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THE INVERSE DEFORMATION PROBLEM

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ABSTRACT. Given a commutative complete local noetherian ring A with finite residue field $\boldsymbol{k}$, we show that there is a topologically finitely generated profinite group Γ and an absolutely irreducible continuous representation $\overline{\rho} : \Gamma \to GL_n(\boldsymbol{k})$ such that A is a universal deformation ring for $\Gamma, \overline{\rho}$.

1. INTRODUCTION

Let Λ be a commutative complete noetherian local ring with residue field $\boldsymbol{k}$ of positive characteristic p. We write $\mathcal{C}_\Lambda$ for the category of commutative complete noetherian local Λ-algebras with residue field $\boldsymbol{k}$; morphisms in $\mathcal{C}_\Lambda$ are continuous Λ-algebra homomorphisms inducing the identity on the residue field $\boldsymbol{k}$. The aim of this paper is to give characterisations of objects in $\mathcal{C}_\Lambda$ which can be realised as a universal deformation ring of some residual representation. The problem, often referred to as the *inverse deformation problem*, originated from a question of Flach; the above formulation is due to Bleher, Chinburg and De Smit. (See [5], [6],[7].)

To describe the main results of this paper, fix Λ, $\boldsymbol{k}$ and $\mathcal{C}_\Lambda$ as above. We refer the reader to [12], [13] for details on deformations of representations. Suppose we are given a profinite group Γ together with a continuous representation $\overline{\rho} : \Gamma \longrightarrow GL_n(\boldsymbol{k})$. The representation $\overline{\rho}$ will be referred to as the residual representation. Given a Λ-algebra A in $\mathcal{C}_\Lambda$ with maximal ideal $\mathfrak{m}_A$, recall that

- a continuous homomorphism $\rho : \Gamma \longrightarrow GL_n(A)$ is said to be a *lifting* of $\overline{\rho}$ if $\overline{\rho} = \rho \pmod{\mathfrak{m}_A}$; and,
- two liftings $\rho_1, \rho_2 : \Gamma \longrightarrow GL_n(A)$ of $\overline{\rho}$ are *strictly equivalent* if there exists a matrix $X \in GL_n(A)$ such that $X \equiv I \pmod{\mathfrak{m}_A}$ and $X\rho_1(g)X^{-1} = \rho_2(g)$ for all $g \in \Gamma$.

Strict equivalence is an equivalence relation; a *deformation* of $\overline{\rho}$ is a strict equivalence class of liftings. Note that if $f : A \to B$ is a morphism in $\mathcal{C}_\Lambda$ and $\rho : \Gamma \longrightarrow GL_n(A)$ is a lifting of $\overline{\rho}$, then $f \circ \rho : \Gamma \longrightarrow GL_n(B)$ is also a lifting of $\overline{\rho}$. Furthermore, the association

$$A \to \mathrm{Def}_{\overline{\rho},\Lambda}(A) := \text{the set of deformations of } \overline{\rho} \text{ to } A$$

is a functor from $\mathcal{C}_\Lambda$ to **Sets**.

Now assume that the profinite group Γ and the residual representation $\overline{\rho}$ satisfy the following two hypotheses:

(H1) (p-finiteness condition.) Each open subgroup U of Γ admits only finitely many continuous homomorphisms to $\mathbb{Z}/p\mathbb{Z}$.

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(H2) The residual representation $\overline{\rho} : \Gamma \longrightarrow GL_n(\boldsymbol{k})$ admits no non-scalar centraliser i.e. if $X \in GL_n(\boldsymbol{k})$ satisfies $X\overline{\rho}(g) = \overline{\rho}(g)X$ for all $g \in \Gamma$ then $X = \lambda I$ for some $\lambda \in \boldsymbol{k}$.

The functor $\mathrm{Def}_{\overline{\rho},\Lambda} : \boldsymbol{\mathcal{C}}_\Lambda \to \mathbf{Sets}$ is then representable under these assumptions (see [9, Proposition 7.1], [12, Proposition 1]). We recall that this means we can find an object R of $\boldsymbol{\mathcal{C}}_\Lambda$ and a continuous representation $\rho : \Gamma \to GL_n(R)$ lifting $\overline{\rho}$ with the following universal property: if $A \in \mathrm{Ob}(\boldsymbol{\mathcal{C}}_\Lambda)$ and $\rho_A : \Gamma \to GL_n(A)$ is a lifting of $\overline{\rho}$ then there is a unique morphism $f : R \to A$ in $\boldsymbol{\mathcal{C}}_\Lambda$ such that ρ_A is strictly equivalent to $f \circ \rho$. We shall refer to the pair (R, ρ)—which is unique up to a canonical isomorphism and strict equivalence—as the *universal deformation* for $\overline{\rho}$ in $\boldsymbol{\mathcal{C}}_\Lambda$.

Definition. A Λ-algebra $A \in \mathrm{Ob}(\boldsymbol{\mathcal{C}}_\Lambda)$ is said to be a *universal deformation ring* in $\boldsymbol{\mathcal{C}}_\Lambda$ if we can find a profinite group Γ and a residual representation $\overline{\rho} : \Gamma \to GL_n(\boldsymbol{k})$ satisfying (H1) and (H2) with universal deformation (R, ρ) in $\boldsymbol{\mathcal{C}}_\Lambda$ such that R and A are isomorphic objects in $\boldsymbol{\mathcal{C}}_\Lambda$.

We can now state the main result of this paper.

Theorem 1.1. *Let $\boldsymbol{k}$ be a finite field of characteristic p and let $W(\boldsymbol{k})$ be its Witt ring. Then every object of $\boldsymbol{\mathcal{C}}_{W(\boldsymbol{k})}$ is a universal deformation ring.*

More precisely, let A be an object of $\boldsymbol{\mathcal{C}}_{W(\boldsymbol{k})}$ and let $\Gamma := SL_n(A)$ where n is a positive integer subject to the following restrictions.

- *If $\boldsymbol{k}$ has at least 7 elements or $\boldsymbol{k}$ is $\mathbb{F}_4$, then $n \geq 2$.*
- *If $\boldsymbol{k}$ is $\mathbb{F}_2$, then $n \geq 5$.*
- *If $\boldsymbol{k}$ is either $\mathbb{F}_3$ or $\mathbb{F}_5$, then $n \geq 3$.*

Let $\overline{\rho} : \Gamma \to GL_n(\boldsymbol{k})$ be the reduction of the standard representation $\rho_A : \Gamma \to SL_n(A)$ modulo the maximal ideal of A. Then the pair (A, ρ_A) is the universal deformation for $\overline{\rho}$ in $\boldsymbol{\mathcal{C}}_{W(\boldsymbol{k})}$.

The restrictions on n in Theorem 1.1 are in place because our method relies on a structure result for subgroups of SL_n over commutative complete noetherian local rings (see Proposition 4.1). The said result is used, in the first instance, to construct an isomorphism of local rings, and then to show that the isomorphism constructed is a morphism in $\boldsymbol{\mathcal{C}}_{W(\boldsymbol{k})}$. Unfortunately Proposition 4.1 fails in the excluded cases. In fact, Theorem 1.1 fails in all but one of the excluded cases (see Remark 4.6).

We now give an overview of the developments concerning the inverse deformation problem; for a more detailed account see [6]. As indicated at the start of this section, the inverse deformation problem originated from a question by Flach in [7] which asked if it is possible for a universal deformation ring to not be a complete intersection ring. The motivation behind this question was that up to that point, although there had been many explicit calculations, all known universal deformation rings were complete intersection rings.

The first example of a universal deformation ring which was not a complete intersection was given by Bleher and Chinburg in [3], where they showed $W(\boldsymbol{k})[[t]]/(t^2, 2t)$ is a universal deformation ring when $p = 2$. (See also [4]). This example was greatly generalised by Bleher, Chinburg and de Smit in [5] to provide a positive answer to the inverse deformation problem for all rings of the form $W(\boldsymbol{k})[[t]]/(p^n t, t^2)$. Bleher, Chinburg and de Smit further gave a categorisation of all possible pairs $(\Gamma, \overline{\rho})$ which have $W(\boldsymbol{k})[[t]]/(p^n t, t^2)$ as its universal deformation ring in [6]. Another class of non complete intersection rings which are universal deformation rings is given by $\mathbb{Z}_p[[t]]/(p^n, p^m t)$ where $p > 3$ and $n > m$ are positive integers. This is due to Rainone (see [16]).

We note that Krzysztof Dorobisz has independently proved results similar to Theorem 1.1 in [10]. His methods are more linear algebraic (in that skillful use of matrix identities are employed) while we rely on cohomological arguments.

This article is organised as follows. In Section 2 we present preparatory material concerning the structure of SL_n which will be essential to the proof of Theorem 1.1. The results here are either well known or elementary. We then prove Theorem 1.1 in Section 3, assuming a key result on subgroups of SL_n holds (Assumption 3.2). Section 4 then discusses verification of Assumption 3.2 following a simplified form of the argument used in [11].

Notation and conventions. Throughout this paper, all rings are assumed to have a multiplicative identity. Moreover, *all local rings are assumed to be commutative.* The maximal ideal of a local ring A will be denoted by $\mathfrak{m}_A$.

If $\boldsymbol{k}$ is a perfect field of positive characteristic then $W(\boldsymbol{k})$ denotes its ring of Witt vectors. We will often write W instead of $W(\boldsymbol{k})$ when it is clear what the residue field $\boldsymbol{k}$ is. If A is a complete noetherian local ring with residue field $\boldsymbol{k}$ then A is canonically a W-algebra. We write $\iota_A : W \to A$ for the corresponding natural homomorphism and set $W_A := \iota_A(W)$. Note that W_A is the smallest closed subring of A which is local with residue field $\boldsymbol{k}$.

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2. Preliminaries: Some properties of SL_n.

In this section, we describe certain aspects of the structure of special linear groups which play a key role in the proof of Theorem 1.1. *Throughout this section, A denotes a commutative ring.* Recall that if $x \in A$, and $1 \leq i, j \leq n$ with $i \neq j$, the elementary matrix $E_{ij}(x) \in SL_n(A)$ is the n by n matrix whose (i, j)-th entry is x, whose diagonal entries are all 1, and whose remaining entries are all 0.

Lemma 2.1.

(i) The n by n elementary matrices in $SL_n(A)$ satisfy the following Steinberg relations.
   (a) $E_{ij}(x)E_{ij}(y) = E_{ij}(x + y)$,
   (b) $[E_{ij}(x), E_{jk}(y)] = E_{ik}(xy)$ if $i \neq k$
   (c) $[E_{ij}(x), E_{kl}(y)] = 1$ if $i \neq l, j \neq k$

(ii) Let $a, b, c, d \in A$. Then the relation

$$\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \tag{2.1}$$

holds if and only if $a = c$, $b = d$ and $ab = cd = 1$.

(iii) If A is a local ring and $n \geq 2$, then $SL_n(A)$ is generated by the elementary matrices $E_{ij}(x)$, $x \in A$.

The first part is well known (see [17] or [21], for instance). For the second part of the lemma, multiplying out the left hand side transforms relation (2.1) into

$$\begin{pmatrix} 2a-a^2b & 1-ab \\ ab-1 & b \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$$

and the conclusion follows. The third part of Lemma 2.1 is essentially covered by the discussion following Example 1.6 in [21].

Lemma 2.1 implies the following proposition (which will be used in determining the image of liftings of residual representations in deformation problems).

Proposition 2.2. *Let A be a local ring and let $n \geq 2$ be an integer. In addition, assume that the residue field of A has at least 4 elements when $n = 2$. Then the following holds: For any commutative ring B and positive integer m, the image of a group homomorphism $\rho : SL_n(A) \to GL_m(B)$ is in fact a subgroup of $SL_m(B)$.*

Proof. By Lemma 2.1 the image of ρ is generated by the images of elementary matrices and so we need to verify that the image of an elementary matrix has determinant 1.

First suppose $n \geq 3$. Given an elementary matrix $E_{ij}(x)$ in $SL_n(A)$, pick an integer k between 1 and n distinct from i, j. The relation $[E_{ik}(x), E_{kj}(1)] = E_{ij}(x)$ then implies that the determinant of $\rho(E_{ij}(x))$ must be 1.

Suppose now that $n = 2$. Our assumption on the size of the residue field allows us to fix a unit $u \in A$ such that $u^2 \not\equiv 1 \pmod{\mathfrak{m}_A}$. Thus $1-u^2$ is a unit in A. The desired conclusion then follows from the commutator relation

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} 1 & x(1-u^2) \\ 0 & 1 \end{pmatrix}, \tag{2.2}$$

and a similar one for lower triangular matrices, valid for all $x \in A$. □

We now highlight a class of signed permutation matrices which can be used to conjugate the elementary matrix $E_{1n}(x)$ to another elementary matrix $E_{ij}(x)$.

Definition 2.3. Let n be a positive integer. The diagonal matrix in $GL_n(\mathbb{Z})$ obtained by replacing the (i,i)-th entry of the identity matrix by -1 is denoted by D_i. For $1 \leq i, j \leq n$ with $i \neq j$, the permutation matrix in $GL_n(\mathbb{Z})$ obtained by interchanging the i-th and j-th rows of the identity matrix will be denoted by $P_{(ij)}$. The matrices D_i and $P_{(ij)}$ have determinant -1.

Finally, given $1 \leq i, j \leq n$ with $i \neq j$, define the signed permutation matrix $T_{ij} \in SL_n(\mathbb{Z})$ by

$$T_{ij} := \begin{cases} I & \text{if } (i,j) = (1,n), \\ D_2 P_{(1n)} & \text{if } (i,j) = (n,1), \\ D_n P_{(jn)} & \text{if } i = 1 \text{ and } j \neq n, \\ D_1 P_{(1i)} & \text{if } i \neq 1 \text{ and } j = n, \\ P_{(1i)} P_{(nj)} & \text{if } i \neq 1 \text{ and } j \neq n \text{ and } (i,j) \neq (n,1). \end{cases} \tag{2.3}$$

If $X \in GL_n(\mathbb{Z})$ then its image in $GL_n(A)$ under the unique ring homomorphism $\mathbb{Z} \to A$ will also be denoted by X. We then have the following lemma.

Lemma 2.4. *Let $n \geq 2$ be an integer.*

(i) Suppose $X \in GL_n(A)$. Then $XE_{ij}(1) = E_{ij}(1)X$ for all elementary matrices $E_{ij}(1)$ with $1 \leq i < j \leq n$ if and only if $X = \lambda E_{1n}(x)$ for some $\lambda \in A^\times$, $x \in A$.

(ii) $T_{ij}E_{1n}(x)T_{ij}^{-1} = E_{ij}(x)$ for all $1 \leq i \neq j \leq n$ and $x \in A$.

We give a brief sketch of the proof. That $\lambda E_{1n}(x)$ commutes with $E_{ij}(1)$ when $1 \leq i < j \leq n$ is clear from the Steinberg relations (Lemma 2.1, part $i(c)$). For the other direction, let e_{st} denote the n by n matrix whose (s,t)-th entry is 1 and whose all other entries are 0. If x_{lk} denotes the (l,k)-th entry of X, then the relation $E_{ij}(1)X = XE_{ij}(1)$ implies

$$\sum_{m=1}^{n} x_{jm}e_{im} = \sum_{m=1}^{n} x_{mi}e_{mj}$$

and the desired conclusion follows. The second part is a straightforward calculation which we skip.

3. Proof of Theorem 1.1

We shall now show that every complete local noetherian ring with finite residue field is a universal deformation ring.

Throughout this section, $\boldsymbol{k}$ is a finite field of characteristic $p > 0$ and W denotes its Witt ring. Recall that $\boldsymbol{\mathcal{C}}_W$ is the category of complete noetherian local W-algebras with residue field $\boldsymbol{k}$. If A is a W-algebra in $\boldsymbol{\mathcal{C}}_W$, we define $W_A := \iota_A(W)$ where $\iota_A : W \to A$ is the canonical structure map.

We now turn to the proof of Theorem 1.1. Fix a W-algebra A in $\boldsymbol{\mathcal{C}}_W$. Fix also an integer n subject to the following conditions.

Assumption 3.1.

- *If the cardinality of $\boldsymbol{k}$ is at least 7 or $\boldsymbol{k} = \mathbb{F}_4$, then $n \geq 2$.*
- *If $\boldsymbol{k}$ is $\mathbb{F}_2$, then $n \geq 5$.*
- *If $\boldsymbol{k}$ is either $\mathbb{F}_3$ or $\mathbb{F}_5$, then $n \geq 3$.*

Set $\Gamma := SL_n(A)$. We write $\rho_A : \Gamma \to SL_n(A)$ for the standard representation of Γ, and define the residual representation $\overline{\rho} := \rho_A \pmod{\mathfrak{m}_A}$. Thus

$$\overline{\rho} : \Gamma \to SL_n(\boldsymbol{k}) \hookrightarrow GL_n(\boldsymbol{k}).$$

Note that Γ is topologically finitely generated and $\overline{\rho}$ is clearly surjective. Thus Γ and $\overline{\rho}$ satisfy hypotheses (H1) and (H2) from Section 1, and hence $\overline{\rho}$ has a universal deformation in $\boldsymbol{\mathcal{C}}_W$.

We will show that the pair (A, ρ_A) is in fact the universal deformation for $\overline{\rho}$ in $\boldsymbol{\mathcal{C}}_W$. For clarity the argument is split into four steps.

Step 1. We begin by observing some characteristics of the universal deformation. Let R together with $\rho_R : \Gamma \to GL_n(R)$ be the universal deformation for

$$\overline{\rho} : \Gamma \to SL_n(\boldsymbol{k}) \hookrightarrow GL_n(\boldsymbol{k}).$$

By Proposition 2.2, the restrictions imposed on n imply that ρ_R takes values in $SL_n(R)$.

We now make the following critical assumption which will only be justified in Section 4.

Assumption 3.2. *There exists an $X \in GL_n(R)$ with $X \equiv I \pmod{\mathfrak{m}_R}$ such that $X\rho_R(\Gamma)X^{-1} \supseteq SL_n(W_R)$.*

We continue with the proof of Theorem 1.1. Assumption 3.2 allows us to derive the following consequence: Replacing ρ_R with a strictly equivalent representation if necessary, we may assume that $\rho_R(\Gamma)$ contains a copy of $SL_n(W_R)$.

Step 2. Let $\pi : R \to A$ be the unique W-algebra homomorphism in $\boldsymbol{\mathcal{C}}_W$ associated with ρ_A by the universality of (R, ρ_R). Thus $\pi \circ \rho_R$ is strictly equivalent to ρ_A and π is compatible with the W-algebra structure morphisms ι_A and ι_R i.e. the diagram

$$\begin{array}{ccc} W & = & W \\ \big\downarrow{\scriptstyle \iota_R} & & \big\downarrow{\scriptstyle \iota_A} \\ R & \xrightarrow{\pi} & A \end{array} \tag{3.1}$$

commutes. We now make the following observations.

Proposition 3.3.

(i) $\rho_R : \Gamma \to SL_n(R)$ is injective and $\pi : \rho_R(\Gamma) \to SL_n(A)$ is an isomorphism.
(ii) The map $\pi : R \to A$ is surjective.
(iii) The restriction $\pi|_{W_R} : W_R \to W_A$ is an isomorphism.

Proof. Part (i) follows from the observations that $\pi \circ \rho_R$ is strictly equivalent to ρ_A, and that ρ_A is an isomorphism. Part (ii) is then immediate.

We now consider part (iii). By Assumption 3.2, we can pick a γ in Γ with $\rho_R(\gamma) = E_{12}(1)$. Now $\rho_A(\gamma)$ and $E_{12}(\pi(1))$ have the same order (as they are conjugates), and so we may conclude that the restriction $\pi|_{W_R} : W_R \to W_A$ must be an isomorphism. □

The above assertion allows us to identify W_R and W_A. Henceforth, *we will not differentiate between $\iota_R(x)$ and $\iota_A(x)$ for $x \in W$.*

Step 3. We shall now show that under the group isomorphism $\pi : \rho_R(\Gamma) \to SL_n(A)$, the preimage of an elementary matrix in $SL_n(A)$ is an elementary matrix in $SL_n(R)$. This allows us to construct a local W-algebra homomorphism $A \to R$ which is a section for $\pi : R \to A$.

We first observe the following lemma.

Lemma 3.4. *For each $x \in A$ there exist a unique λ_x in $R^\times$ and a unique $s(x)$ in R such that the following holds: $\lambda_x E_{1n}(s(x)) \in \rho_R(\Gamma)$ and $\pi(\lambda_x E_{1n}(s(x))) = E_{1n}(x)$.*

The above association has the following additional properties.

(i) Let $x \in A$ and let $1 \le i, j \le n$ with $i \ne j$. Then the preimage of $E_{ij}(x)$ under the isomorphism $\pi : \rho_R(\Gamma) \to SL_n(A)$ is the matrix $\lambda_x E_{ij}(s(x))$.
(ii) If $x \in W_A$ then $\lambda_x = 1$ and $s(x) = x$.

Proof. Uniqueness is immediate from Proposition 3.3(i). For existence, let $X \in \rho_R(\Gamma)$ satisfy $\pi(X) = E_{1n}(x)$. Now $E_{1n}(x)$ commutes with the elementary matrices $E_{ij}(1)$ where $1 \le i < j \le n$. Then by Proposition 3.3 and our identification of W_R with W_A, the elementary matrices $E_{ij}(1)$ with $1 \le i < j \le n$ are in $\rho_R(\Gamma)$ and commute with X. Hence by Lemma 2.4 we must have $X = \lambda_x E_{1n}(s(x))$ for some $s(x)$ in R and λ_x in $R^\times$.

Now for the first part of the two properties. Let $x \in A$ and let $1 \le i, j \le n$ with $i \ne j$. From the preceding two steps, the signed permutation matrix T_{ij}, as defined by the relations (2.3), is in $\rho_R(\Gamma)$. Since $\lambda_x E_{ij}(s(x)) = T_{ij} \lambda_x E_{1n}(s(x)) T_{ij}^{-1}$ by Lemma 2.4, we see that $\lambda_x E_{ij}(s(x))$ is in $\rho_R(\Gamma)$ and is the unique preimage of $E_{ij}(x)$. The second property is immediate as we are identifying W_A and W_R. $\square$

Keep the notation of Lemma 3.4. We will now show that λ_x is in fact 1 for all $x \in A$. This can be derived from the Steinberg relations when $n \ge 3$ as follows. Let i, j, k be three distinct integers in $\{1, 2, \dots, n\}$. By considering their preimages in $\rho_R(\Gamma)$, the relation $E_{ij}(x) = E_{ik}(x)E_{kj}(1)E_{ik}(x)^{-1}E_{kj}(1)^{-1}$ then implies that

$$\begin{aligned} \lambda_x E_{ij}(s(x)) &= \lambda_x E_{ik}(s(x)) E_{kj}(1) \lambda_x^{-1} E_{ik}(s(x))^{-1} E_{kj}(1)^{-1} \\ &= E_{ij}(s(x)), \end{aligned}$$

and hence $\lambda_x = 1$.

We now consider the case when $n = 2$. Note the following claim.

Claim 3.5. *Under the isomorphism $\pi : \rho_R(\Gamma) \to SL_2(A)$, the preimage of a diagonal matrix in $SL_2(A)$ is also a diagonal matrix.*

To see the claim, let $X \in \rho_R(\Gamma) \subseteq SL_2(R)$ be the preimage of the diagonal matrix $D \in SL_2(A)$. Then $DE_{12}(1)D^{-1} = E_{12}(u)$ for some $u \in A^\times$, and so $XE_{12}(1)X^{-1} = \lambda_u E_{12}(s(u))$. Similarly X conjugates $E_{21}(1)$ to a lower triangular matrix (in fact $XE_{21}(1)X^{-1} = \lambda_{u^{-1}} E_{21}(s(u^{-1}))$). We can therefore conclude that the preimage X is a diagonal matrix in $\rho_R(\Gamma)$.

Now let $x \in A$. As $\boldsymbol{k}$ has at least 4 elements, we can find $y \in A$, $u \in A^\times$ so that

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

(Use the commutator relation (2.2).) Using Lemma 3.4, Claim 3.5 and taking preimages, we get

$$\lambda_y \begin{pmatrix} 1 & s(y) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \left(\lambda_y \begin{pmatrix} 1 & s(y) \\ 0 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}^{-1} = \lambda_x \begin{pmatrix} 1 & s(x) \\ 0 & 1 \end{pmatrix},$$

for some $v \in R^\times$, and we obtain $\lambda_x = 1$.

We can now define the desired section of $\pi : R \to A$.

Proposition 3.6. *The function $s : A \to R$ characterised by the following property is well defined:*

> *If $x \in A$ then $s(x)$ is the unique element in R such that $\pi(s(x)) = x$ and the elementary matrix $E_{ij}(s(x))$ is a matrix in $\rho_R(\Gamma)$ for all $1 \le i, j \le n$, $i \ne j$.*

Moreover, the map $s : A \to R$ is in fact a morphism in $\mathcal{C}_W$.

Proof. The characterising property that defines $s : A \to R$ has been covered in Lemma 3.4 and the discussion following it.

We shall now show that the map $s : A \to R$ is a morphism in $\mathcal{C}_W$. It follows immediately from the construction and Lemma 2.1, part $i(a)$, that $s(x + y) = s(x) + s(y)$ for all $x, y \in A$. Moreover, $s|_{W_A}$ is the inverse to $\pi|_{W_R}$, and $\pi \circ s$ is the identity on A. By construction $s(\mathfrak{m}_A) \subseteq \mathfrak{m}_R$ and s induces the identity on $A/\mathfrak{m}_A = R/\mathfrak{m}_R = \boldsymbol{k}$. Thus if we can show that $s(xy) = s(x)s(y)$ for all $x, y \in A$ then $s : A \to R$ will be a morphism in $\mathcal{C}_W$. This follows from the Steinberg relations when $n \geq 3$: if $1 \leq i, j, k \leq n$ are three distinct integers, then the commutator relation $[E_{ij}(s(x)), E_{jk}(s(y))] = E_{ik}(s(x)s(y))$ shows that $s(xy) = s(x)s(y)$.

We now consider the multiplicativity of s when $n = 2$. If $u \in A^\times$ then, using relation (2.1), we have

$$\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}.$$

Since the preimage of a diagonal matrix is also diagonal by Claim 3.5, the above relation implies that

$$\begin{pmatrix} 1 & -s(u) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s(u^{-1}) & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s(u) & 1 \end{pmatrix} = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$$

for some $v \in R^\times$. By part (b) of Lemma 2.1, we must have $v = s(u)$ i.e. the preimage of $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ under $\pi : \rho_R(\Gamma) \to SL_2(A)$ is $\begin{pmatrix} s(u) & 0 \\ 0 & s(u)^{-1} \end{pmatrix}$.

It now follows $s(xy) = s(x)s(y)$ if both x, y are units in A. If $x \in A^\times$ and $y \notin A^\times$ then, as $1 + y$ is a unit, we get $s(x(1 + y)) = s(x)s(1 + y)$. Using additivity of $s : A \to R$ and $s(1) = 1$, we obtain $s(xy) = s(x)s(y)$. The other two remaining cases are treated similarly. □

Step 4. To complete the proof of the theorem, we have to verify that the pair $(A, \rho_A : \Gamma \to SL_n(A))$ is the universal deformation. Since the elementary matrices $E_{ij}(x)$ generate $SL_n(A)$ by Lemma 2.1, the elementary matrices $E_{ij}(s(x))$ must generate $\rho_R(\Gamma)$. As $\pi \circ s$ is the identity on A, we can now conclude that $s \circ \pi \circ \rho_R = \rho_R$. By universality, the homomorphism $s \circ \pi : R \to R$ must be the identity on R. Thus $\pi : R \to A$ is an isomorphism with inverse $s : A \to R$ and $\pi \circ \rho_R$, respectively $s \circ \rho_A$, is strictly equivalent to ρ_A, respectively ρ_R. This concludes the proof of Theorem 1.1.

4. Subgroups of SL_n over complete noetherian local rings and Assumption 3.2

In this section, we justify that Assumption 3.2, which we made in step (1) of the proof of Theorem 1.1, is valid. We retain the notation and assumptions made at the start of Section 3. Thus W is the Witt ring of the finite field $\boldsymbol{k}$ and n is a positive integer satisfying the restrictions made in Assumption 3.1. Assumption 3.2 is then a consequence of the following proposition.

Proposition 4.1. *Let the finite field $\boldsymbol{k}$ and positive integer n be as above. Thus $n \geq 2$ and the pair $(n, |\boldsymbol{k}|)$ is not one of the following: $(2, 2)$, $(2, 3)$, $(2, 5)$, $(3, 2)$, or $(4, 2)$. Suppose we are given a W-algebra A in $\boldsymbol{\mathcal{C}}_W$ and a closed subgroup G of $SL_n(A)$ with full residual image i.e. $G \pmod{\mathfrak{m}_A} = SL_n(\boldsymbol{k})$. Then there exists an $X \in GL_n(A)$ with $X \equiv I \pmod{\mathfrak{m}_A}$ such that $XGX^{-1} \supseteq SL_n(W_A)$.*

When $\boldsymbol{k}$ is not equal to either $\mathbb{F}_2$ or $\mathbb{F}_3$, or when $\boldsymbol{k} = \mathbb{F}_4$ and $n \neq 3$, Proposition 4.1 is covered by the main theorem of [11]. The argument in *loc. cit.* required certain cohomological properties of $SL_n(\boldsymbol{k})$ which followed from results of Cline, Parshall and Scott in [8], and Quillen in [15]. In this paper we shall indicate how the same argument may be recovered in the excluded cases by using results of Sah in [18] and [19]. To this end, we begin by setting out the following assumptions and notations.

Assumption 4.2. *Throughout this section:*

- *The finite field* $\boldsymbol{k}$ *is either* $\mathbb{F}_2$ *or* $\mathbb{F}_3$ *or* $\mathbb{F}_4$*,* p *denotes its characteristic, and* $W_m := W/p^m$*.*
- n *is a fixed integer subject to the following conditions.*
  - *If* $\boldsymbol{k} = \mathbb{F}_2$ *then* $n \geq 5$*.*
  - *If* $\boldsymbol{k} = \mathbb{F}_3$ *then* $n \geq 3$*.*
  - *If* $\boldsymbol{k} = \mathbb{F}_4$ *then* $n = 3$*.*
- $\mathbb{M}$*, resp.* $\mathbb{M}_0$*, denotes the space of* n *by* n *matrices over* $\boldsymbol{k}$*, resp. the space of* n *by* n *matrices over* $\boldsymbol{k}$ *with trace* 0*. When* $p|n$*, we set* $\mathbb{S} := \boldsymbol{k}I$ *and* $\mathbb{V} = \mathbb{M}_0/\mathbb{S}$*.*

We remark that if A is a W-algebra in $\boldsymbol{\mathcal{C}}_W$, then $GL_n(A)$ acts on $\mathbb{M}$ and $\mathbb{M}_0$ by conjugation. We make free use of standard results on group extensions and cohomology (see [2], [14]); what will be needed here is covered by [11, Section 2].

The following proposition gathers various properties of $SL_n(W_m)$ that will be needed in the proof of Proposition 4.1.

Proposition 4.3. *Let* $\boldsymbol{k}$ *and* n *be as in Assumption 4.2.*

(i) If $p \nmid n$ *then* $\mathbb{M}_0$ *is an irreducible* $SL_n(\boldsymbol{k})$*-module. If* $p|n$ *then* $\mathbb{S}$ *is the unique non-trivial* $SL_n(\boldsymbol{k})$*-submodule of* $\mathbb{M}_0$*. Moreover,*

$$Hom_{SL_n(\boldsymbol{k})}(\mathbb{M}_0, \mathbb{M}_0) \cong \boldsymbol{k} \cong Hom_{SL_n(\boldsymbol{k})}(\mathbb{V}, \mathbb{V}).$$

(ii) Let $\Gamma_m := \ker(SL_n(W_{m+1}) \xrightarrow{\text{mod } p^m} SL_n(W_m))$*. Then the extension*

$$I \to \Gamma_m \to SL_n(W_{m+1}) \to SL_n(W_m) \to I$$

does not split.

(iii) Suppose $p|n$*. Then* $H^1(SL_n(\boldsymbol{k}), \boldsymbol{k})$ *and* $H^2(SL_n(\boldsymbol{k}), \boldsymbol{k})$ *are both* (0)*. Furthermore* $H^1(SL_n(W_m), \boldsymbol{k}) = (0)$ *for all* $m \geq 1$*.*

(iv) The inflation map $H^1(SL_n(W_m), \mathbb{M}_0) \to H^1(SL_n(W_{m+1}), \mathbb{M}_0)$ *is an isomorphism. Consequently*

$$H^1(SL_n(W_m), \mathbb{M}_0) \cong H^1(SL_n(\boldsymbol{k}), \mathbb{M}_0) = \begin{cases} (0) & \text{if} \quad p \nmid n, \\ \boldsymbol{k} & \text{if} \quad p|n. \end{cases}$$

(v) Suppose $p|n$*.*

(a) If Z_m *denotes the subgroup of scalar matrices in* Γ_m*, then the extension*

$$I \to \Gamma_m/Z_m \to SL_n(W_{m+1})/Z_m \xrightarrow{\text{mod } p^m} SL_n(W_m) \to I. \tag{4.1}$$

does not split.

(b) The inflation map $H^1(SL_n(W_m), \mathbb{V}) \to H^1(SL_n(W_{m+1}), \mathbb{V})$ *is an isomorphism.*

(c) The map $H^2(SL_n(W_m), \mathbb{S}) \to H^2(SL_n(W_m), \mathbb{M}_0)$ *induced by the inclusion* $\mathbb{S} \subset \mathbb{M}_0$ *is an injection.*

(d) $H^1(SL_n(W_m), \mathbb{M}) = (0)$ *for all* $m \geq 1$*.*

Proof. Part (i): See [11, Lemma 3.3].

Part (ii): When $m \geq 2$, the non-splitting is covered by the argument in [11, Proposition 3.7]. (See the paragraphs above and around the displayed relation (3.5) there, *loc. cit.*) We give an indication of the proof.

For a contradiction, assume the sequence splits. Then the image in $SL_n(W_{m+1})$ of the elementary matrix $E_{12}(1) \in SL_n(W_m)$ under the section splitting the sequence can be written in the form $(I+p^m X)E_{12}(1)$, and this must have order p^m. By induction, it follows that

$$((I+p^m X)E_{12}(1))^k = \Big(I + p^m \sum_{j=0}^{k-1} E_{12}(j) X E_{12}(-j)\Big) E_{12}(k)$$

for all integers $k \geq 1$. If we write $E_{12}(1)$ as $I+N$ (so $E_{12}(j) = I + jN$), the above relation then becomes

$$((I+p^m X)E_{12}(1))^k = \big(I + p^m(kX + a_k(NX - XN) - b_k NXN)\big) E_{12}(k)$$

where $a_k = 1 + \ldots + (k-1)$ and $b_k = 1^2 + \ldots + (k-1)^2$. If $k = p^m$ and $m \geq 2$ then p divides a_k and b_k, and we get

$$((I+p^m X)E_{12}(1))^{p^m} = E_{12}(p^m).$$

Since this equality does not hold in $SL_n(W_{m+1})$, we obtain the desired contradiction.

So now assume $m = 1$. When $\boldsymbol{k} = \mathbb{F}_2$ or $\mathbb{F}_3$, the sequence is non-split by [18, Theorem II.7]. Thus, by Assumption 4.2, we only need to show that the sequence does not split when $\boldsymbol{k} = \mathbb{F}_4$ and $n = 3$. Now if Γ denotes the kernel of the reduction map $GL_3(W_2) \to GL_3(\boldsymbol{k})$, then the sequence

$$I \to \Gamma \to GL_3(W_2) \to GL_3(\boldsymbol{k}) \to I \tag{4.2}$$

is non-split by [19, Proposition 0.3]. Let $\widetilde{G}$ be the subgroup of $GL_3(W_2)$ consisting of matrices with determinant 1 modulo p. Since $H^1(SL_3(\boldsymbol{k}), \mathbb{M}) = (0)$ (see [19, Proposition 3.4]), the restriction map

$$H^2(GL_3(\boldsymbol{k}), \mathbb{M}) \to H^2(SL_3(\boldsymbol{k}), \mathbb{M})$$

is injective. Therefore the non-splitting of (4.2) implies that

$$I \to \Gamma \to \widetilde{G} \to SL_3(\boldsymbol{k}) \to I$$

is non-split. Consequently $I \to \Gamma_1 \to SL_3(W_2) \to SL_3(\boldsymbol{k}) \to I$ can not be split.

Part (iii): Assumption 4.2 and the hypothesis $p|n$ imply that $\boldsymbol{k}$ is either $\mathbb{F}_2$ or $\mathbb{F}_3$. The first part is then covered by [18, Theorem III.5] and [18, Proposition III.7].

For the second part, first identify Γ_m and $\mathbb{M}_0$ using the isomorphism $\phi : \Gamma_m \to \mathbb{M}_0$ given by $\phi(I + p^m M) := M \pmod{p}$. Then

$$H^1(\Gamma_m, \boldsymbol{k})^{SL_n(W_m)} \cong \mathrm{Hom}_{SL_n(\boldsymbol{k})}(\mathbb{M}_0, \boldsymbol{k}) = (0)$$

by part (i) above. An induction argument using inflation–restriction then implies that $H^1(SL_n(W_m), \boldsymbol{k}) = (0)$ for all $m \geq 1$.

Part (iv): As in the proof of part (iii), we use the identification $\phi : \Gamma_m \to \mathbb{M}_0$ given by $\phi(I + p^m M) := M \pmod{p}$. The transgression map

$$\delta : H^1(\Gamma_m, \mathbb{M}_0)^{SL_n(W_m)} \to H^2(SL_n(W_m), \mathbb{M}_0)$$

sends $-\phi$ to the class of the extension

$$0 \to \mathbb{M}_0 \xrightarrow{\phi^{-1}} SL_n(W_{m+1}) \to SL_n(W_m) \to 1$$

in $H^2(SL_n(W_m), \mathbb{M}_0)$ (see [11, Proposition 2.1]). Since $H^1(\Gamma_m, \mathbb{M}_0)^{SL_n(W_m)}$ has dimension 1 as a $\boldsymbol{k}$-vector space by part (i), and $\delta(-\phi) \neq 0$ as the above extension is non-split by part (ii), the transgression map δ is injective. Hence the inflation map $H^1(SL_n(W_m), \mathbb{M}_0) \to H^1(SL_n(W_{m+1}), \mathbb{M}_0)$ is an isomorphism. Consequently $H^1(SL_n(W_m), \mathbb{M}_0) \cong H^1(SL_n(\boldsymbol{k}), \mathbb{M}_0)$.

We now turn to calculation of the cohomology group $H^1(SL_n(\boldsymbol{k}), \mathbb{M}_0)$ under our assumptions on $\boldsymbol{k}$ and n. Since $H^1(SL_3(\mathbb{F}_4), \mathbb{M}_0) = (0)$ by [8, Theorem 4.2], we shall assume that $\boldsymbol{k}$ is either $\mathbb{F}_2$ or $\mathbb{F}_3$. By [18, Theorem III.5], we have $H^1(SL_n(\boldsymbol{k}), \mathbb{M}) = (0)$. When $p \nmid n$ the direct sum decomposition $\mathbb{M} = \mathbb{M}_0 \oplus \boldsymbol{k}I$ then implies that $H^1(SL_n(\boldsymbol{k}), \mathbb{M}_0) = (0)$. If $p|n$ the exact sequence $0 \to \mathbb{M}_0 \to \mathbb{M} \to \boldsymbol{k} \to 0$ along with part (iii) implies that the connecting map $H^0(SL_n(\boldsymbol{k}), \boldsymbol{k}) \to H^1(SL_n(\boldsymbol{k}), \mathbb{M}_0)$ is an isomorphism, and so $H^1(SL_n(\boldsymbol{k}), \mathbb{M}_0) \cong \boldsymbol{k}$.

Part (v): We give a brief sketch; see [11, Section 3.3] for details.

We first claim that for a fixed positive integer m, the non-splitting of extension (4.1) is equivalent to the inflation map $H^1(SL_n(W_m), \mathbb{V}) \to H^1(SL_n(W_{m+1}), \mathbb{V})$ being an isomorphism. This is done by an argument similar to part (iv) as follows. The identification $\phi : \Gamma_m \to \mathbb{M}_0$ induces an isomorphism $\psi : \Gamma_m/Z_m \to \mathbb{V}$, and the image of $-\psi$ under the transgression map

$$\delta : H^1(\Gamma_m/Z_m, \mathbb{V})^{SL_n(W_m)} \to H^2(SL_n(W_m), \mathbb{V})$$

is the cohomology class of extension (4.1). Since

$$H^1(\Gamma_m/Z_m, \mathbb{V})^{SL_n(W_m)} \cong \mathrm{Hom}_{SL_n(\boldsymbol{k})}(\mathbb{V}, \mathbb{V}) \cong \boldsymbol{k},$$

it follows that δ is injective if and only if $\delta(-\psi) \neq 0$ i.e. the extension (4.1) is non-split. But δ is injective if and only if the inflation map

$$H^1(SL_n(W_m), \mathbb{V}) \to H^1(SL_n(W_{m+1})/Z_m, \mathbb{V})$$

is an isomorphism. Now, the third term in the inflation–restriction exact sequence

$$0 \to H^1(SL_n(W_{m+1})/Z_m, \mathbb{V}) \to H^1(SL_n(W_{m+1}), \mathbb{V}) \to H^1(Z_m, \mathbb{V})^{SL_n(W_{m+1})}$$

is isomorphic to $\mathrm{Hom}_{SL_n(\boldsymbol{k})}(\mathbb{S}, \mathbb{V})$, which vanishes. Hence the inflation map

$$H^1(SL_n(W_{m+1})/Z_m, \mathbb{V}) \to H^1(SL_n(W_{m+1}), \mathbb{V})$$

is an isomorphism, and the desired equivalence follows.

The non-splitting of extension (4.1) when $m \geq 2$ is covered by [11, Lemma 3.9]. (The proof is similar to the one we sketched in part (ii) above, except that we need to work modulo the central subgroup Z_m.) By the claim above, the inflation map $H^1(SL_n(W_m), \mathbb{V}) \to H^1(SL_n(W_{m+1}), \mathbb{V})$ is an isomorphism whenever $m \geq 2$.

To see that the extension (4.1) does not split when $m = 1$, consider the commutative diagram

$$\begin{array}{ccc} H^1(\Gamma_1, \mathbb{M}_0)^{SL_n(\boldsymbol{k})} & \xrightarrow{\delta} & H^2(SL_n(\boldsymbol{k}), \mathbb{M}_0) \\ \downarrow & & \downarrow \\ H^1(\Gamma_1, \mathbb{V})^{SL_n(\boldsymbol{k})} & \xrightarrow{\delta} & H^2(SL_n(\boldsymbol{k}), \mathbb{V}) \end{array}$$

where the vertical maps come from $0 \to \mathbb{S} \to \mathbb{M}_0 \to \mathbb{V} \to 0$ and the horizontal maps are transgressions. The left hand arrow is an isomorphism by part (i), the right hand arrow is an injection by part (iii), and the top arrow is an injection by part (iv). The bottom arrow is therefore an injection and consequently the inflation map $H^1(SL_n(\boldsymbol{k}), \mathbb{V}) \to H^1(SL_n(W_2), \mathbb{V})$ is an isomorphism. Hence, by our claim above, extension (4.1) is non-split when $m = 1$ as well, and this completes the proofs of parts (a) and (b).

We now verify the injectivity of $H^2(SL_n(W_m), \mathbb{S}) \to H^2(SL_n(W_m), \mathbb{M}_0)$. Observe that we have isomorphisms

$$H^1(SL_n(W_m), \mathbb{V}) \cong H^1(SL_n(\boldsymbol{k}), \mathbb{V}) \cong H^1(SL_n(\boldsymbol{k}), \mathbb{M}_0) \cong \boldsymbol{k}$$

for all $m \geq 1$. (The first isomorphism is by part (b) above. The second isomorphism follows from the short exact sequence $0 \to \mathbb{S} \to \mathbb{M}_0 \to \mathbb{V} \to 0$ using part (iii). The third isomorphism holds by part (iv).) Writing $H^i(-)$ for the cohomology group $H^i(SL_n(W_m), -)$, we obtain the following long exact sequence

$$H^1(\mathbb{S}) \to H^1(\mathbb{M}_0) \to H^1(\mathbb{V}) \to H^2(\mathbb{S}) \to H^2(\mathbb{M}_0)$$

from $0 \to \mathbb{S} \to \mathbb{M}_0 \to \mathbb{V} \to 0$. Now $H^1(\mathbb{S}) = (0)$ by part (iii). Also, $H^1(\mathbb{M}_0)$ and $H^1(\mathbb{V})$ are both isomorphic to $\boldsymbol{k}$ (by part (iv) and from the observation above). Hence the map $H^1(\mathbb{M}_0) \to H^1(\mathbb{V})$ is an isomorphism, and part (c) follows.

For the final part, consider the long exact sequence

$$0 \to \mathbb{S} \to \mathbb{S} \to \boldsymbol{k} \to H^1(\mathbb{M}_0) \to H^1(\mathbb{M}) \to H^1(\boldsymbol{k})$$

obtained from $0 \to \mathbb{M}_0 \to \mathbb{M} \to \boldsymbol{k} \to 0$. Since the map $\mathbb{M} \to \boldsymbol{k}$ is the trace map, the sequence $0 \to \boldsymbol{k} \to H^1(\mathbb{M}_0) \to H^1(\mathbb{M}) \to H^1(\boldsymbol{k})$ is exact. Part (d) now follows because $\dim_{\boldsymbol{k}} H^1(\mathbb{M}_0) = 1$ by part (iv) and $H^1(\boldsymbol{k}) = 0$ by part (iii). □

We now return to the proof of Proposition 4.1. We begin by indicating how the result for general complete noetherian local rings can be deduced from the artinian case.

Suppose we are given a complete local noetherian W-algebra A in $\boldsymbol{\mathcal{C}}_W$ and a closed subgroup $G \subseteq SL_n(A)$ with full residual image. Note that the filtration $\mathfrak{m}_A \supseteq \mathfrak{m}_A^2 \supseteq \dots$ can be refined to a filtration $J_1 \supseteq J_2 \supseteq \dots$ by closed ideals satisfying the following conditions.

- For all i, the quotient A/J_i is artinian, and $J_1 = \mathfrak{m}_A$.
- For all i the surjection $A/J_{i+1} \to A/J_i$ is small.

Recall that a surjective morphism $f : B \to C$ of local artinian rings is *small* if $\ker(f)$ is a principal ideal killed by $\mathfrak{m}_B$.

Now suppose we can find, for each positive integer i, an invertible matrix $X_i \in GL_n(A/J_i)$ such that

- $X_i G(i) X_i^{-1} \supseteq SL_n(W_{A/J_i})$ where $G(i) = G \pmod{J_i}$, and
- $X_1 \in SL_n(\boldsymbol{k})$ is the identity and $X_{i+1} \pmod{J_i} = X_i$ for all i.

Then, since $A \cong \varprojlim A/J_i$, we can find $X \in GL_n(A)$ such that $X \equiv X_i \pmod{J_i}$ and $XGX^{-1} \supseteq SL_n(W_A)$. Thus Proposition 4.1 will follow from the result below.

Proposition 4.4. *Keep the hypotheses and notations of Assumption 4.2. Let A be an artinian W-algebra in $\boldsymbol{\mathcal{C}}_W$ and let t be a non-zero element of A killed by its maximal ideal i.e. $t\,\mathfrak{m}_A = (0)$.*

Suppose G is a subgroup of $SL_n(A)$ such that $G \pmod{tA} = SL_n(W_{A/tA})$. Then there is an $X \in GL_n(A)$ with $X \equiv I \pmod{tA}$ such that $SL_n(W_A) \subseteq XGX^{-1}$.

Proof. We set $B := A/tA$ and $\pi : A \to B$ to be reduction modulo tA. Then we have an exact sequence

$$0 \to \mathbb{M}_0 \xrightarrow{\varepsilon} SL_n(A) \xrightarrow{\pi} SL_n(B) \to I \tag{4.3}$$

where the map $\varepsilon : \mathbb{M}_0 \to SL_n(A)$ is constructed as follows: lift $x \in \mathbb{M}_0$ to an n by n matrix $\tilde{x}$ over A and take $\varepsilon(x) := I + t\tilde{x}$. Denote by $\widetilde{G}$ the preimage of $SL_n(W_B)$ in $SL_n(A)$. Thus

$$0 \to \mathbb{M}_0 \xrightarrow{\varepsilon} \widetilde{G} \xrightarrow{\pi} SL_n(W_B) \to I \tag{4.4}$$

is exact and G, $SL_n(W_A)$ are subgroups of $\widetilde{G}$. There are then the following three possibilities to consider:

- $G = \widetilde{G}$, in which case there is nothing to prove;
- $\pi : G \to SL_n(W_B)$ is an isomorphism; or,
- G fits into an exact sequence $0 \to \mathbb{S} \to G \to SL_n(W_B) \to I$.

Suppose $\pi : G \to SL_n(W_B)$ is an isomorphism. Then the sequence (4.4) splits. Consequently $\pi : SL_n(W_A) \to SL_n(W_B)$ must also be an isomorphism (otherwise $\widetilde{G} = SL_n(W_A)$ and the splitting of sequence (4.4) contradicts Proposition 4.3(ii)). It follows that G is a twist of $SL_n(W_A)$ by an element of $H^1(SL_n(W_B), \mathbb{M}_0)$.

If p and n are coprime then $H^1(SL_n(W_B), \mathbb{M}_0) = (0)$ by Proposition 4.3(iv), and we can find $X \in SL_n(A)$ with $\pi(X) = I$ such that $XGX^{-1} = SL_n(W_A)$. If p divides n then $H^1(SL_n(W_B), \mathbb{M}) = (0)$ by Proposition 4.3(v)(d). In this case, we can find $X \in GL_n(A)$ with $X \equiv I \pmod{tA}$ such that $XGX^{-1} = SL_n(W_A)$.

We now consider the case when

$$0 \to \mathbb{S} \to G \to SL_n(W_B) \to I \tag{4.5}$$

is exact. Assumption 4.2 then implies that $\boldsymbol{k}$ is either $\mathbb{F}_2$ or $\mathbb{F}_3$ and that $p|n$. Since $H^2(SL_n(W_B), \mathbb{S}) \to H^2(SL_n(W_B), \mathbb{M}_0)$ is injective by Proposition 4.3(v)(c), the sequence (4.5) splits if and only if the sequence (4.4) splits. Now, if $W_A = W_B$ then the sequence (4.4) splits. Consequently, the sequence (4.5) also splits. Hence G contains a subgroup which is isomorphic to $SL_n(W_B)$ under the reduction map π, and we are in the set up covered by the second case, which was discussed in the previous paragraph.

We are now left with the possibility that $W_A = W_{m+1}$ and $W_B = W_m$. In this case, we must have $\widetilde{G} = SL_n(W_A)$. The image of $\mathbb{M}_0$, resp. $\mathbb{S}$, under the map ε is precisely the subgroup Γ_m, resp. Z_m, defined in Proposition 4.3, parts (ii) and (v) respectively. Now sequence (4.5) implies that $G/Z_m \to SL_n(W_m)$ is an isomorphism splitting the sequence

$$I \to \Gamma_m/Z_m \to SL_n(W_{m+1})/Z_m \to SL_n(W_m) \to I.$$

This contradicts Proposition 4.3(v)(a), completing the proof. $\square$

Remark 4.5. The results of Sah ([18], [19]) used in the proof of Proposition 4.3 are precisely the cohomological results needed to make the main argument of [11] carry over when $\boldsymbol{k}$ and n satisfy the conditions set out in Assumption (4.2). We leave the precise verification to the interested reader, and state the following extension of Proposition 4.1 (and the main theorem of [11]).

Let A be a complete local noetherian ring with maximal ideal $\mathfrak{m}_A$ and finite residue field $A/\mathfrak{m}_A$ of characteristic p. Suppose we are given a subfield $\boldsymbol{k}$ of $A/\mathfrak{m}_A$ and a closed subgroup G of $GL_n(A)$ such that

- *$n \geq 2$ and the pair $(n, |\boldsymbol{k}|)$ is not one of the following: $(2,2)$, $(2,3)$, $(2,5)$, $(3,2)$, or $(4,2)$;*
- *$G \pmod{\mathfrak{m}_A} \supseteq SL_n(\boldsymbol{k})$.*

Then G contains a conjugate of $SL_n(W_A)$.

Remark 4.6. We now discuss the necessity of the restrictions on $(n, |\boldsymbol{k}|)$ in Proposition 4.1, and also in Theorem 1.1.

The standard representation of S_3 shows that $SL_2(\mathbb{F}_2)$ lifts to $GL_2(\mathbb{Z}_2)$. Since there is no ring homomorphism from $\mathbb{F}_2$ to $\mathbb{Z}_2$, Theorem 1.1 fails for (A, ρ_A) when $A = \mathbb{F}_2$ and $\Gamma = SL_2(\mathbb{F}_2)$ (with $\bar{\rho}$ and ρ_A as in the statement of Theorem 1.1). Also, $SL_2(\mathbb{Z}_2)$ contains a double cover of $SL_2(\mathbb{F}_2)$ (see the exercises at the end of [20, Chapter IV(3)]), and this double cover shows Proposition 4.1 fails when $(n, |\boldsymbol{k}|) = (2, 2)$.

Similarly, the following two observations imply the necessity of excluding the cases when $(n, |\boldsymbol{k}|)$ is one of $(2, 3)$, $(3, 2)$ or $(2, 5)$ from Proposition 4.1 and Theorem 1.1.

- The reduction map $SL_n(\mathbb{Z}/p^2\mathbb{Z}) \to SL_n(\mathbb{Z}/p\mathbb{Z})$ has a section when (n, p) is either $(2, 3)$ or $(3, 2)$. (See [18, Theorem II.7].)
- $SL_2(\mathbb{F}_5)$ has a lift to $SL_2(\mathbb{Z}_5[\zeta])$ where $\zeta^5 = 1, \zeta \neq 1$. (See Section 11.3.3, particularly Proposition 11.3.6 and the paragraph preceding it, of [1].)

Finally, it is known (see [22]) that $SL_4(\mathbb{F}_2)$ has a double cover inside $SL_4(\mathbb{Z}/4\mathbb{Z})$ and, from their orders, we see that this double cover cannot contain a conjugate of $SL_4(\mathbb{Z}/4\mathbb{Z})$. The restriction $(n, |\boldsymbol{k}|) \neq (4, 2)$ is therefore necessary in Proposition 4.1. However, although our proof breaks down, Dorobisz (see [10]) has shown that the conclusion of Theorem 1.1 still holds in this case.

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