Gleason-type Theorems from Cauchy's Functional Equation

Victoria J Wright and Stefan Weigert
Department of Mathematics, University of York
York YO10 5DD, United Kingdom
vw550@york.ac.uk, stefan.weigert@york.ac.uk

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Abstract

Gleason-type theorems derive the density operator and the Born rule formalism of quantum theory from the measurement postulate, by considering additive functions which assign probabilities to measurement outcomes. Additivity is also the defining property of solutions to Cauchy's functional equation. This observation suggests an alternative proof of the strongest known Gleason-type theorem, based on techniques used to solve functional equations.

1 Introduction

Gleason's theorem [11] is a fundamental result in the foundations of quantum theory simplifying the axiomatic structure upon which the theory is based. The theorem shows that quantum states must correspond to density operators if they are to consistently assign probabilities to the outcomes of projective measurements in Hilbert spaces of dimension three or larger.[1]

More explicitly, let $\mathcal{P}(\mathcal{H})$ be the lattice of self-adjoint projections onto closed subspaces of a separable Hilbert space $\mathcal{H}$ of dimension at least three. Consider functions $f : \mathcal{P}(\mathcal{H}) \to [0,1]$, which are *additive* for projections P_1 and P_2 onto *orthogonal* subspaces of $\mathcal{H}$, i.e.

$$f(P_1) + f(P_2) = f(P_1 + P_2) \ . \tag{1}$$

[1] By a *consistent* assignment of probabilities we mean one in which the probabilities for all outcomes of a given measurement sum to one.

Gleason concluded that the solutions of Eq. (1) necessarily admit an expression

$$f(\cdot) = \mathrm{Tr}(\rho\cdot), \tag{2}$$

for some positive-semidefinite self-adjoint operator ρ on $\mathcal{H}$.

The result does not hold, however, in Hilbert spaces of dimension two since the constraints (1) degenerate in this case: the projections lack the "intertwining" property [11] present in higher dimensions. In 2003, Busch [3] and then Caves et al. [5] extended Gleason's theorem to dimension two by considering *generalised* quantum measurements described by positive operator-valued measures, or POMs. In analogy with Gleason's original requirement, a state is now defined as an additive probability assignment not only on projections but on a larger set of operators, the space $\mathcal{E}(\mathcal{H})$ of *effects*[2] defined on a separable Hilbert space. Then, any function $f : \mathcal{E}(\mathcal{H}) \to [0,1]$ satisfying additivity,

$$f(E_1) + f(E_2) = f(E_1 + E_2), \tag{3}$$

for effects $E_1, E_2 \in \mathcal{E}(\mathcal{H})$ such that

$$E_1 + E_2 \in \mathcal{E}(\mathcal{H}), \tag{4}$$

is found to necessarily admit an expression of the form given in Eq. (2).[3] The effects E_1 and E_2 are said to *coexist* since the condition in Eq. (4) implies that they occur in the range of a *single* POM. More recently, it has been shown that this *Gleason-type theorem*[4] also follows from weaker assumptions: it is sufficient to require Eq. (3) hold only for effects E_1 and E_2 that coexist in *projective-simulable* measurements obtained by mixing projective measurements [16].

Additive functions were first given serious consideration in 1821 when Cauchy [4] attempted to find all solutions of the equation

$$f(x) + f(y) = f(x + y), \tag{5}$$

for real variables $x, y \in \mathbb{R}$. In addition to the obvious linear solutions, non-linear solutions to *Cauchy's functional equation* are known to exist [12]. However, the non-linear functions f satisfying Eq. (5) cannot be Lebesgue measurable [2], continuous at a single point [8] or bounded on any set of positive measure [13]. Similar results also hold for Cauchy's functional equation with arguments more general than real numbers, reviewed in [1], for example.

[2]An effect E on $\mathcal{H}$ is a self-adjoint operator satisfying $0 \leq \langle\psi|E\psi\rangle \leq \langle\psi|\psi\rangle$ for all vectors $|\psi\rangle \in \mathcal{H}$.

[3]This result does *not* imply Gleason's result since in dimensions greater than two the requirement (3) is stronger than the requirement (1).

[4]It is important to clearly distinguish Gleason-*type* theorems from Gleason's original theorem.

Recalling that the Hermitian operators on $\mathbb{C}^d$ form a real vector space, it becomes clear that the Gleason-type theorems described above can be viewed as results about the solutions of Cauchy's functional equation for *vector-valued arguments*: additive functions on subsets of a real vector space, subject to some additional constraints, are necessarily linear. Taking advantage of this connection, we use results regarding Cauchy's functional equation to present an alternative proof of known Gleason-type theorems.

In Sec. 2, we spell out the conditions which single out *linear* solutions to Cauchy's functional equation defined on a finite interval of the real line. The main result of this paper—an alternative method to derive Busch's Gleason-type theorem—is presented in Sec. 3. We conclude with a summary and a discussion of the results in Sec. 4.

2 Cauchy's functional equation on a finite interval

In 1821 Cauchy [4] showed that a *continuous* function over the real numbers satisfying Eq. (5) is necessarily linear. It is important to note, however, that relaxing the continuity restriction does allow for non-linear solutions [12], as pathological as they may be.[5] Other conditions known to ensure linearity of an additive function include Lebesgue measurability [2], positivity on small numbers [9] or continuity at a single point [8]. We begin by proving a slight extension of these results, in which the domain of the function is restricted to an interval, as opposed to the entire real line.

Theorem 1. *Let $a \geq 1$ and $f : [0, a] \to \mathbb{R}$ be a function which satisfies*

$$f(x) + f(y) = f(x+y) \,, \tag{6}$$

for all $x, y \in [0, a]$ such that $(x + y) \in [0, a]$. The function f is necessarily linear, i.e.

$$f(x) = f(1)\, x \,, \tag{7}$$

if it satisfies any of the following four conditions:

(i) *$f(x) \leq b$ for some $b \geq 0$ and all $x \in [0, a]$;*

(ii) *$f(x) \geq c$ for some $c \leq 0$ and all $x \in [0, a]$;*

(iii) *f is continuous at zero;*

(iv) *f is Lebesgue-measurable.*

[5] The existence of non-linear solutions depends on the existence of Hamel bases and, thus, on the axiom of choice.

Theorem 1 says that non-linear solutions of Eq. (6) cannot be bounded from below or above, continuous at zero or Lebesgue measurable. We will now prove the linearity of f for Case (i). The proofs for the remaining cases are given in Appendix A.

Proof. For any real number $x \in [0, a]$, Eq. (6) implies that

$$f(x) = f\left(\frac{n}{n}x\right) = nf\left(\frac{1}{n}x\right), \tag{8}$$

where n is a positive integer. If we choose an integer $m \in \mathbb{N}$ with $m/n \in [0, a]$, then we have

$$f\left(\frac{m}{n}x\right) = mf\left(\frac{1}{n}x\right) = \frac{m}{n}f(x) . \tag{9}$$

Any real number $r \in (0, a/n)$ satisfies $nr \in (0, a)$ implying that

$$nf(r) = f(nr) \leq b \tag{10}$$

holds. Given any real number $x \in [0, a]$ choose a *rational* number $q_x \in [0, a]$ close to it, $0 < (x - q_x) < a/n$. The difference between the additive function $f(x)$ and the linear function $f(1)x$ can be written as

$$\begin{aligned} f(x) - f(1)x &= f(x - q_x + q_x) - f(1)x \\ &= f(x - q_x) + f(q_x) - f(1)x \\ &= f(x - q_x) + (q_x - x) f(1) . \end{aligned} \tag{11}$$

By letting $r = (x - q_x)$ in Eq. (10), it follows that the modulus of the difference is given by

$$|f(x) - f(1)x| \leq |f(x - q_x)| + |(q_x - x) f(1)| < \frac{(a+1)b}{n}, \tag{12}$$

which, upon taking the limit $n \to \infty$, implies that $f(x)$ must have the form given in Eq. (7). □

3 When Cauchy meets Gleason: additive functions on effect spaces

The first Gleason-type theorem discovered in 2003 assumes additivity of the frame function not only on projections which occur in the same PVM but on the larger set of effects which coexist in the same POM.

Theorem 2 (Busch [3])**.** *Let $\mathcal{E}_d$ be the space of effects on $\mathbb{C}^d$ and I_d be the identity operator on $\mathbb{C}^d$. Any function $f : \mathcal{E}_d \to [0,1]$ satisfying*

$$f(\mathrm{I}_d) = 1\,, \tag{13}$$

and

$$f(E_1) + f(E_2) = f(E_1 + E_2)\,, \tag{14}$$

for all $E_1, E_2 \in \mathcal{E}_d$ such that $(E_1 + E_2) \in \mathcal{E}_d$, admits an expression

$$f(E) = \mathrm{Tr}\,(E\rho)\,, \tag{15}$$

for some density operator ρ, and all effects $E \in \mathcal{E}_d$.

Theorem 2 rephrases the (finite-dimensional case of the) theorem proved by Busch [3] and the theorem due to Caves et al. [5]. Busch uses the positivity of the frame function f to directly establish its homogeneity whereas Caves et al. derive homogeneity by showing that the frame function f must be continuous at the zero operator. These arguments seem to run in parallel with Cases (ii) and (iii) of Theorem 1 presented in the previous section. In Sec. 3.2, we will give an alternative proof of Theorem 2 which can be based on any of the four cases of Theorem 1.

3.1 Preliminaries

To begin, let us introduce a number of useful concepts and establish a suitable notation. Throughout this section we will make use of the fact that the Hermitian operators on $\mathbb{C}^d$ constitute $\mathbb{H}^{d^2}$, a real vector space of dimension d^2. We may therefore employ the standard inner product $\langle A, B\rangle = \mathrm{Tr}\,(AB)$, for Hermitian operators A and B, in our reasoning as well as the norm $\|\cdot\|$ which it induces.

A discrete POM on $\mathbb{C}^d$ is described by its range, i.e. by a sequence of effects $[\![E_1, E_2, \dots]\!]$ that sum to the identity operator on $\mathbb{C}^d$. A *minimal informationally-complete* (MIC) POM $\mathcal{M}$ on $\mathbb{C}^d$ consists of exactly d^2 linearly independent effects, $\mathcal{M} = [\![M_1, \dots, M_{d^2}]\!]$. Hence, MIC-POMs constitute bases of the vector space of Hermitian operators, and it is known that they exist in all finite dimensions [6].

Positive linear combinations of effects will play an important role below, giving rise to the following definition.

Definition 1. The *positive cone* of a set of Hermitian operators $S = \{H_j, j \in J\}$ on $\mathbb{C}^d$, where J is some index set, consists of the operators

$$\mathcal{C}(S) = \left\{ H = \sum_{j\in J} a_j H_j\,,\ a_j \geq 0 \right\}. \tag{16}$$

Next, we introduce so-called "augmented" bases of the space $\mathbb{H}^{d^2}$ which are built around sets of d projections $\{|e_1\rangle\langle e_1|, \ldots, |e_d\rangle\langle e_d|\}$ where the vectors $\{|e_1\rangle, \ldots |e_d\rangle\}$ form an orthonormal basis of $\mathbb{C}^d$.

Definition 2. An *augmented basis* of the Hermitian operators on $\mathbb{C}^d$ is a set of d^2 linearly independent rank-one effects $\mathcal{B} = \{B_{1,}, \ldots, B_{d^2}\}$ satisfying

(i) $B_j = c|e_j\rangle\langle e_j|$ for $1 \leq j \leq d$, with $0 < c < 1$ and an orthonormal basis $\{|e_1\rangle, \ldots |e_d\rangle\}$ of $\mathbb{C}^d$;

(ii) $\sum_{j=1}^{d^2} B_j \in \mathcal{E}_d$.

Given any orthonormal basis $\{|e_1\rangle, \ldots, |e_d\rangle\}$ of $\mathbb{C}^d$, we can construct an augmented basis for the space of operators acting on it. First, complete the d projectors

$$\Pi_j = |e_j\rangle\langle e_j| \,, \qquad j = 1 \ldots d \,, \tag{17}$$

into a basis $\left\{\Pi_j, 1 \leq j \leq d^2\right\}$ of the Hermitian operators on $\mathbb{C}^d$, by adding $d(d-1)$ further rank-one projections; this is always possible [6]. The sum

$$G = \sum_{j=1}^{d^2} \Pi_j \,, \tag{18}$$

is necessarily a positive operator. The relation $\operatorname{Tr} G = d^2$ implies that G must have at least one eigenvalue larger than 1. If $\Gamma > 1$ is the largest eigenvalue of G, then G/Γ is an effect since it is a positive operator with eigenvalues less than or equal to one. Defining

$$B_j = \Pi_j/\Gamma \,, \qquad j = 1 \ldots d^2 \,, \tag{19}$$

the set $\mathcal{B} = \{B_1, \ldots, B_{d^2}\}$ turns into an augmented basis. One can show that $\mathcal{B}$ can never correspond to a POM. Nevertheless, the effects B_j are *coexistent*, in the sense that they can occur in one single POM, for example $[\![B_1, \ldots, B_{d^2}, \mathrm{I} - G/\Gamma]\!]$.

Given an effect, one can always represent it as a positive linear combination of elements in a suitable augmented basis.

Lemma 1. *For any effect $E \in \mathcal{E}_d$ there exists an augmented basis $\mathcal{B}$ such that E is in the positive cone of $\mathcal{B}$.*

Proof. By the spectral theorem we may write

$$E = \sum_{j=1}^{d} \lambda_j |e_j\rangle\langle e_j| \,, \qquad \lambda_j \in [0,1] \,, \tag{20}$$

for an orthonormal basis $\{|e_j\rangle, 1 \leq j \leq d\}$ of $\mathbb{C}^d$. Take $\mathcal{B}$ to be an augmented basis with

$$B_j = c|e_j\rangle\langle e_j| \,, \tag{21}$$

for $1 \leq j \leq d$ and some $c \in (0,1)$. Then we may express E as the linear combination

$$E = \sum_{j=1}^{d^2} e_j B_j \,, \tag{22}$$

with non-negative coefficients

$$e_j = \begin{cases} \frac{1}{c}\lambda_j & j = 1 \ldots d \,, \\ 0 & j = (d+1) \ldots d^2 \,, \end{cases} \tag{23}$$

showing that the positive cone of the basis $\mathcal{B}$ indeed contains the effect E. □

Finally, we need to establish that the intersection of the positive cones associated with an augmented basis and a MIC POM, respectively, has dimension d^2.

Lemma 2. *Let $\mathcal{B} = \{B_1, \ldots, B_{d^2}\}$ be an augmented basis and $\mathcal{M} = [\![M_1, \ldots, M_{d^2}]\!]$ a MIC-POM on $\mathbb{C}^d$. The effects in the intersection $\mathcal{C}(\mathcal{B}) \cap \mathcal{C}(\mathcal{M})$ of the positive cones of $\mathcal{B}$ and $\mathcal{M}$ span the real vector space $\mathbb{H}^{d^2}$ of Hermitian operators on $\mathbb{C}^d$.*

Proof. Since the effects in a POM sum to the identity, we have

$$\frac{1}{d}\,\mathrm{I}_d = \sum_{j=1}^{d^2} \frac{1}{d} M_j \,. \tag{24}$$

With each of the coefficients in the unique decomposition on the right-hand side being finite and positive (as opposed to non-negative), the effect I_d/d is seen to be an *interior* point of the positive cone $\mathcal{C}(\mathcal{M})$. At the same time, the effect I_d/d is located on the *boundary* of the cone $\mathcal{C}(\mathcal{B})$ since its expansion in an augmented basis has only d non-zero terms. Let us define the operator

$$E_\delta = \frac{1}{d}\,\mathrm{I}_d + \delta \sum_{j=d+1}^{d^2} B_j = \frac{1}{cd} \sum_{j=1}^{d} B_j + \delta \sum_{j=d+1}^{d^2} B_j \,, \tag{25}$$

which, for any positive $\delta > 0$, is an *interior* point of the cone $\mathcal{C}(\mathcal{B})$: each of the positive coefficients in its unique decomposition in terms of the augmented basis $\mathcal{B}$ is non-zero; we have used Property 1 of Def. 2 to express the identity I_d in terms of

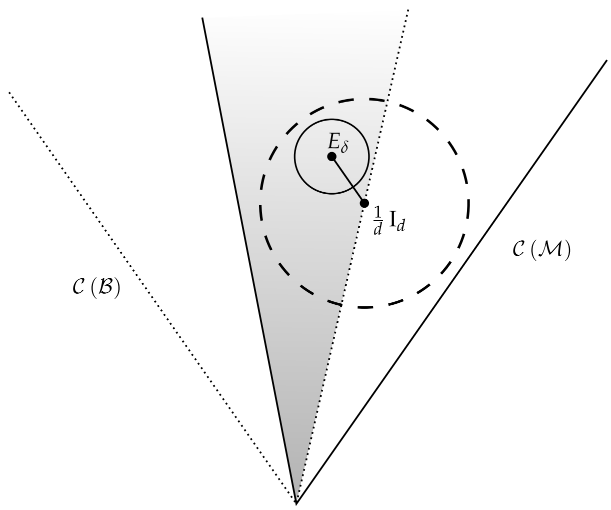

Figure 1: Sketch of the construction of the open ball $\mathfrak{B}_\gamma(E_\delta)$ of dimension d^2; the positive cones $\mathcal{C}(\mathcal{M})$ (solid border) and $\mathcal{C}(\mathcal{B})$ (dotted border) intersect in the cone $\mathcal{C}(\mathcal{B}) \cap \mathcal{C}(\mathcal{M})$ (shaded cone); the intersection entirely contains the d^2-dimensional ball $\mathfrak{B}_\gamma(E_\delta)$ around E_δ (solid circle) sitting inside the ball $\mathfrak{B}_\varepsilon(\mathrm{I}_d/d)$ of radius ε around I_d/d (dashed circle); the distance between E_δ and I_d/d (solid line) is given in Eq. (26) .

the basis $\mathcal{B}$. For sufficiently small values of δ, the operator E_δ is also an interior point of the open ball $\mathfrak{B}_\varepsilon(\mathrm{I}_d/d)$ with radius ε about the point I_d/d since

$$\left\| E_\delta - \frac{1}{d}\mathrm{I}_d \right\| = \delta \left\| \sum_{j=d+1}^{d^2} B_j \right\| < \varepsilon \tag{26}$$

holds whenever

$$0 < \delta < \varepsilon \left\| \sum_{j=d+1}^{d^2} B_j \right\|^{-1} . \tag{27}$$

Being an interior point of both the positive cones $\mathcal{C}(\mathcal{B})$ and $\mathcal{C}(\mathcal{M})$, the operator E_δ is at the center of an open ball $\mathfrak{B}_\gamma(E_\delta)$, located entirely in the intersection $\mathcal{C}(\mathcal{B}) \cap \mathcal{C}(\mathcal{M})$ (cf. Fig. 1). Since the ball $\mathfrak{B}_\gamma(E_\delta)$ has dimension d^2, the effects contained in it must indeed span the real vector space $\mathbb{H}^{d^2}$ of Hermitian operators. □

Combining Theorem 1 with Lemmata 1 and 2 will allow us to present a new proof of Busch's Gleason-type theorem.

3.2 An alternative proof of Busch's Gleason-type theorem

Recalling that the trace of the product of two Hermitian operators constitutes an inner product on the vector space of Hermitian operators, Theorem 2 essentially states that the frame function f acting on an effect can be written as the inner product of that effect with a fixed density operator. To underline the connection with the inner product we adopt the following notation. Let $\mathcal{A} = \{A_1, \ldots A_{d^2}\}$ be a basis for the Hermitian operators on $\mathbb{C}^d$. We describe the effect E by the "effect vector" $\mathbf{e} = (e_1, \ldots, e_{d^2})^T \in \mathbb{R}^{d^2}$, given by its expansion coefficients in this basis,

$$E = \sum_{j=1}^{d^2} e_j A_j \equiv \mathbf{e} \cdot \mathbf{A}\,, \tag{28}$$

where $\mathbf{A}$ is an operator-valued vector with d^2 components. Theorem 2 now states that the frame function is given by a scalar product,

$$f\,(E) = \mathbf{e} \cdot \mathbf{c}\,, \tag{29}$$

between the effect vector $\mathbf{e}$ and a *fixed* vector $\mathbf{c} \in \mathbb{R}^{d^2}$. Let us determine the relation between the density matrix ρ in (15) in the theorem and the vector $\mathbf{c}$ in (29). Consider any orthonormal basis $\mathcal{W} = \{W_j\}$ of the Hermitian operators on $\mathbb{C}^d$ and let $\mathbf{e}' \in \mathbb{R}^{d^2}$ be the vector such that $E = \mathbf{e}' \cdot \mathbf{W}$. Then we may write

$$\begin{aligned} f\,(E) = \mathbf{e} \cdot \mathbf{c} = \mathbf{e}' \cdot \mathbf{c}' &= \mathrm{Tr}\left(\sum_{j=1}^{d^2} e'_j W_j \sum_{k=1}^{d^2} c'_k W_k \right) \\ &= \mathrm{Tr}\left(E \sum_{j=1}^{d^2} c'_j W_j \right)\ ; \end{aligned} \tag{30}$$

here $\mathbf{c}' \in \mathbb{R}^{d^2}$ is a fixed vector given by $\mathbf{c}' = C^{-T}\mathbf{c}$ and C^{-T} is the inverse transpose of the change-of-basis matrix C between the bases $\mathcal{B}$ and $\mathcal{W}$, i.e. the matrix satisfying $C\mathbf{h} = \mathbf{h}'$ for all Hermitian operators $H = \mathbf{h} \cdot \mathbf{B} = \mathbf{h}' \cdot \mathbf{W}$. By the definition of a frame function the operator

$$\rho \equiv \sum_{j=1}^{d^2} c'_j W_j = \sum_{j=1}^{d^2} \left(C^{-T}\right)_{jk} c_k W_j \tag{31}$$

must be positive semi-definite (since f is positive) and have unit trace (due to Eq. (13)) i.e. be a density operator.

We will now prove that a frame function always admits an expression as in Eq. (29).

Proof. By Lemma 1, there exists an augmented basis $\mathcal{B} = \{B_1, \ldots, B_{d^2}\}$ for any $E \in \mathcal{E}_d$ such that

$$E = \mathbf{e} \cdot \mathbf{B} \equiv \sum_{j=1}^{d^2} e_j B_j \,, \tag{32}$$

with coefficients $e_j \geq 0$, as in Eq. (28).

For each value $j \in \{1, \ldots, d^2\}$, we write the restriction of the frame function f to the set of effects of the form xB_j, for $x \in \mathbb{R}$, as

$$f\left(xB_j\right) = F_j\left(x\right) \,, \tag{33}$$

where $F_j : \left[0, a_j\right] \rightarrow [0,1]$ and $a_j = \max\left\{x | xB_j \in \mathcal{E}_d\right\}$. By Eq. (14) we have that F_j satisfies Cauchy's functional equation, i.e. $F_j\left(x+y\right) = F_j\left(x\right) + F_j\left(y\right)$. Due to the assumption in Theorem 2 that $f : \mathcal{E}_d \rightarrow [0,1]$, each F_j must satisfy Condition (i) of Theorem 1 which implies

$$f\left(xB_j\right) = F_j\left(x\right) = F_j\left(1\right) x = f\left(B_j\right) x \,. \tag{34}$$

Thus we find

$$f\left(E\right) = \sum_{j=1}^{d^2} f\left(e_j B_j\right) = \sum_{j=1}^{d^2} e_j f\left(B_j\right) = \mathbf{e} \cdot \mathbf{f}_{\mathcal{B}} \,, \tag{35}$$

where the j-th component of $\mathbf{f}_{\mathcal{B}} \in \mathbb{R}^{d^2}$ is given by $f\left(B_j\right)$, by repeatedly using additivity and Eq. (34). Note that Eq. (35) is not yet in the desired form of Eq. (29) since the vector $\mathbf{f}_{\mathcal{B}}$ depends on the basis $\mathcal{B}$ and thus the effect E.

Let $\mathcal{M} = [\![M_1, \ldots, M_{d^2}]\!]$ be a MIC-POM on $\mathbb{C}^d$. Since the elements of $\mathcal{M}$ are a basis for the space $\mathbb{H}^{d^2}$, the Hermitian operators on $\mathbb{C}^d$, we have for any $E \in \mathcal{E}_d$

$$E = \mathbf{e}'' \cdot \mathbf{M} \,, \tag{36}$$

for coefficients $e''_j \in \mathbb{R}$ which may be negative. There exists a fixed change-of-basis matrix D such that

$$D\mathbf{e} = \mathbf{e}'' \,, \tag{37}$$

for all $E \in \mathcal{E}_d$. Now we have

$$\begin{aligned} f\left(E\right) &= \mathbf{e} \cdot \mathbf{f}_{\mathcal{B}} \\ &= \left(D\mathbf{e}\right) \cdot \left(D^{-T} \mathbf{f}_{\mathcal{B}}\right) \\ &= \mathbf{e}'' \cdot \left(D^{-T} \mathbf{f}_{\mathcal{B}}\right) . \end{aligned} \tag{38}$$

Any effect G in the intersection of the positive cones $\mathcal{C}\left(\mathcal{B}\right)$ and $\mathcal{C}\left(\mathcal{M}\right)$ can be expressed in two ways,

$$G = \mathbf{g} \cdot \mathbf{B} = \mathbf{g}'' \cdot \mathbf{M} \,, \tag{39}$$

where both effect vectors $\mathbf{g}$ and $\mathbf{g}''$ have only non-negative components. Eqs. (35) and (38) imply that

$$\mathbf{g}'' \cdot \mathbf{f}_{\mathcal{M}} = f(G) = \mathbf{g}'' \cdot \left(D^{-T} \mathbf{f}_{\mathcal{B}} \right) . \tag{40}$$

Since, by Lemma 2, there are d^2 linearly independent effects G in the intersection $\mathcal{C}(\mathcal{M}) \cap \mathcal{C}(\mathcal{B})$, we conclude that

$$D^{-T} \mathbf{f}_{\mathcal{B}} = \mathbf{f}_{\mathcal{M}} . \tag{41}$$

Combining this equality with Equation (38) we find, for a fixed MIC-POM $\mathcal{M} = [\![M_1, \ldots, M_{d^2}]\!]$ and any effect $E \in \mathcal{E}_{\mathbb{C}^d}$, that the frame function f takes the form

$$f(E) = \mathbf{e}'' \cdot \mathbf{f}_{\mathcal{M}} . \tag{42}$$

Here $\mathbf{f}_{\mathcal{M}} \equiv \mathbf{c}$ is a *fixed* vector since it does not depend on E. □

Note that Eq. (34) may also be found using the other three cases of Theorem 1. For Case (ii), we observe that each of the functions $F_j, j = 1 \ldots d^2$, is *non-negative* by definition. Alternatively, each function F_j can be shown to be *continuous* at zero (Case (iii)) using the following argument which is similar to the one given in [6]. Assume F_j is not continuous at zero.Then there exists a number $\varepsilon > 0$ such that for all $\delta > 0$ we have

$$F_j(x_0) > \varepsilon , \tag{43}$$

for some $0 < x_0 < \delta < 1$. For any given ε choose $\delta = 1/n < \varepsilon$, there is a value of $x_0 < \delta$ such that $F_j(x_0) > \varepsilon$. However, $nx_0 < 1$, which leads to

$$F_j(nx_0) = nF_j(x_0) > n\varepsilon > 1 , \tag{44}$$

contradicting the the existence of an upper bound of one on values of F_j. Finally, each of the functions F_j is *Lebesgue measurable* (Case (iv)) which follows from the monotonicity of the function.

4 Summary and discussion

We are aware of two papers linking Gleason's theorem and Cauchy's functional equation. Cooke et al. [7] used Cauchy's functional equation to demonstrate the necessity of the boundedness of frame functions in proving Gleason's theorem. Dvurečenskij [10] introduced frame functions defined on effect algebras but did not proceed to derive a Gleason-type theorem in the context of quantum theory.

In this paper, we have exploited the fact that additive functions are central to both Gleason-type theorems and Cauchy's functional equation. Gleason-type theorems are based on the assumption that states assign probabilities to measurement outcomes via additive functions, or *frame functions,*on the effect space. Linearity of the frame functions has been shown to follow from positivity and other assumptions, which are well-known in the context of Cauchy's functional equation. Altogether, the result obtained here amounts to an alternative proof of the extension of Gleason's theorem to dimension two given by Busch [3] and Caves et al. [6].

Other Gleason-type theorems are known which are *stronger,* in the sense that they depend on assumptions *weaker* than those of Theorem 2. The smallest known set of assumptions requires Eq. (14) to only be valid for effects E_1 and E_2 which coexist in a *projective-simulable* POM [15], i.e. a POM which may be simulated using only classic mixtures of projective measurements, as opposed to any POM. Since the proof given in [16] relies on Theorem 2, the alternative proof presented in Sec. 3.2 also gives rise to a new proof of the strongest existing Gleason-type theorem.

We have not been able to exploit the structural similarity between the requirements on frame functions and on the solutions of Cauchy's functional equation in order to yield a new proof of Gleason's original theorem. Additivity of frame functions defined on projections instead of effects does not provide us with the type of continuous parameters which are necessary for the argument developed here. It remains an intriguing open question whether such a proof does exist.

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A Proofs of Cases (ii), (iii) and (iv) of Theorem 1

It is shown that each of the conditions given in Cases (ii) to (iv) imply Theorem 1 which states that an additive function on a particular interval must be linear.

Proof. Case (ii): Suppose that there exists a *non-linear* function f satisfying Eq. (6) and Case (ii) of Theorem 1. Then the function $g : [0, a] \to \mathbb{R}$ defined by $g(x) = -f(x)$ is non-linear but satisfies Eq. (6) and $g(x) \leq b$ and $b \geq 0$, with $b = -c$, contradicting Case (i). □

Proof. Case (iii): Since f is continuous at zero and $f(0) = 0$, as follows from Eq. (6), we have that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x)| < \varepsilon$ for all x satisfying $|x| < \delta$. Let $x, x_0 \in [0, a]$ be such that $|x - x_0| < \delta$. First consider the case $x < x_0$. Using additivity,

$$f(x) + f(x_0 - x) = f(x + x_0 - x) = f(x_0) \ , \tag{45}$$

we find

$$|f(x) - f(x_0)| = |f(x_0 - x)| < \varepsilon \, . \tag{46}$$

On the other hand, if $x > x_0$ we have

$$f(x) = f(x - x_0 + x_0) = f(x - x_0) + f(x_0) \ , \tag{47}$$

and then

$$|f(x) - f(x_0)| = |f(x - x_0)| < \varepsilon \, . \tag{48}$$

It follows that f is continuous on $[0, a]$. As in the proof for Case (i), Eqs. (8) and (9) show that

$$f(q) = f(1)\, q \, , \tag{49}$$

for rational $q \in [0, a]$. Therefore, if $(q_1, q_2, \ldots)$ is a sequence of rational numbers converging to x, the function $f(x)$ must be linear in x:

$$f(x) = \lim_{j \to \infty} f\left(q_j\right) = \lim_{j \to \infty} f(1)\, q_j = f(1)\, x \, . \tag{50}$$

□

In Case (iv), where f is Lebesgue measurable, the proof of the analogous result for functions on the full real line by Banach [2] remains valid, as we will now show. Given Case (iii), it suffices to prove that f is continuous at 0, i.e. that for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(h) - f(0)| = |f(h)| < \varepsilon \tag{51}$$

holds for all $0 < h < \delta$.

Proof. Case (iv): Lusin's theorem [14] states that, for a Lebesgue measurable function g on an interval J of Lesbesgue measure $\mu(J) = m$, there exists a compact subset of any measure $m' < m$ such that the restriction of g to this subset is continuous. Thus we may find a compact set $F \subset [0,1]$ with $\mu(F) \geq 2/3$ on which f is continuous. Let $\varepsilon > 0$ be given. Since F is compact, f is uniformly continuous on F and there exists a $\delta \in (0, 1/3)$ such that

$$|f(x) - f(y)| < \varepsilon \tag{52}$$

is valid for two numbers $x, y \in F$ such that $|x - y| < \delta$. Let $h \in (0, \delta)$. Suppose F and $F - h = \{x - h | x \in F\}$ were disjoint. Then we would have

$$1 + h = \mu([-h, 1]) \geq \mu(F \cup (F - h)) = \mu(F) + \mu(F - h) \geq \frac{4}{3}, \tag{53}$$

which contradicts $h < \delta < 1/3$. Taking a point $x \in F \cap (F - h)$ then a number $\delta \in (0, 1/3)$ can be found such that

$$|f(h)| = |f(x) - f(x) - f(h)| = |f(x) - f(x + h)| < \varepsilon, \tag{54}$$

for $h \in (0, \delta)$. Hence, the function $f(x)$ is continuous at $x = 0$. □