+\varepsilon}\Big) \tag{9}$$

for μ-almost all $x \in F$, where

$$\Psi(N) := \sum_{n=1}^{N} \psi(q_n)\,. \tag{10}$$

Remark 3. By definition, $\Psi(N) = rN$ when ψ is the constant function $\psi(n) = r$ and so the theorem trivially implies (6). Indeed, it implies (6) with an error term. However, note that to apply the theorem we need to assume a faster logarithmic decay rate than that given by (5) which suffices to conclude (6).

Hopefully, it is pretty clear that the theory of uniform distribution, in particular the theorem of Davenport, Erdös & LeVeque, is a key motivating factor towards establishing statements such as Theorem 1. Another key motivating factor, which we now bring to the forefront, is the theory of Diophantine approximation on manifolds; also known as Diophantine approximation of dependent quantities. In short, this theory refers to the study of Diophantine properties of points in $\mathbb{R}^n$ whose coordinates are confined by functional relations or equivalently are restricted to a sub-manifold of $\mathbb{R}^n$. Over the last twenty years, the theory has developed at some considerable pace with the catalyst undoubtedly being the pioneering work of Kleinbock & Margulis on the Baker-Sprindžuk conjecture (see [3, §6]). Given a real number $\gamma \in \mathbb{I}$, a function $\psi : \mathbb{N} \to \mathbb{I}$ and a sequence $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ of natural numbers, consider the set

$$W_{\mathcal{A}}(\gamma;\psi) := \{x \in \mathbb{I} : \|q_n x - \gamma\| \leq \psi(q_n) \text{ for infinitely many } n \in \mathbb{N}\}\,. \tag{11}$$

By definition, $x \in W_{\mathcal{A}}(\gamma;\psi)$ if and only if the inequality

$$\Big|x - \frac{p+\gamma}{q}\Big| \leq \frac{\psi(q)}{q}$$

is satisfied for infinitely many $(p,q) \in \mathbb{Z} \times \mathcal{A}$. In other words, $W_{\mathcal{A}}(\gamma;\psi)$ is the standard set of inhomogeneous ψ-well approximable real numbers in which the denominators q of the shifted rational approximates $(p+\gamma)/q$ are restricted to the set $\mathcal{A}$. When $\mathcal{A} = \mathbb{N}$, we will drop the subscript $\mathcal{A}$ from $W_{\mathcal{A}}(\gamma;\psi)$. The fundamental theorem of Khintchine in the theory of metric Diophantine approximation, provides an elegant criterion for the 'size' of the set $W(\gamma;\psi)$ expressed in terms of Lebesgue measure m.

Theorem KS. *Let $\gamma \in \mathbb{I}$ and $\psi : \mathbb{N} \to \mathbb{I}$ be a real, positive decreasing function. Then*

$$m\big(W(\gamma;\psi)\big) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \psi(n) < \infty\,, \\ 1 & \text{if } \sum_{n=1}^{\infty} \psi(n) = \infty\,. \end{cases}$$

To be accurate, Khintchine [16] proved the homogeneous statement (i.e. when $\gamma = 0$) in 1924. Szüsz [26] generalized Khintchine's result to the inhomogeneous case in 1954. Ten years later, Schmidt in his far-reaching paper [24], established the quantitative strengthening of Theorem KS. In short, the main theorem in [24] implies that with ψ decreasing and $\mathcal{A} = \mathbb{N}$, the asymptotic counting statement (9) is valid for m-almost all $x \in \mathbb{I}$. In the case $\mathcal{A}$ is a lacunary sequence of natural numbers, the assumption that ψ is decreasing can be dropped and so the precise analogue of Theorem 1 holds for Lebesgue measure m – see [11, Theorem 7.3]. Motivated by the classical theory of Diophantine approximation on manifolds, suppose we restrict the points of interest in $W_{\mathcal{A}}(\gamma;\psi)$ to lie in some subset F of $\mathbb{I}$. Assume that $m(F) = 0$ – this is trivially the case if $\dim F < 1$. Then, the Lebesgue measure statements just described provide no information regarding the 'size' of the set of inhomogeneous ψ-well approximable real numbers restricted to F; we always have that

$$m\big(W_{\mathcal{A}}(\gamma;\psi) \cap F\big) = 0$$

irrespective of γ, ψ and $\mathcal{A}$. With this in mind, let μ be a non-atomic probability measure supported on F. Then, under some natural conditions on F, μ and $\mathcal{A}$, the goal is to obtain an analogue of Theorem KS for the 'size' of $W_{\mathcal{A}}(\gamma;\psi) \cap F$ expressed in terms of the measure μ. Indeed, generically it would not be unreasonable to expect that

$$\mu\big(W_{\mathcal{A}}(\gamma;\psi) \cap F\big) = 0 \;\; (\text{resp.} = 1) \quad \text{if} \quad \sum_{n=1}^{\infty} \psi(q_n) < \infty \;\; (\text{resp.} = \infty). \tag{12}$$

Such a statement would be precisely in line with the conjectured 'Dream Theorem' [3, §6.1.3] for Diophantine approximation on non-degenerate manifolds. For a basic introduction to the theory of metric Diophantine approximation including the manifold theory, see [3] and references within.

The following statement concerning the 'size' of $W_{\mathcal{A}}(\gamma, \psi) \cap F$ is a direct consequence of Theorem 1. It simply makes use of the fact that the theorem implies that for μ-almost all $x \in F$, the quantity $R(x, N)$ is bounded if $\Psi(N)$ is bounded and will tend to infinity if $\Psi(N)$ tends to infinity.

Corollary 1. *Let μ be a probability measure supported on a subset F of $\mathbb{I}$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be a lacunary sequence of natural numbers. Let $\gamma \in \mathbb{I}$ and $\psi : \mathbb{N} \to \mathbb{I}$ be a real, positive function. Suppose there exists a constant $A > 2$, so that* (8) *is satisfied. Then*

$$\mu\big(W_{\mathcal{A}}(\gamma;\psi) \cap F\big) = \begin{cases} 0 & \text{if} \quad \sum_{n=1}^{\infty} \psi(q_n) < \infty\ , \\ 1 & \text{if} \quad \sum_{n=1}^{\infty} \psi(q_n) = \infty\ . \end{cases}$$

Remark 4. A consequence of the general convergence result stated in §1.2, is that we can get away with $A > 1$ in the convergence case of the above corollary. In fact, the following decay rate suffices: for some $\epsilon > 0$ arbitrarily small

$$\widehat{\mu}(t) = O\left((\log|t|)^{-1}(\log\log|t|)^{-(1+\epsilon)}\right) \qquad \text{as} \quad |t| \to \infty\,.$$

Remark 5. Note that in view of Remark 1, whenever we are in the divergence case of the corollary we are able to conclude that μ-almost all numbers in $W_{\mathcal{A}}(\gamma, \psi) \cap F$ are normal.

The upshot of the results presented so far is that if $\mathcal{A}$ is a lacunary sequence of natural numbers and if $\widehat{\mu}$ decays quickly enough, then we are in pretty good shape with our understanding of the set $W_{\mathcal{A}}(\gamma;\psi) \cap F$ – both in terms of counting solutions (cf. Theorem 1) and size (cf. Corollary 1). The obvious question that now comes to mind is: what can we say if the growth of the sequence is slower than lacunary? More specifically, is the statement of Theorem 1 valid for the sequence $\mathcal{A} = \{2^a3^b : a, b \in \mathbb{Z}_{\geq 0}\}$ if we choose the decay rate constant A in (8) large enough? Before addressing this, it is worth comparing the above lacunary results with previous related works.

1.1.1 Connection to previous works

Let $x = [a_1, a_2, ...]$ represent the regular continued fraction expansion of $x \in \mathbb{I}$, and as usual let $p_n/q_n := [a_1, a_2, ..., a_n]$ denote its n–th convergent. Recall that $q_n\|q_nx\| \leq 1$ for any $n \in \mathbb{N}$. Given $M \in \mathbb{N}$, let F_M denote the set of real numbers in the unit interval with partial quotients bounded above by M. Thus

$$F_M := \{x \in \mathbb{I} : x = [a_1, a_2, ...] \ \text{ with } \ a_i \leq M \quad \text{for all } \ i \in \mathbb{N}\}\,.$$

It is well known that any F_M is a subset of the set **Bad** of badly approximable numbers; indeed

$$\bigcup_{M\in\mathbb{N}} F_M \ = \ \mathbf{Bad} := \big\{x \in \mathbb{I} : \liminf_{q\to\infty} q\|qx\| > 0\big\}\,.$$

In a pioneering paper [13], R. Kaufman showed that for any $M \geq 3$ the set F_M is an M_0-set. More precisely, he constructed a probability measure μ supported on F_M satisfying the decay property:

$$\widehat{\mu}(t) = O\left(|t|^{-0.0007}\right) \qquad \text{as} \quad |t| \to \infty\,. \tag{13}$$

Kaufman's construction was subsequently refined by Queffeléc & Ramaré [23]. In particular, they showed that F_2 also supports a probability measure with polynomial decay.

Remark 6. For any $M \geq 2$, the Kaufman measure μ supported on F_M trivially satisfies the decay condition (5) and so it follows (see Remark 1) that μ–almost all numbers in F_M (and thus in **Bad**) are normal. This observation is attributed to R.C. Baker – see [19, Appendix: Problem 45] for further details.

Remark 7. By construction, for any $M \geq 2$ the Kaufman measure μ supported on F_M also satisfies the following desirable property: for any $s < \dim F_M$, there exist constants $c, r_0 > 0$ such that $\mu(B) \leq c\,r^s$ for any ball B with radius $r < r_0$. This together with the Mass Distribution Principle [8, §4.1] implies that if $F \subset \mathbb{I}$ is such that $\mu(F) > 0$, then

$$\dim F \geq \dim F_M\,. \tag{14}$$

A well known conjecture of Littlewood dating back to the nineteen thirties states that

$$\liminf_{q\to\infty} q\|qx\|\,\|qy\| = 0 \quad \forall\; x, y \in \mathbb{I}\,.$$

This statement is trivially true if either x or y are not in **Bad**. The decay property (13) of the Kaufman measure was successfully utilized in [21] to prove the following statement for badly approximable numbers: *given* $x \in$ **Bad***, there exists a subset* $\mathbb{G}(x)$ *of* **Bad** *with full dimension such that for any* $y \in \mathbb{G}(x)$*,*

$$q\|qx\|\,\|qy\| \;\leq\; 1/\log q \quad \textit{for infinitely many } q \in \mathbb{N}\,. \tag{15}$$

Trivially, such $x, y \in$ **Bad** satisfy Littlewood's conjecture with an explicit 'rate of approximation' of $1/\log q$. The strategy behind the proof is simple enough. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be the sequence of denominators of the convergents p_n/q_n of the given $x \in$ **Bad**. Then (15) holds for the given x and any $y \in \mathbb{G}(x)$, where

$$\mathbb{G}(x) := \{y \in \textbf{Bad} : \|q_n y\| \leq 1/\log q_n \text{ for infinitely many } n \in \mathbb{N}\}$$

The crux of the proof (see [21, §3] for the details) boils down to showing that for any $M \geq 3$

$$\mu\big(W_{\mathcal{A}}(\gamma;\psi) \cap F_M\big) > 0 \quad \text{with} \quad \gamma = 0 \quad \text{and} \quad \psi(q) = 1/\log q\,, \tag{16}$$

where μ is the Kaufman measure supported on F_M. Note that by definition, $\mathbb{G}(x) = W_{\mathcal{A}}(\gamma;\psi) \cap$ **Bad** and so the desired full dimension statement follows on combining (16), (14) and the fact that $\dim F_M \to 1$ as $M \to \infty$. The proof of (16) given in [21] makes use of the explicit choices of ψ and γ, and the fact that the Fourier transform of the Kaufmann measure has polynomial decay. It also explicitly exploits the fact that for any $x \in$ **Bad**, the sequence $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ arising from its convergents p_n/q_n is not only lacunary but satisfies the additional property that

$$\frac{q_{n+1}}{q_n} \leq K^* \quad (n \in \mathbb{N})\,, \tag{17}$$

where $K^* := K^*(x) > 1$ is a constant. Corollary 1 improves the work carried out in [21] on four key fronts:

(a) It gives a full μ-measure statement rather than just a positive μ-measure statement.

(b) It is for any function $\psi : \mathbb{N} \to \mathbb{I}$ and any $\gamma \in \mathbb{I}$ rather than the explicit choices given by (16).

(c) It is for any lacunary sequence $\mathcal{A}$ of natural numbers rather than those satisfying the additional property (17).

(d) It is for any subset F of $\mathbb{I}$ that supports a probability measure μ with sufficient logarithmic transform decay rather than requiring polynomial decay.

Indeed, Corollary 1 restricted to the homogeneous case ($\gamma = 0$) establishes the zero-one law claim made in [21, §3.3]. Indeed, with $\psi(q) = 1/(\log q \, \log\log q)$ it enables us to replace (15) by the stronger statement that $\liminf_{q\to\infty} q \, \log q \, \|qx\| \, \|qy\| = 0$. To the best of our knowledge, the quantitative strengthening of the corollary, namely Theorem 1, is completely new. Within the context of Littlewood's conjecture it implies the following statement: *given* $x \in \mathbf{Bad}$ *and* $\gamma \in \mathbb{I}$, *there exists a subset* $\mathbb{G}(x,\gamma)$ *of* $\mathbf{Bad}$ *with full dimension such that for any* $y \in \mathbb{G}(x,\gamma)$,

$$\#\big\{1 \le q \le N : q \, \|qx\| \, \|qy - \gamma\| \ \le \ 1/\log q\big\} \gg \log\log N \, . \tag{18}$$

With $\gamma = 0$, this establishes the followup quantitative claim made in [21, §3.3]. To some extent, in the inhomogeneous case it would be more natural and desirable to establish (18) in which $\mathbb{G}(x,\gamma)$ is defined as a subset of

$$\mathbf{Bad}_\gamma := \big\{x \in \mathbb{I} : \liminf_{q\to\infty} q\|qx - \gamma\| > 0\big\} \, .$$

The point is that for $y \notin \mathbf{Bad}_\gamma$, the corresponding inhomogeneous version of Littlewood's conjecture; namely

$$\liminf_{q\to\infty} q \, \|qx\| \, \|qy - \gamma\| = 0 \qquad \text{for all } x, y \in \mathbb{I} \, ,$$

is trivially true. The major obstacle preventing us from establishing the desired inhomogeneous statement (18) with $\mathbb{G}(x,\gamma) \subset \mathbf{Bad}_\gamma$ is that we are unable to prove the existence of a probability measure μ supported on a subset of $\mathbf{Bad}_\gamma$ with transform decay as in Theorem 1. With this in mind, we suspect that a statement of the following type is true.

Claim 1. *Let* $\gamma \in \mathbb{I}$. *Then for any sufficiently small constant* $c > 0$, *the set*

$$\mathbf{Bad}_\gamma(c) := \big\{x \in \mathbb{I} : q\|qx - \gamma\| > c \ \ \forall \ q \in \mathbb{N}\big\}$$

supports a probability measure μ *with* $\widehat{\mu}$ *satisfying* (8) *for some* $A > 2$.

Trivially, $\mathbf{Bad}_\gamma(c)$ is a subset of $\mathbf{Bad}_\gamma$ for any $c > 0$. We suspect that the above claim is true with $\widehat{\mu}$ satisfying polynomial decay as in the homogeneous case. Having said this, as far as we are aware, we do not even know that the sets $\mathbf{Bad}_\gamma(c)$ are M_0-sets. Establishing this is of independent interest and can be regarded as a first step towards Claim 1.

Note that if Claim 1 is true then it follows (see Remark 1) that μ–almost all numbers in $\mathbf{Bad}_\gamma(c)$ are normal. Proving the existence of normal numbers in $\mathbf{Bad}_\gamma$ is an interesting problem in its own right. To the best of our knowledge, currently we do not know of any such numbers when γ is irrational.

Claim 2. *For any $\gamma \in \mathbb{I}$, there exists at least one normal number in* $\mathbf{Bad}_\gamma$.

We suspect that the set of normal numbers in $\mathbf{Bad}_\gamma$ is of full dimension irrespective of the choice of γ. This is true in the homogenous case.

The paper [12] further develops the ideas from [21]. Within the setting of the inhomogeneous version of Littlewood's conjecture stated above, Haynes, Jensen & Kristensen establish the following result.

Theorem HJK. *Fix $\epsilon > 0$ and let X be a countable subset of* **Bad**. *Then there exists a subset $\mathbb{G}(X)$ of* **Bad** *with full dimension such that for any $y \in \mathbb{G}(X)$, $x \in X$, and $\gamma \in \mathbb{I}$*

$$q\|qx\|\,\|qy-\gamma\| \;\leq\; 1/(\log q)^{1/2-\epsilon} \quad \text{for infinitely many } q \in \mathbb{N}\,. \tag{19}$$

A few comments are in order. As we shall see in a moment, the fact that the statement is true for a countable subset $X \subset \mathbf{Bad}$ is not the meat. The strength of the theorem lies in the fact that set $\mathbb{G}(X)$ is independent of the inhomogeneous factor γ. With this in mind, we briefly describe the key step in the proof of Theorem HJK. Fix $\lambda \in (0,1]$, and let $\psi_\lambda(q) := 1/(\log q)^\lambda$ and $\Psi_\lambda(N) := \sum_{n=1}^{N} \psi_\lambda(q_n)$. Note that $\Psi_\lambda(N) \to \infty$ as $N \to \infty$ for $\lambda \leq 1$. It is this that is exploited in the proof rather than actually 'counting solutions' – see Remark 9 below. Fix $M \geq 3$, and let $x \in F_M$ and $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be the sequence of natural numbers arising from the denominators of the convergents p_n/q_n of x. Finally, let μ be the Kaufman measure supported on F_M and

$$\mathbb{G}_M(x,\psi_\lambda) := \big\{y \in F_M : R(y,N;\gamma,\psi_\lambda,\mathcal{A}) \geq 2\Psi_\lambda(N) \;\forall\, N \gg 1,\, \forall\, \gamma \in \mathbb{I}\big\}\,, \tag{20}$$

where R is the counting function given by (7). The key to the proof of the theorem lies in showing that for any $\lambda \in (0,1/2)$,

$$\mu(\mathbb{G}_M(x,\psi_\lambda)) = 1\,. \tag{21}$$

It follows that $\mu\big(\cap_{x\in X}\,\mathbb{G}_M(x,\psi_\lambda)\big) = 1$ for any countable set $X \subset F_M$. In turn, this leads to a theorem that is valid for any countable set of badly approximable numbers. The proof of (21) makes nifty use of the Erdős-Turán inequality [19, Corollary 1.1] and a standard effective version of the (divergent) Borel-Cantelli Lemma (see Lemma 2.3 in [11]) which we shall exploit too. As in [21], the proof also makes use of the explicit nature of the function ψ_λ and the fact that the Fourier transform of the Kaufmann measure has polynomial decay.

In order to compare the above work of Haynes, Jensen & Kristensen [12] to our work, fix $\gamma \in \mathbb{I}$ and with (20) in mind, let

$$\mathbb{G}_M(x,\psi_\lambda,\gamma) := \big\{y \in F_M : R(y,N;\gamma,\psi_\lambda,\mathcal{A}) \geq 2\Psi_\lambda(N) \;\forall\, N \gg 1\big\}\,. \tag{22}$$

Then,

$$\mathbb{G}_M(x,\psi_\lambda) = \bigcap_{\gamma\in\mathbb{I}} \mathbb{G}_M(x,\psi_\lambda,\gamma)\,. \tag{23}$$

A simple consequence of Theorem 1 is that for any $\gamma \in \mathbb{I}$ and any $\lambda \in (0,1]$,

$$\mu(\mathbb{G}_M(x,\psi_\lambda,\gamma)) = 1\,. \tag{24}$$

Indeed, this statement with $\lambda = 1$ is at the heart of establishing (18). Also, note that to apply Theorem 1 we only need μ to have sufficient logarithmic transform decay. However, since the intersection in (23) is uncountable we are unable to conclude the full measure statement (21). Just to reiterate, the approach taken in [12] yields (21) as long as $\lambda < 1/2$. Unfortunately, it is not at all clear how to modify the method for $\lambda \geq 1/2$ even within the context of establishing the weaker statement (24).

Remark 8. Since $\Psi_\lambda(N) \to \infty$ as $N \to \infty$ for $\lambda \leq 1$, it follows via (21) that for any $\lambda \in (0, 1/2)$

$$\mu\Big(\bigcap_{\gamma\in\mathbb{I}} W_{\mathcal{A}}(\gamma,\psi_\lambda) \cap F_M\Big) = 1\,,$$

and so $\mu\big(W_{\mathcal{A}}(\gamma,\psi_\lambda) \cap F_M\big) = 1$ for any $\gamma \in \mathbb{I}$. Clearly, Corollary 1 implies the latter full measure statement for any $\lambda \in (0, 1]$ but it cannot be exploited to yield the former.

Remark 9. In [12, §4], the authors suggest that the approximating function $1/(\log q)^{1/2-\epsilon}$ in the right hand side of the inequality in (19) can in fact be replaced by $1/\log q$. This would follow if we could prove (21) with $\lambda = 1$. In fact, it would suffice to prove this for a weaker form of the set $\mathbb{G}_M(x, \psi_\lambda)$ in which the growth condition on the counting function R appearing in (20) is replaced by the condition that $R(y, N; \gamma, \psi_\lambda, \mathcal{A}) \to \infty$ as $N \to \infty$.

For the sake of completeness, we finish with a short discussion on Salem sets. For both consistency and simplicity, we restrict the discussion to subsets F of $\mathbb{I}$. The *Fourier dimension* of $F \subset \mathbb{I}$ is defined by

$$\dim_F F := \sup\Big\{0 \leq \eta \leq 1 : \exists\, \mu \in M_1^+(F) \ \text{ with } \ \widehat{\mu}(t) = O(|t|^{-\eta/2}) \text{ as } |t| \to \infty\ \Big\}.$$

where $M_1^+(F)$ denotes the set of all positive Borel probability measures with support in F. A simple consequence of the classical result of Frostman mentioned right at the start of the paper (namely, if $\widehat{\mu}(t) = O(|t|^{-\eta/2})$ then $\dim F \geq \min\{1, \eta\}$), is that the Fourier dimension is bounded above by the Hausdorff dimension. A set F with $\dim_F F = \dim F$ is called a *Salem set*. Observe that Theorem 1 and its corollary are applicable to any Salem set with strictly positive dimension. In fact, this is also true for the non-lacunary results presented in the next section. To the best of our knowledge, it is unknown whether or not the badly approximable subsets F_M are Salem sets. However, the story is quite different for well approximable subsets of $\mathbb{I}$. To start with, given $\tau \geq 1$, let $\psi_\tau(q) := q^{-\tau}$ and consider the classical homogeneous set $W(0, \psi_\tau)$ of τ-well approximable numbers. By definition, this corresponds to $W_{\mathcal{A}}(\gamma, \psi)$ given by (11) with $\mathcal{A} = \mathbb{N}$, $\gamma = 0$ and $\psi = \psi_\tau$. By Dirichlet's theorem, $W(0, \psi_\tau) = \mathbb{I}$ when $\tau = 1$ and for $\tau > 1$, a classical theorem of Jarnik and Besicovitch (see [3, §1.3.2]) states that $\dim W(0, \psi_\tau) = 2/(\tau + 1)$. In another elegant paper [14], Kaufman constructed a probability measure μ supported on $W(0, \psi_\tau)$ for any $\tau > 1$ satisfying the decay property:

$$\widehat{\mu}(t) = |t|^{-\frac{1}{\tau+1}}\ o\left(\log|t|\right) \qquad \text{as} \quad |t| \to \infty\,. \tag{25}$$

The upshot is that $W(0, \psi_\tau)$ is a Salem set for any $\tau > 1$. Bluhm [4] subsequently generalised this statement to arbitrary decreasing functions ψ. In short, for ψ-well approximable sets $W(0, \psi)$ the quantity τ in the Jarnik-Besicovitch theorem and in (25) is replaced by the quantity $\lambda(\psi) := \liminf_{q\to\infty} -\log\psi(q)/\log q$; namely the lower order at infinity of the function $1/\psi$. In the last couple of years, Hambrook [9] and independently Zafeiropoulos [27] have extended Bluhm's work to the inhomogeneous setup. Thus, for any $\gamma \in \mathbb{I}$ and any real, positive decreasing function $\psi : \mathbb{N} \to \mathbb{I}$ with $\lambda(\psi) > 1$, we now know that the set $W(\gamma, \psi)$ is a Salem set. Hambrook actually does a lot more; for example, he obtains results for the higher dimensional analogues of the restricted 'denominators' sets $W_{\mathcal{A}}(\gamma, \psi)$.

The main purpose of this section was to compare our results for lacunary sequences (namely, Theorem 1 and Corollary 1) with previous related works. As far as we are aware, if the growth of the sequence is slower than lacunary, then nothing is known even within the context of Corollary 1, let alone Theorem 1.

1.2 Beyond lacunarity

With reference to Corollary 1, in the case of convergence we are able to prove the following stronger statement for general sequences.

Theorem 2. *Let μ be a probability measure supported on a subset F of $\mathbb{I}$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers. Let $\gamma \in \mathbb{I}$ and $\psi : \mathbb{N} \to \mathbb{I}$ be a real, positive function. Suppose that at least one of the following two conditions is satisfied:*

$$\sum_{n=1}^{\infty} \max_{k\in\mathbb{Z}/\{0\}} |\hat{\mu}(kq_n)| < \infty \tag{26}$$

$$\sum_{n=1}^{\infty} \sum_{k\in\mathbb{Z}/\{0\}} \frac{|\widehat{\mu}(kq_n)|}{|k|} < \infty. \tag{27}$$

Then

$$\mu(W_{\mathcal{A}}(\gamma;\psi)) = 0 \quad \text{if} \quad \sum_{n=1}^{\infty} \psi(q_n) < \infty. \tag{28}$$

It is easily verified that Theorem 2 implies the convergence case of Corollary 1. Indeed, if $\mathcal{A} = (q_n)_{n=1}^{\infty}$ is lacunary and μ satisfies condition (8) for some $A > 1$, then

$$\begin{aligned} \sum_{n=1}^{\infty} \max_{k\in\mathbb{Z}/\{0\}} |\hat{\mu}(kq_n)| \;&\ll\; \sum_{n=1}^{\infty} \frac{1}{(\log q_n)^A} \\ &\ll\; \sum_{n=1}^{\infty} \frac{1}{n^A} \;<\; \infty \end{aligned}$$

and so condition (26) of Theorem 2 is satisfied. The upshot is that Theorem 2 implies the convergence case of Corollary 1 under the weaker assumption that (8) is satisfied for some $A > 1$. As pointed out above in Remark 4 we can actually get away with even slightly less.

In the case of divergence and within the context of Theorem 1, we are able to go beyond lacunarity if we restrict the prime divisors of the elements in the given integer sequence to lie in a finite set. More precisely, fix $k \in \mathbb{N}$ and let

$$\mathcal{S} := \{p_1, \ldots, p_k\} \tag{29}$$

be a set of k distinct primes $p_1, \dots, p_k$. In turn, given $\mathcal{S}$ let

$$\mathcal{A}_{\mathcal{S}} := \left\{ \prod_{i=1}^{k} p_i^{a_i} \; : \; a_1, \dots, a_k \in \mathbb{Z}_{\geq 0} \right\} \tag{30}$$

be the set of positive integers with prime divisors restricted to $\mathcal{S}$. In other words, $\mathcal{A}_{\mathcal{S}}$ is precisely the set of smooth numbers over $\mathcal{S}$. Obviously, if p is the largest prime among $\mathcal{S}$ then by definition, every integer in $\mathcal{A}_{\mathcal{S}}$ is p-smooth.

The following constitutes our main result for sequences that are not necessarily lacunary.

Theorem 3. *Let μ be a probability measure supported on a subset F of $\mathbb{I}$. Let*

$$\mathcal{A} = (q_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}_{\mathcal{S}}$$

be an increasing sequence of natural numbers. Let $\gamma \in \mathbb{I}$ and $\psi : \mathbb{N} \to \mathbb{I}$ be a real, positive function. Suppose there exists a constant $A > 2k$ so that (8) is satisfied. Then, for any $\epsilon > 0$ the counting function $R(x, N)$ satisfies

$$R(x, N) = 2\Psi(N) + O\Big(\Psi(N)^{1/2} \left(\log\left(\Psi(N) + 2\right)\right)^{2+\varepsilon}\Big) \tag{31}$$

for μ-almost all $x \in F$, where $\Psi(N)$ is given by (10).

Theorem 3 answers the question raised at the end of §1.1 concerning the specific sequence $\mathcal{A} = \{2^a 3^b \; : \; a, b \in \mathbb{Z}_{\geq 0}\}$. Namely, the analogue of Theorem 1 is valid for this specific $\mathcal{A}$ if the decay rate constant A in (8) is strictly larger than four. Note that in the case of one prime p, the corresponding sequence $\mathcal{A} = \{p^n : n \in \mathbb{N}\}$ is trivially lacunary. However, due to the fact that $\mathcal{S}$ is a finite set of primes the above theorem does not cover arbitrary lacunary sequences unlike Theorem 1. Nevertheless, Theorem 3 does give a better error term than Theorem 1.

The following statement is a direct consequence of Theorem 3. It follows in exactly the same way as Corollary 1 follows from Theorem 1.

Corollary 2. *Let μ be a probability measure supported on a subset F of $\mathbb{I}$. Let $\mathcal{A} = (q_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}_{\mathcal{S}}$ be an increasing sequence of natural numbers. Let $\gamma \in \mathbb{I}$ and $\psi : \mathbb{N} \to \mathbb{I}$ be a real, positive function. Suppose there exists a constant $A > 2k$, so that (8) is satisfied. Then*

$$\mu\big(W_{\mathcal{A}}(\gamma; \psi) \cap F\big) = \begin{cases} 0 & \textit{if} \quad \sum_{n=1}^{\infty} \psi(q_n) < \infty \, , \\ 1 & \textit{if} \quad \sum_{n=1}^{\infty} \psi(q_n) = \infty \, . \end{cases}$$

It can be verified that any given increasing sequence $\mathcal{A} = (q_n)_{n=1}^{\infty} \subseteq \mathcal{A}_{\mathcal{S}}$ satisfies the growth condition

$$\log q_n \; > \; C \, n^{1/B} \qquad \forall \; n \geq 2 \, , \tag{32}$$

for some constants $B \geq 1$ and $C > 0$. Indeed, we can always get away with $B = k$ and $C = (\log 2)/2$ irrespective of the choice of $\mathcal{A} \subseteq \mathcal{A}_{\mathcal{S}}$ since $\mathcal{A}_{\mathcal{S}}$ satisfies (32) with these choices of B and C (see Appendix B of §7). Hence,

$$\sum_{n=1}^{\infty} \frac{1}{(\log q_n)^A} \;\ll\; \sum_{n=1}^{\infty} \frac{1}{n^{A/k}} \;<\; \infty$$

for any $A > k$ and it follows via Theorem 2 that we only require that $A > k$ in the convergence case of the above corollary. The stronger condition that $A > 2k$ on the decay rate constant A is required to establish Theorem 3. This condition then carries over to the corollary since the divergence case is directly deduced from the theorem. We suspect that within the context of the corollary, $A > k$ would suffice even in the divergence case.

Remark 10. As the following example indicates, it is necessary to impose some condition (either directly or indirectly) on the growth of the sequence in Theorem 3 and indeed its corollary. With reference to §1.1.1, let μ be the Kaufman measure supported on the badly approximable subset $F_M \subset \mathbf{Bad}$ for some $M \geq 2$. Let $\mathcal{A} = \mathbb{N}$, $\gamma = 0$ and $\psi(q) = 1/(q \log q)$. Then the sum appearing in Corollary 2 diverges but in view of the definition of **Bad**, we trivially have that

$$W_{\mathcal{A}}(0; \psi) \cap F_M = \emptyset\,.$$

We will deduce Theorem 3 from a general statement for sequences satisfying the growth condition (32) and the following 'separation' condition. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers and let $\alpha \in (0, 1)$ be a real number. We say that $\mathcal{A}$ is *α-separated* if there exists a constant $m_0 \in \mathbb{N}$ so that for any integers $m_0 \leq m < n$, if

$$1 \leq |sq_m - tq_n| < q_m^{\alpha}$$

for some $s, t \in \mathbb{N}$, then

$$s > m^{12}.$$

Note that when $\mathcal{A}$ is lacunary then the growth condition (32) is trivially satisfied with $B = 1$ but $\mathcal{A}$ need not be α-separated[1]. Thus, we can not deduce Theorem 1 directly from the following result.

Theorem 4. *Let μ be a probability measure supported on a subset F of $\mathbb{I}$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers that (i) satisfies the growth condition* (32) *for some constants $B \geq 1$ and $C > 0$, and (ii) is α-separated. Let $\gamma \in \mathbb{I}$ and $\psi : \mathbb{N} \to \mathbb{I}$ be a real, positive function. Suppose there exists a constant*

$$A \; > \; 2B\,, \tag{33}$$

so that (8) *is satisfied. Then, for any $\epsilon > 0$ the counting function $R(x, N)$ satisfies*

$$R(x, N) \;=\; 2\Psi(N) + O\Big(\left(\Psi(N) + E(N)\right)^{1/2} \left(\log(\Psi(N) + E(N) + 2)\,\right)^{2+\varepsilon}\Big) \tag{34}$$

for μ-almost all $x \in F$, where $\Psi(N)$ is given by (10) *and*

$$E(N) \;:=\; \sum\sum_{1\leq m<n\leq N} (q_m, q_n)\;\min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right)\,. \tag{35}$$

[1]For example, consider the sequence $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ given by $q_n := 2^n + \varepsilon_n$, where $\varepsilon_n = 1$ for even n and $\varepsilon_n = 0$ for odd n. This sequence is clearly lacunary with $q_{n+1}/q_n \geq 8/5$ (the minimum of the left hand side is attained at $n = 2$). Moreover, $q_n - 2q_{n-1} = 1$ whenever n is even and so $\mathcal{A}$ is not α-separated.

We have already mentioned that any increasing sequence $\mathcal{A} \subseteq \mathcal{A}_{\mathcal{S}}$ satisfies the growth condition (32) with $B = k$. Thus, Theorem 3 will follow from Theorem 4 on showing that any such sequence is α-separated and that the gcd term $E(N)$ appearing in the 'error' term is less than the 'main' term $\Psi(N)$. This will be the subject of §6 and is very much self-contained. In short, the key ingredient towards showing α-separation is an explicit linear forms in logarithms result by Baker and Wüstholz, while the key to dealing with the the gcd term $E(N)$ is to show that the sum

$$\sum_{m=1}^{n-1} \frac{(q_m, q_n)}{q_n} \qquad (n \geq 2) \tag{36}$$

is bounded from above by an absolute constant that depends only on the set $\mathcal{S}$ of primes.

Remark 11. The conclusion of Theorem 4 is in line with its Lebesgue measure counterpart [11, Theorem 3.1]. The latter essentially dates back to a paper of LeVeque from 1959.

Remark 12. Observe that under the hypotheses of Theorem 4, if $\Psi(N) \to \infty$ as $N \to \infty$ and

$$E(N) \;=\; O\Big(\Psi(N)^{2-\epsilon}\Big) \tag{37}$$

for some $\epsilon > 0$, then $R(x, N)$ is asymptotically equal to $2\,\Psi(N)$ for μ-almost all $x \in F$. Clearly, this together with Theorem 2 trivially implies the conclusion of Corollary 2 under the significantly weaker assumptions of Theorem 4.

Remark 13. As already mentioned, we will deduce Theorem 3 from Theorem 4 and in order to do so we show that any sequence $\mathcal{A} \subseteq \mathcal{A}_{\mathcal{S}}$ is α-separated. This is precisely the statement of Proposition 3 in §6.1 and its proof makes direct use of the fact that (32) automatically holds for such sequences. We shall show in §6.1.1, that the proof can be easily adapted to prove that if an increasing sequence $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ of natural numbers satisfies:

(i) the growth condition (32) for some constants $B \geq 1$ and $C > 0$,

(ii) there exist constants $n_0 \in \mathbb{N}$ and $D \in \mathbb{N}$, such that for any $n \geq n_0$

  (a) $\#\{\, p \text{ prime} : \, p|q_n\} \;\leq\; D$

  (b) if prime $p|q_n$, then $\log p \;\leq\; (\log q_n)^{\frac{1-\epsilon}{2D}}$ for some $\epsilon > 0$,

then $\mathcal{A}$ is α-separated. We say that $\mathcal{A}$ *satisfies Property D* if the growth condition (i) and condition (ii) on its prime divisors are satisfied. It is clear that if $\mathcal{A} \subseteq \mathcal{A}_{\mathcal{S}}$ then it automatically satisfies Property D. Indeed the latter significantly broadens the explicit class of sequences for which Theorem 4 applies. The draw back is that for an arbitrary sequence satisfying Property D it is not possible to control the gcd term $E(N)$ appearing in the 'error' term of (34). Indeed, there exist (see Appendix C of §7) sequences satisfying Property D and associated functions $\psi : \mathbb{N} \to \mathbb{I}$ such that for any $T > 0$

$$E(N) \;\gg\; \Psi(N)^T \qquad \forall \;\; N \in \mathbb{N}. \tag{38}$$

As a consequence, in general we are not even able to show (37) with $\epsilon = 0$ let alone with $\epsilon = 1$ (that enables us to reduce (34) to (31)) as in the situation when $\mathcal{A} \subseteq \mathcal{A}_{\mathcal{S}}$. Nevertheless, it is relatively straightforward to construct explicit sequences $\mathcal{A} \nsubseteq \mathcal{A}_{\mathcal{S}}$ (for any $\mathcal{S}$) that satisfy Property D and for which $E(N) = O(\Psi(N))$. We stress that for such sequences, the counting function $R(x, N)$ satisfies (31) for μ-almost all $x \in F$. We now show that perturbing a given sequence $\mathcal{A}_{\mathcal{S}} = (q_n)_{n\in\mathbb{N}}$ in the following manner has the desired effect. Let $\mathcal{P}$ be a infinite set of distinct primes not in $\mathcal{S}$ such that $\sum_{p\in\mathcal{P}} p^{-1} \leq 1$. Now choose $p_1 \in \mathcal{P}$ such that $p_1 > q_1$ and p_1 is larger than any of the k primes in $\mathcal{S}$. Then choose s sufficiently large so that

$$\log p_1 \ \leq\ (\log q_s)^{\frac{1}{3(k+1)}}\ , \tag{39}$$

and let t_1 be the unique integer such that $q_{t_1} < \widetilde{q}_{t_1} := q_s p_1 < q_{t_1+1}$. Now replace $q_{t_1} \in \mathcal{A}_{\mathcal{S}}$ by $\widetilde{q}_{t_1}$ and observe that (32) holds for $\widetilde{q}_{t_1}$ since it holds for any element of $\mathcal{A}_{\mathcal{S}}$ with $B = k$ and $C = (\log 2)/2$. The previous terms $q_1, \ldots, q_{t_1-1} \in \mathcal{A}_{\mathcal{S}}$ remain unchanged. Next, choose $p_2 \in \mathcal{P}$ such that $p_2 > q_{t_1+1}$ and choose s sufficiently large so that (39) holds with p_1 replaced by p_2. Let t_2 be the unique integer such that $q_{t_2} < \widetilde{q}_{t_2} := q_s p_2 < q_{t_2+1}$. Now replace $q_{t_2} \in \mathcal{A}_{\mathcal{S}}$ by $\widetilde{q}_{t_2}$ and keep the previous terms $q_{t_1+1}, \ldots, q_{t_2-1} \in \mathcal{A}_{\mathcal{S}}$ unchanged. On repeating the above procedure indefinitely, we end up with a sequence $\widetilde{\mathcal{A}} = (\widetilde{q}_n)_{n\in\mathbb{N}}$ which by construction satisfies Property D with $D = k+1$ and contains arbitrarily large prime divisors. In view of the latter, $\widetilde{\mathcal{A}}$ cannot be a subsequence of smooth numbers over a finite set of primes. It now remains to show that $E(N) = O(\Psi(N))$ and as already mentioned earlier this follows on showing that (36) is bounded above by an absolute constant $\widetilde{K}$ for $\widetilde{\mathcal{A}}$. The latter is not hard to show assuming it holds (which it does, see Theorem 5 in §6.2) for the sequence $\mathcal{A}_{\mathcal{S}}$. To see this, fix $n \geq 2$ and first assume that $\widetilde{q}_n = q_n \in \mathcal{A}_{\mathcal{S}}$; that is, $n \neq t_i$ for some $i \in \mathbb{N}$. Then, it follows that

$$\sum_{m=1}^{n-1} \frac{(\widetilde{q}_m, \widetilde{q}_n)}{\widetilde{q}_n} \ \leq\ \sum_{m=1}^{n-1} \frac{(q_m, q_n)}{q_n} \ +\ \sum_{\substack{m=1: \\ \widetilde{q}_m \notin \mathcal{A}_{\mathcal{S}}}}^{n-1} \frac{(\widetilde{q}_m, q_n)}{q_n} \ \leq\ K + \sum_{p\in\mathcal{P}} p^{-1} \ \leq\ K + 1\,, \tag{40}$$

where K is the absolute constant associated with $\mathcal{A}_{\mathcal{S}}$ that upper bounds (36). Here we use the fact that if $\widetilde{q}_m \notin \mathcal{A}_{\mathcal{S}}$, then by definition $\widetilde{q}_m = q_s p$ for some $s < m$ and $p \in \mathcal{P}$, and it follows that $(\widetilde{q}_m, q_n) = (q_s, q_n) \leq q_s$. Now suppose that $\widetilde{q}_n \neq q_n \in \mathcal{A}_{\mathcal{S}}$; that is $\widetilde{q}_n = q_s p$ for some $s < n$ and $p \in \mathcal{P}$. Let p_* be the smallest prime amongst those in $\mathcal{S}$ and let u be the unique integer such that $p_*^u < p < p_*^{u+1}$. Then, it follows that $q_s p_*^u < \widetilde{q}_n < q_{n_*} := q_s p_*^{u+1} \in \mathcal{A}_{\mathcal{S}}$ for some $n_* > n$ and also note that $(\widetilde{q}_m, \widetilde{q}_n) \leq (\widetilde{q}_m, q_{n_*})$. This together with (40) implies that

$$\sum_{m=1}^{n-1} \frac{(\widetilde{q}_m, \widetilde{q}_n)}{\widetilde{q}_n} \ \leq\ p_* \sum_{m=1}^{n_*-1} \frac{(\widetilde{q}_m, q_{n_*})}{q_{n_*}} \ \leq\ p_*(K+1)\,.$$

The upshot is that (36) is bounded above by the absolute constant $\widetilde{K} := p_*(K+1)$ for all $n \geq 2$.

Remark 14. In the homogeneous case ($\gamma = 0$), it is possible to give a direct proof of Corollary 2 that enables us to replace the condition that $\mathcal{A} \subseteq \mathcal{A}_{\mathcal{S}}$ by the significantly milder condition that $\mathcal{A}$ satisfies Property D. This will be the subject of a forthcoming note. In short, the overall strategy is to establish local quasi-independence on average – see [3, Equation (2.6)] which, in itself, is a consequence of [2, Propositions 1-3]. A key ingredient, that is potentially of independent interest, is to first show that the decay property (8) holds locally in the following sense. Let μ be a probability measure supported on a subset F of $\mathbb{I}$ and suppose there is a constant $A > 1$ so that (8) is satisfied. Then for any ball $B \subset \mathbb{I}$ with $\mu(B) > 0$

$$\widehat{\mu}_B(t) = \frac{1}{\mu(B)} O\left((\log |t|)^{-A}\right) ,$$

where μ_B is the normalised restriction of μ to B.

2 Basic estimates and establishing Theorem 2

In this section we present various basic estimates that will be required in proving our main results. Indeed, the estimates provided will be enough to deduce the general convergence statement Theorem 2. Given $\gamma \in \mathbb{I}$, $\psi : \mathbb{N} \to \mathbb{I}$ and $q \in \mathbb{N}$, let

$$E_q^\gamma = E_q^\gamma(\psi) := \{x \in \mathbb{I} : \|qx - \gamma\| \leq \psi(q)\} . \tag{41}$$

By definition, given any increasing sequence of natural numbers $\mathcal{A} = (q_n)_{n=1}^{\infty}$, we have that

$$R(x, N) = \#\{1 \leq n \leq N : x \in E_{q_n}^\gamma\}$$

where $R(x, N)$ is the counting function given by (7). Also, the set $W_{\mathcal{A}}(\gamma; \psi)$ defined by (11) is precisely the set of real numbers in $\mathbb{I}$ which lie in infinitely many of the sets $E_{q_n}^\gamma$; that is,

$$W_{\mathcal{A}}(\gamma; \psi) \; = \; \limsup_{n \to \infty} E_{q_n}^\gamma .$$

Thus the sets E_q^γ with $q \in \mathbb{N}$ can be regarded as being the 'building blocks' of the basic objects studied in this paper. Let μ be a probability measure supported on a subset F of $\mathbb{I}$. We now proceed to estimate the μ-measure of these building blocks.

2.1 Estimating $\mu(E_q^\gamma)$

Let ε and δ be real numbers such that $0 < \varepsilon \leq 1$ and $0 < \delta < 1/4$. Let $\chi_\delta : \mathbb{I} \to \mathbb{R}$ be the characteristic function defined by

$$\chi_\delta(x) := \begin{cases} 1 & \text{if } \|x\| \leq \delta \\ 0 & \text{if } \|x\| > \delta , \end{cases}$$

and let $\chi_{\delta,\varepsilon}^+ : \mathbb{I} \to \mathbb{R}$ and $\chi_{\delta,\varepsilon}^- : \mathbb{I} \to \mathbb{R}$ be the continuous upper and lower approximations of χ_δ given by

$$\chi_{\delta,\varepsilon}^+(x) := \begin{cases} 1 & \text{if } \|x\| \leq \delta, \\ 1 + \dfrac{1}{\delta\varepsilon}(\delta - \|x\|) & \text{if } \delta < \|x\| \leq (1+\varepsilon)\delta \\ 0 & \text{if } \|x\| > (1+\varepsilon)\delta , \end{cases}$$

and

$$\chi^{-}_{\delta,\varepsilon}(x) := \begin{cases} 1 & \text{if } \ \|x\| \le (1-\varepsilon)\delta \\ \dfrac{1}{\delta\varepsilon}(\delta - \|x\|) & \text{if } \ (1-\varepsilon)\delta < \|x\| \le \delta \\ 0 & \text{if } \ \|x\| > \delta\,. \end{cases}$$

Clearly, both $\chi^{+}_{\delta,\varepsilon}$ and $\chi^{-}_{\delta,\varepsilon}$ are periodic functions with period 1. Next, given a real positive function $\psi : \mathbb{N} \to \mathbb{I}$ and any integer $q \ge 4$, consider the functions $W^{+}_{q,\gamma,\varepsilon,\psi}$ and $W^{-}_{q,\gamma,\varepsilon,\psi}$ defined by

$$W^{+}_{q,\gamma,\varepsilon}(x) = W^{+}_{q,\gamma,\varepsilon,\psi}(x) \ := \ \Big(\sum_{p=0}^{q-1} \delta_{\frac{p+\gamma}{q}}(x)\Big) * \chi^{+}_{\frac{\psi(q)}{q},\varepsilon}(x) \tag{42}$$

and

$$W^{-}_{q,\gamma,\varepsilon}(x) = W^{-}_{q,\gamma,\varepsilon,\psi}(x) \ := \ \Big(\sum_{p=0}^{q-1} \delta_{\frac{p+\gamma}{q}}(x)\Big) * \chi^{-}_{\frac{\psi(q)}{q},\varepsilon}(x)\,,$$

where as usual $*$ denotes convolution and δ_x denotes the Dirac delta-function at the point $x \in \mathbb{R}$. As alluded to in defining the functions $W^{+}_{q,\gamma,\varepsilon,\psi}$ and $W^{-}_{q,\gamma,\varepsilon,\psi}$, we will often exclude stressing their dependance on the function ψ since it will often be fixed in any given discussion or argument. With this in mind, it is easily verified that

$$W^{+}_{q,\gamma,\varepsilon}(x) = \sum_{p=0}^{q-1} \chi^{+}_{\frac{\psi(q)}{q},\varepsilon}\left(x - \tfrac{p+\gamma}{q}\right)$$

and

$$W^{-}_{q,\gamma,\varepsilon}(x) = \sum_{p=0}^{q-1} \chi^{-}_{\frac{\psi(q)}{q},\varepsilon}\left(x - \tfrac{p+\gamma}{q}\right)\,.$$

It thus follows that for any $0 < \varepsilon \le 1$ and any integer $q \ge 4$,

$$\int_0^1 W^{-}_{q,\gamma,\varepsilon}(x)\mathrm{d}\mu(x) \ \le \ \mu(E^{\gamma}_{q}) \ \le \ \int_0^1 W^{+}_{q,\gamma,\varepsilon}(x)\mathrm{d}\mu(x). \tag{43}$$

We now proceed to evaluate the above integrals by considering the Fourier series expansions of $W^{+}_{q,\gamma,\varepsilon}$ and $W^{-}_{q,\gamma,\varepsilon}$. When there is no risk of confusion, we will simply write $W^{\pm}_{q,\gamma,\varepsilon}$ to mean both the 'upper' and 'lower' functions. Similarly, we will write $\chi^{\pm}_{\delta,\varepsilon}$ when we refer to both $\chi^{+}_{\delta,\varepsilon}$ and $\chi^{-}_{\delta,\varepsilon}$. With this in mind, for $k \in \mathbb{Z}$ let $\widehat{\chi}^{\pm}_{\delta,\varepsilon}(k)$ and $\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(k)$ denote the k–th Fourier coefficient of $\chi^{\pm}_{\delta,\varepsilon}(k)$ and $W^{\pm}_{q,\gamma,\varepsilon}(k)$, respectively. A straightforward calculation yields that

$$\widehat{\chi}^{+}_{\delta,\varepsilon}(k) \ = \ \begin{cases} (2+\varepsilon)\delta & \text{if } \ k = 0 \\ \dfrac{\cos(2\pi k\delta) - \cos(2\pi k\delta(1+\varepsilon))}{2\pi^2 k^2 \delta\varepsilon} & \text{if } \ k \ne 0\,, \end{cases} \tag{44}$$

and

$$\widehat{\chi}^{-}_{\delta,\varepsilon}(k) \;=\; \begin{cases} (2-\varepsilon)\delta & \text{if} \;\; k=0 \\[2ex] \dfrac{\cos(2\pi k\delta(1-\varepsilon))-\cos(2\pi k\delta)}{2\pi^2k^2\delta\varepsilon} & \text{if} \;\; k\neq 0\,. \end{cases} \tag{45}$$

Since the functions $W^{\pm}_{q,\gamma,\varepsilon}$ are defined via convolution, we have that

$$\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(k) = \sum_{p=0}^{q-1} \widehat{\delta}_{\frac{p+\gamma}{q}}(k)\,.\,\widehat{\chi}^{\pm}_{\frac{\psi(q)}{q},\varepsilon}(k)\,.$$

Trivially,

$$\widehat{\delta}_{\frac{p+\gamma}{q}}(k) = \exp\left(-\frac{2\pi i k(p+\gamma)}{q}\right)\,.$$

Thus, it follows from (44) that for $k\neq 0$,

$$\widehat{W}^{+}_{q,\gamma,\varepsilon}(k) \;=\; \begin{cases} \exp\left(-\dfrac{2\pi i k\gamma}{q}\right) \dfrac{q\left(\cos(2\pi k\psi(q)q^{-1})-\cos(2\pi k\psi(q)q^{-1}(1+\varepsilon))\right)}{2\pi^2k^2\psi(q)q^{-1}\varepsilon} & \text{if} \;\; q \mid k \\[2ex] 0 & \text{if} \;\; q\nmid k\,, \end{cases} \tag{46}$$

and for $k=0$,

$$\widehat{W}^{+}_{q,\gamma,\varepsilon}(0) = (2+\varepsilon)\,\psi(q)\,. \tag{47}$$

Similarly, it follows from (45) that for $k\neq 0$,

$$\widehat{W}^{-}_{q,\gamma,\varepsilon}(k) \;=\; \begin{cases} \exp\left(-\dfrac{2\pi i k\gamma}{q}\right) \dfrac{q\left(\cos(2\pi k\psi(q)q^{-1}(1-\varepsilon))-\cos(2\pi k\psi(q)q^{-1})\right)}{2\pi^2k^2\psi(q)q^{-1}\varepsilon} & \text{if} \;\; q \mid k \\[2ex] 0 & \text{if} \;\; q\nmid k\,, \end{cases} \tag{48}$$

and for $k=0$,

$$\widehat{W}^{-}_{q,\gamma,\varepsilon}(0) = (2-\varepsilon)\,\psi(q)\,. \tag{49}$$

It is easily seen that $\sum_{k\in\mathbb{Z}} \left|\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(k)\right| < \infty$, so the Fourier series

$$\sum_{k\in\mathbb{Z}} \widehat{W}^{\pm}_{q,\gamma,\varepsilon}(k)\exp(2\pi k i x)$$

converges uniformly to $W^{\pm}_{q,\gamma,\varepsilon}(x)$ for all $x\in\mathbb{I}$. Hence, it follows that

$$\int_0^1 W^{\pm}_{q,\gamma,\varepsilon}(x)\;\mathrm{d}\mu(x) \;=\; \sum_{k\in\mathbb{Z}} \widehat{W}^{\pm}_{q,\gamma,\varepsilon}(k)\;\widehat{\mu}(-k)\,.$$

This together with (43), (47), (49) and the fact that $\widehat{\mu}(0) = 1$, implies that

$$\begin{aligned} \mu(E_q^\gamma) \ &\leq \ (2+\varepsilon)\,\psi(q) \ + \sum_{k\in\mathbb{Z}\setminus\{0\}} \widehat{W}^+_{q,\gamma,\varepsilon}(k)\,\widehat{\mu}(-k) \\ \mu(E_q^\gamma) \ &\geq \ (2-\varepsilon)\,\psi(q) \ + \sum_{k\in\mathbb{Z}\setminus\{0\}} \widehat{W}^-_{q,\gamma,\varepsilon}(k)\,\widehat{\mu}(-k)\,. \end{aligned} \tag{50}$$

We now proceed to estimate the Fourier coefficient $\widehat{W}^\pm_{q,\gamma,\varepsilon}(k)$ when $k \neq 0$. In view of (46) and (48), we only need to consider the case when k is a multiple of q. With this in mind, for any $s \in \mathbb{Z}\setminus\{0\}$ we claim that

$$\left|\widehat{W}^\pm_{q,\gamma,\varepsilon}(sq)\right| \ \leq \ (2+\varepsilon)\,\psi(q), \tag{51}$$

$$\left|\widehat{W}^\pm_{q,\gamma,\varepsilon}(sq)\right| \ \leq \ \frac{1}{\pi^2 s^2 \psi(q)\varepsilon}. \tag{52}$$

The upper bound (52) follows from (46) and (48) by using the trivial fact that $|\cos(x)| \leq 1$ for all $x \in \mathbb{R}$. Note that (52) is stronger than (51) for large values of $|s|$; namely when

$$s^2 \geq \frac{1}{\pi^2 \psi(q)^2 \varepsilon(2+\varepsilon)}\,.$$

In order to establish the upper bound (51), note that

$$\begin{aligned} \left|\cos\left(2\pi k\psi(q)q^{-1}\right) - \cos\left(2\pi k\psi(q)q^{-1}(1+\varepsilon)\right)\right| \ &= \ \left|\int_{\frac{2\pi k\psi(q)}{q}}^{\frac{2\pi k\psi(q)}{q}(1+\varepsilon)} \sin(x)\,\mathrm{d}x\right| \\ &\leq \ \int_{\frac{2\pi k\psi(q)}{q}}^{\frac{2\pi k\psi(q)}{q}(1+\varepsilon)} |x|\,\mathrm{d}x \\ &= \ 2\pi^2 k^2 \psi(q)^2 q^{-2}\,\varepsilon(2+\varepsilon) \end{aligned}$$

This together with (46) yields (51) for the upper function $W^+_{q,\gamma,\varepsilon}$. The proof for the lower function follows the same steps, with appropriate modifications such as using (48) instead of (46).

The above estimates enable us to prove the following two useful lemmas.

Lemma 1. *Let $0 < \varepsilon, \tilde{\varepsilon} \leq 1$. Then, for any integers $q, r \geq 4$*

$$\sum_{s\in\mathbb{Z}} \left|\widehat{W}^\pm_{q,\gamma,\varepsilon}(sq)\right| \ < \ \frac{3}{\varepsilon^{1/2}} \tag{53}$$

and

$$\sum_{s\in\mathbb{Z}} \sum_{t\in\mathbb{Z}} \left|\widehat{W}^\pm_{q,\gamma,\varepsilon}(sq)\right| \left|\widehat{W}^\pm_{r,\gamma,\tilde{\varepsilon}}(tr)\right| \ \leq \ \frac{9}{\varepsilon^{1/2}\,.\,\tilde{\varepsilon}^{1/2}}\,. \tag{54}$$

Proof. For any $N \in \mathbb{N}$, we trivially have that

$$\sum_{s=-N}^{N} \left|\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(sq)\right| \;\leq\; 2 \sum_{0 \leq s \leq \frac{1}{\sqrt{2}\pi\psi(q)\varepsilon^{1/2}}} \left|\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(sq)\right| \;+\; 2 \sum_{\frac{1}{\sqrt{2}\pi\psi(q)\varepsilon^{1/2}} < s \leq N} \left|\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(sq)\right|$$

On using (47) and (51) to estimate the first sum appearing on the right hand side of the above inequality and (52) for the second, it follows that

$$\sum_{s=-N}^{N} \left|\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(sq)\right| \leq 2 \sum_{0 \leq s \leq \frac{1}{\sqrt{2}\pi\psi(q)\varepsilon^{1/2}}} 3\psi(q) \;+\; 2 \sum_{\frac{1}{\sqrt{2}\pi\psi(q)\varepsilon^{1/2}} < s \leq N} \frac{1}{\pi^2 s^2 \psi(q)\varepsilon}$$

$$\leq \; \frac{3\sqrt{2}}{\pi\varepsilon^{1/2}} \;+\; \frac{2\sqrt{2}}{\pi\varepsilon^{1/2}} \;<\; \frac{3}{\varepsilon^{1/2}}\,.$$

The desired inequality (53) now follows on letting $N \to \infty$. The other inequality (54), trivially follows from (53) and the fact that

$$\sum_{s\in\mathbb{Z}} \sum_{t\in\mathbb{Z}} \left|\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(sq)\right| \left|\widehat{W}^{\pm}_{r,\gamma,\tilde{\varepsilon}}(tr)\right| \;\leq\; \Big(\sum_{s\in\mathbb{Z}} \left|\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(sq)\right|\Big) \Big(\sum_{t\in\mathbb{Z}} \left|\widehat{W}^{\pm}_{r,\gamma,\tilde{\varepsilon}}(tr)\right|\Big)\,.$$

□

Lemma 2. *Let μ be a probability measure supported on a subset F of $\mathbb{I}$. Let E^{γ}_q be given by* (41)*. Then, for any integer $q \geq 4$*

$$\mu(E^{\gamma}_q) \;\leq\; 3\psi(q) + 3 \max_{s\in\mathbb{N}} |\widehat{\mu}(sq)| \tag{55}$$

$$\mu(E^{\gamma}_q) \;\leq\; 3\psi(q) + 2 \sum_{s=1}^{\infty} \frac{|\widehat{\mu}(sq)|}{s}\,. \tag{56}$$

Proof. It follows from (50) with $\varepsilon = 1$ and (46), that

$$\mu(E^{\gamma}_q) \;\leq\; 3\psi(q) + \sum_{s\in\mathbb{Z}\setminus\{0\}} \widehat{W}^{+}_{q,\gamma,1}(sq)\; \widehat{\mu}(-sq). \tag{57}$$

The desired estimate (55) now immediately follows from this, together with the fact that by Lemma 1,

$$\Big|\sum_{s\in\mathbb{Z}\setminus\{0\}} \widehat{W}^{+}_{q,\gamma,1}(sq)\; \widehat{\mu}(-sq)\Big| \;\overset{(53)}{\leq}\; 3 \max_{s\in\mathbb{N}} |\widehat{\mu}(sq)|\,.$$

To prove (56), note that the geometric mean of the inequalities (51) and (52) implies that

$$\left|\widehat{W}^{\pm}_{q,\gamma,1/2}(sq)\right| \;\leq\; \frac{\sqrt{3}}{\pi|s|} \;\leq\; \frac{1}{|s|}\,.$$

This together with (57) implies (56) . □

2.2 Proof of Theorem 2

Armed with Lemma 2, it is easily seen that Theorem 2 is a straightforward consequence of the classical Borel-Cantelli Lemma [11, Chapter 1] from probability theory.

Lemma 3 (Borel-Cantelli)**.** *Let $(X, \mathcal{B}, \mu)$ be a probabilty space and $(A_n)_{n=1}^{\infty} \subseteq \mathcal{B}$ be a sequence of subsets of X. If*

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty$$

then

$$\mu\left(\limsup_{n\to\infty} A_n\right) = 0.$$

To establish Theorem 2, we first observe that Lemma 2 together with the hypotheses of the theorem guarantees that

$$\sum_{n=1}^{\infty} \mu(E_{q_n}^{\gamma}) < \infty\,.$$

Hence, the Borel-Cantelli Lemma with $A_n := E_{q_n}^{\gamma}$ implies that

$$\mu\big(\limsup_{n\to\infty} E_{q_n}^{\gamma}\big) = 0\,.$$

This completes the proof since by definition $W_{\mathcal{A}}(\gamma;\psi) = \limsup_{n\to\infty} E_{q_n}^{\gamma}$.

We now move onto proving the counting results: Theorem 1 and Theorem 4. As already mentioned, Theorem 3 is deduced from Theorem 4 in §6.

3 Establishing the counting results modulo 'independence'

To begin with, we observe that Theorem 2 (which we have already proved) implies both Theorems 1 & 4 if $\Psi(N)$ is bounded; that is, if

$$\textstyle\sum_{n=1}^{\infty} \psi(q_n) < \infty\,.$$

Indeed, it is easily verified that the hypotheses on the measure μ and the sequence $\mathcal{A}$ within the statements of Theorems 1 & 4 guarantees the convergence condition (26), and so Theorem 2 implies that for μ-almost all $x \in F$

$$R(x, N) = R(x, N; \gamma, \psi, \mathcal{A}) < \infty \quad \text{as} \quad N \to \infty\,.$$

This is consistent with the conclusions of the counting theorems; namely the statements associated with (9) and (34). Thus, during the course of proving Theorems 1 & 4 we can assume that $\Psi(N)$ is unbounded.

Fact 1. *Without loss of generality, we can assume that*

$$\textstyle\sum_{n=1}^{\infty} \psi(q_n) = \infty \quad \textit{or equivalently} \quad \Psi(N) \to \infty \;\; \text{as} \;\; N \to \infty\,.$$

Before describing the main mechanism for establishing the desired counting results, we mention three more useful facts that we can assume during the course of their proofs.

Fact 2. *Without loss of generality, we can assume that for any given* $\tau > 1$

$$\psi(q_n) \geq 3\, n^{-\tau} \quad \forall \quad n \in \mathbb{N}\,. \tag{58}$$

It is easily seen that this follows on showing that there is no loss of generality in assuming that

$$\psi(q_n) \geq \omega(q_n) \quad \forall \quad n \in \mathbb{N}\,, \tag{59}$$

where $\omega : \mathbb{N} \to \mathbb{I}$ is any real, positive function such that $\sum_{n=1}^{\infty} \omega(q_n) < \infty$. With this in mind, consider the auxiliary function

$$\psi^* : q_n \to \psi^*(q_n) := \max(\psi(q_n), \omega(q_n))\,.$$

Trivially, the function ψ^* satisfies (59) and by the definition of the counting function (see (7)), we have that

$$R(x, N; \gamma, \psi, \mathcal{A}) \leq R(x, N; \gamma, \psi^*, \mathcal{A}) \leq R(x, N; \gamma, \psi, \mathcal{A}) + R(x, N; \gamma, \omega, \mathcal{A})\,.$$

For μ-almost all $x \in F$, Theorem 2 implies that $R(x, N; \gamma, \omega, \mathcal{A})$ remains bounded as $N \to \infty$ and so it follows that

$$R(x, N; \gamma, \psi^*, \mathcal{A}) = R(x, N; \gamma, \psi, \mathcal{A}) + O(1)\,.$$

Now in view of Fact 1, we can assume that the sum $\sum_{n=1}^{\infty} \psi(q_n)$ diverges. Hence $\sum_{n=1}^{\infty} \psi^*(q_n)$ diverges and so the desired statements associated with (9) and (34) for the function ψ are equivalent to the analogous statements for the modified function ψ^*. In short, within the context of the right hand sides of (9) and (34), for μ-almost all $x \in F$ the additional contribution from the counting function associated with ω is negligible. Hence, without loss of generality we can assume (59) and thus (58).

Fact 3. *Without loss of generality, we can assume that for any given increasing sequence* $\mathcal{A} = (q_n)_{n \in \mathbb{N}}$ *of natural numbers,* $q_1 > 4$.

To see this, simply observe that there are only a finite number of terms $q_n \in \mathcal{A}$ with $q_n \leq 4$. Thus removing these 'small' terms from $\mathcal{A}$ and working with the resulting sequence introduces at most an additional $O(1)$ term on the right hand sides of (9) and (34). However, this is negligible since by Fact 1 we are assuming that $\Psi(N) \to \infty$ as $N \to \infty$.

Fact 4. *Without loss of generality, we can assume that for any given* α*-separated increasing sequence* $\mathcal{A} = (q_n)_{n \in \mathbb{N}}$ *of natural numbers, the associated implicit constant* $m_0 = 1$. *In other words, we can assume that for any integers* $1 \leq m < n$, *if* $1 \leq |sq_m - tq_n| < q_m^{\alpha}$ *for some* $s, t \in \mathbb{N}$, *then* $s > m^{12}$.

To see this, if $m_0 \geq 2$ we simply remove the first $m_0 - 1$ terms of $\mathcal{A}$ and observe that the resulting sequence $(q_{n+(m_0-1)})_{n \in \mathbb{N}}$ has the desired properties. We have already seen in the justification of Fact 3, that removing a finite number of terms of $\mathcal{A}$ is negligible within the context of establishing (9) and (34).

3.1 A mechanism for establishing counting results

The following statement [11, Lemma 1.5] represents an important tool in the theory of metric Diophantine approximation for establishing counting statements (along the lines of Theorems 1 & 4). It has its bases in the familiar variance method of probability theory and can be viewed as the quantitative form of the (diveregnce) Borel-Cantelli Lemma [3, Lemma 2.2].

Lemma 4. *Let $(X, \mathcal{B}, \mu)$ be a probability space, let $(f_n(x))_{n\in\mathbb{N}}$ be a sequence of non-negative μ-measurable functions defined on X, and $(f_n)_{n\in\mathbb{N}}$, $(\phi_n)_{n\in\mathbb{N}}$ be sequences of real numbers such that*

$$0 \le f_n \le \phi_n \qquad (n = 1, 2, \ldots).$$

Suppose that for arbitrary $a, b \in \mathbb{N}$ with $a < b$, we have

$$\int_X \left(\sum_{n=a}^{b} \big(f_n(x) - f_n \big) \right)^2 \mathrm{d}\mu(x) \;\le\; C \sum_{n=a}^{b} \phi_n \tag{60}$$

for an absolute constant $C > 0$. Then, for any given $\varepsilon > 0$, we have

$$\sum_{n=1}^{N} f_n(x) \;=\; \sum_{n=1}^{N} f_n \;+\; O\left(\Phi(N)^{1/2} \log^{\frac{3}{2}+\varepsilon} \Phi(N) + \max_{1\le k\le N} f_k \right) \tag{61}$$

for μ-almost all $x \in X$, where $\Phi(N) := \sum_{n=1}^{N} \phi_n$.

Remark 15. Note that in statistical terms, f_n is the mean of $f_n(x)$ and (60) deals with the variance.

Given a real number $\gamma \in \mathbb{I}$, a real, positive function $\psi : \mathbb{N} \to \mathbb{I}$ and an increasing sequence of natural numbers $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$, we consider Lemma 4 with

$$X := \mathbb{I}\,, \qquad f_n(x) := \chi_{E^{\gamma}_{q_n}}(x) \qquad \text{and} \qquad f_n = 2\psi(q_n)\,, \tag{62}$$

where $\chi_{E^{\gamma}_{q_n}}$ is the characteristic function of the set $E^{\gamma}_{q_n}$ given by (41). Then, clearly for any $x \in \mathbb{I}$ and $N \in \mathbb{N}$ we have that the

$$\text{l.h.s. of (61)} \;=\; R(x, N)\,,$$

where $R(x, N)$ is the counting function given by (7). Also, the main term on the r.h.s. of (61) is precisely $2\Psi(N)$ where $\Psi(N)$ is the partial sum given by (10). Furthermore, it is easily verified that for any $a, b \in \mathbb{N}$ with $a < b$

$$\begin{aligned}
\left(\sum_{n=a}^{b} (f_n(x) - f_n) \right)^2 &= \left(\sum_{n=a}^{b} f_n(x) \right)^2 + \left(\sum_{n=a}^{b} f_n \right)^2 - 2 \sum_{n=a}^{b} f_n(x) \cdot \sum_{n=a}^{b} f_n \\
&= \sum_{n=a}^{b} f_n(x) \;+\; 2 \sum\sum_{a\le m<n\le b} f_m(x) f_n(x) \;+\; \left(\sum_{n=a}^{b} f_n \right)^2 \;-\; 2 \sum_{n=a}^{b} f_n \cdot \sum_{n=a}^{b} f_n(x)\,,
\end{aligned}$$

and so it follows that

$$\text{l.h.s. of (60)} \;=\; \sum_{n=a}^{b} \mu(E^{\gamma}_{q_n}) \;+\; 2 \sum\sum_{a\le m<n\le b} \mu(E^{\gamma}_{q_m} \cap E^{\gamma}_{q_n}) \\ \;-\; 4\sum_{n=a}^{b} \psi(q_n) \left(\sum_{n=a}^{b} \mu(E^{\gamma}_{q_n}) - \sum_{n=a}^{b} \psi(q_n) \right) \tag{63}$$

where μ is a non-atomic probability measure supported on a subset of $\mathbb{I}$. The upshot of this is that in view of Lemma 4, the proof of Theorem 1 and Theorem 4 boils down to 'appropriately' estimating the right hand side of (63). Estimating the measure of the intersection of the sets $E^{\gamma}_{q_n}$ is where the main difficulty lies. In short we need to show that these 'building block' sets are independent on average with an acceptable error term. Suppose for the moment that there was no error term; in other words

$$2 \sum\sum_{a\le m<n\le b} \mu(E^{\gamma}_{q_m} \cap E^{\gamma}_{q_n}) \;\le\; \left(\sum_{n=a}^{b} \mu(E^{\gamma}_{q_n}) \right)^2 . \tag{64}$$

Then, and as we shall see in a moment, it is easy to deduce the desired counting statements (9) and (34) from Lemma 4 once we have established the following statement.

Lemma 5. *Let μ be a probability measure supported on a subset F of $\mathbb{I}$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers that satisfies the growth condition* (32) *for some constants $B \ge 1$ and $C > 0$. Furthermore, assume that $q_1 > 4$. Let $\gamma \in \mathbb{I}$ and $\psi : \mathbb{N} \to \mathbb{I}$ be a real, positive function that satisfies* (58)*. Suppose there exists a constant $A > 2B$ so that* (8) *is satisfied. Then, for arbitrary $a, b \in \mathbb{N}$ with $a < b$, we have*

$$\sum_{n=a}^{b} \mu(E^{\gamma}_{q_n}) = 2\sum_{n=a}^{b} \psi(q_n) \;+\; O\left(\min\left(1, \sum_{n=a}^{b} \psi(q_n)\right)\right). \tag{65}$$

Note that in view of Facts 1-3 at the start of this section, we can assume the hypotheses of Theorems 1 & 4 satisfy those of the lemma. Hence, if we genuinely had independence on average (i.e., (64) holds), then Lemma 5 would imply that

$$\begin{aligned} \text{r.h.s. of (63)} \;&\le\; \sum_{n=a}^{b} \mu(E^{\gamma}_{q_n}) + \left(\sum_{n=a}^{b} \mu(E^{\gamma}_{q_n}) \right)^2 - 4\sum_{n=a}^{b} \psi(q_n) \left(\sum_{n=a}^{b} \mu(E^{\gamma}_{q_n}) - \sum_{n=a}^{b} \psi(q_n) \right) \\ &=\; O\left(\sum_{n=a}^{b} \psi(q_n) \right) . \end{aligned}$$

Thus, we conclude that

$$\text{l.h.s. of (60)} \ll 2\sum_{n=a}^{b} \psi(q_n) \;:=\; \sum_{n=a}^{b} f_n$$

and (9) and (34) follow on applying Lemma 4 with $\phi_n := f_n$. In fact, we would be able to improve the associated error terms. However, we do not have (64) and establishing the following estimates is at the heart of proving the desired counting statements.

Throughout the paper, we use the notation $\log^+(x) := \max(0, \log(x))$.

Proposition 1. *Let F, μ, $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$, γ and ψ be as in Theorem 1. Furthermore, assume ψ satisfies* (58) *and that $q_1 > 4$. Then, for arbitrary $a, b \in \mathbb{N}$ with $a < b$, we have*

$$2\sum\sum_{a\le m<n\le b} \mu(E^\gamma_{q_m} \cap E^\gamma_{q_n}) \;\le\; \left(\sum_{n=a}^{b} \mu(E^\gamma_{q_n})\right)^2 + O\left(\left(\sum_{n=a}^{b}\psi(q_n)\right)^{4/3}\left(\log^+\left(\sum_{n=a}^{b}\psi(q_n)\right)+1\right) + \sum_{n=a}^{b}\psi(q_n)\right). \quad (66)$$

Proposition 2. *Let F, μ, $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$, γ and ψ be as in Theorem 4. Furthermore, assume ψ satisfies* (58)*, $q_1 > 4$ and that $\mathcal{A}$ is α-separated with the implicit constant $m_0 = 1$. Then, for arbitrary $a, b \in \mathbb{N}$ with $a < b$, we have*

$$2\sum\sum_{a\le m<n\le b} \mu(E^\gamma_{q_m} \cap E^\gamma_{q_n}) \;\le\; \left(\sum_{n=a}^{b} \mu(E^\gamma_{q_n})\right)^2 + O\left(\sum\sum_{a\le m<n\le b}(q_m, q_n)\min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right)\right) + O\left(\left(\sum_{n=a}^{b}\psi(q_n)\right)\log^+\left(\sum_{n=a}^{b}\psi(q_n)\right) + \sum_{n=a}^{b}\psi(q_n)\right). \quad (67)$$

Remark 16. We stress that within the context of establishing Theorems 1 & 4, there is no harm in assuming the additional hypotheses imposed in the statements of Propositions 1 & 2. The justification for this is provided by Facts 1-4 at the start of this section.

The proofs of the above propositions are technically a little involved and will be the subject of §4 and §5. In the remaining part of this section we shall first prove Lemma 5 and then go onto completing the proofs of Theorems 1 & 4 modulo the above propositions.

3.2 Proof of Lemma 5

For any given sequence of real numbers $(\varepsilon_n)_{n=a}^{b}$ in $(0, 1]$, it follows via (50) that

$$\left|\sum_{n=a}^{b}\mu(E_{q_n}) - 2\sum_{n=a}^{b}\psi(q_n)\right| \le \sum_{n=a}^{b}\psi(q_n)\varepsilon_n + \max_{\circ\in\{+,-\}}\sum_{n=a}^{b}\left|\sum_{k\in\mathbb{Z}\setminus\{0\}}\widehat{W}^{\circ}_{q_n,\gamma,\varepsilon_n}(k)\widehat{\mu}(-k)\right|. \quad (68)$$

Now, in view of (46) and (48), we have that $\widehat{W}^{\pm}_{q_n,\gamma,\varepsilon_n}(k) = 0$ unless $k = sq_n$ for some integer s. Also, on using (8) and (32) it can be verified that $|\hat{\mu}(sq_n)| \ll n^{-A/B}$ for any non-zero integer s and $n \in \mathbb{N}$. Hence by Lemma 1, it follows that

$$\left|\sum_{n=a}^{b} \mu(E_{q_n}) - 2\sum_{n=a}^{b} \psi(q_n)\right| \;\leq\; \sum_{n=a}^{b} \psi(q_n)\varepsilon_n \;+\; \sum_{n=a}^{b} \frac{3}{n^{A/B}\varepsilon_n^{1/2}}\,. \tag{69}$$

In turn, this together with (58) and the fact that $A > 2B$ implies that

$$\left|\sum_{n=a}^{b} \mu(E_{q_n}) - 2\sum_{n=a}^{b} \psi(q_n)\right| \leq \sum_{n=a}^{b} \psi(q_n)\varepsilon_n + \sum_{n=a}^{b} \frac{\psi(q_n)}{n^{A/2B}\varepsilon_n^{1/2}}.$$

On letting $\varepsilon_n = 1$ for all $a \leq n \leq b$, we obtain the upper bound

$$\left|\sum_{n=a}^{b} \mu(E_{q_n}) - 2\sum_{n=a}^{b} \psi(q_n)\right| \ll \sum_{n=a}^{b} \psi(q_n)\,. \tag{70}$$

To complete the proof of the lemma, it remains to establish the 'other' upper bound; that is

$$\left|\sum_{n=a}^{b} \mu(E_{q_n}) - 2\sum_{n=a}^{b} \psi(q_n)\right| \ll 1. \tag{71}$$

With this in mind, for any $n \in \mathbb{N}$ let

$$\Psi(n) := \sum_{k=1}^{n} \psi(q_k) \quad \text{and} \quad \varepsilon_n := \min\left(1, \Psi(n)^{-2}\right).$$

By definition, $|\psi(q_n)| \leq 1$ and so $\varepsilon_n^{-1} \leq n^2$. Hence, with reference to the second term on the r.h.s. of (69) we have that

$$\sum_{n=a}^{b} \frac{3}{n^{A/B}\varepsilon_n^{1/2}} \;\leq\; 3\cdot\sum_{n=1}^{\infty} \frac{1}{n^{\frac{A}{B}-1}} \;<\; \infty\,. \tag{72}$$

The last inequality makes use of the fact that $A > 2B$. We now turn out attention to the first term on the r.h.s. of (69). A straight forward application of Lemma D4 in Appendix D of §7 with $s_n = \psi(q_n)$ and $\gamma = 1$, yields that

$$\sum_{n=a}^{b} \psi(q_n)\varepsilon_n \;<\; \sum_{n=1}^{\infty} \frac{\psi(q_n)}{\max\left(1, \Psi(n)^2\right)} \;<\; 3\,. \tag{73}$$

Combining the inequalities (69), (72) and (73) gives the desired upper bound (71).

3.3 Proof of Theorem 1 modulo Proposition 1

On using (65) and (66) on the right hand side of (63), we find that for any $a, b \in \mathbb{N}$ with $a < b$

$$\text{l.h.s. of (60)} \;=\; O\left(\left(\sum_{n=a}^{b}\psi(q_n)\right)^{4/3}\left(\log^+\left(\sum_{n=a}^{b}\psi(q_n)\right)+1\right)+\sum_{n=a}^{b}\psi(q_n)\right). \tag{74}$$

We now estimate the above term on the right and show that

$$\left(\sum_{n=a}^{b}\psi(q_n)\right)^{4/3}\left(\log^+\left(\sum_{n=a}^{b}\psi(q_n)\right)+1\right)+\sum_{n=a}^{b}\psi(q_n) \;\leq\; 6\sum_{n=a}^{b}\phi_n\,, \tag{75}$$

where

$$\phi_n := \psi(q_n)\Psi(n)^{1/3}\left(\log^+\Psi(n)+1\right)+2\psi(q_n)\,. \tag{76}$$

To this end, denote by $m \in \mathbb{N}$ the smallest integer satisfying $a \leq m \leq b$ such that

$$\sum_{n=a}^{m}\psi(q_n) \;\geq\; \frac{1}{2}\sum_{n=a}^{b}\psi(q_n). \tag{77}$$

Note that by the definition of m, we have that

$$\sum_{n=m}^{b}\psi(q_n) \;\geq\; \frac{1}{2}\sum_{n=a}^{b}\psi(q_n) \tag{78}$$

and that for any integer n such that $m \leq n \leq b$

$$2\Psi(n) \;\geq\; \sum_{k=a}^{b}\psi(q_k). \tag{79}$$

Then, by using (78) and then (79), it is easily verified that

$$\begin{aligned}
&\left(\sum_{n=a}^{b}\psi(q_n)\right)^{4/3}\left(\log^+\left(\sum_{n=a}^{b}\psi(q_n)\right)+1\right)+\sum_{n=a}^{b}\psi(q_n)\\
&\quad\leq\; 2\sum_{n=m}^{b}\left(\psi(q_n)\left(\sum_{k=a}^{b}\psi(q_k)\right)^{1/3}\left(\log^+\left(\sum_{k=a}^{b}\psi(q_k)\right)+1\right)\right)+\sum_{n=a}^{b}\psi(q_n)\\
&\quad\leq\; 2\sum_{n=m}^{b}\Big(\psi(q_n)\left(2\Psi(n)\right)^{1/3}\left(\log^+\left(2\Psi(n)\right)+1\right)\Big)+\sum_{n=a}^{b}\psi(q_n)\\
&\quad\leq\; 4\cdot 2^{1/3}\cdot\sum_{n=m}^{b}\Big(\psi(q_n)\Psi(n)^{1/3}\left(\log^+\left(\Psi(n)\right)+1\right)\Big)+\sum_{n=a}^{b}\psi(q_n)\\
&\quad\leq\; 6\sum_{n=a}^{b}\Big(\psi(q_n)\Psi(n)^{1/3}\left(\log^+\left(\Psi(n)\right)+1\right)\Big)+\sum_{n=a}^{b}\psi(q_n)\,.
\end{aligned}$$

This establishes (75) which together with (74) implies that condition (60) of Lemma 4 is satisfied with $X, f_n(x)$ and f_n given by (62) and ϕ_n by (76). Also note that for any $n \in \mathbb{N}$, we trivially have that $f_n \leq \phi_n$, $f_n \leq 2$ and

$$\begin{aligned}\Phi(N) := \sum_{n=1}^{N} \phi_n \ &\leq \ \sum_{n=1}^{N} \psi(q_n)\Psi(N)^{1/3}\big(\log^+ \Psi(N) + 1\big) + \sum_{n=1}^{N} \psi(q_n) \\ &= \ \Psi(N)^{4/3}\big(\log^+ \Psi(N) + 1\big) + \Psi(N).\end{aligned}$$

Hence, Lemma 4 implies that for any $\varepsilon > 0$

$$\begin{aligned}R(x, N) \ &= \ \text{l.h.s. of (61)} \\ &= \ 2\Psi(N) + O\Big(\Psi(N)^{2/3}\left(\log \Psi(N) + 2\right)^{2+\varepsilon}\Big),\end{aligned}$$

and this completes the proof of Theorem 1 assuming the truth of Proposition 1.

3.4 Proof of Theorem 4 modulo Proposition 2

The proof is similar to that in the previous subsection. On using (65) and (67) on the right hand side of (63), we find that for any $a, b \in \mathbb{N}$ with $a < b$

$$\begin{aligned}\text{l.h.s. of (60)} \ = O\Bigg(&\Big(\sum_{n=a}^{b} \psi(q_n)\Big)\Big(\log^+\Big(\sum_{n=a}^{b} \psi(q_n)\Big) + 1\Big) \\ &+ \sum\sum_{a \leq m < n \leq b} (q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right)\Bigg)\end{aligned} \tag{80}$$

We estimate the above term on the right and show that

$$\begin{aligned}&\left(\sum_{n=a}^{b} \psi(q_n)\right)\left(\log^+\left(\sum_{n=a}^{b} \psi(q_n)\right) + 1\right) \\ &\quad + \sum\sum_{a \leq m < n \leq b} (q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right) \ \leq \ 2\sum_{n=a}^{b} \phi_n\,,\end{aligned} \tag{81}$$

where

$$\phi_n := \psi(q_n)\big(\log^+ \Psi(n) + 2\big) + \sum_{m=1}^{n-1} (q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right). \tag{82}$$

Clearly, this will immediately follow on showing that

$$\left(\sum_{n=a}^{b} \psi(q_n)\right)\left(\log^+\left(\sum_{n=a}^{b} \psi(q_n)\right) + 1\right) \ \leq \ 2\sum_{n=a}^{b} \psi(q_n)\big(\log^+ \Psi(n) + 2\big). \tag{83}$$

As in the previous subsection, let $m \in \mathbb{N}$ be the smallest integer satisfying $a \leq m \leq b$ such that (77) holds. Then, by using (78) and (79), it is easily verified that

$$\begin{aligned}\left(\sum_{n=a}^{b}\psi(q_n)\right)\left(\log^+\left(\sum_{n=a}^{b}\psi(q_n)\right)+1\right) &\leq\ 2\left(\sum_{n=m}^{b}\psi(q_n)\right)\left(\log^+\left(\sum_{n=a}^{b}\psi(q_n)\right)+1\right)\\ &\leq\ 2\left(\sum_{n=m}^{b}\psi(q_n)\left(\log^+\left(2\Psi(n)\right)+1\right)\right)\\ &\leq\ 2\left(\sum_{n=m}^{b}\psi(q_n)\left(\log^+\Psi(n)+2\right)\right)\\ &\leq\ 2\left(\sum_{n=a}^{b}\psi(q_n)\left(\log^+\Psi(n)+2\right)\right).\end{aligned}$$

This establishes (83) and so (81) follows. The upshot is that (81) together with (80) implies that condition (60) of Lemma 4 is satisfied with $X, f_n(x)$ and f_n given by (62) and ϕ_n by (82). Also note that for any $n \in \mathbb{N}$, we trivially have that $f_n \leq \phi_n$, $f_n \leq 2$ and

$$\Phi(N) := \sum_{n=1}^{N}\phi_n\ \leq\ \Psi(N)\left(\log^+\Psi(N)+2\right)\ +\ E(N)\,,$$

where $E(N)$ is given by (35). The desired counting statement (34) now follows on applying Lemma 4. Hence, this completes the proof of Theorem 4 assuming the truth of Proposition 2.

4 Preliminaries for 'independence'

In this section we set out the ground work for establishing the desired estimates for the measure of the intersection of the sets $E^{\gamma}_{q_n}$; namely Propositions 1 & 2. Recall, that these estimates are at the heart of proving our main counting results; namely Theorems 1 & 4.

Throughout this section $(\varepsilon_n)_{n\in\mathbb{N}}$ will be a fixed sequence of real numbers in $(0,1]$. Also, with reference to (42), for any $m, n \in \mathbb{N}$ with $m < n$ let

$$W^+_{m,n} := W^+_{q_m,\gamma,\varepsilon_m}W^+_{q_n,\gamma,\varepsilon_n}\,. \tag{84}$$

Then, by definition

$$\begin{aligned}\mu(E^{\gamma}_{q_m}\cap E^{\gamma}_{q_n}) &\leq \int_0^1 W^+_{q_m,\gamma,\varepsilon_m}(x)W^+_{q_n,\gamma,\varepsilon_n}(x)\,\mathrm{d}\mu(x)\\ &= \int_0^1 W^+_{m,n}(x)\mathrm{d}\mu(x)\end{aligned}$$

Our aim is to obtain a sufficiently strong upper bound for the above integral. This we do by considering the Fourier series expansion of the function $W^+_{m,n}$. It is easily verified that for any $k \in \mathbb{Z}$,

$$\begin{aligned} \widehat{W}^+_{m,n}(k) &:= \int_0^1 W^+_{q_m,\gamma,\varepsilon_m}(x) W^+_{q_n,\gamma,\varepsilon_n}(x)\, \exp(-2\pi kix)\mathrm{d}x \\ &= \sum_{j\in\mathbb{Z}} \widehat{W}^+_{q_m,\gamma,\varepsilon_m}(k)\ \widehat{W}^+_{q_n,\gamma,\varepsilon_n}(k-j)\,. \end{aligned} \tag{85}$$

Moreover, it is easily seen that $\sum_{k\in\mathbb{Z}} \left|\widehat{W}^+_{m,n}(k)\right| < \infty$, so the Fourier series

$$\sum_{k\in\mathbb{Z}} \widehat{W}^+_{m,n}(k)\exp(2\pi kix)$$

converges uniformly to $W^+_{m,n}(x)$ for all $x \in \mathbb{I}$. Hence, it follows that

$$\int_0^1 W^+_{m,n}(x)(x)\ \mathrm{d}\mu(x) \ = \ \sum_{k\in\mathbb{Z}} \widehat{W}^+_{m,n}(k)\ \widehat{\mu}(-k)\,.$$

The upshot of this is that

$$\begin{aligned} \mu(E^\gamma_{q_m} \cap E^\gamma_{q_n}) &\le \sum_{k\in\mathbb{Z}} \widehat{W}^+_{m,n}(k)\widehat{\mu}(-k) \\ &= \widehat{W}^+_{m,n}(0) + \sum_{k\in\mathbb{Z}\setminus\{0\}} \widehat{W}^+_{m,n}(k)\widehat{\mu}(-k). \end{aligned} \tag{86}$$

To proceed, we consider the two terms on the right hand side of (86) separately. By definition, for any pair of natural numbers m, n we have that

$$\begin{aligned} \widehat{W}^+_{m,n}(0) &:= \int_0^1 W^+_{q_m,\gamma,\varepsilon_m}(x) W^+_{q_n,\gamma,\varepsilon_n}(x)\mathrm{d}x \\ &\le |(1+\varepsilon_m)E^\gamma_{q_m} \cap (1+\varepsilon_n)E^\gamma_{q_n}|\,, \end{aligned} \tag{87}$$

where $|\,.\,|$ is Lebesgue measure and with reference to (41), for any constant $\kappa > 0$ we let $\kappa E^\gamma_q := E^\gamma_q(\kappa\psi)$. It is relatively straightforward to verify (see for example [11, Equation 3.2.5][2] for the details) that for any $q, q' \in \mathbb{N}$

$$|E^\gamma_q \cap E^\gamma_{q'}| \ = \ 4\psi(q)\psi(q') + O\left((q,q')\ \min\left(\frac{\psi(q)}{q}, \frac{\psi(q')}{q'}\right)\right)\,.$$

Hence, it follows that

$$\begin{aligned} \left|(1+\varepsilon_m)E^\gamma_{q_m} \cap (1+\varepsilon_n)E^\gamma_{q_n}\right| \ &= \ 4(1+\varepsilon_m)(1+\varepsilon_n)\psi(q_m)\psi(q_n) \\ &\quad + \ O\left((q_m,q_n)\min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right)\right)\,. \end{aligned}$$

[2]Equation 3.2.5 in [11] as stated is not correct – the 'big O' error term is missing.

This together with (87) implies that

$$\widehat{W}_{m,n}^{+}(0) \ \leq \ 4(1+\varepsilon_m)(1+\varepsilon_n)\psi(q_m)\psi(q_n) + O\left((q_m,q_n)\min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right)\right) . \tag{88}$$

We now turn our attention to the second term appearing on the right hand side of (86) which for convenience we will denote by $S_{m,n}$. Note that in view of (46) and (85), it follows that

$$\begin{aligned} S_{m,n} \ &:= \ \sum_{k\in\mathbb{Z}\setminus\{0\}} \widehat{W}_{m,n}^{+}(k)\widehat{\mu}(-k) \\ &= \ \sum_{\substack{s,t\in\mathbb{Z} \\ sq_m - tq_n \neq 0}}\sum \widehat{W}_{q_m,\gamma,\varepsilon_m}^{+}(sq_m)\widehat{W}_{q_n,\gamma,\varepsilon_n}^{+}(tq_n)\widehat{\mu}\left(-(sq_m+tq_n)\right). \end{aligned} \tag{89}$$

4.1 Estimates for $S_{m,n}$

We start by providing a general upper bound estimate for $S_{m,n}$ which is applicable to integer sequences associated with both Propositions 1 & 2.

Lemma 6. *Let μ be a probability measure supported on a subset F of $\mathbb{I}$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers that satisfies the growth condition (32) for some constants $B \geq 1$ and $C > 0$. Furthermore, assume that $q_1 > 4$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of real numbers in $(0,1]$, $\alpha \in (0,1)$, $\gamma \in \mathbb{I}$ and $\psi : \mathbb{N} \to \mathbb{I}$ be a real, positive function. Suppose there exists a constant $A > 2B$ so that (8) is satisfied. Then, for any $m,n \in \mathbb{N}$ with $m < n$, we have*

$$|S_{m,n}| \ll \ \frac{\psi(q_m)}{n^{\frac{A}{B}}\varepsilon_n^{1/2}} \ + \ \left(1+\frac{1}{\alpha^A}\right)\frac{\psi(q_n)}{m^{\frac{A}{B}}\varepsilon_m^{1/2}} \ + \ \frac{1}{n^{\frac{A}{B}}\varepsilon_m^{1/2}\varepsilon_n^{1/2}} \ + \ |T(m,n)|,$$

where

$$T(m,n) \ := \ \sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\} \\ 1\leq |sq_m - tq_n| < q_m^\alpha}}\sum \ \widehat{W}_{q_m,\gamma,\varepsilon_m}^{+}(sq_m)\widehat{W}_{q_n,\gamma,\varepsilon_n}^{+}(tq_n)\widehat{\mu}\left(sq_m - tq_n\right). \tag{90}$$

The next two lemmas provide estimates on the average size of the quantity $T(m,n)$ appearing in Lemma 6. The first deals with lacunary sequences (i.e. the context of Proposition 1) and the second deals with α-separated sequences (i.e. the context of Proposition 2).

Lemma 7. *Let μ be a probability measure supported on a subset F of $\mathbb{I}$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be a lacunary sequence of natural numbers. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing sequence of real numbers in $(0,1]$, $\alpha \in (0,1)$, $\gamma \in \mathbb{I}$ and $\psi : \mathbb{N} \to \mathbb{I}$ be a real, positive function. Then, for arbitrary $a,b \in \mathbb{N}$ with $a < b$, we have*

$$\sum_{a\leq m<n\leq b}\sum |T(m,n)| \ \ll \ \sum_{n=a}^{b} \frac{\psi(q_n)}{\varepsilon_n^{1/2}},$$

where $T(m,n)$ is defined by (90).

Lemma 8. *Let μ be a probability measure supported on a subset F of $\mathbb{I}$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an α-separated increasing sequence of natural numbers with the implicit constant $m_0 = 1$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of real numbers in $(0, 1]$, $\gamma \in \mathbb{I}$ and $\psi : \mathbb{N} \to \mathbb{I}$ be a real, positive function. Suppose that for any $n \in \mathbb{N}$*

$$\psi(q_n) \geq n^{-9} \tag{91}$$

and

$$\varepsilon_n^{-1} \leq 2n\,. \tag{92}$$

Then, for arbitrary $a, b \in \mathbb{N}$ with $a < b$, we have

$$\sum\sum_{a\leq m<n\leq b} |T(m,n)| \ll \sum_{n=a}^{b} \psi(q_n), \tag{93}$$

where $T(m,n)$ is defined by (90).

The rest of this section will be devoted to proving the above three lemmas.

Proof of Lemma 6. We start by decomposing $S_{m,n}$ into three sums:

$$S_{m,n} = S_1(m,n) + S_2(m,n) + S_3(m,n),$$

where

$$\begin{aligned}
S_1(m,n) &:= \sum_{t\in\mathbb{Z}\setminus\{0\}} \widehat{W}^+_{q_m,\gamma,\varepsilon_m}(0)\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)\widehat{\mu}(-tq_n),\\
S_2(m,n) &:= \sum_{s\in\mathbb{Z}\setminus\{0\}} \widehat{W}^+_{q_n,\gamma,\varepsilon_n}(0)\widehat{W}^+_{q_m,\gamma,\varepsilon_m}(sq_m)\widehat{\mu}(-sq_m),\\
S_3(m,n) &:= \sum\sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\}\\ sq_m+tq_n\neq 0}} \widehat{W}^+_{q_m,\gamma,\varepsilon_m}(sq_m)\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)\widehat{\mu}\left(-(sq_m+tq_n)\right).
\end{aligned}$$

In order to find an upper bound for $S_1(m,n)$, first note that by making use of (8) and (32) it follows that

$$|\widehat{\mu}(-tq_n)| \ll (\log|tq_n|)^{-A} \ll n^{-\frac{A}{B}}\,.$$

This together with (47) and (53) implies that

$$|S_1(m,n)| \ll \frac{(2+\varepsilon_m)\psi(q_m)}{n^{\frac{A}{B}}}\sum_{t\in\mathbb{Z}}\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n) \ll \frac{\psi(q_m)}{n^{\frac{A}{B}}\varepsilon_n^{1/2}}\,.$$

Similarly, we find that

$$|S_2(m,n)| \ll \frac{\psi(q_n)}{m^{\frac{A}{B}}\varepsilon_m^{1/2}}\,.$$

To deal with $S_3(m,n)$, we decompose further into two sums:

$$S_3(m,n) \;=\; S_4(m,n) + S_5(m,n),$$

where

$$S_4(m,n) \;:=\; \sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\} \\ |sq_m - tq_n|\geq q_n/2}}\sum \widehat{W}^+_{q_m,\gamma,\varepsilon_m}(sq_m)\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)\widehat{\mu}\,(sq_m - tq_n)$$

and

$$S_5(m,n) \;:=\; \sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\} \\ 1\leq|sq_m - tq_n|< q_n/2}}\sum \widehat{W}^+_{q_m,\gamma,\varepsilon_m}(sq_m)\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)\widehat{\mu}\,(sq_m - tq_n)\,. \tag{94}$$

Regarding $S_4(m,n)$, by making use of (8), (32) and the restriction $|sq_m - tq_n| \geq q_n/2$ imposed on $s,t \in \mathbb{Z}\setminus\{0\}$, it follows that

$$|\widehat{\mu}\,(sq_m - tq_n)\,| \ll n^{-\frac{A}{B}}.$$

This together with (54) implies that

$$|S_4(m,n)| \ll \frac{1}{n^{\frac{A}{B}}\varepsilon_m^{1/2}\varepsilon_n^{1/2}}.$$

To deal with $S_5(m,n)$, we decompose further into two sums:

$$S_5(m,n) \;=\; S_6(m,n) \;+\; T(m,n)\,,$$

where

$$S_6(m,n) \;:=\; \sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\} \\ q_m^{\alpha}\leq|sq_m - tq_n|< q_n/2}}\sum \widehat{W}^+_{q_m,\gamma,\varepsilon_m}(sq_m)\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)\widehat{\mu}\,(sq_m - tq_n)$$

and $T(m,n)$ is defined by (90). Regarding $S_6(m,n)$, we first make use of (8), (32) and the lower bound restriction $|sq_m - tq_n| \geq q_m^{\alpha}$ imposed on $s,t \in \mathbb{Z}\setminus\{0\}$, to find that

$$|\widehat{\mu}\,(sq_m - tq_n)\,| \;\ll\; \alpha^{-A}\, m^{-\frac{A}{B}}.$$

Next, we observe that the upper bound restriction $|sq_m - tq_n| \leq q_n/2$ imposed on the non-zero integers s,t is equivalent to

$$\left|s\frac{q_m}{q_n} - t\right| < \frac{1}{2}\,. \tag{95}$$

It is now easy to see that if s and t satisfy (95) then both necessarily must have the same sign and also that for each fixed integer s there exists at most one non-zero integer $t = t_s$ satisfying (95). Thus,

$$|S_6(m,n)| \;\;\ll\;\; \frac{1}{\alpha^A\, m^{A/B}} \sum_{\substack{s\in\mathbb{N}: \\ t_s \text{ exists}}} |\widehat{W}^+_{q_m,\gamma,\varepsilon_m}(sq_m)|\;|\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(t_s q_n)|$$

and on using (51) to bound $|\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(t_s q_n)|$ and (53) to bound $|\widehat{W}^+_{q_m,\gamma,\varepsilon_n}(sq_m)|$, we find that

$$|S_6(m,n)| \ll \frac{1}{\alpha^A}\,\frac{\psi(q_n)}{m^{\frac{A}{B}}\varepsilon_m^{1/2}}\,.$$

The above upper bounds for the absolute values of $S_1(m,n), S_2(m,n), S_4(m,n)$ and $S_6(m,n)$ together with the fact that

$$|S_{m,n}| \;\le\; |S_1(m,n)| + |S_2(m,n)| + |S_4(m,n)| + |S_6(m,n)| + |T(m,n)|\,,$$

completes the proof of the proposition. □

Proof of Lemma 7. The strategy is similar to that used above to estimate $S_6(m,n)$. To start with, observe that the restriction $|sq_m - tq_n| \le q_m^\alpha$ imposed on the non-zero integers s,t associated with $T(m,n)$ implies that

$$\left| s - t\frac{q_n}{q_m} \right| < 1\,. \tag{96}$$

Hence, if s and t satisfy (96) then both necessarily must have the same sign and also for each fixed integer t there exists a set S_t of at most two non-zero integers s satisfying (96). Thus, on using the trivial bound $|\widehat{\mu}(t)| \le 1$, it follows that

$$|T(m,n)| \;\le\; \sum_{\substack{t\in\mathbb{N}:\\ s\in S_t}} |\widehat{W}^+_{q_m,\gamma,\varepsilon_m}(sq_m)|\;|\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)|\,.$$

Also note that (96) and the fact that $tq_n/q_m > 1$ implies

$$\frac{1}{2}\frac{q_m}{q_n}s \le \frac{q_m}{q_n}\max\{1,(s-1)\} \;\le\; t \;\le\; \frac{q_m}{q_n}(s+1) \le 2\frac{q_m}{q_n}s.$$

On using this together with (51) to bound $\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)$ and both (51) and (52) to bound $\widehat{W}^+_{q_m,\gamma,\varepsilon_n}(s_t q_m)$, we find that for any integers $1 \le m < n$

$$\begin{aligned}
|T(m,n)| \;&\ll\; \sum_{t\in\mathbb{N}} \min\left(\frac{q_m^2}{q_n^2}\cdot\frac{1}{t^2\psi(q_m)\varepsilon_m}, \psi(q_m)\right)\psi(q_n)\\
&\ll\; \sum_{1\le t\le \frac{q_m}{q_n\psi(q_m)\varepsilon_m^{1/2}}} \psi(q_m)\psi(q_n) \;+\; \sum_{t>\frac{q_m}{q_n\psi(q_m)\varepsilon_m^{1/2}}} \frac{q_m^2}{q_n^2}\cdot\frac{1}{t^2\psi(q_m)\varepsilon_m}\psi(q_n)\\
&\ll\; \frac{q_m}{q_n}\frac{\psi(q_n)}{\varepsilon_m^{1/2}} \;\le\; \frac{q_m}{q_n}\frac{\psi(q_n)}{\varepsilon_n^{1/2}}\,.
\end{aligned}$$

The last inequality makes use of the fact that $(\varepsilon_n)_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers. Now the fact that $(q_n)_{n\in\mathbb{N}}$ is lacunary implies that

$$\sum_{1\le m<n} q_m/q_n \ll 1,$$

and so it follows that for arbitrary $a, b \in \mathbb{N}$ with $a < b$, we have

$$\sum_{a \le m < n \le b} \sum T(m,n) \ \ll \ \sum_{n=a}^{b} \sum_{m=a}^{n-1} \frac{q_m}{q_n} \frac{\psi(q_n)}{\varepsilon_n^{1/2}} \ \ll \ \sum_{n=a}^{b} \frac{\psi(q_n)}{\varepsilon_n^{1/2}} .$$

□

Proof of Lemma 8. To start with, observe that the restriction $|sq_m - tq_n| \le q_m^{\alpha}$ imposed on the non-zero integers s, t associated with $T(m,n)$ implies that

$$\left| s\frac{q_m}{q_n} - t \right| < 1 . \tag{97}$$

Hence, if s and t satisfy (96) then both necessarily must have the same sign and also for each fixed integer s there exists a set T_s of at most two non-zero integers t satisfying (97). Thus, we can decompose $T(m,n)$ into two sums:

$$T(m,n) = T_1(m,n) + T_2(m,n),$$

where

$$T_1(m,n) := \sum_{\substack{s,t \in \mathbb{N} \\ 1 \le s \le m^3/\psi(q_m) \\ 1 \le |sq_m - tq_n| < q_m^{\alpha}}} \sum \widehat{W}^{+}_{q_m,\gamma,\varepsilon_m}(sq_m) \widehat{W}^{+}_{q_n,\gamma,\varepsilon_n}(tq_n) \widehat{\mu}\left(sq_m - tq_n\right)$$

and

$$T_2(m,n) := \sum_{\substack{s,t \in \mathbb{N} \\ s > m^3/\psi(q_m) \\ 1 \le |sq_m - tq_n| < q_m^{\alpha}}} \sum \widehat{W}^{+}_{q_m,\gamma,\varepsilon_m}(sq_m) \widehat{W}^{+}_{q_n,\gamma,\varepsilon_n}(tq_n) \widehat{\mu}\left(sq_m - tq_n\right) .$$

In view of (91) the condition $s \le m^3/\psi(q_m)$ in the definition of $T_1(m,n)$ implies that $s \le m^{12}$. In turn, this together with the fact that $(q_n)_{n \in \mathbb{N}}$ is α-separated with the implicit constant $m_0 = 1$ implies that $T_1(m,n)$ is an empty sum. Thus,

$$T_1(m,n) = 0 .$$

Regarding $T_2(m,n)$, on using the trivial bound $|\widehat{\mu}(t)| \le 1$ together with (51) to bound $|\widehat{W}^{+}_{q_n,\gamma,\varepsilon_n}(tq_n)|$ and (52) to bound $|\widehat{W}^{+}_{q_m,\gamma,\varepsilon_n}(sq_m)|$, we obtain that for any integers $1 \le m < n$

$$\begin{aligned} |T(m,n)| \ = \ |T_2(m,n)| \ &\ll \ \sum_{\substack{s > m^3/\psi(q_m): \\ t \in T_s}} |\widehat{W}^{+}_{q_m,\gamma,\varepsilon_m}(sq_m)| \ |\widehat{W}^{+}_{q_n,\gamma,\varepsilon_n}(tq_n)| \\ &\ll \ \sum_{s > m^3/\psi(q_m)} \frac{1}{s^2 \psi(q_m)\, \varepsilon_m} \psi(q_n) \ \ll \ \frac{\psi(q_n)}{m^3\, \varepsilon_m} . \end{aligned}$$

Now on making use of (92), it follows that

$$|T_2(m,n)| \ \ll \ \frac{\psi(q_n)}{m^2}$$

and so in turn, for arbitrary $a, b \in \mathbb{N}$ with $a < b$, we have

$$\sum_{a\le m<n\le b}\sum |T_2(m,n)| \ll \sum_{n=a}^{b}\psi(q_n).$$

$\square$

5 Establishing Propositions 1 and 2

To start with we work under the hypotheses of Lemma 6 which clearly both Proposition 1 and Proposition 2 satisfy. With this in mind, for arbitrary $a, b \in \mathbb{N}$ with $a < b$, we have via (86) that

$$\sum_{a\le m<n\le b}\sum \mu(E^{\gamma}_{q_m} \cap E^{\gamma}_{q_n}) \le \sum_{a\le m<n\le b}\sum \widehat{W}^{+}_{m,n}(0) + \sum_{a\le m<n\le b}\sum S_{m,n} \tag{98}$$

where $W^{+}_{m,n}$ and $S_{m,n}$ are given by (84) and (89) respectively. Now let

$$\varepsilon_n := \min\left(2^{-\delta}, \left(\sum_{k=a}^{n}\psi(q_k)\right)^{-\delta}\right), \tag{99}$$

where $0 < \delta \le 1$ is a parameter to be determined later. By definition, it follows that

$$\varepsilon_n^{-1} \le \max\left(2^{\delta}, n^{\delta}\right) < 2n \tag{100}$$

and so (92) associated with Lemma 8 is satisfied. We also observe that (33) together with (100) implies that

$$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{A}{B}}\varepsilon_n^{1/2}} < \infty\,.$$

Thus, for any fixed $\alpha > 0$, it follows on using Lemma 6 that

$$\sum_{a\le m<n\le b}\sum |S_{m,n}| \ll \sum_{n=a}^{b}\psi(q_n) + \sum_{a\le m<n\le b}\sum \frac{1}{n^{\frac{A}{B}}\varepsilon_m^{1/2}\varepsilon_n^{1/2}} + \sum_{a\le m<n\le b}\sum |T(m,n)|\,, \tag{101}$$

where the implied constant is dependent on α. We now estimate the second sum on the right hand side of (101) by considering two cases.

Case 1: $\sum\limits_{k=a}^{b}\psi(q_k) > 2$. It follows that

$$\frac{1}{n^{\frac{A}{B}}\varepsilon_m^{1/2}\varepsilon_n^{1/2}} \le \frac{1}{n^{\frac{A}{B}}}\left(\sum_{k=a}^{b}\psi(q_k)\right)^{\delta},$$

and so

$$\sum_{a\le m<n\le b}\sum \frac{1}{n^{\frac{A}{B}}\varepsilon_m^{1/2}\varepsilon_n^{1/2}} \le \sum_{a\le m<n\le b}\sum \frac{1}{n^{\frac{A}{B}}}\left(\sum_{k=a}^{b}\psi(q_k)\right)^{\delta} \ll \left(\sum_{k=a}^{b}\psi(q_n)\right)^{\delta} \le \sum_{k=a}^{b}\psi(q_n). \tag{102}$$

Case 2: $\sum_{k=a}^{b} \psi(q_k) \leq 2$. It follows that $\varepsilon_n = 2^{-\delta}$ for all $a \leq n \leq b$, hence

$$\frac{1}{n^{\frac{A}{B}} \varepsilon_m^{1/2} \varepsilon_n^{1/2}} \ll \frac{1}{n^{\frac{A}{B}}}.$$

On using (58) with $\tau := A/2B$, it follows that

$$\frac{1}{n^{\frac{A}{B}}} \leq \frac{\psi(q_n)}{n^{\frac{A}{2B}}} \qquad \forall\, n \in \mathbb{N}$$

and so

$$\sum\sum_{a \leq m < n \leq b} \frac{1}{n^{\frac{A}{B}} \varepsilon_m^{1/2} \varepsilon_n^{1/2}} \ll \sum\sum_{a \leq m < n \leq b} \psi(q_n) n^{-\frac{A}{2B}} \leq \sum\sum_{a \leq m < n \leq b} \psi(q_n) m^{-\frac{A}{2B}} \ll \sum_{n=a}^{b} \psi(q_n). \tag{103}$$

The upshot of (102) and (103) is that both cases give rise to the same estimate which together with (101) gives

$$\sum\sum_{a \leq m < n \leq b} |S_{m,n}| \ll \sum_{n=a}^{b} \psi(q_n) + \sum\sum_{a \leq m < n \leq b} |T(m,n)|\,.$$

This, in turn with (98) yields the estimate

$$\sum\sum_{a \leq m < n \leq b} \mu(E_m^\gamma \cap E_n^\gamma) \leq \sum\sum_{a \leq m < n \leq b} \widehat{W}_{m,n}^{+}(0) + \sum\sum_{a \leq m < n \leq b} |T(m,n)| + O\left(\sum_{n=a}^{b} \psi(q_n)\right). \tag{104}$$

We now turn our attention to estimating the first term on the right hand of (104). For this we will make use of (88). With this in mind, first note that by definition $(\varepsilon_n)_{n \in \mathbb{N}}$ is decreasing and so $\varepsilon_n \psi(q_m)\psi(q_n) \leq \varepsilon_m \psi(q_m)\psi(q_n)$ for all $m < n$. Moreover, $\varepsilon_n < 1$ for all n and so $\varepsilon_m \varepsilon_n \psi(q_m)\psi(q_n) < \varepsilon_m \psi(q_m)\psi(q_n)$. Thus

$$4(1+\varepsilon_m)(1+\varepsilon_n)\psi(q_m)\psi(q_n) \;\leq\; 4\psi(q_m)\psi(q_n) + 12\varepsilon_m \psi(q_m)\psi(q_n)\,. \tag{105}$$

We estimate the second sum on the right hand side of the above by considering two cases.

Case 1: $\sum_{k=a}^{b} \psi(q_k) < 2$. It follows that

$$\sum\sum_{a \leq m < n \leq b} \varepsilon_m \psi(q_m)\psi(q_n) \leq \sum\sum_{a \leq m < n \leq b} \psi(q_m)\psi(q_n) < \left(\sum_{n=a}^{b} \psi(q_n)\right)^2 < 2\sum_{n=a}^{b} \psi(q_n). \tag{106}$$

Case 2: $\sum_{k=a}^{b} \psi(q_k) \geq 2$. It follows that

$$\sum\sum_{a\leq m<n\leq b} \varepsilon_m \psi(q_m)\psi(q_n) = \sum\sum_{a\leq m<n\leq b} \psi(q_m)\psi(q_n) \ \min\left(2^{-\delta}, \left(\sum_{k=a}^{m}\psi(q_k)\right)^{-\delta}\right)$$

$$\leq \ \max\left(2^{1-\delta}, \left(\sum_{n=a}^{b}\psi(q_n)\right)^{1-\delta}\right) \sum\sum_{a\leq m<n\leq b} \frac{\psi(q_m)\psi(q_n)}{\max\left(2, \sum_{k=a}^{m}\psi(q_k)\right)} \tag{107}$$

$$\leq \ \left(\sum_{n=a}^{b}\psi(q_n)\right)^{1-\delta} \sum_{a\leq n\leq b}\psi(q_n) \sum_{a\leq m\leq b} \frac{\psi(q_m)}{\max\left(2, \sum_{k=a}^{m}\psi(q_k)\right)} .$$

On using Lemma D2 in Appendix D of §7 with $\gamma = 2$, $s_k := \psi(q_{k-a+1})$ and a and b replaced by 1 and $b - a + 1$ respectively, we infer that

$$\sum_{a\leq m\leq b} \frac{\psi(q_m)}{\max\left(2, \sum_{k=a}^{m}\psi(q_k)\right)} \ \leq \ \frac{3}{2} + \frac{1}{2\log\frac{3}{2}} \log\left(\sum_{n=a}^{b}\psi(q_n)\right) .$$

This together with (107) implies that

$$\sum\sum_{a\leq m<n\leq b} \varepsilon_m \psi(q_m)\psi(q_n) \ \leq \ \left(\sum_{n=a}^{b}\psi(q_n)\right)^{1-\delta} \left(\sum_{n=a}^{b}\psi(q_n)\right)\left(\frac{3}{2} + \frac{1}{2\log\frac{3}{2}} \log\left(\sum_{n=a}^{b}\psi(q_n)\right)\right)$$

$$\ll \ \left(\sum_{n=a}^{b}\psi(q_n)\right)^{2-\delta} \log\left(\sum_{n=a}^{b}\psi(q_n)\right) . \tag{108}$$

Hence, on combining the estimates (88), (105), (106) and (108) with find that

$$\sum\sum_{a\leq m<n\leq b} W^{+}_{m,n}(0) \leq 4\left(\sum_{n=a}^{b}\psi(q_n)\right)^{2} + O\left(\left(\sum_{n=a}^{b}\psi(q_n)\right)^{2-\delta} \log^{+}\left(\sum_{n=a}^{b}\psi(q_n)\right)\right)$$

$$+ \ O\left(\sum\sum_{a\leq m<n\leq b} (q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right)\right) . \tag{109}$$

We stress that the above estimates, in particular (104) and (109) are valid under the hypotheses of both Proposition 1 and Proposition 2.

Completing the proof of Proposition 1. Working under the hypotheses of Proposition 1, we can employ Lemma 7 with $\alpha = 1/2$ to obtain that

$$\sum\sum_{a\le m<n\le b} |T(m,n)| \ll \sum_{n=a}^{b} \frac{\psi(q_n)}{\varepsilon_n^{1/2}} \ll \frac{1}{\varepsilon_b^{1/2}} \sum_{n=a}^{b} \psi(q_n) \ll \left(\sum_{n=a}^{b} \psi(q_n)\right)^{1+\frac{\delta}{2}} . \tag{110}$$

Also, since the sequence $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ is lacunary, there exists a constant $K > 1$ such that for any integers $m < n$

$$q_n \ \ge \ K^{n-m}\, q_m \,,$$

and so

$$\begin{aligned} \sum\sum_{a\le m<n\le b} (q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right) \ &\le \ \sum\sum_{a\le m<n\le b} q_m \frac{\psi(q_n)}{q_n} \\ &\le \ \sum_{n=a}^{b} \psi(q_n) \sum_{m=1}^{n-1} \frac{q_m}{q_n} \ \ll \ \sum_{n=a}^{b} \psi(q_n) \,. \end{aligned} \tag{111}$$

Hence, on combining the estimates (104), (109), (110) and (111) we find that

$$\begin{aligned} \sum\sum_{a\le m<n\le b} \mu(E^{\gamma}_{q_m} \cap E^{\gamma}_{q_n}) \ \le \ 4 \left(\sum_{n=a}^{b} \psi(q_n)\right)^2 \ + \ O\Bigg(\left(\sum_{n=a}^{b} \psi(q_n)\right)^{2-\delta} \log^{+}\left(\sum_{n=a}^{b} \psi(q_n)\right) \\ + \ \left(\sum_{n=a}^{b} \psi(q_n)\right)^{1+\frac{\delta}{2}} \Bigg). \end{aligned}$$

To complete the proof of Proposition 1, we set $\delta = 2/3$ in the above and apply Lemma 5. □

Completing the proof of Proposition 2. Working under the hypotheses of Proposition 2, we can employ Lemma 8 with α given by the α-separated sequence $\mathcal{A}$. Note that condition (91) on ψ is guaranteed by (58) while (100) shows that condition (92) on ε_n is satisfied. On combining (93), (104) and (109) we find that

$$\begin{aligned} \sum\sum_{a\le m<n\le b} \mu(E^{\gamma}_{q_m} \cap E^{\gamma}_{q_n}) \ \le \ 4 \left(\sum_{n=a}^{b} \psi(q_n)\right)^2 + O\left(\sum\sum_{a\le m<n\le b} (q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right)\right) \\ + O\left(\left(\sum_{n=a}^{b} \psi(q_n)\right)^{2-\delta} \log^{+}\left(\sum_{n=a}^{b} \psi(q_n)\right) + \sum_{n=a}^{b} \psi(q_n)\right). \end{aligned}$$

To complete the proof of Proposition 2, we set $\delta = 1$ in the above and use Lemma 5. □

6 Deducing Theorem 3 from Theorem 4

Recall, that any increasing sequence $\mathcal{A} \subseteq \mathcal{A}_{\mathcal{S}}$ satisfies the growth condition (32) with $B = k$ and $C = (\log 2)/2$ – see Appendix B of §7 for the details. Thus, Theorem 3 will follow from Theorem 4 on showing that:

(i) any such sequence is α-separated, and

(ii) the gcd term $E(N)$ appearing in the 'error' term of (34) is less than the 'main' term $\Psi(N)$; that is to say

$$E(N) = O(\Psi(N)). \tag{112}$$

The point is once we have (i) the hypotheses of Theorem 4 are verified and the theorem implies that the counting function $R(x, N)$ satisfies (34) for μ-almost all $x \in F$. On the other hand, (34) trivially coincides with (31) once we have (ii) and thus completes the proof of Theorem 3.

6.1 Showing that any $\mathcal{A} \subseteq \mathcal{A}_{\mathcal{S}}$ is α-separated

The goal of this section is to prove the following statement.

Proposition 3. *Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_{\mathcal{S}}$ be an increasing sequence of natural numbers. Then, $\mathcal{A}$ is α-separated for any $\alpha \in (0, 1)$.*

The proof of the proposition we will make essential use of a fundamental theorem due to Baker & Wüstholz [1] in the theory of linear forms in logarithms. The following statement is a simplified version of that appearing in [1]. It is more than adequate for the application we have in mind.

Theorem BW. *Let $n \in \mathbb{N}$, $b_1 \dots b_n \in \mathbb{Z}$ and $a_1 \dots a_n \in \mathbb{N}$. Suppose that*

$$\Lambda := \sum_{k=1}^{n} b_k \log a_k \neq 0.$$

Then,

$$\log|\Lambda| > -C(n) \cdot \prod_{k=1}^{n} \max(1, \log a_k) \cdot \log\Big(\max(1, |b_1|, \dots, |b_n|)\Big)$$

where

$$C(n) := 18(n+1)!\, n^{n+1}(32)^{n+2} \log(2n). \tag{113}$$

Proof of Proposition 3. Let $\alpha \in (0, 1)$. The aim is to show that there exists a constant $m_0 \in \mathbb{N}$ so that for any natural numbers $m < n$, if

$$1 \;\; \leq \;\; |sq_m - tq_n| < q_m^{\alpha} \tag{114}$$

for some $s, t \in \mathbb{N}$ with

$$s \leq m^{12}, \tag{115}$$

then $m \le m_0$. With this in mind, first of all note that (115) and (114) imply that

$$t < \frac{q_m}{q_n} s + \frac{1}{2} \le \frac{q_m}{q_n} m^{12} + \frac{1}{2} < m^{12} + \frac{1}{2}. \tag{116}$$

Hence, taking into account that m and t are integers, we have that

$$t \le m^{12}. \tag{117}$$

From the second inequality appearing in (116) and the fact that $t \in \mathbb{N}$, it follows that

$$1 \le t < \frac{q_m}{q_n} m^{12} + \frac{1}{2}.$$

Hence,

$$q_n < 2m^{12} q_m$$

which together with the fact that $q_m < q_n$, implies that

$$\log q_m < \log q_n < \log q_m + 12 \log m + \log 2. \tag{118}$$

Note that $sq_m \ne tq_n$ because of (114) and assume for the moment that $sq_m < tq_n$. Then, on using the fact that $\exp(x) - 1 \ge x$ for any $x > 0$, it follows that

$$\begin{aligned} |sq_m - tq_n| &= tq_n - sq_m \\ &= sq_m \left(\exp\left(\log t + \log q_n - \log s - \log q_m\right) - 1\right) \\ &\ge sq_m \left(\log t + \log q_n - \log s - \log q_m\right). \end{aligned} \tag{119}$$

We now proceed to estimate the quantity involving the logarithm terms. On using the fact that $q_m, q_n \in \mathcal{A}_{\mathcal{S}}$, it follows via Theorem BW that

$$|\log t + \log q_n - \log s - \log q_m| \ge \exp\Big(- C(k+2) \log s \log t \prod_{p \in \mathcal{S}} \log p \log \max_{a \in \mathcal{E}_{m,n}} a \Big). \tag{120}$$

Here $C(k+2)$ is the constant associated with Theorem BW and $\mathcal{E}_{m,n}$ is the set of exponents of prime powers in the canonical factorisation of the integers $q_m, q_n \in \mathcal{A}_{\mathcal{S}}$. By definition, for any $a \in \mathcal{E}_{m,n}$ we have $2^a \le q_n$ and so $a \le \log q_n / \log 2$. This together with the upper bound estimates (115) and (117), implies that

$$\begin{aligned} \log s \log t \prod_{p \in \mathcal{S}} \log p \log \max_{a \in \mathcal{E}_{m,n}} a &\le 144 \left(\log m\right)^2 \Big(\prod_{p \in \mathcal{S}} \log p \Big) \log\left(\log q_n / \log 2\right) \\ &\ll k^2 \left(\log \log q_m\right)^3, \end{aligned} \tag{121}$$

where in the last step we have also used (118) and (32) with $B = k$. Hence

$$|\log t + \log q_n - \log s - \log q_m| \ge \exp\left(-\tilde{C} \left(\log \log q_m\right)^3\right), \tag{122}$$

where the constant $\tilde{C}$ depends on the set $\mathcal{S}$ only. This together with (119) yields that

$$|sq_m - tq_n| = tq_n - sq_m \ge \exp\left(\log q_m - \tilde{C} \left(\log \log q_m\right)^3\right). \tag{123}$$

Now if $sq_m > tq_n$, the above above argument can easily be modified to show that (123) still holds. Indeed, on using the fact that $q_m < q_n$, we find that

$$
\begin{aligned}
|sq_m - tq_n| &= sq_m - tq_n \\
&\geq tq_n \left(\log s + \log q_m - \log t - \log q_n\right) \\
&\overset{(122)}{\geq} tq_n \exp\left(- \tilde{C}\,(\log\log q_m)^3 \right) \\
&\geq \exp\left(\log q_m - \tilde{C}(\log\log q_m)^3 \right).
\end{aligned}
$$

Comparing (123) with (114), we find that

$$\log q_m - \tilde{C}\,(\log\log q_m)^3 \leq \alpha \log q_m.$$

We can rewrite this as

$$\tilde{C}\,(\log\log q_m)^3 \geq (1-\alpha)\log q_m. \tag{124}$$

As $\alpha < 1$, it is evident that this inequality can only hold for m not exceeding some integer m_0 that depends on $\mathcal{S}$ and α only. This completes the proof of the proposition.

□

6.1.1 A stronger version of Proposition 3 involving Property D

In this section we show that the proof of Proposition 3 can be easily adapted to prove the analogous statement for sequences $\mathcal{A}$ satisfying Property D – see Remark 13 of §1.2 for the defintiion. Note that in the proof of Proposition 3 we made direct use of the fact that the growth condition (32) is satisfied for $\mathcal{A} \subseteq \mathcal{A}_{\mathcal{S}}$. By definition, this condition automatically holds for sequences satisfying Property D.

Formally, we establish the following generalisation of Proposition 3.

Proposition 3A. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers that satisfies Property D. Then, $\mathcal{A}$ is α-separated for any $\alpha \in (0,1)$.

Proof (sketch). The proof is exactly the same as that of Proposition 3 up to and including the inequality given by (119). The main modifications after that are as follows:

- The product $\prod_{p\in\mathcal{S}}$ appearing in (120) and thereafter needs to be replaced by $\prod_{p\in\mathcal{P}_{m,n}}$ in which $\mathcal{P}_{m,n}$ denotes the set of all prime divisors of $q_m\,q_n$. Also the constant associated with Theorem BW is $C(2D+2)$ where D is constant coming from Property D (namely part (a) of condition (ii)). Note that since $\mathcal{A}$ satisfies Property D, we have that for $n_0 < m < n$ (which, without loss of generality, we can assume)

$$\#\mathcal{P}_{m,n} \leq 2 \cdot D\,. \tag{125}$$

- The upper bound for $p \in \mathcal{P}_{m,n}$ coming from Property D (namely part (b) of condition (ii)) and (125) have to be added to the list of the upper bound inequalities (115) and (117) used to derive the analogue of (121); namely, for $n_0 < m < n$ with m sufficiently large

$$\log s \log t \prod_{p\in\mathcal{P}_{m,n}} \log p \ \log \max_{k\in\mathcal{E}_{m,n}} k \leq \frac{144}{(\log 2)^{2D}} (\log m)^2 \left((\log q_n)^{\frac{1-\epsilon}{2D}} \right)^{2D} \log(\log q_n - \log 2)$$

$$\ll B^2 (\log q_m)^{1-\epsilon} (\log\log q_m)$$

$$\ll (\log q_m)^{1-\epsilon/2},$$

where B is the constant associated with the growth condition (32).

With the above main modifications in mind, we continue exactly as in the proof of Proposition 3 and obtain the following analogue of (124)

$$\tilde{C} (\log q_m)^{1-\epsilon/2} \geq (1-\alpha) \log q_m ,$$

where $\tilde{C}$ is a constant that depends on B and D only. As $\alpha < 1$, it is evident that this inequality can only hold for m not exceeding some integer m_0 that depends only on the constants associated with Property D, α and ϵ. This completes the proof of the proposition. □

6.2 Showing that $E(N) = O\big(\Psi(N)\big)$

The goal of this section is to establish (112). As we shall soon see, this is an immediate consequence of the following statement.

Theorem 5. *Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_\mathcal{S}$ be an increasing sequence of natural numbers. Then, there exists a constant C which depends only on the cardinality k of $\mathcal{S}$, such that for any integer $n \geq 2$*

$$\sum_{m=1}^{n-1} \frac{(q_m, q_n)}{q_n} \leq C . \tag{126}$$

Clearly, Theorem 5 implies that

$$E(N) := \sum\sum_{1\leq m<n\leq N} (q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n} \right)$$

$$\leq \sum\sum_{1\leq m<n\leq N} \frac{(q_m, q_n)}{q_n} \psi(q_n) \ \ll \ \sum_{n=1}^{N} \psi(q_n) := \Psi(N)$$

and so yields the desired goal. Note the left hand side of (126) can only increase if we enlarge our sequence $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$. So, without loss of generality, we can assume that $\mathcal{A} = \mathcal{A}_\mathcal{S}$ during the course of establishing Theorem 5. With this in mind, we start by proving a couple of useful lemmas.

Lemma 9. *For any $s \in \mathbb{R}$, there exists a constant C_s, which depends on s only, such that, for any $r \in \mathbb{N}$, and any integers $n_1, \dots, n_r \geq 2$,*

$$\sum_{t_1=1}^{\infty} \cdots \sum_{t_r=1}^{\infty} \frac{(t_1 \log_2 n_1 + \cdots + t_r \log_2 n_r)^s}{n_1^{t_1} \cdot \dots \cdot n_r^{t_r}} < C_s.$$

Proof. First note that the function defined for $x \geq 1$ by $x \to \dfrac{(\log_2 x)^s}{\sqrt{x}}$ is bounded above by a constant. Define

$$C_s = \sup_{x \geq 1} \frac{(\log_2 x)^s}{\sqrt{x}} > 0\,.$$

Then,

$$\begin{aligned} \sum_{t_1=1}^{\infty} \cdots \sum_{t_r=1}^{\infty} \frac{(t_1 \log_2 n_1 + \cdots + t_r \log_2 n_r)^s}{n_1^{t_1} \cdot \dots \cdot n_r^{t_r}} &\leq \sum_{t_1=1}^{\infty} \cdots \sum_{t_r=1}^{\infty} \frac{C_s}{n_1^{t_1/2} \cdot \dots \cdot n_r^{t_r/2}} \\ &\leq C_s \cdot \frac{1}{(\sqrt{n_1} - 1) \cdot \dots \cdot (\sqrt{n_r} - 1)} \\ &\leq C_s, \end{aligned}$$

and this proves the lemma. □

Lemma 10. *Let $K \geq 1$ be a real number and $n \in \mathbb{N}$. Then*

$$\sum_{\substack{q_m | q_n \\ q_m \cdot K < q_n}} \frac{(q_m, q_n)}{q_n} \leq \frac{(\log_2 K + 2)^{k-1}}{K} \cdot \prod_{i=1}^{k} \frac{p_i}{p_i - 1}. \tag{127}$$

Proof. The statement is obviously true when $n = 1$. Observe that the condition that $q_m | q_n$ and $q_m K < q_n$ is equivalent to

$$\frac{q_n}{q_m} \equiv \prod_{i=1}^{k} p_i^{a_i} > K \tag{128}$$

for some integers $a_1, \dots, a_k \geq 0$. Since n is fixed, the k–tuple $(a_1, \dots, a_k)$ depends exclusively on m. We will say that the divisor q_m of q_n is *maximal* if and only if (128) holds and for any other divisor q_l of q_n such that $q_l K < q_n$ and $q_m | q_l$ we necessarily have that $q_m = q_l$. It then follows that

$$\begin{aligned}
\sum_{\substack{q_m|q_n\\ q_mK<q_n}} \frac{(q_m,q_n)}{q_n} \quad &\leq \quad \sum_{q_m \text{maximal}} \sum_{q_l|q_m} \frac{(q_l,q_n)}{q_n} \;=\; \sum_{q_m \text{maximal}} \sum_{q_l|q_m} \frac{q_l}{q_n} \\
&= \quad \sum_{q_m \text{maximal}} \sum_{q_l|q_m} \frac{q_l}{q_m}\frac{q_m}{q_n} \;\leq\; \sum_{q_m \text{maximal}} \frac{1}{K} \sum_{q_l|q_m} \frac{q_l}{q_m} \\
&\leq \quad \sum_{q_m \text{maximal}} \frac{1}{K} \sum_{c_1=0}^{\infty} \cdots \sum_{c_k=0}^{\infty} \frac{1}{p_1^{c_1}\cdots p_k^{c_k}} \\
&\leq \quad \sum_{q_m \text{maximal}} \frac{1}{K} \prod_{i=1}^{k} \frac{p_i}{p_i-1}\,,
\end{aligned}$$

where the outer sum on the right hand side is over all maximal divisors q_m of q_n. Thus, the proof of the lemma is reduced to showing that the number of such divisors is bounded above by $(\lfloor \log_2 K \rfloor + 2)^{k-1}$; that is

$$\sum_{q_m \text{maximal}} 1 \;\leq\; (\lfloor \log_2 K \rfloor + 2)^{k-1}\,. \tag{129}$$

With this in mind, observe that (128) gives

$$\sum_{i=1}^{k} a_i \log p_i > \log K \tag{130}$$

and so q_m is maximal if and only if the corresponding solution $(a_1,\ldots,a_k) \in \mathbb{Z}_{\geq 0}^k$ to inequality (130) is *minimal*, in the sense that for any other solution $(b_1,\ldots,b_k) \in \mathbb{Z}_{\geq 0}^k$ with $b_i \leq a_i$ for all $i = 1,\ldots,k$, we necessarily have that $b_i = a_i$ for all $i = 1,\ldots,k$. It is easily versified that if $(a_1,\ldots,a_k)$ is a minimal solution to (130), then

$$a_1 + \cdots + a_k \leq \log_2 K + 1. \tag{131}$$

Indeed, to show that this is so, assume on the contrary that $(a_1,\ldots,a_k)$ is minimal and $a_1 + \ldots + a_k > \log_2 K + 1$. Then, without loss of generality, we may assume $a_1 \geq 1$. Then

$$(a_1-1)\log p_1 + a_2 \log p_2 + \ldots + a_k \log p_k \geq (a_1-1) + a_2 + \ldots + a_k \geq \log_2 K.$$

This means that $(a_1-1, a_2,\ldots,a_k)$ is a solution to (130) and thus contradicts the fact that $(a_1,\ldots,a_k)$ is minimal.

By definition, (129) is equivalent to the statement that number of minimal solutions to (130) is bounded above by $(\lfloor \log_2 K \rfloor + 2)^{k-1}$. This we now proceed to prove. Define the map from the set of minimal solutions $(a_1,\ldots,a_k)\in\mathbb{Z}_{\geq 0}^k$ of (130) to the set of k-tuples $(b_1,\ldots,b_k)\in\mathbb{Z}_{\geq 0}^k$ satisfying $b_1+\ldots+b_k=\lfloor\log_2 K\rfloor+1$ by

$$(a_1,\ldots,a_k)\to(b_1,\ldots,b_k):=\left(a_1,\ldots,a_{k-1},\lfloor\log_2 K\rfloor+1-\sum_{i=1}^{k-1}a_i\right). \tag{132}$$

In view of (131), $b_k\geq 0$ and so the map is well-defined. The map is also injective. Indeed, assume $(x_1,\ldots,x_k)$ and $(y_1,\ldots,y_k)$ are two distinct minimal solutions to (130) with the same image under the map (132). Then necessarily $x_i=y_i$ for all $i=1,\ldots,k-1$, and since the solutions are distinct, either $x_k<y_k$ or $x_k>y_k$. This means that one of the solutions is not minimal, which is a contradiction. Thus the map defined via (132) is injective, whence the number of minimal solutions to (130) is at most equal to the number of k-tuples satisfying $b_1+\ldots+b_k=\lfloor\log_2 K\rfloor+1$; namely

$$\binom{\lfloor\log_2 K\rfloor+k}{k-1}=\frac{(\lfloor\log_2 K\rfloor+k)\cdot\cdots\cdot(\lfloor\log_2 K\rfloor+1)}{(k-1)!}\leq(\lfloor\log_2 K\rfloor+2)^{k-1}.$$

This thereby completes the proof of the lemma.

□

Proof of Theorem 5. As already mentioned, it suffices to proves the theorem with $\mathcal{A}=\mathcal{A}_{\mathcal{S}}$. With this in mind, we are given an integer $n\geq 2$ and so this fixes

$$q_n=\prod_{i=1}^{k}p_i^{a_i}\qquad(a_1,\ldots,a_k\geq 0). \tag{133}$$

For any integer $m<n$, we write

$$q_m=\prod_{i=1}^{k}p_i^{b_i}\qquad(b_1,\ldots,b_k\geq 0)\,, \tag{134}$$

where the exponents $b_1,\ldots,b_k$ depend on the index m. Obviously, since $m<n$

$$\sum_{i=1}^{k}b_i\log p_i<\sum_{i=1}^{k}a_i\log p_i\,, \tag{135}$$

and for each such m we set

$$\mathcal{P}(q_m):=\{1\leq i\leq k: b_i<a_i\}\,.$$

It then follows that

$$\begin{aligned}\sum_{m=1}^{n-1}\frac{(q_m,q_n)}{q_n} &= \sum_{\substack{(b_1,\ldots,b_k)\in\mathbb{Z}_{\geq 0}^k:\\ (135)\text{ holds}}}\frac{p_1^{\min(a_1,b_1)}\cdots p_k^{\min(a_k,b_k)}}{p_1^{a_1}\cdots p_k^{a_k}}\\ &= \sum_{\mathcal{T}\subseteq\{1,\ldots,k\}}\ \sum_{\substack{(b_1,\ldots,b_k)\in\mathbb{Z}_{\geq 0}^k:\\ (135)\text{ holds }\&\\ \mathcal{P}(q_m)=\mathcal{T}}}\frac{\prod_{i\in\mathcal{T}}p_i^{b_i}\cdot\prod_{i\notin\mathcal{T}}p_i^{a_i}}{\prod_{i\in\mathcal{T}}p_i^{a_i}\cdot\prod_{i\notin\mathcal{T}}p_i^{a_i}}\,.\end{aligned}$$

In view of the fact that (135) is imposed as a condition on the inner sum, we can assume that $\mathcal{T} \neq \emptyset$. With this in mind, it follows that

$$\sum_{m=1}^{n-1} \frac{(q_m, q_n)}{q_n} \leq \sum_{\mathcal{T} \subset \{1,\dots,k\}} \sum_{b_i \geq a_i :\, i \notin \mathcal{T}} \sum_{\substack{b_i < a_i :\, i \in \mathcal{T} \ \& \\ \prod_{i\in\mathcal{T}} p_i^{b_i} \cdot K < \prod_{i\in\mathcal{T}} p_i^{a_i}}} \frac{\prod_{i\in\mathcal{T}} p_i^{b_i}}{\prod_{i\in\mathcal{T}} p_i^{a_i}}$$

$$+ \sum_{\substack{b_i < a_i :\, i \in \mathcal{T} = \{1,\dots,k\} \ \& \\ \prod_{i\in\mathcal{T}} p_i^{b_i} < \prod_{i\in\mathcal{T}} p_i^{a_i}}} \frac{\prod_{i\in\mathcal{T}} p_i^{b_i}}{\prod_{i\in\mathcal{T}} p_i^{a_i}} \, ,$$

where $K := \prod_{i \notin \mathcal{T}} p_i^{b_i - a_i}$. Now on appealing to Lemma 10 and then Lemma 9, we find that

$$\begin{aligned}
\sum_{m=1}^{n-1} \frac{(q_m, q_n)}{q_n} &\leq \sum_{\mathcal{T} \subset \{1,\dots,k\}} \sum_{b_i \geq a_i :\, i \notin \mathcal{T}} \frac{(\log_2 K + 2)^{k-1}}{K} \prod_{i \in \mathcal{T}} \frac{p_i}{p_i - 1} \\
&\quad + 2^{k-1} \prod_{i=1}^{k} \frac{p_i}{p_i - 1} \\
&\leq \sum_{\mathcal{T} \subset \{1,\dots,k\}} \sum_{b_i \geq a_i :\, i \notin \mathcal{T}} \frac{(2 \log_2 K)^{k-1}}{K} \cdot 2^k + 2^{k-1} \cdot 2^k \\
&\leq 2^{k-1} \cdot 2^k \Big(C_{k-1} \sum_{\mathcal{T} \subset \{1,\dots,k\}} 1 + 1 \Big) ,
\end{aligned}$$

where $C_{k-1} > 0$ is the constant associated with Lemma 9. This together with the fact that there are at most 2^k different subsets $\mathcal{T}$ of $\{1, \dots, k\}$ implies the desired statement. □

7 Appendices

Appendix A: Theorem DEL $\Longrightarrow$ Corollary DEL

The goal is to deduce Corollary DEL from Theorem DEL – the fundamental theorem of Davenport, Erdös & LeVeque in the theory of uniform distribution.

We are given that $\mathcal{A} = (q_n)_{n \in \mathbb{N}}$ is a lacunary sequence of natural numbers. Thus, there exists a constant $K > 1$ such that any integers $m < n$

$$q_n - q_m = q_m \left(\frac{q_n}{q_m} - 1 \right) \geq K^{n-m} - 1 \, . \tag{136}$$

Now let f be as in Corollary DEL and consider the associated function $F : \mathbb{N} \to \mathbb{R}^+$ given by

$$F(n) := f(K^n - 1) \, .$$

Note that F is decreasing (since f is decreasing) and so it follows that the convergence condition (3) is equivalent to the condition that

$$\sum_{n=1}^{\infty} \frac{F(n)}{n} \; < \; \infty\,. \tag{137}$$

Also, by the decay condition (4) and the fact that $\widehat{\mu}(0) = 1$, it follows that for any integer $h \neq 0$

$$\begin{aligned}
\sum_{m,n=1}^{N} \widehat{\mu}(h(q_n - q_m)) \quad &= \quad \sum_{n=1}^{N} \widehat{\mu}(0) \; + \; 2 \sum\sum_{1 \le m < n \le N} \widehat{\mu}(h(q_n - q_m)) \\
&\overset{(4)}{\ll} \quad N \; + \; \sum\sum_{1 \le m < n \le N} f\big(|h(q_n - q_m)|\big) \\
&\le \quad N \; + \; \sum\sum_{1 \le m < n \le N} f(q_n - q_m) \\
&\overset{(136)}{\le} \quad N \; + \; \sum\sum_{1 \le m < n \le N} F(n - m) \\
&\le \quad N \; + \sum_{n=1}^{N} \sum_{m=1}^{n} F(m) \;\; = \;\; N \; + \; \sum_{n=1}^{N} (N + 1 - n) F(n) \\
&\ll \quad N \sum_{n=1}^{N} F(n)\,.
\end{aligned}$$

The upshot of this is that

$$\begin{aligned}
\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{m,n=1}^{N} \widehat{\mu}(h(q_n - q_m)) \quad &\ll \quad \sum_{N=1}^{\infty} \frac{1}{N^2} \sum_{n=1}^{N} F(n) = \sum_{n=1}^{\infty} \sum_{N=n}^{\infty} \frac{F(n)}{N^2} \\
&\ll \quad \sum_{n=1}^{\infty} \frac{F(n)}{n} \overset{(137)}{<} \; \infty\,.
\end{aligned}$$

Thus, Theorem DEL implies that the sequence $\mathcal{A} = (q_n)_{n \in \mathbb{N}}$ is uniformly distributed modulo one for μ–almost all $x \in F$. This completes the proof of Corollary DEL.

Appendix B: The growth rate of $\mathcal{A}_{\mathcal{S}}$

The goal is to show the sequence $\mathcal{A}_{\mathcal{S}} = (q_n)_{n \in \mathbb{N}}$ defined by (30), satisfies the growth condition

$$\log q_n \; > \; C \, n^{1/k} \qquad \forall \; n \ge 2\,,$$

where k is the cardinality of the finite set $\mathcal{S} := \{p_1, \ldots, p_k\}$ of distinct primes and $C = (\log 2)/2$. It is easily seen that this is an immediate consequence of the following counting statement: for $X \geq 2$

$$\pi_{\mathcal{S}}(X) := \#\{q \in \mathcal{A}_{\mathcal{S}} : q \leq X\} \ \leq \ C^{-1}(\log X)^k . \tag{138}$$

Indeed, given $q_n \in \mathcal{A}_{\mathcal{S}}$ with $n \geq 2$, put $X = q_n$. Then $X \geq 2$ and (138) implies that

$$\pi_{\mathcal{S}}(q_n) \ = \ n \leq C^{-1}(\log q_n)^k$$

and we are done since $C^{1/k} \geq C$.

We prove (138) by induction on k. When $k = 1$, we have only one prime p and by definition $\pi_{\mathcal{S}}(X) := \#\left\{p^{n-1} : p^{n-1} \leq X\right\}$. The condition that $p^{n-1} \leq X$ implies that

$$n \leq \frac{\log X}{\log p} + 1 \ \leq \ \frac{2}{\log 2} \log X \ = \ C^{-1} \log X \ .$$

This verifies (138) when $k = 1$. Now assume (138) is true for any set of k distinct primes and let $\mathcal{S} = \{p_1, \ldots, p_k, p_{k+1}\}$ be a set of $k+1$ distinct primes. Write $\mathcal{S}_k$ for the set $\{p_1, \ldots, p_k\}$ of k distinct primes. It follows that

$$\begin{aligned}
\pi_{\mathcal{S}}(X) \ &:= \ \sum_{\substack{q \leq X \\ q \in \mathcal{A}_{\mathcal{S}}}} 1 \ = \sum_{\substack{(a_1,\ldots,a_{k+1}) \in \mathbb{Z}_{\geq 0}^{k+1}: \\ \prod_{i=1}^{k+1} p_i^{a_i} \leq X}} 1 \ = \sum_{\substack{(a_1,\ldots,a_{k+1}) \in \mathbb{Z}_{\geq 0}^{k+1}: \\ \sum_{i=1}^{k+1} a_i \log p_i \leq \log X}} 1 \\
&\leq \sum_{0 \leq a_{k+1} \leq \frac{\log X}{\log p_{k+1}}} \ \sum_{\substack{(a_1,\ldots,a_k) \in \mathbb{Z}_{\geq 0}^{k}: \\ \sum_{i=1}^{k} a_i \log p_i \leq \log X}} 1 \ = \sum_{0 \leq a_{k+1} \leq \frac{\log X}{\log p_{k+1}}} \pi_{\mathcal{S}_k}(X) \\
&\leq \sum_{0 \leq a_{k+1} \leq \frac{\log X}{\log p_{k+1}}} C^{-1}(\log X)^k \qquad \text{(by the induction hypothesis)} \\
&\leq \ \frac{C^{-1}}{\log p_{k+1}}(\log X)^{k+1} \ < \ C^{-1}\,(\log X)^{k+1} .
\end{aligned}$$

This completes the inductive step and establishes (138) for arbitrary $k \in \mathbb{N}$.

Appendix C: Example of 'bad' sequences satisfying Property D

The goal is to construct an increasing sequence $\mathcal{A} = (q_n)_{n \in \mathbb{N}}$ of natural numbers satisfying Property D and an associated function $\psi : \mathbb{N} \to \mathbb{I}$ such that for all integers $N \geq N_0$

$$\sum_{j=1}^{N} \psi(q_j) \sum_{m=1}^{j} \frac{(q_m, q_j)}{q_j} > \exp\left(c \sum_{j=1}^{N} \psi(q_j)\right) . \tag{139}$$

Here $c > 0$ and $N_0 \geq 1$ are absolute constants. This 'strongly' implies the claim associated with (38) in Remark 13. Thus with reference to Theorem 4, for arbitrary sequences satisfying Property D, we can not reduce (34) to (31) as in the situation when $\mathcal{A} \subseteq \mathcal{A}_{\mathcal{S}}$.

Step 1: Constructing the sequence $\mathcal{A}$. To start with, let $(n_t)_{t\in\mathbb{N}}$ be an increasing sequence of natural numbers satisfying the following conditions:

- The integer n_1 is large enough so that

$$\ln\ln n_1 > 2\ln 2 + 1 \tag{140}$$

  and

$$2\log n + 2\log\log n < n^{1/5} \qquad \forall\ \ n \geq n_1\,. \tag{141}$$

- For all $t\in\mathbb{N}$,

$$2n_t \leq n_{t+1}. \tag{142}$$

It can be easily verified that (141) implies (140) but it will be useful to have both explicitly stated. Also, note that in view of (140) and (142), it follows that

$$n_t \geq 2^{t-1} n_1 \geq 2^{t-1} e^{4e} \qquad \forall\ \ t\in\mathbb{N}\,. \tag{143}$$

In particular, this implies

$$\ln n_t \geq t\ln 2 + 4e - \ln 2 > (t+14)\ln 2. \tag{144}$$

Next, let $\mathcal{P}$ denote the set of all prime numbers and for $t\in\mathbb{N}$, let

$$\mathcal{P}_t := \{p\in\mathcal{P}\ :\ 3\leq p\leq n_t\log n_t\}\,.$$

It follows from Rosser's theorem [22], that

$$\#\mathcal{P}_t\ <\ n_t. \tag{145}$$

Also a simple consequence of the well known lower bound estimate

$$\sum_{\substack{p\in\mathcal{P}\\ p\leq n}} \frac{1}{p}\ \geq\ \ln\ln(n+1) - \ln(\pi^2/6)\,,$$

is that

$$\sum_{p\in\mathcal{P}_t} \frac{1}{p}\ \geq\ \ln\ln n_t - \ln 2 - \frac{1}{2}. \tag{146}$$

Now, for each $t\in\mathbb{N}$ define

$$\tilde{q}_t := 2^{n_t}\,,$$

and in turn, for any $p\in\mathcal{P}_t$ let

$$\tilde{q}_{t,p} :=\ \tilde{q}_t\, 2^{-u_p-1}\, p\ =\ 2^{n_t-u_p-1}\, p \tag{147}$$

where $u_p := \lfloor\log_2 p\rfloor$. Then, by definition $2^{u_p}\leq p < 2^{u_p+1}$ and it follows that

$$\frac{1}{2}\tilde{q}_t < \tilde{q}_{t,p} < \tilde{q}_t. \tag{148}$$

Also, in view of (141) and (140) we have that for every $t \in \mathbb{N}$ and $p \in \mathcal{P}_t$

$$\log p \leq \log n_t + \log\log n_t \leq \frac{n_t^{1/5}}{2} = \frac{(\log \tilde{q}_t)^{1/5}}{2(\log 2)^{1/5}} \leq (\log \tilde{q}_{t,p})^{1/5}. \tag{149}$$

Trivially, the above upper bound estimate also holds for $p = 2$. Also note that since $q_1 \geq 2$, for every $t \in \mathbb{N}$

$$(\log 2)^5 < \log 2 \ \leq\ \log \tilde{q}_1 \ \leq\ \log \tilde{q}_t$$

and so

$$\log 2 < (\log \tilde{q}_t)^{1/5}. \tag{150}$$

The desired sequence $\mathcal{A} := (q_j)_{j\in\mathbb{N}}$ is precisely the elements of the set

$$\{\tilde{q}_t : t \in \mathbb{N}\} \ \cup\ \{\tilde{q}_{t,p} \mid t \in \mathbb{N},\, p \in \mathcal{P}_t\}$$

listed in increasing order of size.

Step 2: Verifying $\mathcal{A}$ satisfies Property D. By construction, each element of $\mathcal{A}$ trivially has at most two prime divisors. Also, in view of (149) and (150), any prime divisor p of an element $q_j \in \mathcal{A}$ satisfies

$$\log p \leq (\log q_j)^{1/5}. \tag{151}$$

This verifies condition (ii) of Property D with $D = 2$. It now remains to verify condition (i) of Property D. By construction, every element of the sequence $(q_j)_{j\in\mathbb{N}}$ is either equal to $\tilde{q}_t$ for some $t \in \mathbb{N}$ or equal to $\tilde{q}_{t,p}$ for some $t \in \mathbb{N}$ and $p \in \mathcal{P}_t$. Denote by π the bijective map from the set of integers $j \in \mathbb{N}$ to the set of couples (t,p) with $t \in \mathbb{N}$ and $p \in (\mathcal{P}_t \cup \{2\})$ so that

$$q_j = \tilde{q}_{\pi(j)} :=\ \tilde{q}_{t,p},$$

Here and throughout, we use the notation $\tilde{q}_{t,2} := \tilde{q}_t$. Note that for any $t \in \mathbb{N}$, we have that $\tilde{q}_{t,p} \leq \tilde{q}_t$ for every $p \in (\mathcal{P}_t \cup \{2\})$. Thus for any $j \in \mathbb{N}$, given $\pi(j) = (t,s)$ it follows that

$$j \ \leq\ \sum_{k=1}^{t} \#\mathcal{P}_k \ \overset{(145)}{<}\ \sum_{k=1}^{t} n_k \ \overset{(142)}{<}\ 2n_t\,. \tag{152}$$

On the other hand,

$$q_j = \tilde{q}_{t,p} \ \overset{(148)}{\geq}\ \frac{1}{2}\tilde{q}_t$$

and so

$$\log q_j \geq \log(\tilde{q}_t/2) \geq n_t - 1 \geq n_t/2. \tag{153}$$

On combining (152) and (153), we obtain that

$$\log q_j > j/4. \tag{154}$$

In other words, $\mathcal{A}$ satisfies the growth condition (32) with $B = 1$ and $C = 1/4$. This verifies condition (i) of Property D.

Step 3: A useful gcd estimate. Let $j \in \mathbb{N}$ be such that $\pi(j) = (t, 2)$ for some $t \in \mathbb{N}$; that is, $q_j = \tilde{q}_t = 2^{n_t}$. Note that for any $t \in \mathbb{N}$, we have that $\tilde{q}_{t,p} < \tilde{q}_t$ for every $p \in \mathcal{P}_t$. Hence,

$$\sum_{m=1}^{j} \frac{(q_m, q_j)}{q_j} \geq \sum_{p \in \mathcal{P}_t} \frac{(\tilde{q}_{t,p}, \tilde{q}_t)}{\tilde{q}_t} \geq \sum_{p \in \mathcal{P}_t} \frac{2^{n_t - u_p - 1}}{2^{n_t}} = \frac{1}{2} \sum_{p \in \mathcal{P}_t} \frac{1}{2^{u_p}}$$

$$\geq \frac{1}{2} \sum_{p \in \mathcal{P}_t} \frac{1}{p} \overset{(146)}{\geq} \frac{1}{2} \ln \ln n_t - \frac{1}{2} \ln 2 - \frac{1}{4} \overset{(140)}{\geq} \frac{1}{4} \ln \ln n_t \,.$$

The upshot of this is that whenever $j \in \mathbb{N}$ is such that $q_j = \tilde{q}_t$ for some $t \in \mathbb{N}$, then

$$\sum_{m=1}^{j} \frac{(q_m, q_j)}{q_j} \geq \frac{1}{4} \ln \ln n_t. \tag{155}$$

Step 4: Constructing the function ψ. Working with the sequence $\mathcal{A} = (q_j)_{j \in \mathbb{N}}$ coming from Step 1, the goal is to construct a suitable function ψ so that (139) is satisfied. To begin with we split $\mathcal{A}$ into two classes. We define $\mathcal{I}_1$ to be the set of indices $j \in \mathbb{N}$ such that $\pi(j) = (t, p)$ for some $t \in \mathbb{N}$ and $p \in \mathcal{P}_t$. In other words, $j \in \mathcal{I}_1$ if and only if the corresponding element $q_j \in \mathcal{A}$ is not a power of 2. We let $\mathcal{I}_2 := \mathbb{N} \setminus \mathcal{I}_1$. Thus, $\mathcal{I}_2$ is the set of indices $j \in \mathbb{N}$ such that $\pi(j) = (t, 2)$ for some $t \in \mathbb{N}$. For any index $j \in \mathcal{I}_2$, in order to emphasize the dependence on j, let us denote by t_j the unique integer associated with $\pi(j)$. Thus, by definition

$$q_j = \tilde{q}_{t_j} = 2^{n_{t_j}}.$$

Note that in view of (155), for any $j \in \mathcal{I}_2$

$$\sum_{m=1}^{j} \frac{(q_m, q_j)}{q_j} \geq \frac{1}{4} \ln \ln n_{t_j} \overset{(144)}{>} \frac{1}{4} \ln \left((t_j + 14) \ln 2 \right) . \tag{156}$$

We define the function $\psi : \mathbb{N} \to \mathbb{I}$ on the sequence $\mathcal{A} = (q_j)_{j \in \mathbb{N}}$ as follows:

- For $j \in \mathcal{I}_1$, we let
$$\psi(q_j) := 2^{-j}$$
.

- For $j \in \mathcal{I}_2$, we let
$$\psi(q_j) := \begin{cases} 1 & \text{if} \quad t_j = 1 \, , \\ \frac{1}{t_j \log t_j} & \text{if} \quad t_j \geq 2 \, . \end{cases}$$

First of all, note that

$$\sum_{j \in \mathcal{I}_1} \psi(q_j) \leq \sum_{j \in \mathbb{N}} 2^{-j} \leq 1 \, .$$

Then, it follows that for any integer $N \in \mathcal{I}_2$

$$\sum_{j=1}^{N} \psi(q_j) = \sum_{\substack{1\le j\le N\\ j\in\mathcal{I}_1}} \psi(q_j) + \sum_{\substack{1\le j\le N\\ j\in\mathcal{I}_2}} \psi(q_j) \le 2 + \sum_{\substack{1\le j\le N\\ j\in\mathcal{I}_2}} \frac{1}{t_j \log t_j}$$

$$= 2 + \sum_{i=2}^{t_N} \frac{1}{i\log i} \ll \max\{1, \log\log t_N\}, \tag{157}$$

where t_N is the unique integer associated with $\pi(N)$ so that $q_N = \tilde{q}_{t_N}$. On the other hand, it follows that for any integer $N \in \mathcal{I}_2$

$$\sum_{j=1}^{N} \psi(q_j) \sum_{m=1}^{j} \frac{(q_m, q_j)}{q_j} > \sum_{\substack{1\le j\le N\\ j\in\mathcal{I}_2}} \psi(q_j) \sum_{m=1}^{j} \frac{(q_m, q_j)}{q_j}$$

$$\overset{(156)}{\ge} \ln(15\ln 2) + \sum_{\substack{2\le j\le N\\ j\in\mathcal{I}_2}} \frac{\ln(t_j \ln 2)}{t_j \log t_j}$$

$$\gg \sum_{\substack{1\le j\le N\\ j\in\mathcal{I}_2}} \frac{1}{t_j} = \sum_{i=1}^{t_N} \frac{1}{i} \gg \max\{1, \log t_N\}. \tag{158}$$

This together with (157) implies the desired estimate (139) for any integer $N \in \mathcal{I}_2$. We now show that inequalities (157) and (158) are valid for any integer N satisfying

$$N \ge n_{100}. \tag{159}$$

With this in mind, given such an N, define $k \in \mathbb{N}$ by

$$n_k \le \log_2 q_N < n_{k+1}. \tag{160}$$

Now let N_1 be the integer such that $q_{N_1} = \tilde{q}_k = 2^{n_k}$ and let N_2 be the integer such that $q_{N_2} = \tilde{q}_{k+1} = 2^{n_{k+1}}$. Note that by definition, both $N_1, N_2 \in \mathcal{I}_2$ and $t_{N_1} = k$ and $t_{N_2} = k+1$. Also, in view of (160)

$$q_{N_1} \le q_N < q_{N_2}.$$

Thus,

$$\sum_{j=1}^{N} \psi(q_j) < \sum_{j=1}^{N_2} \psi(q_j) \overset{(157)}{\ll} \log\log t_{N_2} = \log\log(k+1) \ll \log\log k, \tag{161}$$

where in the last step we use the fact that (159) implies $k \ge 100$. On the other hand, it follows that

$$\sum_{j=1}^{N} \psi(q_j) \sum_{m=1}^{j} \frac{(q_m, q_j)}{q_j} > \sum_{j=1}^{N_1} \psi(q_j) \sum_{m=1}^{j} \frac{(q_m, q_j)}{q_j} \overset{(158)}{\gg} \log t_{N_1} = \log k. \tag{162}$$

On combining (161) and (162) we obtain the desired inequality (139) for all $N \ge N_0 := n_{100}$.

Appendix D: Some basic results on sums of sequences

In this appendix, we collect together various elementary lemmas concerning sums of sequences that are used at various points in the main body of the paper; in particular, during the course of establishing Lemma 5 and Propositions 1 & 2.

Lemma D1. *Let* $(s_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers contained in* $\mathbb{I}$ *and let*

$$S_n := \sum_{k=1}^{n} s_k.$$

Let $a, b \in \mathbb{N}$ *with* $2 \le a < b$ *and let* $\gamma > 0$. *Suppose that*

$$S_{a-1} \ge \gamma. \tag{163}$$

Then,

$$\frac{\gamma}{\gamma+1}\big(\log S_b - \log S_{a-1}\big) \;\le\; \sum_{k=a}^{b}\frac{s_k}{S_k} \;\le\; \frac{1}{\gamma\log\left(\frac{\gamma+1}{\gamma}\right)}\big(\log S_b - \log S_{a-1}\big)\,.$$

Proof. For any integer $n \ge a$, let

$$a_n := \frac{s_n}{S_n} \tag{164}$$

and

$$b_n := \log S_n - \log S_{n-1} \;=\; \log(1+\frac{s_n}{S_{n-1}})\,. \tag{165}$$

The proof of the lemma will follow on showing that

$$\frac{\gamma}{\gamma+1}\,b_n \;\le\; a_n \;\le\; \frac{1}{\gamma\log\left(\frac{\gamma+1}{\gamma}\right)}\,b_n\,. \tag{166}$$

First note that $b_n = 0$ if and only if $s_n = 0$, which, in turn, is true if and only if $a_n = 0$. Thus, (166) is trivially true if $b_n = 0$. We can therefore assume that $b_n > 0$. Then,

$$\frac{a_n}{b_n} \;=\; \frac{\frac{s_n}{S_n}}{\log\left(1+\frac{s_n}{S_{n-1}}\right)} \;=\; \frac{S_{n-1}}{S_n}\cdot\frac{\frac{s_n}{S_{n-1}}}{\log\left(1+\frac{s_n}{S_{n-1}}\right)}\,.$$

By (163), for all $n \ge a$ we have that

$$\frac{\gamma}{\gamma+1} \le \frac{S_{n-1}}{S_n} \le 1$$

and together with the fact that the function $x \to \frac{x}{\log(1+x)}$ is monotonically increasing for all $x > 0$, it follows that

$$1 \le \frac{\frac{s_n}{S_{n-1}}}{\log\left(1+\frac{s_n}{S_{n-1}}\right)} \le \frac{1}{\gamma\log\left(\frac{\gamma+1}{\gamma}\right)}\,.$$

On multiplying the last two double inequalities, we find that

$$\frac{\gamma}{\gamma+1} \leq \frac{S_{n-1}}{S_n} \cdot \frac{\frac{s_n}{S_{n-1}}}{\log\left(1+\frac{s_n}{S_{n-1}}\right)} \leq \frac{1}{\gamma \log\left(\frac{\gamma+1}{\gamma}\right)}.$$

In other words,

$$\frac{\gamma}{\gamma+1} \leq \frac{a_n}{b_n} \leq \frac{1}{\gamma \log\left(\frac{\gamma+1}{\gamma}\right)}$$

and the desired statement (166) follows. □

Lemma D2. *Let $(s_n)_{n\in\mathbb{N}}$ and S_n be as in Lemma D1. Let $\gamma > 0$ and let*

$$\tilde{S}_n := \max(\gamma, S_n).$$

Then, for any $a, b \in \mathbb{N}$ with $a < b$, we have that

$$\sum_{k=a}^{b} \frac{s_k}{\tilde{S}_k} < 1 + \frac{1}{\gamma} + \frac{\log S_b - \log S_a}{\gamma \log\left(\frac{\gamma+1}{\gamma}\right)}. \tag{167}$$

Proof. Denote by $m \in \mathbb{N}$ the smallest integer such that $\tilde{S}_m > \gamma$. This implies that $S_m > \gamma$, $S_n \leq \gamma$ if $n < m$ and

$$S_m = s_m + S_{m-1} \leq \gamma + 1. \tag{168}$$

We now split the proof into three cases depending on the size of m.

(i) If $m \leq a-1$, then it follows that $a \geq 2$ and that $S_{a-1} > \gamma$. Thus, Lemma D1 implies that

$$\sum_{k=a}^{b} \frac{s_k}{\tilde{S}_k} \leq \sum_{k=a}^{b} \frac{s_k}{S_k} \leq \frac{\log S_b - \log S_a}{\gamma \log\left(\frac{\gamma+1}{\gamma}\right)}$$

and this proves (167).

(ii) If $m \geq b$, then it follows that $S_b \leq \gamma + 1$. Hence,

$$\sum_{k=a}^{b} \frac{s_k}{\tilde{S}_k} = \sum_{k=a}^{b} \frac{s_k}{\gamma} \leq \frac{S_b}{\gamma} \leq 1 + \frac{1}{\gamma} \tag{169}$$

and this proves (167).

(iii) If $a \leq m < b$, the previous two cases can naturally be utilised to yield that

$$\sum_{k=a}^{b} \frac{s_k}{\tilde{S}_k} = \sum_{k=a}^{m} \frac{s_k}{\tilde{S}_k} + \sum_{k=m+1}^{b} \frac{s_k}{\tilde{S}_k} \leq 1 + \frac{1}{\gamma} + \frac{\log S_b - \log S_{m+1}}{\gamma \log\left(\frac{\gamma+1}{\gamma}\right)}.$$

This together with the fact that $S_{m+1} \geq S_a$ proves (167). □

Lemma D3. *Let $(s_n)_{n\in\mathbb{N}}$ and S_n be as in Lemma D1. Let $a, b \in \mathbb{N}$ with $2 \le a < b$ and suppose that $S_{a-1} > 0$. Then*

$$\sum_{k=a}^{b} \frac{s_k}{S_k^2} \le \frac{1}{S_{a-1}} - \frac{1}{S_b}.$$

Proof. The desired inequality immediately follows from the observation that for any integer $k \ge 2$,

$$\frac{1}{S_{k-1}} - \frac{1}{S_k} = \frac{s_k}{S_{k-1}S_k} = \frac{S_k}{S_{k-1}} \cdot \frac{s_k}{S_k^2}$$

and so

$$\frac{s_k}{S_k^2} \le \frac{1}{S_{k-1}} - \frac{1}{S_k}.$$

□

Lemma D4. *Let $(s_n)_{n\in\mathbb{N}}$ and S_n be as in Lemma D1. Let $\gamma > 0$ and let $\tilde{S}_n := \max(\gamma, S_n)$. Then*

$$\sum_{k=1}^{\infty} \frac{s_k}{\tilde{S}_k^2} < \frac{2\gamma + 1}{\gamma^2}.$$

Proof. As in the proof of Lemma D2, let $m \in \mathbb{N}$ be the smallest integer such that $\tilde{S}_m > \gamma$. This implies that $S_m > \gamma$, $S_n \le \gamma$ if $n < m$ and $S_m \le \gamma + 1$. Then, by making use of Lemma D3 with $a = m + 1$, it is easily verified that

$$\sum_{k=1}^{\infty} \frac{s_k}{\tilde{S}_k^2} = \sum_{k=1}^{m} \frac{s_k}{\tilde{S}_k^2} + \sum_{k=m+1}^{\infty} \frac{s_k}{\tilde{S}_k^2} \le \frac{\gamma+1}{\gamma^2} + \frac{1}{S_m} < \frac{\gamma+1}{\gamma^2} + \frac{1}{\gamma} = \frac{2\gamma+1}{\gamma^2}.$$

□

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