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# Weak solutions of the Stochastic Landau-Lifschitz-Gilbert Equations with non-zero anisotrophy energy

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**Abstract:** We study a stochastic Landau-Lifschitz-Gilbert Equations with non-zero anisotrophy energy and multidimensional noise. We prove the existence and some regularities of weak solution proved. Our paper is motivated by finite-dimensional study of stochastic LLGEs or general stochasric differential equations with constraints studied by Kohn et al [17] and Lelièvre et al [19].

KEY WORDS stochastic partial differential equations, ferromagnetism, anisotrophy, heat flow

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## 1 Introduction

The ferromagnetism theory was first studied by Weiß in 1907 and then further developed by Landau and Lifshitz [18] and Gilbert [15]. According to their theory there is a characteristic of the material called the Curie's temperature, whence below this critical temperature, large ferromagnetic bodies would break up into small uniformly magnetized regions separated by thin transition layers. The small uniformly magnetized regions are called Weiß domains and the transition layers are called Bloch walls. This fact is taken into account by imposing the following constraint:

$$|u(t,x)|_{\mathbb{R}^3} = 1. \tag{1.1}$$

Moreover the magnetization in a domain $D \subset \mathbb{R}^3$ at time $t > 0$ given by $u(t,x) \in \mathbb{R}^3$ satisfies the following Landau-Lifschitz equation:

$$\frac{\mathrm{d}u(t,x)}{\mathrm{d}t} = \lambda_1 u(t,x) \times \rho(t,x) - \lambda_2 u(t,x) \times (u(t,x) \times \rho(t,x)). \tag{1.2}$$

The $\rho$ in equation (1.2) is called the effective magnetic field and defined by

$$\rho = -\nabla_u \mathcal{E}, \tag{1.3}$$

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where the $\mathcal{E}$ is the so called total electro-magnetic energy which composed by anisotropy energy, exchange energy and electronic energy.

In order to describe phase transitions between different equilibrium states induced by thermal fluctuations of the effective magnetic field $\rho$, Brzeźniak and Goldys and Jegaraj [9] introduced the Gaussian noise into the Landau-Lifschitz-Gilbert (LLG) equation to perturb $\rho$ and then the stochastic Landau-Lifschitz-Gilbert (SLLG) equation have the following form:

$$\mathrm{d}u(t) = (\lambda_1 u(t) \times \rho(t) - \lambda_2 u(t) \times (u(t) \times \rho(t)))\ \mathrm{d}t + (u(t) \times h) \circ\ \mathrm{d}W(t), \tag{1.4}$$

where $h \in L^\infty(D; \mathbb{R}^3)$. Their total energy contains only the exchange energy $\frac{1}{2}\|\nabla u\|_{\mathbb{L}^2}$, and hence their equation has the following form:

$$\begin{cases} \mathrm{d}u(t) = (\lambda_1 u(t) \times \Delta u(t) - \lambda_2 u(t) \times (u(t) \times \Delta u(t)))\,\mathrm{d}t + (u(t) \times h) \circ\ \mathrm{d}W(t), \\ \frac{\partial u}{\partial n}(t,x) = 0, \qquad t > 0, x \in \partial D, \\ u(0,x) = u_0(x), \qquad x \in D. \end{cases} \tag{1.5}$$

They concluded the existence of the weak solution of (1.5) and also proved some regularities of the solution.

There is also some research about the numerical schemes of equation (1.5), such as Baňas, Brzeźniak, and Prohl [5], Baňas, Brzeźniak, Neklyudov, and Prohl [6], Baňas, Brzeźniak, Neklyudov, and Prohl [7], Goldys, Le, and Tran [16] and Alouges, de Bouard and Hocquet [4]. The last paper differs from all previous papers as it deals with the LLGEs in the so called Gilbert form, see [15] and [3] for some related deterministic results, and with an infinite dimensional noise (correlated in space).

In the present paper we consider the SLLG equation with the total energy $\mathcal{E}$ consisting of the exchange and anisotropy energies and hence it defined as:

$$\mathcal{E}(u) = \mathcal{E}_{an}(u) + \mathcal{E}_{ex}(u) = \int_D \left( \phi(u(x)) + \frac{1}{2}|\nabla u(x)|^2 \right) \mathrm{d}x,$$

where $\mathcal{E}_{an}(u) := \int_D \phi(u(x))\,\mathrm{d}x$ stands for the anisotropy energy and $\mathcal{E}_{ex}(u) := \frac{1}{2}\int_D |\nabla u(x)|^2\,\mathrm{d}x$ stands for the exchange energy.

Our study is motivated by finite-dimensional study of stochastic LLGEs or general stochastic differential equations with constraints studied by Kohn et al [17] and Lelièvre et al [19]. An essential feature of the model studies in [17] was the presence of anisotropy energy (while the exchange energy was absent). So far none of the papers, apart from [10] which treats only one-dimensional domains, on the stochastic LLGEs considered nonzero anisotropy energy. Therefore there is a need to fill this literature gap and that is what we have achieved in the current work.

The main novelty of the current paper lies in being able to study of LLGEs with energy including the anisotropy energy. As we have mentioned earlier, both the papers by the first named authour, Goldys and Jegaraj and by Alouges, De Bouard and Hocquet, treat purely exchange energy. Our success was possible because we have been able to find uniform *a priori* estimates for the appropriately chosen finite dimensional approximations of the full problem. This in turn was possible because our suitable approximations satisfy equalities (1.9) and (1.9) which lead to equation (1.11), a similar one to equation (1.8) for the full Stochastic LLGEs equations. It turns out that the *a priori* estimates derived from the latter equalities are exactly what is needed in order to prove the existence of a weak martingale solution to the full Stochastic LLGEs equations.

So the SLLG equation we are going to study in this paper has the form:

$$\begin{cases} du(t) & = \Big[\lambda_1 u(t) \times (\Delta u(t) - \nabla\phi(u(t))) \\ & \quad -\lambda_2 u(t) \times \Big(u(t) \times (\Delta u(t) \quad - \nabla\phi(u(t)))\Big)\Big]\,dt + \displaystyle\sum_{j=1}^{N} (u(t) \times h_j) \circ \, dW_j(t), \\ \frac{\partial u}{\partial n}\Big|_{\Gamma} & = 0, \\ u(0) & = u_0, \end{cases} \tag{1.6}$$

where $h_j \in L^\infty(D;\mathbb{R}^3) \cap \mathbb{W}^{1,3}$, for $j = 1, \cdots, N$ and some $N \in \mathbb{N}$; see Assumption 2.2.

Let me describe on a heuristic level the idea of the proof. For this let us denote by $M$ the set of all functions $u \in \mathrm{H} = L^2(D;\mathbb{R}^3)$ such that $u(x) \in \mathbb{S}^2$ for a.a. $x \in D$, where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$. Since for $u \in H^2(D;\mathbb{R}^3) \cap M$ the H-orthogonal projection from H to $T_uM$ is equal to the map $\mathrm{H} \ni z \mapsto -u \times \big(u \times (z)\big) \in T_uM$, and $\Delta u - \nabla\phi(u)$ is equal to $-\nabla_{\mathrm{H}}\mathcal{E}(u)$, the $-$H gradient of the total energy $\mathcal{E}$, the second deterministic term on the RHS of (1.6) is equal to $-\lambda_2\nabla_M\mathcal{E}(u)$, the -gradient of the total energy $\mathcal{E}$ with respect to the riemannian structure of $M$ inherited from H. Similarly, the first deterministic term on the RHS of (1.6) is equal to $-\lambda_1 u \times (-\nabla_M\mathcal{E}(u))$ and in particular is perpendicular to $\nabla_M\mathcal{E}(u)$. Note also that for each $j$, $M \ni u \mapsto u \times h_j \in T_uM$, so that the function $u \times h_j$ could be seen as a (tangent!) vector field on $M$. Therefore, the the first equation of the system (1.6) could be written in the following geometric form

$$du(t) \quad = \quad \Big[\lambda_1 u \times (\nabla_M\mathcal{E}(u)) - \lambda_2\nabla_M\mathcal{E}(u) + \frac{1}{2}\sum_{j=1}^{N}(u \times h_j) \times h_j\Big]\,dt + \sum_{j=1}^{N} (u(t) \times h_j)\, dW_j(t). \tag{1.7}$$

Thus, on a purely heuristics level, applying the Itô Lemma, which is a generalisation of a deterministic result from [20] or [22], to the function $\mathcal{E}$ and a solution $u$ to (1.6), or equivalently to (1.7), we get

$$\begin{aligned}
d\mathcal{E}(u(t)) &= \lambda_1\langle\nabla_M\mathcal{E}(u), u\times(\nabla_M\mathcal{E}(u))\rangle\,dt - \lambda_2\langle\nabla_M\mathcal{E}(u),\nabla_M\mathcal{E}(u)\rangle\,dt \\
&+ \frac{1}{2}\sum_{j=1}^{N}\langle\nabla_M\mathcal{E}(u),(u\times h_j)\times h_j\rangle\,dt + \sum_{j=1}^{N}\langle\nabla_M\mathcal{E}(u),u\times h_j\rangle\,dW_j \\
&+ \frac{1}{2}\sum_{j=1}^{N}\langle\nabla_M^2\mathcal{E}(u)(u\times h_j),u\times h_j\rangle\,dt \\
&= -\lambda_2|\nabla_M\mathcal{E}(u)|^2\,dt \\
&+ \frac{1}{2}\sum_{j=1}^{N}\langle\nabla_M\mathcal{E}(u),(u\times h_j)\times h_j\rangle\,dt + \sum_{j=1}^{N}\langle\nabla_M\mathcal{E}(u),u\times h_j\rangle\,dW_j \\
&+ \frac{1}{2}\sum_{j=1}^{N}\langle\nabla_M^2\mathcal{E}(u)(u\times h_j),u\times h_j\rangle\,dt \qquad (1.8)
\end{aligned}$$

The above equality could naturally lead to *a priori* estimates but two problems. Firstly, we do not have a solution and secondly, even if we had it, it might not be strong or regular enough for the applicability of the Itô Lemma. A standard procedure is to approximate the full equation by some simpler problems. In the paper [9] we used Galerkin approximation, in a series of works with Banas, Prohl and Neklyudov culminating in a monograph [7], we used the finite element approximation. Here We follow the same method as used in Brzeźniak, Goldys and Jegaraj's paper [9] but with one important addition. We introduce, as in [9], finite dimensional subspaces $\mathrm{H}_n$ of the Hilbert space H. However, contrary to the finite element approximation used in [7], the set $M_n = M\cap \mathrm{H}$ is usually empty and an analog of equation (1.7) doesn't make sense. However, if $\mathcal{E}_n$ is the energy function $\mathcal{E}$ restricted to $\mathrm{H}_n$, the gradient $\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n)$ makes sense and, by the properties of the vector product, if $\pi_n : \mathrm{H}\to\mathrm{H}_n$ is the orthogonal projection, then

$$\begin{aligned}
&\Big\langle\pi_n[u_n\times(u_n\times(\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n)))],\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n)\Big\rangle_{\mathrm{H}_n} = \Big\langle[u_n\times(u_n\times(\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n)))],\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n)\Big\rangle_{\mathrm{H}} \qquad (1.9)\\
&= \int_D\Big\langle u_n(x)\times[u_n(x)\times(\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n)(x))],\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n)(x)\Big\rangle_{\mathbb{R}^3}\,dx \\
&= -\int_D\Big|u_n(x)\times(\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n)(x))\Big|^2_{\mathbb{R}^3}\,dx = -\Big|u_n\times(\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n))\Big|^2_{\mathrm{H}}
\end{aligned}$$

and

$$\begin{aligned}
&\Big\langle\pi_n(u_n\times(\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n))),\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n)\Big\rangle_{\mathrm{H}_n} = \Big\langle(u_n\times(\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n))),\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n)\Big\rangle_{\mathrm{H}} \qquad (1.10)\\
&= \int_D\Big\langle u_n(x)\times(\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n)(x)),\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n)(x)\Big\rangle_{\mathbb{R}^3}\,dx = \int_D 0\,dx = 0.
\end{aligned}$$

The above two equalities suggest the the correct finite dimensional approximation of equation (1.6), or (1.7), is an equation in the spirit of the former one, i.e.

$$\begin{aligned} \mathrm{d}u_n(t) &= \Big[\lambda_1\pi_n[u_n\times(u_n\times(\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n)))] - \lambda_2\pi_n(u_n\times(\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n)) \\ &+ \frac{1}{2}\sum_{j=1}^{N}\pi_n\Big((\pi_n(u_n\times h_j))\times h_j\Big)\Big]\,\mathrm{d}t + \sum_{j=1}^{N}\pi_n(u_n(t)\times h_j)\,\mathrm{d}W_j. \end{aligned} \tag{1.11}$$

Equation (1.11) is nothing else but equation (3.5) or (3.11). Now, the above problem is a Stochastic Differential Equation in a finite dimensional space $\mathrm{H}_n$ and hence it has a unique local maximal solution $u_n$. Applying the, now correct, Itô lemma to process $u_n$ and the function $\mathcal{E}_n$ we get an analog of identity

$$\begin{aligned} d\mathcal{E}_n(u_n(t)) &+ \lambda_2|\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n)|^2\,dt \\ &= \frac{1}{2}\sum_{j=1}^{N}\langle\nabla_{\mathrm{H}_n}\mathcal{E}(u_n),\pi_n\Big((\pi_n(u_n\times h_j))\times h_j\Big)\rangle\,dt + \sum_{j=1}^{N}\langle\nabla_{\mathrm{H}_n}\mathcal{E}_n(u_n),\pi_n(u_n\times h_j)\rangle\,dW_j \\ &+ \frac{1}{2}\sum_{j=1}^{N}\langle\nabla^2_{\mathrm{H}_n}\mathcal{E}_n(u_n)(\pi_n(u_n\times h_j)),\pi_n(u_n\times h_j)\rangle\,dt \end{aligned} \tag{1.12}$$

As a byproduct of out method, we prove that the solutions to the finite-dimensional stochastic Landau-Lifshitz-Gilbert equations (1.11) converge, after taking a subsequence and modulo a change of probability space, to a solution of the full (infinite-dimensional) stochastic Landau-Lifshitz-Gilbert equations (1.6).

In particular, our results give an alternative proof of the existence result from Brzeźniak, Goldys and Jegaraj's paper [10], where large deviations principle for stochastic LLG equation on a 1-dimensional domain has been studied. Our method of using the tightness criteria, the Skorokhod Theorem and the construction of the Wiener process is related but different from those applied to related problems in [11] and [12].

This paper is organized as follows. In Section 2 we introduce the notations and formulate the main result, i.e. Theorem 2.6, on the existence of the weak solution of the Equation (1.6) as well as some regularities. In Section 3 we introduce the finite dimensional approximation and prove the existence of the global solutions $\{u_n\}$ of the approximate equation of (1.6). In our main technical Section 4 we prove that our solutions to the approximate equations satisfy some *a priori* estimates. In Section 5 we state that the *a priori* estimates from the previous section are sufficient to prove that the laws of the solutions $\{u_n\}$ are tight on a suitable path space. The proof of this claim is omitted since it only a relatively simple modification of the proof of a corresponding result from [9].

In Section 6 we use the tightness results and the Skorohod's Theorem to construct a new probability space and some processes $\{u_n'\}$ with the same laws as $\{u_n\}$ such that $\{u_n'\}$ converges a.s to a limit process $u'$. In Section 7 we show that the path space from section Section 5 is small enough so that the process $u'$ is a weak solution to equation (1.6). In Section 8 we prove $u'$ takes values in the sphere $\mathbb{S}^2$ and so conclude the proof of Theorem 2.6.

Let us finish the introduction by remarking that all our results are formulated for $D\subset\mathbb{R}^d$, $d=3$, but they are also valid for $d=1$ or $d=2$.

*Remark.* This paper is from a part of the Ph.D. thesis at the University of York in UK of the second named author .

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## 2 Notation and the formulation of the main result

*Notation* 2.1. Let us denote the classical spaces:

$$\mathbb{L}^p := L^p(D;\mathbb{R}^3) \text{ or } L^p(D;\mathbb{R}^{3\times 3}),$$

$$\mathbb{W}^{k,p} := W^{k,p}(D;\mathbb{R}^3),\ \mathbb{H}^k := H^k(D;\mathbb{R}^3) = W^{k,2}(D;\mathbb{R}^3),\ \text{and } \mathbb{V} := \mathbb{W}^{1,2}.$$

The dual brackets between a space $X$ and its dual $X^*$ will be denoted ${}_{X^*}\langle\cdot,\cdot\rangle_X$. A scalar product in a Hilbert space $H$ will be denoted $\langle\cdot,\cdot\rangle_H$ and its associated norm $\|\cdot\|_H$.

**Assumption 2.2.** *Let $D$ be an open and bounded domain in $\mathbb{R}^3$ with $C^2$ boundary $\Gamma := \partial D$. $n$ is the outward normal vector on $\Gamma$. $\lambda_1 \in \mathbb{R}$, $\lambda_2 > 0$, $h_j \in \mathbb{L}^\infty \cap \mathbb{W}^{1,3}$, for $j = 1,\dots,N$. $\phi : \mathbb{R}^3 \to \mathbb{R}^+ \cup \{0\}$ is in $C^4$ and $\phi$, $\nabla\phi$, $\phi''$ and $\phi^{(3)}$ are bounded. $\nabla\phi$ is also globally Lipschitz.*

**Assumption 2.3.** *We assume that $(\Omega,\mathcal{F},\mathbb{F}=(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ a filtered probability space satisfying the so called **usual conditions**, i.e.*

*(i) $\mathbb{P}$ is complete on $(\Omega,\mathcal{F})$,*

*(ii) for each $t \geq 0$, $\mathcal{F}_t$ contains all $(\mathcal{F},\mathbb{P})$-null sets,*

*(iii) the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous,*

*and that $(W(t))_{t\geq 0} = ((W_j)_{j=1}^N(t))_{t\geq 0}$ is a $\mathbb{R}^N$-valued, $\mathbb{F}$-Wiener process defined on $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$.*

In this paper we are going to study the following equation,

$$\begin{cases} \mathrm{d}u(t) &= \Big[\lambda_1 u(t)\times(\Delta u(t) - \nabla\phi(u(t))) \\ &\quad -\lambda_2 u(t)\times\Big(u(t)\times(\Delta u(t) - \nabla\phi(u(t)))\Big)\Big]\mathrm{d}t + \sum_{j=1}^N (u(t)\times h_j)\circ \mathrm{d}W_j(t), \quad t\geq 0, \\ \frac{\partial u}{\partial n}\Big|_\Gamma &= 0, \quad u(0) = u_0. \end{cases} \tag{2.1}$$

*Remark* 2.4. Since the function $\phi : \mathbb{R}^3 \to \mathbb{R}$ is of $C^4$ class, it's Fréchet derivative $\mathrm{d}_x\phi : \mathbb{R}^3 \to \mathbb{R}$, at $x \in \mathbb{R}^3$, can be identified with a vector $\nabla\phi(x) \in \mathbb{R}^3$ such that

$$\langle \nabla\phi(x), y\rangle_{\mathbb{R}^3} = \mathrm{d}_x\phi(y), \quad y \in \mathbb{R}^3.$$

**Definition 2.5.** A weak solution of (2.1), with $u_0 \in \mathbb{V}$, is system consisting of a filtered probability space $(\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$, an $N$-dimensional $\mathbb{F}'$-Wiener process $W' = (W'_j)_{j=1}^N$ and an $\mathbb{F}'$-progressively measurable process

$$u' = (u'_i)_{i=1}^3 : \Omega' \times [0,T] \to \mathbb{V} \cap \mathbb{L}^\infty$$

such that for all $\psi \in C_0^\infty(D;\mathbb{R}^3)$, $t \in [0,T]$, we have, $\mathbb{P}'$-a.s.,

$$\begin{aligned}
\langle u'(t), \psi\rangle_{\mathbb{L}^2} &= \langle u_0, \psi\rangle_{\mathbb{L}^2} - \lambda_1 \int_0^t \langle \nabla u'(s), \nabla\psi \times u'(s)\rangle_{\mathbb{L}^2}\, ds \\
&+ \lambda_1 \int_0^t \langle u'(s) \times \nabla\phi(u'(s)), \psi\rangle_{\mathbb{L}^2}\, ds \\
&- \lambda_2 \int_0^t \langle \nabla u'(s), \nabla(u' \times \psi)(s) \times u'(s)\rangle_{\mathbb{L}^2}\, ds \\
&+ \lambda_2 \int_0^t \langle u'(s) \times (u'(s) \times \nabla\phi(u'(s)), \psi\rangle_{\mathbb{L}^2}\, ds \\
&+ \sum_{j=1}^N \int_0^t \langle u'(s) \times h_j, \psi\rangle_{\mathbb{L}^2} \circ dW'_j(s).
\end{aligned} \tag{2.2}$$

Next we will formulate the main result of this paper:

**Theorem 2.6.** *Under the assumptions listed in Assumption 2.2, for every $u_0 \in \mathbb{V}$, there exits a a weak solution*

$$(\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}'), W' = (W'_j)_{j=1}^N, u' = (u'_i)_{i=1}^3$$

*of* (2.1) *such that*

$$\mathbb{E} \int_0^T \left\| u'(t) \times \Delta u'(t) - u'(t) \times \nabla\phi(u'(t)) \right\|_{\mathbb{L}^2}^2\, dt < \infty \tag{2.3}$$

$$\begin{aligned}
u'(t) &= u_0 + \lambda_1 \int_0^t \left(u' \times \Delta u' - u' \times \nabla\phi(u')\right)(s)\, ds \\
&\quad -\lambda_2 \int_0^t u'(s) \times \left(u' \times \Delta u' - u' \times \nabla\phi(u')\right)(s)\, ds \\
&\quad + \sum_{j=1}^N \int_0^t (u'(s) \times h_j) \circ dW'_j(s);
\end{aligned}$$

*for every $t \in [0,T]$, in $L^2(\Omega';\mathbb{L}^2)$, and*

$$|u'(t,x)|_{\mathbb{R}^3} = 1, \quad \textit{for Lebesgue a.e. } (t,x) \in [0,T] \times D \textit{ and } \mathbb{P}' - \textit{a.s.}. \tag{2.4}$$

$$u' \in C^\alpha([0,T];\mathbb{L}^2), \qquad \mathbb{P}' - \textit{a.s., for every } \alpha \in (0, \frac{1}{2}). \tag{2.5}$$

*Remark* 2.7. The notation $u' \times \Delta u'$ used in Theorem 2.6 will be defined in the Notation 6.11.

The notation $u' \times (u' \times \Delta u')$ used in Theorem 2.6 will be defined in the Notation 6.12.

*Remark* 2.8. Our results are for the Laplace operator with Neumann boundary conditions. Without any difficult work one could prove the same result for the Laplace operator on a compact manifold without boundary. In particular, for Laplace operator with periodic boundary condition.

## 3 Galerkin approximation

Let $A$ be the $-$Laplace operator in $D$ acting on $\mathbb{R}^3$-valued functions with the Neumann boundary condition, i.e.

$$D(A) = \left\{ u \in \mathbb{H}^2 : \frac{\partial u}{\partial n}\Big|_{\partial D} = 0 \right\} \subset \mathbb{L}^2, \quad Au = -\Delta u, \quad u \in D(A).$$

Sine $A$ is self-adjoint, by ([13, Thm 1, p. 335]), there exists an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of $\mathbb{L}^2$, consisting of eigenvectors of $A$, such that $e_k \in C^\infty(\bar{D})$ for all $k = 1, 2, \ldots,$. We set $H_n = \text{linspan}\{e_1, e_2, \ldots, e_n\}$ and by $\pi_n$ denote the orthogonal projection from $\mathbb{L}^2$ to $H_n$. Put $A_1 := I + A$. Then $\mathbb{V} = D(A_1^{\frac{1}{2}}) = D(A^{\frac{1}{2}})$ and $\|u\|_{\mathbb{V}} = \|A_1^{\frac{1}{2}} u\|_{\mathbb{L}^2}$ for $u \in \mathbb{V}$. The following definition and proposition relate to the fractional powers of $A_1$ and will be frequently used later.

**Definition 3.1.** For any nonnegative real number $\beta$ we define the Hilbert space $X^\beta := D(A_1^\beta)$, which is the domain of the fractional power operator $A_1^\beta$. The dual of $X^\beta$ is denoted by $X^{-\beta}$, see [9].

**Proposition 3.2.** *We have, see [23, 4.3.3],*

$$X^\gamma = D(A_1^\gamma) = \begin{cases} \left\{ u \in \mathbb{H}^{2\gamma} : \frac{\partial u}{\partial n}\Big|_{\partial D} = 0 \right\}, & \text{if } 2\gamma > \frac{3}{2}, \\ \mathbb{H}^{2\gamma}, & \text{if } 2\gamma < \frac{3}{2}. \end{cases}$$

**Proposition 3.3.** *If $u \in D(A)$, $\mathrm{v} \in \mathbb{V}$ then*

$$\langle Au, \mathrm{v} \rangle_{\mathbb{L}^2} = \int_D \langle \nabla u(x), \nabla \mathrm{v}(x) \rangle_{\mathbb{R}^{3\times 3}} \, \mathrm{d}x,$$

$$\int_D \langle u(x) \times Au(x), Au(x) \rangle_{\mathbb{R}^3} \, \mathrm{d}x = 0, \tag{3.1}$$

$$\int_D \langle u(x) \times (u(x) \times Au(x)), Au(x) \rangle_{\mathbb{R}^3} \, \mathrm{d}x = -\int_D |u(x) \times Au(x)|^2 \, \mathrm{d}x, \tag{3.2}$$

$$\int_D \langle u(x) \times Au(x), \mathrm{v}(x) \rangle_{\mathbb{R}^3} \, \mathrm{d}x = \sum_{i=1}^3 \int_D \left\langle \frac{\partial u}{\partial x_i}(x), \frac{\partial \mathrm{v}}{\partial x_i}(x) \times u(x) \right\rangle_{\mathbb{R}^3} \mathrm{d}x, \tag{3.3}$$

$$\int_D \langle u(x) \times (u(x) \times Au(x)), \mathrm{v}(x) \rangle_{\mathbb{R}^3} \, \mathrm{d}x = \sum_{i=1}^3 \int_D \left\langle \frac{\partial u}{\partial x_i}(x), \frac{\partial (\mathrm{v} \times u)}{\partial x_i}(x) \times u(x) \right\rangle_{\mathbb{R}^3} \mathrm{d}x. \tag{3.4}$$

*Proof.* [Proof of (3.3) and (3.4)] The equality (3.3) follows from [9]. Since $\langle u \times (u \times Au), \mathrm{v} \rangle = \langle u \times Au, \mathrm{v} \times u \rangle$ and $\mathrm{v} \times u \in \mathbb{V}$, (3.4) follows from (3.3).

We consider the following equation in $H_n$ ($H_n \subset D(A)$) with all the assumptions in Assumptions 2.2 and 2.3, see the Introduction for the motivation of the system.

$$\begin{cases} \mathrm{d}u_n(t) & = -\pi_n\Big\{\lambda_1 u_n(t) \times \Big[Au_n(t) + \pi_n(\nabla\phi(u_n(t)))\Big] \\ & \quad -\lambda_2 u_n(t) \times \Big(u_n(t) \times \Big[Au_n(t) + \pi_n(\nabla\phi(u_n(t)))\Big]\Big)\Big\}\,\mathrm{d}t \\ & \quad + \sum_{j=1}^N \pi_n\Big[u_n(t) \times h_j\Big] \circ \mathrm{d}W_j(t), \quad t \geq 0, \\ u_n(0) & = \pi_n u_0. \end{cases} \tag{3.5}$$

Let us define the following maps:

$$F_n^1 \quad : \quad H_n \ni u \longmapsto -\pi_n(u \times Au) \in H_n, \tag{3.6}$$

$$F_n^2 \quad : \quad H_n \ni u \longmapsto -\pi_n(u \times (u \times Au)) \in H_n, \tag{3.7}$$

$$F_n^3 \quad : \quad H_n \ni u \longmapsto -\pi_n\,(u \times \pi_n(\nabla\phi(u))) \in H_n, \tag{3.8}$$

$$F_n^4 \quad : \quad H_n \ni u \longmapsto -\pi_n\Big(u \times (u \times \pi_n(\nabla\phi(u)))\Big) \in H_n, \tag{3.9}$$

$$G_{jn} \quad : \quad H_n \ni u \longmapsto \pi_n(u \times h_j) \in H_n, \quad j = 1, \dots, N. \tag{3.10}$$

Since the restriction $A_n$ of $A$ to $H_n$ is linear and bounded operator in $H_n$, and since $H_n \subset D(A) \subset \mathbb{L}^\infty$, we infer that $G_{jn}$ and $F_n^1, F_n^2, F_n^3, F_n^4$ are well defined maps from $H_n$ to $H_n$.

The problem (3.5) can be written in a compact way, see also (3.13),

$$\begin{cases} \mathrm{d}u_n(t) & = \lambda_1\left(F_n^1(u_n(t)) + F_n^3(u_n(t))\right)\,\mathrm{d}t - \lambda_2\,\left(F_n^2(u_n(t)) + F_n^4(u_n(t))\right)\,\mathrm{d}t \\ & \quad + \frac{1}{2}\sum_{j=1}^N G_{jn}^2(u_n(t))\,\mathrm{d}t + \sum_{j=1}^N G_{jn}(u_n(t))\,\mathrm{d}W_j(t), \\ u_n(0) & = \pi_n u_0. \end{cases} \tag{3.11}$$

*Remark* 3.4. In the Equations (2.1) and (3.5), we use the Stratonovich differential and in the Equation (3.11) we use the Itô differential. The following equality relates the two differentials: for a smooth map $G : \mathbb{L}^2 \to \mathbb{L}^2$,

$$(Gu) \circ \,\mathrm{d}W(t) = \frac{1}{2}G'(u)[G(u)]\,\mathrm{d}t + G(u)\,\mathrm{d}W(t), \qquad u \in \mathbb{L}^2.$$

*Remark* 3.5. As in equation (1.3), we also have

$$-\nabla_{H_n}\mathcal{E}_n(u_n) = Au_n + \pi_n\nabla\phi(u_n),$$

so with the projection "$\pi_n$"s in equation (3.5), our approximation keeps as much as possible the structure of equation (2.1), and consequently we will get the *a priori* estimates.

In order to establish solvability of Equation (3.11) we have the following result whose proof is omitted.

**Lemma 3.6.** *The maps $F_n^i$, $i = 1,2,3,4$ are Lipschitz on balls, that is, for every $R > 0$ there exists a constant $C = C(n,R) > 0$ such that whenever $x, y \in H_n$ and $\|x\|_{\mathbb{L}^2} \le R$, $\|y\|_{\mathbb{L}^2} \le R$, we have*

$$\left\| F_n^i(x) - F_n^i(y) \right\|_{\mathbb{L}^2} \le C \|x - y\|_{\mathbb{L}^2}.$$

*The map $G_{jn}$ is linear, $G_{jn}^* = -G_{jn}$ and*

$$\|G_{jn} u\|_{H_n} \le \|u\|_{\mathbb{L}^2} \|h_j\|_{\mathbb{L}^\infty}, \qquad u \in H_n. \tag{3.12}$$

*Moreover for $i = 1,2,3,4$ and $u \in H_n$, we have*

$$\left\langle F_n^i(u), u \right\rangle_{\mathbb{L}^2} = 0.$$

**Corollary 3.7.** *[2] The Equation* (3.5) *has a unique global solution $u_n : [0,T] \to H_n$.*

*Proof.* By Lemma 3.6, the coefficients $F_n^i$, $i = 1,2,3,4$ and $G_{jn}$ are locally Lipschitz and of one sided linear growth. Hence, see e.g. [2], the Equation (3.5) has a unique global solution $u_n : [0,\infty) \to H_n$.

Let us define functions $F_n$ and $\hat{F}_n : H_n \to H_n$ by

$$F_n = \lambda_1 (F_n^1 + F_n^3) - \lambda_2 (F_n^2 + F_n^4), \text{ and } \hat{F}_n = F_n + \frac{1}{2} \sum_{j=1}^N G_{jn}^2.$$

Then the problem (3.5) (or (3.11)) can be written in the following compact way

$$\mathrm{d}u_n(t) = \hat{F}_n(u_n(t))\,\mathrm{d}t + \sum_{j=1}^N G_{jn}(u_n(t))\,\mathrm{d}W_j(t). \tag{3.13}$$

## 4 *A priori* estimates

In this section we will get some properties of the solution $u_n$ of Equation (3.5) especially some *a priori* estimates.

**Theorem 4.1.** *Assume that $n \in \mathbb{N}$. Then for every $t \in [0,\infty)$,*

$$\|u_n(t)\|_{\mathbb{L}^2} = \|u_n(0)\|_{\mathbb{L}^2}, \qquad a.s.. \tag{4.1}$$

*Proof.* Let us consider a $C^\infty$ function $\psi : H_n \ni u \mapsto \frac{1}{2}\|u\|_H^2 \in \mathbb{R}$. Since $\psi'(u)(g) = \langle u, g\rangle_{\mathbb{L}^2}$ and $\psi''(u)(g,k) = \langle k, g\rangle_{\mathbb{L}^2}$, by the Itô Lemma and Lemma 3.6, we get

$$\begin{aligned}
\frac{1}{2}\,\mathrm{d}\|u_n(t)\|_H^2 &= \left(\Big\langle u_n(t), \hat{F}_n(u_n(t))\Big\rangle_{\mathbb{L}^2} + \frac{1}{2}\sum_{j=1}^N \Big\langle G_{jn}(u_n(t)), G_{jn}(u_n(t))\Big\rangle_{\mathbb{L}^2}\right)\mathrm{d}t \\
&\quad + \sum_{j=1}^N \Big\langle u_n(t), G_{jn}(u_n(t))\Big\rangle_{\mathbb{L}^2}\ \mathrm{d}W_j(t) \\
&= \frac{1}{2}\sum_{j=1}^N \left(\Big\langle u_n(t), G_{jn}^2(u_n(t))\Big\rangle_{\mathbb{L}^2} + \sum_{j=1}^N \Big\|G_{jn}(u_n(t))\Big\|_{\mathbb{L}^2}^2\right)\mathrm{d}t + 0\,\mathrm{d}W_j(t) = 0
\end{aligned}$$

Hence the result follows.

**Lemma 4.2.** *Let us define a function $\Phi : H_n \to \mathbb{R}$ by*

$$\Phi(u) := \frac{1}{2}\int_D \|\nabla u(x)\|^2\,\mathrm{d}x + \int_D \phi(u(x))\,\mathrm{d}x, \qquad u \in H_n. \tag{4.2}$$

*Then $\Phi \in C^2(H_n)$ and for $u, g, k \in H_n$,*

$$\begin{aligned}
\mathrm{d}_u\Phi(g) &= \Phi'(u)(g) = \langle \nabla u, \nabla g\rangle_{\mathbb{L}^2} + \int_D \langle \nabla\phi(u(x)), g(x)\rangle_{\mathbb{R}^3}\,\mathrm{d}x && (4.3)\\
&= \langle Au, g\rangle_{\mathbb{L}^2} + \int_D \langle \nabla\phi(u(x)), g(x)\rangle_{\mathbb{R}^3}\,\mathrm{d}x, \\
\Phi''(u)(g,k) &= \langle \nabla g, \nabla k\rangle_{\mathbb{L}^2} + \int_D \phi''(u(x))(g(x), k(x))\,\mathrm{d}x. && (4.4)
\end{aligned}$$

**Proposition 4.3.** *There exist constants $a, b, a_1, b_1 > 0$ such that for all $n \in \mathbb{N}$,*

$$\Big\|\nabla G_{jn}u\Big\|_{\mathbb{L}^2}^2 \le a\Big\|\nabla u\Big\|_{\mathbb{L}^2}^2 + b, \quad u \in H_n, \tag{4.5}$$

*and*

$$\Big\|\nabla G_{jn}^2 u\Big\|_{\mathbb{L}^2}^2 \le a_1\Big\|\nabla u\Big\|_{\mathbb{L}^2}^2 + b_1, \quad u \in H_n. \tag{4.6}$$

*Proof.* Since estimate (4.6) follows from a double application of (4.5) it is sufficient to prove the latter. Since $A_1$ is self-adjoint and $A_1 \ge A$, we have

$$\begin{aligned}
&\Big\|\nabla G_{jn}u\Big\|_{\mathbb{L}^2}^2 = \Big(AG_{jn}(u), G_{jn}(u)\Big)_{\mathbb{L}^2} \le (A_1 G_{jn}(u), G_{jn}(u))_{\mathbb{L}^2} \\
&= \left\|A_1^{\frac{1}{2}}\pi_n(u\times h_j)\right\|_{\mathbb{L}^2}^2 = \left\|\pi_n A_1^{\frac{1}{2}}(u\times h_j)\right\|_{\mathbb{L}^2}^2 \le \left\|A_1^{\frac{1}{2}}(u\times h_j)\right\|_{\mathbb{L}^2}^2 \\
&= \Big\|(u\times h_j)\Big\|_{\mathbb{V}}^2 \le N\Big(\|u\times h_j\|_{\mathbb{L}^2}^2 + \|\nabla(u\times h_j)\|_{\mathbb{L}^2}^2\Big) \\
&\le \left[\|h_j\|_{\mathbb{L}^\infty}^2\Big(\|u\|_{\mathbb{L}^2}^2 + 2\|\nabla u\|_{\mathbb{L}^2}^2\Big) + 2\|\nabla h_j\|_{\mathbb{L}^3}^2\|u\|_{\mathbb{L}^6}^2\right].
\end{aligned}$$

Next, since $\mathbb{L}^6 \hookrightarrow \mathbb{V}$, by equality (4.1) we infer that

$$\left\|\nabla G_{jn}u\right\|_{\mathbb{L}^2}^2 \le a\left\|\nabla u\right\|_{\mathbb{L}^2}^2 + b,$$

for some constants $a$ and $b$ which only depend on $\|h_j\|_{\mathbb{L}^\infty}$, $\|\nabla h_j\|_{\mathbb{L}^3}$ and $\|u_0\|_{\mathbb{L}^2}$, but not on $n$.

*Remark* 4.4. The previous results will be used to prove the following fundamental *a priori* estimates on the sequence $\{u_n\}$ of the solution of Equation (3.5).

**Theorem 4.5.** *Assume that $p \ge 1$, $\beta > \frac{1}{4}$ and $T > 0$. Then there exists a constant $C > 0$, such that for all $n \in \mathbb{N}$,*

$$\mathbb{E} \sup_{r\in[0,t]} \left\{\left\|\nabla u_n(r)\right\|_{\mathbb{L}^2}^2 + \int_D \phi(u_n(r,x))\,\mathrm{d}x\right\}^p \le C, \qquad t \in [0,T], \tag{4.7}$$

$$\mathbb{E}\left[\left(\int_0^T \left\|u_n(t) \times (\Delta u_n(t) - \pi_n \nabla\phi(u_n(t)))\right\|_{\mathbb{L}^2}^2 \mathrm{d}t\right)^p\right] \le C, \tag{4.8}$$

$$\mathbb{E}\left[\left(\int_0^T \left\|u_n(t) \times (u_n(t) \times (\Delta u_n(t) - \pi_n \nabla\phi(u_n(t))))\right\|_{\mathbb{L}^{\frac{3}{2}}(D)}^2 \mathrm{d}t\right)^p\right] \le C, \tag{4.9}$$

$$\mathbb{E}\int_0^T \left\|\pi_n\left(u_n(t) \times (u_n(t) \times (\Delta u_n(t) - \pi_n \nabla\phi(u_n(t))))\right)\right\|_{X^{-\beta}}^2 \,\mathrm{d}t \le C. \tag{4.10}$$

*Proof.* [Proof of (4.7) and (4.8)] By the Itô Lemma applied to the function $\Phi$ defined in (4.2) we get

$$\begin{aligned} \Phi(u_n(t)) - \Phi(u_n(0)) \quad &= \quad \sum_{j=1}^N \int_0^t \Phi'(u_n(s))G_{jn}(u_n(s))\,\mathrm{d}W_j(s) \\ &+ \quad \int_0^t \left(\Phi'(u_n(s))\hat{F}_n(u_n(s)) + \frac{1}{2}\sum_{j=1}^N \Phi''(u_n(s))G_{jn}(u_n(s))^2\right)\mathrm{d}s, \qquad t \in [0,T]. \end{aligned} \tag{4.11}$$

Since

$$\begin{aligned} \Phi'(u)\hat{F}_n(u) = \quad &- \quad \lambda_2\left\|u \times (\Delta u - \pi_n(\nabla\phi(u))\right\|_{\mathbb{L}^2}^2 \\ &- \quad \frac{1}{2}\sum_{j=1}^N \langle \Delta u - \pi_n(\nabla\phi(u)), \pi_n(u \times h_j) \times h_j\rangle_{\mathbb{L}^2} \end{aligned} \tag{4.12}$$

$$\Phi'(u)[G_{jn}(u)] \quad = \quad -\langle \Delta u, u \times h_j\rangle_{\mathbb{L}^2} + \langle \nabla\phi(u), \pi_n(u \times h_j)\rangle_{\mathbb{L}^2}, \tag{4.13}$$

$$\begin{aligned} \Phi''(u)[G_{jn}(u)^2] \quad &= \quad = \|\nabla\pi_n(u \times h_j)\|_{\mathbb{L}^2}^2 \\ &+ \quad \int_D \phi''(u(x))(\pi_n(u \times h_j)(x), \pi_n(u \times h_j)(x))\,\mathrm{d}x. \end{aligned} \tag{4.14}$$

in view of Equation (4.2), we infer that Equation (4.11) transforms to:

$$\begin{aligned}
\frac{1}{2}\|\nabla u_n(t)\|^2_{\mathbb{L}^2} \quad &+ \quad \frac{1}{2}\int_D \phi(u_n(t,x))\,\mathrm{d}x + \lambda_2 \int_0^t \|u_n(s)\times(\Delta u_n(s) - \pi_n \nabla\phi(u_n(s)))\|^2_{\mathbb{L}^2}\,\mathrm{d}s \\
&= \quad \frac{1}{2}\|\nabla u_n(0)\|^2_{\mathbb{L}^2} + \frac{1}{2}\int_D \phi(u_n(0)(x))\,\mathrm{d}x - \frac{1}{2}\sum_{j=1}^N \int_0^t \langle \Delta u_n(s), \pi_n(u_n(s)\times h_j)\times h_j\rangle_{\mathbb{L}^2}\,\mathrm{d}s \qquad (4.15)\\
&+\frac{1}{2}\sum_{j=1}^N \int_0^t \langle \nabla\phi(u_n(s)), \pi_n(u_n(s)\times h_j)\times h_j\rangle_{\mathbb{L}^2}\,\mathrm{d}s + \frac{1}{2}\sum_{j=1}^N \int_0^t \|\nabla \pi_n(u_n(s)\times h_j)\|^2_{\mathbb{L}^2}\,\mathrm{d}s \\
&+\frac{1}{2}\sum_{j=1}^N \int_0^t \int_D \phi''(u_n(s)(x))(\pi_n(u_n(s)\times h_j)(x), \pi_n(u_n(s)\times h_j)(x))\,\mathrm{d}x\,\mathrm{d}s \\
&-\sum_{j=1}^N \int_0^t \langle \Delta u_n(s), u_n(s)\times h_j\rangle_{\mathbb{L}^2}\,\mathrm{d}W_j(s) + \sum_{j=1}^N \int_0^t \langle \nabla\phi(u_n(s), \pi_n(u_n(s)\times h_j)\rangle_{\mathbb{L}^2}\,\mathrm{d}W_j(s).
\end{aligned}$$

Next we will get estimates for some terms on the right hand side of Equation (4.15).

For the first term, we have

$$\|\nabla u_n(0)\|^2_{\mathbb{L}^2} = \|\nabla \pi_n u_0\|^2_{\mathbb{L}^2} \le \|\pi_n u_0\|^2_{\mathbb{V}} = \|A_1^{\frac{1}{2}}\pi_n u_0\|^2_{\mathbb{L}^2} = \|\pi_n A_1^{\frac{1}{2}} u_0\|^2_{\mathbb{L}^2} \le \|A_1^{\frac{1}{2}} u_0\|^2_{\mathbb{L}^2} = \|u_0\|^2_{\mathbb{V}}. \qquad (4.16)$$

Since $\phi$ is bounded, we can find a constant $C_\phi > 0$, such that $\left|\int_D \phi(u_n(0,x))\,\mathrm{d}x\right| \le C_\phi m(D)$.

For the third term, by (4.6) and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\left|\langle \Delta u_n(s), \pi_n(u_n(s)\times h_j)\times h_j\rangle_{\mathbb{L}^2}\right| = \left|\langle \nabla u_n(s), \nabla G_n^2 u_n(s)\rangle_{\mathbb{L}^2}\right| \qquad (4.17)\\
\le \quad & \|\nabla u_n(s)\|_{\mathbb{L}^2}\sqrt{a_1\|\nabla u_n(s)\|^2_{\mathbb{L}^2} + b_1} \le \sqrt{a_1}\|\nabla u_n(s)\|^2_{\mathbb{L}^2} + \frac{b_1}{2\sqrt{a_1}}.
\end{aligned}$$

For the fourth term, by equality (4.1) and Cauchy-Schwartz inequality, we have

$$\langle \nabla\phi(u_n(s)), \pi_n(u_n(s)\times h_j)\times h_j\rangle_{\mathbb{L}^2} \le C m(D)\|u_0\|_{\mathbb{L}^2}\|h_j\|^2_{\mathbb{L}^\infty}. \qquad (4.18)$$

For the fifth term, by (4.5), we have

$$\left\|\nabla \pi_n(u_n(s)\times h_j)(s)\right\|^2_{\mathbb{L}^2} = \left\|\nabla G_{jn}(u_n(s))\right\|^2_{\mathbb{L}^2} \le a\|\nabla u_n(s)\|^2_{\mathbb{L}^2} + b. \qquad (4.19)$$

For the sixth term, we have

$$\begin{aligned}
&\int_D \left|\phi''(u_n(s,x))\left(\pi_n(u_n(s)\times h_j)(x), \pi_n(u_n(s)\times h_j)(x)\right)\right|\,\mathrm{d}x \\
\le \quad & C_{\phi''}\int_D \left|\pi_n(u_n(s)\times h_j)(x))\right|^2\,\mathrm{d}x \le C_{\phi''}\|h_j\|^2_{\mathbb{L}^\infty}\|u_0\|^2_{\mathbb{L}^2}. \qquad (4.20)
\end{aligned}$$

Thus, there exists a constant $C_2 > 0$ such that for all $n \in \mathbb{N}$, $t \in [0, T]$ and $\mathbb{P}$-almost surely:

$$\begin{aligned}
&\|\nabla u_n(t)\|_{\mathbb{L}^2}^2 + \int_D \phi(u_n(t,x))\,\mathrm{d}x + 2\lambda_2 \int_0^t \|u_n(s) \times (\Delta u_n(s) - \pi_n \nabla\phi(u_n(s)))\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\
&\leq C_2 \int_0^t \|\nabla u_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + C_2 + 2\sum_{j=1}^N \int_0^t \left\langle \nabla u_n(s), \nabla G_{jn}(u_n(s))\right\rangle_{\mathbb{L}^2}\,\mathrm{d}W_j(s) \\
&\quad + \sum_{j=1}^N \int_0^t \left\langle \nabla\phi(u_n(s)), G_{jn}(u_n(s))\right\rangle_{\mathbb{L}^2}\,\mathrm{d}W_j(s). \qquad (4.21)
\end{aligned}$$

Let us now fix $p \geq 1$. Then by Hölder the Burkholder-Davis-Gundy inequality, there exists constant $C_i, K > 0$ such that for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\left\{\|\nabla u_n(r)\|_{\mathbb{L}^2}^2 + \int_D \phi(u_n(r,x))\,\mathrm{d}x + 2\lambda_2 \int_0^r \|u_n(s) \times (\Delta u_n(s) - \pi_n \nabla\phi(u_n(s)))\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right\}^p \\
\leq\;& 4^{p-1}C_2^p t^{p-1}\mathbb{E}\left(\int_0^t \|\nabla u_n(s)\|_{\mathbb{L}^2}^{2p}\,\mathrm{d}s\right) \\
&+4^{p-1}2\mathbb{E}\sup_{r\in[0,t]}\left|\sum_{j=1}^N \int_0^r \left\langle \nabla u_n(s), \nabla G_{jn}(u_n(s))\right\rangle_{\mathbb{L}^2}\,\mathrm{d}W_s\right|^p \\
&+4^{p-1}\mathbb{E}\sup_{r\in[0,t]}\left|\sum_{j=1}^N \int_0^r \left\langle \nabla\phi(u_n(s)), G_{jn}(u_n(s))\right\rangle_{\mathbb{L}^2}\,\mathrm{d}W_s\right|^p + 4^{p-1}C_2^p.
\end{aligned}$$

$$\mathbb{E}\sup_{r\in[0,t]}\left|\sum_{j=1}^N \int_0^r \left\langle \nabla u_n(s), \nabla G_{jn}(u_n(s))\right\rangle_{\mathbb{L}^2}\,\mathrm{d}W_s\right|^p \leq K\mathbb{E}\left|\sum_{j=1}^N \int_0^t \left\langle \nabla u_n(s), \nabla G_{jn}(u_n(s))\right\rangle_{\mathbb{L}^2}^2\,\mathrm{d}s\right|^{\frac{p}{2}},$$

$$\mathbb{E}\sup_{r\in[0,t]}\left|\sum_{j=1}^N \int_0^r \left\langle \nabla\phi(u_n(s)), G_{jn}(u_n(s))\right\rangle_{\mathbb{L}^2}\,\mathrm{d}W_s\right|^p \leq K\mathbb{E}\left|\sum_{j=1}^N \int_0^t \left\langle \nabla\phi(u_n(s)), G_{jn}(u_n(s))\right\rangle_{\mathbb{L}^2}^2\,\mathrm{d}s\right|^{\frac{p}{2}}.$$

By the inequality (4.5) we get, for any $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{E}\left|\sum_{j=1}^N \int_0^t \left\langle \nabla u_n(s), \nabla G_{jn}(u_n(s))\right\rangle_{\mathbb{L}^2}^2\,\mathrm{d}s\right|^{\frac{p}{2}} \leq \mathbb{E}\left[\sup_{r\in[0,t]}\|\nabla u_n(r)\|_{\mathbb{L}^2}^p\left(\sum_{j=1}^N \int_0^t \|\nabla G_{jn}(u_n(s))\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^{\frac{p}{2}}\right] \\
\leq\;& \mathbb{E}\left[\varepsilon\sup_{r\in[0,t]}\|\nabla u_n(r)\|_{\mathbb{L}^2}^{2p} + \frac{4}{\varepsilon}\left(\sum_{j=1}^N \int_0^t \|\nabla G_{jn}(u_n(s))\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right] \\
\leq\;& \varepsilon\mathbb{E}\left(\sup_{r\in[0,t]}\|\nabla u_n(r)\|_{\mathbb{L}^2}^{2p}\right) + \frac{4}{\varepsilon}(2t)^{p-1}a^pN^p\mathbb{E}\left(\int_0^t \|\nabla u_n(s)\|_{\mathbb{L}^2}^{2p}\,\mathrm{d}s\right) + \frac{4}{\varepsilon}2^{p-1}(bt)^pN^p.
\end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}\left|\sum_{j=1}^{N}\int_0^t \left\langle \nabla\phi(u_n(s)), G_{jn}(u_n(s))\right\rangle_{\mathbb{L}^2}^2 \,\mathrm{d}s\right|^{\frac{p}{2}} \\
&\quad\le \mathbb{E}\left[\sup_{r\in[0,t]}\|\nabla\phi(u_n(r))\|_{\mathbb{L}^2}^p\left(\sum_{j=1}^{N}\int_0^t \|\nabla G_{jn}(u_n(s))\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^{\frac{p}{2}}\right] \\
&\quad\le \varepsilon\,[Cm(D)]^{2p} + \frac{4}{\varepsilon}(2t)^{p-1}a^pN^p\mathbb{E}\left(\int_0^t \|\nabla u_n(s)\|_{\mathbb{L}^2}^{2p}\,\mathrm{d}s\right) + \frac{4}{\varepsilon}2^{p-1}(bt)^pN^p.
\end{aligned}
$$

Hence we infer that for $t \in [0, T]$,

$$
\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\left|\sum_{j=1}^{N}\int_0^r \left\langle \nabla u_n(s), \nabla G_{jn}(u_n(s))\right\rangle_{\mathbb{L}^2} \,\mathrm{d}W_s\right|^{p} \\
&\quad\le K\varepsilon\mathbb{E}\left(\sup_{r\in[0,t]}\|\nabla u_n(r)\|_{\mathbb{L}^2}^{2p}\right) + \frac{4K}{\varepsilon}(2t)^{p-1}a^pN^p\mathbb{E}\left(\int_0^t \|\nabla u_n(s)\|_{\mathbb{L}^2}^{2p}\,\mathrm{d}s\right) + \frac{4K}{\varepsilon}2^{p-1}(bt)^pN^p, \qquad (4.22)\\
&\mathbb{E}\sup_{r\in[0,t]}\left|\sum_{j=1}^{N}\int_0^r \left\langle \nabla\phi(u_n(s)), G_{jn}(u_n(s))\right\rangle_{\mathbb{L}^2} \,\mathrm{d}W_s\right|^{p} \\
&\quad\le K\varepsilon\,[Cm(D)]^{2p} + \frac{4K}{\varepsilon}(2t)^{p-1}a^pN^p\mathbb{E}\left(\int_0^t \|\nabla u_n(s)\|_{\mathbb{L}^2}^{2p}\,\mathrm{d}s\right) + \frac{4K}{\varepsilon}2^{p-1}(bt)^pN^p. \qquad (4.23)
\end{aligned}
$$

Hence, for every $t \in [0, T]$, we have

$$
\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\left\{\|\nabla u_n(r)\|_{\mathbb{L}^2}^2 + \int_D \phi(u_n(r,x))\,\mathrm{d}x + 2\lambda_2\int_0^r \|u_n(s)\times(\Delta u_n(s) - \pi_n\nabla\phi(u_n(s)))\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right\}^p \\
&\quad\le 4^{p-1}C_2^pt^{p-1}\mathbb{E}\left(\int_0^t \|\nabla u_n(s)\|_{\mathbb{L}^2}^{2p}\,\mathrm{d}s\right) + 4^{p-1}K\varepsilon\mathbb{E}\left(\sup_{r\in[0,t]}\|\nabla u_n(r)\|_{\mathbb{L}^2}^{2p}\right) + 4^{p-1}K\varepsilon\,[Cm(D)]^{2p} \\
&\qquad + \frac{8K}{\varepsilon}(8t)^{p-1}a^pN^p\mathbb{E}\left(\int_0^t \|\nabla u_n(s)\|_{\mathbb{L}^2}^{2p}\,\mathrm{d}s\right) + \frac{K}{\varepsilon}8^p(bt)^pN^p
\end{aligned}
$$

By setting $\varepsilon = \frac{1}{2K4^{p-1}}$, we can find constants $C_3$ and $C_4$ which do not depend on $n$ such that

$$
\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\left\{\|\nabla u_n(r)\|_{\mathbb{L}^2}^2 + \int_D \phi(u_n(r,x))\,\mathrm{d}x\right. \\
&\qquad\left. + 2\lambda_2\int_0^r \|u_n(s)\times(\Delta u_n(s) - \pi_n\nabla\phi(u_n(s)))\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right\}^p = C_3\mathbb{E}\left(\int_0^t \|\nabla u_n(s)\|_{\mathbb{L}^2}^{2p}\,\mathrm{d}s\right) + C_4. \qquad (4.24)
\end{aligned}
$$

Thus by inequality (4.24), we have

$$
\psi_n(t) \le C_3\int_0^t \psi_n(s)\,\mathrm{d}s + C_4, \quad t \in [0, T], \qquad (4.25)
$$

where, for $t \in [0,T]$, we put

$$\psi_n(t) = \mathbb{E} \sup_{s\in[0,t]} \left\{ \left\|\nabla u_n(s)\right\|_{\mathbb{L}^2}^2 + \int_D \phi(u_n(s,x))\,\mathrm{d}x \right.$$
$$\left. +2\lambda_2 \int_0^s \|u_n(\tau) \times (\Delta u_n(r) - \pi_n \nabla\phi(u_n(r)))\|_{\mathbb{L}^2}^2 \,\mathrm{d}r \right\}^p.$$

Observe that $\psi_n$ is a bounded Borel function. Indeed, for $s$ $in[0,T]$ we have $\|\nabla u_n(s)\|_{\mathbb{L}^2} \le \|u_n(s)\|_{\mathbb{V}} \le C_n\|u_n(s)\|_{\mathbb{L}^2} \le C_n\|u_0\|_{\mathbb{L}^2}$ and $\|u_n(s)\|_{\mathbb{L}^\infty} \le C_n\|u_n(s)\|_{\mathbb{L}^2} \le C_n\|u_0\|_{\mathbb{L}^2}$, where $C_n$ is a constant depending on $n$, so that

$$\|u_n(s) \times (\Delta u_n(s) - \pi_n \nabla\phi(u_n(s)))\|_{H_n} C_n\|u_0\|_{\mathbb{L}^2}\left(C_n\|u_0\|_{\mathbb{L}^2} + Cm(D)^{\frac{1}{2}}\right).$$

Therefore by the Gronwall inequality, we infer that

$$\mathbb{E} \sup_{r\in[0,t]} \left\{ \left\|\nabla u_n(r)\right\|_{\mathbb{L}^2}^2 + \int_D \phi(u_n(r,x))\,\mathrm{d}x + 2\lambda_2 \int_0^r \|u_n(\tau) \times (\Delta u_n(\tau) - \pi_n \nabla\phi(u_n(\tau)))\|_{\mathbb{L}^2}^2 \,\mathrm{d}\tau \right\}^p \le C_T,$$

for some $C_T > 0$, and all $t \in [0,T]$. This completes the proof of inequalities (4.7) and (4.8).

*Proof.* [Proof of (4.9)] By the the Sobolev imbedding theorem, see e.g. [1], $\mathbb{V} \hookrightarrow \mathbb{L}^6$, we can find a constant $c > 0$ such that

$$\left\|u_n(t) \times (u_n(t) \times (\Delta u_n(t) - \pi_n \nabla\phi(u_n(t)))\,)\right\|_{\mathbb{L}^{\frac{3}{2}}} \le c\left\|u_n(t)\right\|_{\mathbb{V}}\left\|u_n(t) \times (\Delta u_n(t) - \pi_n \nabla\phi(u_n(t)))\,\right\|_{\mathbb{L}^2}.$$

Therefore, by (4.1), (4.7) and (4.8), we get

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T \left\|u_n(t) \times (u_n(t) \times (\Delta u_n(t) - \pi_n \nabla\phi(u_n(t)))\,)\right\|_{\mathbb{L}^{\frac{3}{2}}}^2 \,\mathrm{d}t\right)^p\right] \\
\le\ & c\left(\mathbb{E}\left[\sup_{r\in[0,T]} \left\|u_n(r)\right\|_{\mathbb{V}}^{4p}\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\left(\int_0^T \left\|u_n(t) \times (\Delta u_n(t) - \pi_n \nabla\phi(u_n(t)))\,\right\|_{\mathbb{L}^2}^2 \mathrm{d}t\right)^{2p}\right]\right)^{\frac{1}{2}} \le C,
\end{aligned}$$

Note that $C$ is independent of $n$. This completes the proof of (4.9).

*Proof.* [Proof of (4.10)] Since $\beta > \frac{1}{4}$ we infer, by the Sobolev imbedding theorem, that $X^\beta \hookrightarrow \mathbb{H}^{2\beta}(D)$ and $\mathbb{H}^{2\beta}(D) \hookrightarrow \mathbb{L}^3$ continuously. Thus $\mathbb{L}^{\frac{3}{2}}(D) \hookrightarrow X^{-\beta}$ continuously. And since for $\xi \in \mathbb{L}^2$,

$$\begin{aligned}
\|\pi_n\xi\|_{X^{-\beta}} &= \sup_{\|\varphi\|_{X^\beta}\le 1} |{}_{X^{-\beta}}\langle \pi_n\xi, \varphi\rangle_{X^\beta}| = \sup_{\|\varphi\|_{X^\beta}\le 1} |\langle \pi_n\xi, \varphi\rangle_{\mathbb{L}^2}| \\
&= \sup_{\|\varphi\|_{X^\beta}\le 1} |\langle \xi, \pi_n\varphi\rangle_{\mathbb{L}^2}| \le \sup_{\|\pi_n\varphi\|_{X^\beta}\le 1} |{}_{X^{-\beta}}\langle \xi, \pi_n\varphi\rangle_{X^\beta}| = \|\xi\|_{X^{-\beta}}.
\end{aligned}$$

Thus there exists some constant $c > 0$ such that

$$\begin{aligned}
&\mathbb{E}\int_0^T \left\|\pi_n(u_n(t)\times(u_n(t)\times(\Delta u_n(t)-\pi_n\nabla\phi(u_n(t)))\,))\right\|^2_{X^{-\beta}}\,\mathrm{d}t \\
&\quad\leq\; c\mathbb{E}\int_0^T \left\|u_n(t)\times(u_n(t)\times(\Delta u_n(t)-\pi_n\nabla\phi(u_n(t)))\,)\right\|^2_{\mathbb{L}^{\frac{3}{2}}}\,\mathrm{d}t.
\end{aligned}$$

Hence (4.10) follows from (4.9).

**Proposition 4.6.** *Let $u_n$, for $n \in \mathbb{N}$, be the solution of equation* (3.5) *and assume that $\alpha \in (0,\frac{1}{2})$, $\beta > \frac{1}{4}$, $p \geq 2$. Then the following estimates holds:*

$$\sup_{n\in\mathbb{N}}\mathbb{E}(\|u_n\|^2_{W^{\alpha,p}(0,T;X^{-\beta})}) < \infty. \tag{4.26}$$

We need the following Lemma to prove (4.26).

**Lemma 4.7** ([14], Lem 2.1)**.** *Assume that $E$ is a separable Hilbert space, $p \in [2,\infty)$ and $a \in (0,\frac{1}{2})$. Then there exists a constant $C$ depending on $T$ and $a$, such that for any progressively measurable process $\xi = (\xi_j)_{j=1}^\infty$, if $I(\xi_j)$ is defined by $I(\xi) := \sum_{j=1}^\infty \int_0^t \xi_j(s)\,\mathrm{d}W_j(s)$, $t \geq 0$, then*

$$\mathbb{E}\,\|I(\xi)\|^p_{W^{a,p}(0,T;E)} \leq C\mathbb{E}\int_0^T \left(\sum_{j=1}^\infty |\xi_j(r)|^2_E\right)^{\frac{p}{2}}\,\mathrm{d}t,$$

*In particular, $\mathbb{P}$-a.s. the trajectories of the process $I(\xi_j)$ belong to $W^{a,2}(0,T;E)$.*

*Proof.* [Proof of (4.26)] Let us fix $\alpha \in (0,\frac{1}{2})$, $\beta > \frac{1}{4}$, $p \geq 2$. By equation (3.11), we get

$$\begin{aligned}
u_n(t) \;&=\; u_{0,n} + \lambda_1\int_0^t \left(F^1_n(u_n(s)) + F^3_n(u_n(s))\right)\,\mathrm{d}s - \lambda_2\int_0^t\left(F^2_n(u_n(s)) + F^4_n(u_n(s))\right)\,\mathrm{d}s \\
&+\; \frac{1}{2}\sum_{j=1}^N\int_0^t G^2_{jn}(u_n(s))\,\mathrm{d}s + \sum_{j=1}^N\int_0^t G_{jn}(u_n(s))\,\mathrm{d}W(s) =: u_{0,n} + \sum_{i=1}^4 u^i_n(t),\quad t\in[0,T].
\end{aligned}$$

By Theorem 4.5, equality (4.1), inequality (3.12) and Lemma 4.7 there exists $C > 0$ such that for all $n \in \mathbb{N}$,

$$\mathbb{E}\left[\|u^1_n\|^2_{W^{1,2}(0,T;\mathbb{L}^2)}\right] \leq C,\quad \mathbb{E}\left[\|u^2_n\|^2_{W^{1,2}(0,T;X^{-\beta})}\right] \leq C,$$

$$\|u^3_n\|^2_{W^{1,2}(0,T;\mathbb{L}^2)} \leq C, \qquad \mathbb{P}-a.s..$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|u_n(t)\|^p_{\mathbb{L}^2}\right] = \mathbb{E}\left[\|u_n(0)\|^p_{\mathbb{L}^2}\right] \leq C.$$

$$\mathbb{E}\left[\|u^4_n\|^p_{W^{\alpha,p}(0,T;X^{-\beta})}\right] \leq C.$$

Therefore since $H^1(0,T;X^{-\beta}) \hookrightarrow W^{\alpha,p}(0,T;X^{-\beta})$ continuously, we get inequality (4.26).

## 5 Tightness of the laws of approximating sequence

In this subsection we will state a result on the tightness, on a suitable path space, the laws $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$. The proof of this result is based the *a priori* estimates (4.1)-(4.10). The proof of this result is omitted since it only a relatively simple modification of the proof of a corresponding result from [9].

**Lemma 5.1.** *If $p \geq 2$, $q \in [2, 6)$ and $\beta > \frac{1}{4}$, then the measures $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ on $L^p(0, T; \mathbb{L}^q(D)) \cap C([0, T]; X^{-\beta})$ are tight.*

## 6 Construction of new Probability Space and Processes

In this section we will use the Skorohod Theorem to obtain another probability space and an almost surely convergent sequence defined on this space whose limit is a weak martingale solution of equation (2.1).
By Lemma 5.1 and the Prokhorov Theorem, we have the following property.

**Proposition 6.1.** *Let us assume that $W$ is a $N$-dimensional Wiener process and $p \in [2, \infty)$, $q \in [2, 6)$ and $\beta > \frac{1}{4}$. Then there is a subsequence of $\{u_n\}$ which we will denote it in the same way as the full sequence, such that the laws $\mathcal{L}(u_n, W)$ converge weakly to a certain probability measure $\mu$ on $L^p(0, T; \mathbb{L}^q(D)) \cap C([0, T]; X^{-\beta}) \times C([0, T]; \mathbb{R}^N)$.*

Now by the the Skorohod Theorem we have:

**Proposition 6.2.** *Let $\mu$ be the measure from Proposition 6.1. There exist a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, and (on this space) sequence $(u_n', W_n')$ of $[L^4(0, T; \mathbb{L}^4) \cap C([0, T]; X^{-\beta})] \times C([0, T]; \mathbb{R}^N)$-valued random variables and an $L^4(0, T; \mathbb{L}^4) \cap C([0, T]; X^{-\beta})] \times C([0, T]; \mathbb{R}^N)$-valued random variable $(u', W')$ such that such that, on $[L^4(0, T; \mathbb{L}^4) \cap C([0, T]; X^{-\beta})] \times C([0, T]; \mathbb{R}^N)$,*

*(a)* $\mathcal{L}(u_n, W) = \mathcal{L}(u_n', W_n')$, $n \in \mathbb{N}$,

*(b)* $\mathcal{L}(u', W') = \mu$,

*and, $\mathbb{P}'$-a.s., (c) $(u_n', W_n') \to (u', W')$ in $[L^4(0, T; \mathbb{L}^4) \cap C([0, T]; X^{-\beta})] \times C([0, T]; \mathbb{R}^N)$.*

*Notation* 6.3. Let us denote by $\mathbb{F}'$ the filtration generated by processes $u'$ and $W'$ on the probability space $(\Omega', \mathcal{F}', \mathbb{P}')$.

From now on we will prove that $u'$ is the weak solution of equation (2.1). And we begin with showing that $\{u_n'\}$ satisfies the same *a priori* estimates as the original sequence $\{u_n\}$. By the Kuratowski Theorem, we have

**Proposition 6.4.** *The Borel subsets of $C([0, T]; H_n)$ are Borel subsets of $L^4(0, T; \mathbb{L}^4) \cap C([0, T]; X^{-\frac{1}{2}})$.*

So we have the following two results.

**Corollary 6.5.** *$u_n'$ takes values in $H_n$ and the laws on $C([0, T]; H_n)$ of $u_n$ and $u_n'$ are equal.*

**Lemma 6.6.** *The sequence $\{u_n'\}$ introduced in Proposition 6.2 satisfies the following estimates:*

$$\sup_{t \in [0,T]} \left\| u_n'(t) \right\|_{\mathbb{L}^2} \leq \left\| u_0 \right\|_{\mathbb{L}^2}, \qquad \mathbb{P}' - a.s., \tag{6.1}$$

$$\sup_{n\in\mathbb{N}}\mathbb{E}'\left[\sup_{t\in[0,T]}\left\|u_n'(t)\right\|_{\mathbb{V}}^{2r}\right]<\infty,\qquad \forall r\geq 1, \tag{6.2}$$

$$\sup_{n\in\mathbb{N}}\mathbb{E}'\left[\left(\int_0^T \|u_n'(t)\times(\Delta u_n'(t)-\pi_n\nabla\phi(u_n'(t)))\|_{\mathbb{L}^2}^2\,\mathrm{d}t\right)^r\right]<\infty,\qquad \forall r\geq 1, \tag{6.3}$$

$$\sup_{n\in\mathbb{N}}\mathbb{E}'\int_0^T \left\|u_n'(t)\times(u_n'(t)\times(\Delta u_n'(t)-\pi_n\nabla\phi(u_n'(t))))\right\|_{\mathbb{L}^{\frac{3}{2}}}^2\,\mathrm{d}t<\infty, \tag{6.4}$$

$$\sup_{n\in\mathbb{N}}\mathbb{E}'\int_0^T \left\|\pi_n\left[u_n'(t)\times(u_n'(t)\times(\Delta u_n'(t)-\pi_n\nabla\phi(u_n'(t))))\right]\right\|_{X^{-\beta}}^2\,\mathrm{d}t<\infty. \tag{6.5}$$

Now we will study some inequalities satisfied by the limiting process $u'$.

**Proposition 6.7.** *Let $u'$ be the process which is defined in Proposition 6.2. Then we have*

$$\operatorname*{ess\,sup}_{t\in[0,T]}\|u'(t)\|_{\mathbb{L}^2}\leq\|u_0\|_{\mathbb{L}^2},\quad \mathbb{P}'-a.s. \tag{6.6}$$

$$\sup_{t\in[0,T]}\|u'(t)\|_{X^{-\beta}}\leq c\|u_0\|_{\mathbb{L}^2},\quad \mathbb{P}'-a.s. \tag{6.7}$$

*Proof.* First we will prove inequality (6.6). Since $u_n'$ converges to $u'$ in $L^4(0,T;\mathbb{L}^4)\cap C([0,T];X^{-\beta})$ $\mathbb{P}'$ a.s. and $\mathbb{L}^4\hookrightarrow\mathbb{L}^2$, we infer that $\mathbb{P}'$ a.s. $u_n'$ converges to $u'$ in $L^2(0,T;\mathbb{L}^2)$. Therefore by (6.1) we deduce (6.6).
Next we will prove inequality (6.7). Since $\mathbb{L}^2\hookrightarrow X^{-\beta}$, in view of (6.1), we have

$$\sup_{t\in[0,T]}\|u_n'(t)\|_{X^{-\beta}}\leq c\sup_{t\in[0,T]}\|u_n'(t)\|_{\mathbb{L}^2}\leq c\|u_0\|_{\mathbb{L}^2},\quad \mathbb{P}'-a.s.$$

Since by Proposition 6.2 $u_n'$ converges to $u'$ in $C([0,T];X^{-\beta})$, we infer that (6.7) holds.

We continue investigating properties of the process $u'$. The next result and it's proof are related to the estimate (6.2).

**Proposition 6.8.** *Let $u'$ be the process which was defined in Proposition 6.2. Then we have*

$$\mathbb{E}'[\operatorname*{ess\,sup}_{t\in[0,T]}\|u'(t)\|_{\mathbb{V}}^{2r}]<\infty,\quad r\geq 2. \tag{6.8}$$

*Proof.*

Since $L^{2r}(\Omega';L^\infty(0,T;\mathbb{V}))$ is isomorphic to $\left(L^{\frac{2r}{2r-1}}(\Omega';L^1(0,T;X^{-\frac{1}{2}}))\right)^*$, by the Banach-Alaoglu Theorem we infer that the sequence $\{u_n'\}$ contains a subsequence, denoted in the same way as the full sequence, and there exists an element $v\in L^{2r}(\Omega';L^\infty(0,T;\mathbb{V}))$ such that $u_n'\to v$ weakly$^*$ in $L^{2r}(\Omega';L^\infty(0,T;\mathbb{V}))$. In particular, we have

$$\langle u_n',\varphi\rangle\to\langle v,\varphi\rangle,\qquad \varphi\in L^{\frac{2r}{2r-1}}(\Omega';(L^1(0,T;X^{-\frac{1}{2}}))).$$

This means that

$$\int_{\Omega'}\int_0^T \langle u_n'(t,\omega),\varphi(t,\omega)\rangle\,\mathrm{d}t\,\mathrm{d}\mathbb{P}'(\omega) \to \int_{\Omega'}\int_0^T \langle v(t,\omega),\phi(t,\omega)\rangle\,\mathrm{d}t\,\mathrm{d}\mathbb{P}'(\omega).$$

On the other hand, if we fix $\varphi \in L^4(\Omega'; L^{\frac{4}{3}}(0,T;\mathbb{L}^{\frac{4}{3}}))$, by inequality (6.2) we have (to avoid too long formulations, we omit some parameters $t$ in the following equations)

$$\begin{aligned}
&\sup_n \int_{\Omega'} \left| \int_0^T {}_{\mathbb{L}^4}\langle u_n'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t \right|^2 \mathrm{d}\mathbb{P}'(\omega) \le \sup_n \int_{\Omega'} \left| \int_0^T \|u_n'\|_{\mathbb{L}^4}\|\varphi\|_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t \right|^2 \mathrm{d}\mathbb{P}'(\omega)\\
\le\ & \sup_n \int_{\Omega'} \|u_n'\|^2_{L^\infty(0,T;\mathbb{L}^4)}\|\varphi\|^2_{L^1(0,T;\mathbb{L}^{\frac{4}{3}})}\,\mathrm{d}\mathbb{P}'(\omega) \le \sup_n \|u_n'\|^2_{L^4(\Omega';L^\infty(0,T;\mathbb{L}^4))}\|\varphi\|^2_{L^4(\Omega';L^1(0,T;\mathbb{L}^{\frac{4}{3}}))} < \infty.
\end{aligned}$$

So the sequence $\int_0^T {}_{\mathbb{L}^4}\langle u_n'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t$ is uniformly integrable on $\Omega'$. Moreover, by the $\mathbb{P}'$ almost surely convergence of $u_n'$ to $u'$ in $L^4(0,T;\mathbb{L}^4)$, we get $\mathbb{P}'$-a.s.

$$\begin{aligned}
&\left| \int_0^T {}_{\mathbb{L}^4}\langle u_n'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t - \int_0^T {}_{\mathbb{L}^4}\langle u'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t \right|\\
&\quad \le \int_0^T \left| {}_{\mathbb{L}^4}\langle u_n'(t)-u'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac{4}{3}}} \right| \mathrm{d}t \le \|u_n'-u'\|_{L^4(0,T;\mathbb{L}^4)}\|\varphi\|_{L^{\frac{4}{3}}(0,T;\mathbb{L}^{\frac{4}{3}})} \to 0.
\end{aligned}$$

Therefore $\int_0^T {}_{\mathbb{L}^4}\langle u_n'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t \to \int_0^T {}_{\mathbb{L}^4}\langle u'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t$ $\mathbb{P}'$-a.s. and thus, by Vitali Theorem,

$$\int_{\Omega'}\int_0^T {}_{\mathbb{L}^4}\langle u_n'(t,\omega),\varphi(t,\omega)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t\,\mathrm{d}\mathbb{P}'(\omega) \to \int_{\Omega'}\int_0^T {}_{\mathbb{L}^4}\langle u'(t,\omega),\varphi(t,\omega)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t\,\mathrm{d}\mathbb{P}'(\omega).$$

Hence we deduce that

$$\int_{\Omega'}\int_0^T {}_{\mathbb{L}^4}\langle v(t,\omega),\varphi(t,\omega)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t\,\mathrm{d}\mathbb{P}'(\omega) = \int_{\Omega'}\int_0^T {}_{\mathbb{L}^4}\langle u'(t,\omega),\varphi(t,\omega)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t\,\mathrm{d}\mathbb{P}'(\omega)$$

By the density of $L^4(\Omega'; L^{\frac{4}{3}}(0,T;\mathbb{L}^{\frac{4}{3}}))$ in $L^{\frac{2r}{2r-1}}(\Omega'; L^1(0,T;X^{-\frac{1}{2}}))$, we infer that $u' = v$ and so by since $v$ satisfies (6.8) we infer that $u'$ also satisfies (6.8). The proof is complete.

Now we will strengthen part (ii) of Proposition 6.2 about the convergence of $u_n'$ to $u'$.

**Proposition 6.9.**

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \|u_n'(t)-u'(t)\|^4_{\mathbb{L}^4}\,\mathrm{d}t = 0. \tag{6.9}$$

*Proof.* Since $u_n' \to u'$ in $L^4(0,T;\mathbb{L}^4) \cap C([0,T];X^{-\beta})$ $\mathbb{P}'$-a.s., by (6.2) and by (6.8),

$$\sup_n \mathbb{E}' \left( \int_0^T \|u_n'(t)-u'(t)\|^4_{\mathbb{L}^4}\,\mathrm{d}t \right)^2 \le 2^7 \sup_n \left( \|u_n'\|^8_{L^4(0,T;\mathbb{L}^4)} + \|u'\|^8_{L^4(0,T;\mathbb{L}^4)} \right) < \infty,$$

we can apply the Vitali Theorem to deduce (6.9). This completes the proof.

By inequality (6.2), the sequence $\{u'_n\}_{n=1}^{\infty}$ is bounded in $L^2(\Omega'; L^2(0,T;\mathbb{H}^1))$. And since $u'_n \to u'$ in $L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$, we infer that

$$D_i u'_n \to D_i u' \text{ weakly in } L^2(\Omega'; L^2(0,T;\mathbb{L}^2)),\ i = 1,2,3. \tag{6.10}$$

**Lemma 6.10.** *There exists a unique* $\Lambda \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$ *such that for every* $v \in L^2(\Omega'; L^2(0,T;\mathbb{W}^{1,4}))$,

$$\mathbb{E}' \int_0^T \langle \Lambda(t), v(t)\rangle_{\mathbb{L}^2}\, dt = \sum_{i=1}^3 \mathbb{E}' \int_0^T \langle D_i u'(t), u'(t) \times D_i v(t)\rangle_{\mathbb{L}^2}\, dt. \tag{6.11}$$

*Proof.* We will omit"$(t)$" in this proof. Let us denote $\Lambda_n := u'_n \times A u'_n$. By the estimate (6.3), there exists a constant $C$ such that

$$\|\Lambda_n\|_{L^2(\Omega';L^2(0,T;\mathbb{L}^2))} \le C, \quad n \in \mathbb{N}.$$

Hence by the Banach-Alaoglu Theorem, there exists $\Lambda \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$ such that $\Lambda_n \to \Lambda$ weakly in $L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$.

Let us fix $v \in L^2(\Omega'; L^2(0,T;\mathbb{W}^{1,4}))$. Since $u'_n(t) \in D(A)$ for almost every $t \in [0,T]$ and $\mathbb{P}'$-almost surely, by the Proposition 3.3 and estimate (6.3) again, we have

$$\mathbb{E}' \int_0^T \langle \Lambda_n, v\rangle_{\mathbb{L}^2}\, dt = \sum_{i=1}^3 \mathbb{E}' \int_0^T \langle D_i u'_n, u'_n \times D_i v\rangle_{\mathbb{L}^2}\, dt.$$

Moreover, by the results: (6.10), (6.2) and (6.9), we have for $i = 1,2,3$,

$$\begin{aligned}
&\left|\mathbb{E}' \int_0^T \langle D_i u', u' \times D_i v\rangle_{\mathbb{L}^2}\, dt - \mathbb{E}' \int_0^T \langle D_i u'_n, u'_n \times D_i v\rangle_{\mathbb{L}^2}\, dt\right| \\
\le\ & \left|\mathbb{E}' \int_0^T \langle D_i u' - D_i u'_n, u' \times D_i v\rangle_{\mathbb{L}^2}\, dt\right| + \left|\mathbb{E}' \int_0^T \langle D_i u'_n, (u' - u'_n) \times D_i v\rangle_{\mathbb{L}^2}\, dt\right| \\
\le\ & \left|\mathbb{E}' \int_0^T \langle D_i u' - D_i u'_n, u' \times D_i v\rangle_{\mathbb{L}^2}\, dt\right| + \Big(\mathbb{E}' \int_0^T \|D_i u'_n\|^2_{\mathbb{L}^2}\, dt\Big)^{\frac{1}{2}} \\
& \times\Big(\mathbb{E}' \int_0^T \|u' - u'_n\|^4_{\mathbb{L}^4}\, dt\Big)^{\frac{1}{4}} \left(\mathbb{E}' \int_0^T \|D_i v\|^4_{\mathbb{L}^4}\, dt\right)^{\frac{1}{4}} \to 0.
\end{aligned}$$

Therefore we infer that

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \langle \Lambda_n, v\rangle_{\mathbb{L}^2}\, dt = \sum_{i=1}^3 \mathbb{E}' \int_0^T \langle D_i u', u' \times D_i v\rangle\, dt.$$

Since on the other hand we have proved $\Lambda_n \to \Lambda$ weakly in $L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$ equality (6.11) follows.

It remains to prove the uniqueness of $\Lambda$, but this follows from the fact that $L^2(\Omega'; L^2(0,T;\mathbb{W}^{1,4}))$ is dense in $L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$ and (6.11). This complete the proof of Lemma 6.10.

*Notation* 6.11. The process $\Lambda$ introduced in Lemma 6.10 will be denoted by $u' \times \Delta u'$ (as explained in the Appendix). Note that $u' \times \Delta u'$ is an element of $L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$ such that for all test functions $v \in L^2(\Omega'; L^2(0,T;\mathbb{W}^{1,4}))$ the following identity holds

$$\mathbb{E}'\int_0^T \langle (u'\times\Delta u')(t), v(t)\rangle_{\mathbb{L}^2}\,\mathrm{d}t = \sum_{i=1}^3 \mathbb{E}'\int_0^T \langle D_i u'(t), u'(t)\times D_i v(t)\rangle_{\mathbb{L}^2}\,\mathrm{d}t.$$

*Notation* 6.12. Since by the estimate (6.8), $u'\in L^2(\Omega', L^\infty(0,T;\mathbb{V}))$ and by Notation 6.11, $\Lambda\in L^2(\Omega';L^2(0,T;\mathbb{L}^2))$, the process $u'\times\Lambda\in L^{\frac{4}{3}}(\Omega';L^2(0,T;\mathbb{L}^{\frac{3}{2}}(D)))$. And $u'\times\Lambda$ will be denoted by $u'\times(u'\times\Delta u')$.

*Notation* 6.13. $\Lambda - u'\times\nabla\phi(u')$ will be denoted by $u'\times(\Delta u' - \nabla\phi(u'))$.

Next we will show that the limits of the following three sequences

$$\begin{aligned}
&\{u'_n\times(\Delta u'_n - \pi_n\nabla\phi(u'_n))\}_n,\\
&\{u'_n\times(u'_n\times(\Delta u'_n - \pi_n\nabla\phi(u'_n)))\}_n,\\
&\{\pi_n(u'_n\times(u'_n\times(\Delta u'_n - \pi_n\nabla\phi(u'_n))))\}_n,
\end{aligned}$$

exist and are equal respectively to

$$\begin{aligned}
&u'\times(\Delta u' - \nabla\phi(u')),\\
&u'\times(u'\times(\Delta u' - \nabla\phi(u'))),\\
&u'\times(u'\times(\Delta u' - \nabla\phi(u'))).
\end{aligned}$$

By inequalities (6.3)-(6.5), the first sequence is bounded in $L^{2r}(\Omega';L^2(0,T;\mathbb{L}^2))$ for $r\geq 1$, the second sequence is bounded in $L^2(\Omega';L^2(0,T;\mathbb{L}^{\frac{3}{2}}))$ and the third sequence is bounded in $L^2(\Omega';L^2(0,T;X^{-\beta}))$. And since the Banach spaces $L^{2r}(\Omega';L^2(0,T;\mathbb{L}^2))$, $L^2(\Omega';L^2(0,T;\mathbb{L}^{\frac{3}{2}}))$ and $L^2(\Omega';L^2(0,T;X^{-\beta}))$ are all reflexive, by the Banach-Alaoglu Theorem, there exist subsequences weakly convergent. So we can assume that there exist

$$\begin{aligned}
&Y\in L^{2r}(\Omega';L^2(0,T;\mathbb{L}^2)),\\
&Z\in L^2(\Omega';L^2(0,T;\mathbb{L}^{\frac{3}{2}})),\\
&Z_1\in L^2(\Omega';L^2(0,T;X^{-\beta})),
\end{aligned}$$

such that

$$u'_n\times(\Delta u'_n - \pi_n\nabla\phi(u'_n))\to Y\quad\text{weakly in } L^{2r}(\Omega';L^2(0,T;\mathbb{L}^2)),\tag{6.12}$$

$$u'_n\times(u'_n\times(\Delta u'_n - \pi_n\nabla\phi(u'_n)))\to Z\quad\text{weakly in } L^2(\Omega';L^2(0,T;\mathbb{L}^{\frac{3}{2}})),\tag{6.13}$$

$$\pi_n\,(u'_n\times(u'_n\times(\Delta u'_n - \pi_n\nabla\phi(u'_n))))\to Z_1\quad\text{weakly in } L^2(\Omega';L^2(0,T;X^{-\beta})).\tag{6.14}$$

*Remark.* Similar argument has been done in [9] for terms not involving $\nabla\phi$. Our main contribution here is to show the validity of such an argument for term containing $\nabla\phi$ (and to be more precise). This works because earlier, see Lemma 6.6, we have been able to prove generalized estimates as in [9].

**Proposition 6.14.** *If $Z$ and $Z_1$ defined as above, then $Z = Z_1 \in L^2(\Omega'; L^2(0,T; X^{-\beta}))$.*

*Proof.* Since $(L^{\frac{3}{2}})^* = L^3$, $X^\beta = \mathbb{H}^{2\beta}$ and $X^\beta \subset L^3$ (as $\beta > \frac{1}{4}$), we infer that $L^{\frac{3}{2}} \subset X^{-\beta}$. Hence

$$L^2(\Omega'; L^2(0,T; \mathbb{L}^{\frac{3}{2}})) \subset L^2(\Omega'; L^2(0,T; X^{-\beta}))$$

and thus $Z \in L^2(\Omega'; L^2(0,T; X^{-\beta}))$ and $Z_1 \in L^2(\Omega'; L^2(0,T; X^{-\beta}))$.
Recall that $X^\beta = D(A_1^\beta)$ and let $X_k^\beta = H_k$ with the norm inherited from $X^\beta$. Then $\bigcup_{k=1}^\infty X_k^\beta$ is dense $X^\beta$ and thus $\bigcup_{k=1}^\infty L^2(\Omega'; L^2(0,T; X_k^\beta))$ is dense in $L^2(\Omega'; L^2(0,T; X^\beta))$. Thus it is sufficient to prove that for any $\psi \in L^2(\Omega'; L^2(0,T; X_k^\beta))$,

$$_{L^2(\Omega';L^2(0,T;X^{-\beta}))}\langle Z_1, \psi\rangle_{L^2(\Omega';L^2(0,T;X^\beta))} = {}_{L^2(\Omega';L^2(0,T;X^{-\beta}))}\langle Z, \psi\rangle_{L^2(\Omega';L^2(0,T;X^\beta))}. \tag{6.15}$$

For this aim let us fix $k, n \in \mathbb{N}$ and $\psi \in L^2(\Omega'; L^2(0,T; X_k^\beta))$. Then we have

$$\begin{aligned}
&{}_{L^2(\Omega';L^2(0,T;X^{-\beta}))}\langle \pi_n(u'_n \times (u'_n \times (\Delta u'_n - \pi_n \nabla\phi(u'_n)))), \psi\rangle_{L^2(\Omega';L^2(0,T;X^\beta))} \\
&= \mathbb{E}' \int_0^T {}_{X^{-\beta}}\langle \pi_n(u'_n(t) \times (u'_n(t) \times (\Delta u'_n(t) - \pi_n \nabla\phi(u'_n(t))))), \psi(t)\rangle_{X^\beta}\, \mathrm{d}t \\
&= \mathbb{E}' \int_0^T {}_{\mathbb{L}^2}\langle \pi_n(u'_n(t) \times (u'_n(t) \times (\Delta u'_n(t) - \pi_n \nabla\phi(u'_n(t))))), \psi(t)\rangle_{\mathbb{L}^2}\, \mathrm{d}t \\
&= \mathbb{E}' \int_0^T {}_{\mathbb{L}^2}\langle u'_n(t) \times (u'_n(t) \times (\Delta u'_n(t) - \pi_n \nabla\phi(u'_n(t)))), \psi(t)\rangle_{\mathbb{L}^2}\, \mathrm{d}t \\
&= \mathbb{E}' \int_0^T {}_{X^{-\beta}}\langle u'_n(t) \times (u'_n(t) \times (\Delta u'_n(t) - \pi_n \nabla\phi(u'_n(t)))), \psi(t)\rangle_{X^\beta}\, \mathrm{d}t \\
&= {}_{L^2(\Omega';L^2(0,T;X^{-\beta}))}\langle u'_n \times (u'_n \times (\Delta u'_n(t) - \pi_n \nabla\phi(u'_n(t)))), \psi\rangle_{L^2(\Omega';L^2(0,T;X^\beta))}.
\end{aligned}$$

Hence by (6.13) and (6.14) we get (6.15) as required and the proof is complete.

**Lemma 6.15.** *For any measurable process $\psi \in L^4(\Omega'; L^4(0,T; \mathbb{W}^{1,4}))$, we have equality*

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}' \int_0^T \langle u'_n(t) \times (\Delta u'_n - \pi_n \nabla\phi(u'_n(t))), \psi(t)\rangle_{\mathbb{L}^2}\, \mathrm{d}t = \mathbb{E}' \int_0^T \langle Y(t), \psi(t)\rangle_{\mathbb{L}^2}\, \mathrm{d}t \\
&= \mathbb{E}' \int_0^T \sum_{i=1}^3 \langle \frac{\partial u'(t)}{\partial x_i}, u'(t) \times \frac{\partial \psi(t)}{\partial x_i}\rangle_{\mathbb{L}^2}\, \mathrm{d}s - \mathbb{E}' \int_0^T \langle u'(t) \times \nabla\phi(u'(t)), \psi\rangle_{\mathbb{L}^2}\, \mathrm{d}t.
\end{aligned}$$

*Proof.* Let us fix $\psi \in L^4(\Omega'; L^4(0,T; \mathbb{W}^{1,4}))$. Firstly, we will prove that

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \langle u'_n(t) \times \Delta u'_n(t), \psi(t)\rangle_{\mathbb{L}^2}\, \mathrm{d}t = \mathbb{E}' \int_0^T \sum_{i=1}^3 \left\langle \frac{\partial u'(t)}{\partial x_i}, u'(t) \times \frac{\partial \psi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \mathrm{d}t. \tag{6.16}$$

For each $n \in \mathbb{N}$ we have

$$\langle u_n'(t) \times \Delta u_n'(t), \psi \rangle_{\mathbb{L}^2} = \sum_{i=1}^{3} \left\langle \frac{\partial u_n'(t)}{\partial x_i}, u_n'(t) \times \frac{\partial \psi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \tag{6.17}$$

for almost every $t \in [0,T]$ and $\mathbb{P}'$ almost surely. Since by Corollary 6.5, $\mathbb{P}(u_n' \in C([0,T];H_n)) = 1$, we infer that for each $i \in \{1,2,3\}$ we can write

$$\begin{aligned}
&\left\langle \frac{\partial u_n'(t)}{\partial x_i}, u_n'(t) \times \frac{\partial \psi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} - \left\langle \frac{\partial u'(t)}{\partial x_i}, u'(t) \times \frac{\partial \psi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \\
&= \left\langle \frac{\partial u_n'(t)}{\partial x_i} - \frac{\partial u'(t)}{\partial x_i}, u'(t) \times \frac{\partial \psi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} + \left\langle \frac{\partial u_n'(t)}{\partial x_i}, (u_n'(t) - u'(t)) \times \frac{\partial \psi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2}.
\end{aligned} \tag{6.18}$$

Since $\mathbb{L}^4 \hookrightarrow \mathbb{L}^2$ and $\mathbb{W}^{1,4} \hookrightarrow \mathbb{L}^2$, there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
&\mathbb{E}' \int_0^T \left| \left\langle \frac{\partial u_n'(t)}{\partial x_i}, (u_n'(t) - u'(t)) \times \frac{\partial \psi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \right| \mathrm{d}t \\
&\leq C_1 \mathbb{E}' \int_0^T \|u_n'(t)\|_{\mathbb{H}^1} \|u_n'(t) - u'(t)\|_{\mathbb{L}^4} \|\psi(t)\|_{\mathbb{W}^{1,4}} \, \mathrm{d}t.
\end{aligned}$$

Moreover by the Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}' \int_0^T \|u_n'(t)\|_{\mathbb{H}^1} \|u_n'(t) - u'(t)\|_{\mathbb{L}^4} \|\psi(t)\|_{\mathbb{W}^{1,4}} \, \mathrm{d}t \\
&\leq T^{\frac{1}{2}} \left( \mathbb{E}' \sup_{t\in[0,T]} \|u_n'(t)\|_{\mathbb{H}^1}^2 \right)^{\frac{1}{2}} \left( \mathbb{E}' \int_0^T \|u_n'(t) - u'(t)\|_{\mathbb{L}^4}^4 \, \mathrm{d}t \right)^{\frac{1}{4}} \left( \mathbb{E}' \int_0^T \|\psi(t)\|_{\mathbb{W}^{1,4}}^4 \, \mathrm{d}t \right)^{\frac{1}{4}}.
\end{aligned}$$

Hence, by (6.2), (6.9) we infer that

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \left| \left\langle \frac{\partial u_n'(t)}{\partial x_i}, (u_n'(t) - u'(t)) \times \frac{\partial \psi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \right| \mathrm{d}t = 0. \tag{6.19}$$

Since both $u'$ and $\frac{\partial \psi}{\partial x_i}$ belong to $L^4(\Omega'; L^4(0,T;\mathbb{L}^4))$, so that $u' \times \frac{\partial \psi}{\partial x_i} \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$, by (6.10) we have

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \left\langle \frac{\partial u_n'(t)}{\partial x_i} - \frac{\partial u'(t)}{\partial x_i}, u'(t) \times \frac{\partial \psi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \mathrm{d}t = 0. \tag{6.20}$$

Therefore by (6.18), (6.19), (6.20), we infer that

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \left\langle \frac{\partial u_n'(t)}{\partial x_i}, u_n'(t) \times \frac{\partial \psi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \mathrm{d}t = \mathbb{E}' \int_0^T \left\langle \frac{\partial u'(t)}{\partial x_i}, u'(t) \times \frac{\partial \psi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \mathrm{d}t \tag{6.21}$$

and consequently by (6.17), we arrive at (6.16).

Secondly, we will show that

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \langle u_n'(t) \times \pi_n \nabla\phi(u_n'(t)), \psi\rangle_{\mathbb{L}^2} \, \mathrm{d}t = \mathbb{E}' \int_0^T \langle u'(t) \times \nabla\phi(u'(t)), \psi\rangle_{\mathbb{L}^2} \, \mathrm{d}t. \tag{6.22}$$

Since

$$\begin{aligned}
&\left|\langle u_n'(t) \times \pi_n \nabla\phi(u_n'(t)), \psi\rangle_{\mathbb{L}^2} - \langle u'(t) \times \nabla\phi(u'(t)), \psi\rangle_{\mathbb{L}^2}\right| \\
&\quad \leq \; \|\psi\|_{\mathbb{L}^2} \|u_n'(t) - u'(t)\|_{\mathbb{L}^2} \|\nabla\phi(u_n'(t))\|_{\mathbb{L}^2} + \|\psi\|_{\mathbb{L}^2} \|u'(t)\|_{\mathbb{L}^2} \|\pi_n \nabla\phi(u_n'(t)) - \nabla\phi(u'(t))\|_{\mathbb{L}^2},
\end{aligned}$$

we have

$$\begin{aligned}
&\left|\mathbb{E}' \int_0^T \langle u_n'(t) \times \pi_n \nabla\phi(u_n'(t)), \psi\rangle_{\mathbb{L}^2} \, \mathrm{d}t - \mathbb{E}' \int_0^T \langle u'(t) \times \nabla\phi(u'(t)), \psi\rangle_{\mathbb{L}^2} \, \mathrm{d}t\right| \\
&\quad \leq \; \left(\mathbb{E}' \int_0^T \|\psi\|_{\mathbb{L}^{1,4}}^4 \, \mathrm{d}t\right)^{\frac{1}{4}} \left(\mathbb{E}' \int_0^T \|u_n'(t) - u'(t)\|_{\mathbb{L}^4}^4 \, \mathrm{d}t\right)^{\frac{1}{4}} \left(\mathbb{E}' \int_0^T \|\nabla\phi(u_n'(t))\|_{\mathbb{L}^2}^2 \, \mathrm{d}t\right)^{\frac{1}{2}} \\
&\qquad + \left(\mathbb{E}' \int_0^T \|\psi\|_{\mathbb{W}^{1,4}}^4 \, \mathrm{d}t\right)^{\frac{1}{4}} \left(\mathbb{E}' \int_0^T \|u'(t)\|_{\mathbb{L}^4}^4 \, \mathrm{d}t\right)^{\frac{1}{4}} \left(\mathbb{E}' \int_0^T \|\pi_n \nabla\phi(u_n'(t)) - \nabla\phi(u'(t))\|_{\mathbb{L}^2}^2 \, \mathrm{d}t\right)^{\frac{1}{4}} \to 0.
\end{aligned}$$

Thus, in order to prove (6.22) we need to prove that

$$\mathbb{E}' \int_0^T \|\pi_n \nabla\phi(u_n'(t)) - \nabla\phi(u'(t))\|_{\mathbb{L}^2}^2 \, \mathrm{d}t \to 0 \tag{6.23}$$

For this aim, we note that since $\nabla\phi$ is global Lipschitz, there exists a constant $C$ such that

$$\begin{aligned}
&\left(\mathbb{E}' \int_0^T \|\pi_n \nabla\phi(u_n'(t)) - \nabla\phi(u'(t))\|_{\mathbb{L}^2}^2 \, \mathrm{d}t\right)^{\frac{1}{2}} \\
&\quad \leq \; \left(\mathbb{E}' \int_0^T \|\pi_n \nabla\phi(u_n'(t)) - \pi_n \nabla\phi(u'(t))\|_{\mathbb{L}^2}^2 \, \mathrm{d}t\right)^{\frac{1}{2}} + \left(\mathbb{E}' \int_0^T \|\pi_n \nabla\phi(u'(t)) - \nabla\phi(u'(t))\|_{\mathbb{L}^2}^2 \, \mathrm{d}t\right)^{\frac{1}{2}} \\
&\quad \leq \; C\left(\mathbb{E}' \int_0^T \|u_n'(t) - u'(t)\|_{\mathbb{L}^2}^2 \, \mathrm{d}t\right)^{\frac{1}{2}} + \left(\mathbb{E}' \int_0^T \|\pi_n \nabla\phi(u'(t)) - \nabla\phi(u'(t))\|_{\mathbb{L}^2}^2 \, \mathrm{d}t\right)^{\frac{1}{2}}.
\end{aligned}$$

By (6.9), the first term on the right hand side of above inequality converges to 0. And since $\|\pi_n \nabla\phi(u'(t)) - \nabla\phi(u'(t))\|_{\mathbb{L}^2}^2 \to 0$ for almost every $(t, \omega) \in [0, T] \times \Omega$, and since $\nabla\phi$ is bounded, $\|\pi_n \nabla\phi(u'(t)) - \nabla\phi(u'(t))\|_{\mathbb{L}^2}^2$ is uniformly integrable, hence the second term of right hand side also converges to 0 as $n \to \infty$. This proves (6.23) and consequently also (6.22).

Therefore by equalities (6.16) and (6.22), we have

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}'\int_0^T \langle u_n'(t)\times[\Delta u_n'(t)-\pi_n\nabla\phi(u_n'(t))],\psi(t)\rangle_{\mathbb{L}^2}\,\mathrm{d}t \qquad (6.24)\\
=\;& \mathbb{E}'\int_0^T\sum_{i=1}^3\left\langle\frac{\partial u'(t)}{\partial x_i},u'(t)\times\frac{\partial\psi(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2}\mathrm{d}t+\mathbb{E}'\int_0^T\langle u'(t)\times\nabla\phi(u'(t)),\psi\rangle_{\mathbb{L}^2}\,\mathrm{d}t.
\end{aligned}$$

Moreover, by (6.12), for every $\psi\in L^2(\Omega';L^2(0,T;\mathbb{L}^2))$,

$$\lim_{n\to\infty}\mathbb{E}'\int_0^T\langle u_n'(t)\times(\Delta u_n'(t)-\pi_n\nabla\phi(u_n'(t))),\psi\rangle_{\mathbb{L}^2}\,\mathrm{d}t=\mathbb{E}'\int_0^T\langle Y(t),\psi\rangle_{\mathbb{L}^2}\,\mathrm{d}t. \qquad (6.25)$$

Hence by (6.24) and (6.25), we deduce that

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}'\int_0^T\langle u_n'(t)\times(\Delta u_n'(t)-\pi_n\nabla\phi(u_n'(t))),\psi(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}t\\
=\;&\mathbb{E}'\int_0^T\langle Y(t),\psi(t)\rangle_{\mathbb{L}^2}\,\mathrm{d}t\\
=\;&\mathbb{E}'\int_0^T\sum_{i=1}^3\langle\frac{\partial u'(t)}{\partial x_i},u'(t)\times\frac{\partial\psi(t)}{\partial x_i}\rangle_{\mathbb{L}^2}\,\mathrm{d}t+\mathbb{E}'\int_0^T\langle u'(t)\times\nabla\phi(u'(t)),\psi(t)\rangle_{\mathbb{L}^2}\,\mathrm{d}t.
\end{aligned}$$

This completes the proof of Lemma 6.15.

**Lemma 6.16.** *For any process $\psi\in L^4(\Omega';L^4(0,T;\mathbb{L}^4))$ we have*

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}'\int_0^T {}_{\mathbb{L}^{\frac32}}\langle u_n'(s)\times(u_n'(s)\times(\Delta u_n'-\pi_n\nabla\phi(u_n'(t)))),\psi(s)\rangle_{\mathbb{L}^3}\,\mathrm{d}s\\
&\quad=\mathbb{E}'\int_0^T {}_{\mathbb{L}^{\frac32}}\langle Z(s),\psi(s)\rangle_{\mathbb{L}^3}\,\mathrm{d}s \qquad (6.26)\\
&\quad=\mathbb{E}'\int_0^T {}_{\mathbb{L}^{\frac32}}\langle u'(s)\times Y(s),\psi(s)\rangle_{\mathbb{L}^3}\,\mathrm{d}s. \qquad (6.27)
\end{aligned}$$

*Proof.* Let us take $\psi\in L^4(\Omega';L^4(0,T;\mathbb{L}^4))$. For $n\in\mathbb{N}$, put $Y_n:=u_n'\times(\Delta u_n'+\nabla\phi(u_n'))$. Since $L^4(\Omega';L^4(0,T;\mathbb{L}^4))\subset L^2(\Omega';L^2(0,T;\mathbb{L}^3))=\left[L^2(\Omega';L^2(0,T;\mathbb{L}^{\frac32}))\right]'$, we deduce that (6.13) implies that (6.26) holds.

So it remains to prove equality (6.27). Since by the Hölder's inequality

$$\begin{aligned}
\|\psi\times u'\|_{\mathbb{L}^2}^2 &= \int_D|\psi(x)\times u'(x)|^2\,\mathrm{d}x\le\int_D|\psi(x)|^2|u'(x)|^2\,\mathrm{d}x\\
&\le\|\psi\|_{\mathbb{L}^4}^2\|u'\|_{\mathbb{L}^4}^2\le\|\psi\|_{\mathbb{L}^4}^4+\|u'\|_{\mathbb{L}^4}^4.
\end{aligned}$$

And since by (6.9), $u' \in L^4(\Omega'; L^4(0,T;\mathbb{L}^4))$, we infer that

$$\mathbb{E}' \int_0^T \|\psi \times u'\|_{\mathbb{L}^2}^2 \, \mathrm{d}t \leq \mathbb{E}' \int_0^T \|\psi\|_{\mathbb{L}^4}^4 \, \mathrm{d}t + \mathbb{E}' \int_0^T \|u'\|_{\mathbb{L}^4}^4 \, \mathrm{d}t < \infty.$$

This proves that $\psi \times u' \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$ and similarly $\psi \times u'_n \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$.
Thus since by (6.12), $Y_n \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$, we infer that

$$\begin{aligned} {}_{\mathbb{L}^{\frac{3}{2}}}\langle u'_n \times Y_n, \psi \rangle_{\mathbb{L}^3} &= \int_D \langle u'_n(x) \times Y_n(x), \psi(x) \rangle \, \mathrm{d}x \\ &= \int_D \langle Y_n(x), \psi(x) \times u'_n(x) \rangle \, \mathrm{d}x = \langle Y_n, \psi \times u'_n \rangle_{\mathbb{L}^2}. \end{aligned} \tag{6.28}$$

Similarly, since by (6.12), $Y \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$, we have

$$\begin{aligned} {}_{\mathbb{L}^{\frac{3}{2}}}\langle u' \times Y, \psi \rangle_{\mathbb{L}^3} &= \int_D \langle u'(x) \times Y(x), \psi(x) \rangle \, \mathrm{d}x \\ = \int_D \langle Y(x), \psi(x) \times u'(x) \rangle \, \mathrm{d}x &= \langle Y, \psi \times u' \rangle_{\mathbb{L}^2}. \end{aligned} \tag{6.29}$$

Thus by (6.28) and (6.29), we get

$$\begin{aligned} {}_{\mathbb{L}^{\frac{3}{2}}}\langle u'_n \times Y_n, \psi \rangle_{\mathbb{L}^3} - {}_{\mathbb{L}^{\frac{3}{2}}}\langle u' \times Y, \psi \rangle_{\mathbb{L}^3} &= \langle Y_n, \psi \times u'_n \rangle_{\mathbb{L}^2} - \langle Y, \psi \times u' \rangle_{\mathbb{L}^2} \\ &= \langle Y_n - Y, \psi \times u' \rangle_{\mathbb{L}^2} + \langle Y_n, \psi \times (u'_n - u') \rangle_{\mathbb{L}^2}. \end{aligned}$$

In order to prove (6.27), we are aiming to prove that the expectation of the left hand side of the above equality goes to 0 as $n \to \infty$. By (6.12), since $\psi \times u' \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$,

$$\lim_{n \to \infty} \mathbb{E}' \int_0^T \langle Y_n(s) - Y(s), \psi(s) \times u'(s) \rangle_{\mathbb{L}^2} \, \mathrm{d}s = 0.$$

By the Cauchy-Schwartz inequality and equation (6.9), we have

$$\begin{aligned} &\mathbb{E}' \int_0^T \langle Y_n(s), \psi(s) \times (u'_n(s) - u'(s)) \rangle_{\mathbb{L}^2}^2 \, \mathrm{d}s \leq \mathbb{E}' \int_0^T \|Y_n(s)\|_{\mathbb{L}^2}^2 \|\psi(s) \times (u'_n(s) - u'(s))\|_{\mathbb{L}^2}^2 \, \mathrm{d}s \\ &\leq \mathbb{E}' \int_0^T \left\|Y_n(s)\right\|_{\mathbb{L}^2} \left\|\psi(s)\right\|_{\mathbb{L}^4} \left\|u'_n(s) - u'(s)\right\|_{\mathbb{L}^4} \mathrm{d}s \\ &\leq \left(\mathbb{E}' \int_0^T \left\|Y_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^{\frac{1}{2}} \left(\mathbb{E}' \int_0^T \left\|\psi(s)\right\|_{\mathbb{L}^4}^4 \mathrm{d}s\right)^{\frac{1}{4}} \left(\mathbb{E}' \int_0^T \left\|u'_n(s) - u'(s)\right\|_{\mathbb{L}^4}^4 \mathrm{d}s\right)^{\frac{1}{4}} \to 0. \end{aligned}$$

Therefore, we infer that

$$\lim_{n\to\infty}\mathbb{E}'\int_0^T {}_{\mathbb{L}^{\frac{3}{2}}}\langle u_n'(s)\times(u_n'(s)\times\Delta u_n'(s)),\psi(s)\rangle_{\mathbb{L}^3}\,\mathrm{d}s=\mathbb{E}'\int_0^T {}_{\mathbb{L}^{\frac{3}{2}}}\langle u'(s)\times Y(s),\psi(s)\rangle_{\mathbb{L}^3}\,\mathrm{d}s.$$

This completes the proof of Lemma 6.16.

The next result will be used, see Theorem 8.1, to show that the process $u'$ satisfies the condition $|u'(t,x)|_{\mathbb{R}^3}=1$ for all $t\in[0,T]$, $x\in D$ and $\mathbb{P}'$-almost surely.

**Lemma 6.17.** *For any bounded measurable function $\psi:D\to\mathbb{R}$ we have*

$$\langle Y(s,\omega),\psi u'(s,\omega)\rangle_{\mathbb{L}^2}=0,$$

*for almost every $(s,\omega)\in[0,T]\times\Omega'$.*

*Proof.* Let $B\subset[0,T]\times\Omega'$ be an arbitrary progressively measurable set.

$$\begin{aligned}
&\left|\mathbb{E}'\int_0^T 1_B(s)\,\langle u_n'(s)\times(\Delta u_n'(s)-\pi_n\nabla\phi(u_n'(s)))\,,\psi u_n'(s)\rangle_{\mathbb{L}^2}\ \mathrm{d}s-\mathbb{E}'\int_0^T 1_B(s)\langle Y(s),\psi u'(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}s\right|\\
\leq\ &\left|\mathbb{E}'\int_0^T 1_B(s)\,\langle u_n'(s)\times(\Delta u_n'(s)-\pi_n\nabla\phi(u_n'(s)))\,,\psi(u_n'(s)-u'(s))\rangle_{\mathbb{L}^2}\ \mathrm{d}s\right|\\
&+\left|\mathbb{E}'\int_0^T 1_B(s)\,\langle u_n'(s)\times(\Delta u_n'(s)-\pi_n\nabla\phi(u_n'(s)))-Y(s),\psi u'(s)\rangle_{\mathbb{L}^2}\ \mathrm{d}s\right|.
\end{aligned}$$

Next we will show that both terms in the right hand side of the above inequality will converge to 0. For the first term, by the boundness of $\psi$, (6.3) and (6.9), we have

$$\begin{aligned}
&\left|\mathbb{E}'\int_0^T 1_B(s)\,\langle u_n'(s)\times(\Delta u_n'(s)-\pi_n\nabla\phi(u_n'(s)))\,,\psi(u_n'(s)-u'(s))\rangle_{\mathbb{L}^2}\ \mathrm{d}s\right|\\
\leq\ &\mathbb{E}'\int_0^T\left\|u_n'(s)\times(\Delta u_n'(s)-\pi_n\nabla\phi(u_n'(s)))\,\psi\right\|_{\mathbb{L}^2}\left\|u_n'(s)-u'(s)\right\|_{\mathbb{L}^2}\ \mathrm{d}s\\
\leq\ &\left\|u_n'(s)\times(\Delta u_n'(s)-\pi_n\nabla\phi(u_n'(s)))\,\psi\right\|_{L^2(\Omega';L^2(0,T;\mathbb{L}^2))}\left\|u_n'-u'\right\|_{L^2(\Omega';L^2(0,T;\mathbb{L}^2))}\to 0.
\end{aligned}$$

For the second term, since $1_B\psi u'\in L^2(\Omega';L^2(0,T;\mathbb{L}^2))$, by (6.9) and (6.12), we have

$$\left|\mathbb{E}'\int_0^T 1_B(s)\,\langle u_n'(s)\times(\Delta u_n'(s)-\pi_n\nabla\phi(u_n'(s)))-Y(s),\psi u'(s)\rangle_{\mathbb{L}^2}\ \mathrm{d}s\right|\to 0.$$

Therefore we infer that

$$\begin{aligned}
0\ &=\ \lim_{n\to\infty}\mathbb{E}'\int_0^T 1_B(s)\,\langle u_n'(s)\times(\Delta u_n'(s)-\pi_n\nabla\phi(u_n'(s)))\,,\psi u_n'(s)\rangle_{\mathbb{L}^2}\ \mathrm{d}s\\
&=\ \mathbb{E}'\int_0^T 1_B(s)\langle Y(s),\psi u'(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}s,
\end{aligned}$$

where the first equality follows from the fact that $\langle a \times b, a\rangle = 0$. By the arbitrariness of $B$, this concludes the proof of Lemma 6.17.

## 7 Conclusion of the proof of the existence of a weak solution

Our aim in this section is to prove that the process $u'$ from Proposition 6.2 is a weak solution of equation (2.1) according to the definition 2.5. Because the argument is quite analogous to the one in [9] we will try to omit the details leaving only the structure of the proof.

First we define a sequence of $\mathbb{L}^2$-valued process $(M_n(t))_{t\in[0,T]}$ on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\begin{aligned} M_n(t) \quad := \quad & u_n(t) - u_n(0) - \lambda_1 \int_0^t \pi_n\left(u_n(s) \times (\Delta u_n(s) - \pi_n \nabla\phi(u_n(s)))\right) \mathrm{d}s \\ & + \lambda_2 \int_0^t \pi_n\left(u_n(s) \times (u_n(s) \times (\Delta u_n(s) - \pi_n \nabla\phi(u_n(s))))\right) \mathrm{d}s \\ & - \frac{1}{2} \sum_{j=1}^N \int_0^t \pi_n\Big((\pi_n(u_n(s) \times h_j)) \times h_j\Big) \mathrm{d}s. \end{aligned} \tag{7.1}$$

Since $u_n$ is the solution of the Equation (3.5), we infer that

$$M_n(t) = \sum_{j=1}^N \int_0^t \pi_n(u_n(s) \times h_j)\, \mathrm{d}W_j(s), \qquad t \in [0, T]. \tag{7.2}$$

The proof $u'$ is a weak solution of the Equation (2.1) is 2 steps:

**Step 1** : Define a process $M'(t)$ by formula (7.1), but with $u'$ instead of $u_n$.

**Step 2** : Prove equality (7.2) but with $u'$ instead of $u_n$ and $W_j'$ instead of $W_j$.

### 7.1 Step 1

We define a sequence of $\mathbb{L}^2$-valued process $(M_n'(t))_{t\in[0,T]}$ on the new probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ by a formula similar as (7.1).

$$\begin{aligned} M_n'(t) \quad := \quad & u_n'(t) - u_n'(0) - \lambda_1 \int_0^t \pi_n\left(u_n'(s) \times (\Delta u_n'(s) - \pi_n \nabla\phi(u_n'(s)))\right) \mathrm{d}s \\ & + \lambda_2 \int_0^t \pi_n\left(u_n'(s) \times (u_n'(s) \times (\Delta u_n'(s) - \pi_n \nabla\phi(u_n'(s))))\right) \mathrm{d}s \\ & - \frac{1}{2} \sum_{j=1}^N \int_0^t \pi_n[(\pi_n(u_n'(s) \times h_j)) \times h_j]\, \mathrm{d}s. \end{aligned} \tag{7.3}$$

In the following result we show that the sequence $\{M_n'\}$ is convergent.

**Lemma 7.1.** *For each $t \in [0,T]$ the sequence of random variables $M'_n(t)$ is weakly convergent in $L^2(\Omega'; X^{-\beta})$ and it's limit $M'(t)$ satisfies the following equality.*

$$\begin{aligned} M'(t) \quad := \quad & u'(t) - u_0 - \lambda_1 \int_0^t (u'(s) \times (\Delta u'(s) - \nabla\phi(u'(s)))) \, \mathrm{d}s \\ & + \lambda_2 \int_0^t (u'(s) \times (u'(s) \times (\Delta u'(s) - \nabla\phi(u'(s))))) \, \mathrm{d}s \\ & - \frac{1}{2} \sum_{j=1}^N \int_0^t (u'(s) \times h_j) \times h \, \mathrm{d}s. \end{aligned}$$

*Proof.* Let $t \in (0,T]$ and $U \in L^2(\Omega'; X^\beta)$.

Since $u'_n \to u'$ in $C([0,T]; X^{-\beta})$ $\mathbb{P}'$-a.s. we infer that

$$\lim_{n\to\infty} {}_{X^{-\beta}}\langle u'_n(t), U\rangle_{X^\beta} = {}_{X^{-\beta}}\langle u'(t), U\rangle_{X^\beta}, \quad \mathbb{P}' - a.s.$$

Since $\mathbb{L}^2 \hookrightarrow X^{-\beta}$, by (6.1) there exists a constant $C$ such that

$$\sup_n \mathbb{E}'\left[\left|{}_{X^{-\beta}}\langle u'_n(t), U\rangle_{X^\beta}\right|^2\right] \le \sup_n \mathbb{E}'\|U\|^2_{X^\beta}\mathbb{E}'\|u'_n(t)\|^2_{X^{-\beta}} \le C\mathbb{E}'\|U\|^2_{X^\beta}\mathbb{E}'\|u_0\|^2_{\mathbb{L}^2} < \infty$$

and thus by the Vitali Theorem this implies that

$$\lim_{n\to\infty} \mathbb{E}'[{}_{X^{-\beta}}\langle u'_n(t), U\rangle_{X^\beta}] = \mathbb{E}'[{}_{X^{-\beta}}\langle u'(t), U\rangle_{X^\beta}].$$

By (6.12) and (6.14) we infer that

$$\lim_{n\to\infty} \mathbb{E}' \int_0^t \langle u'_n(s) \times (\Delta u'_n(s) - \pi_n \nabla\phi(u'_n(s))) \, , \pi_n U\rangle_{\mathbb{L}^2} \, \mathrm{d}s = \mathbb{E}' \int_0^t \langle Y(s), U\rangle_{\mathbb{L}^2}.$$

$$\lim_{n\to\infty} \mathbb{E}' \int_0^t {}_{X^{-\beta}}\langle \pi_n \left(u'_n(s) \times (u'_n(s) \times (\Delta u'_n(s) - \pi_n \nabla\phi(u'_n(s))))\right), U\rangle_{X^\beta} \, \mathrm{d}s = \mathbb{E}' \int_0^t \langle Z(s), U\rangle_{X^\beta} \, \mathrm{d}s.$$

Moreover, by the Hölder inequality and (6.9) we get

$$\begin{aligned} & \mathbb{E}' \int_0^t \left|\langle \pi_n((u'_n(s) - u'(s)) \times h_j) \times h_j, \pi_n U\rangle_{\mathbb{L}^2}\right| \mathrm{d}s \\ & \le \quad \|h_j\|^2_{\mathbb{L}^\infty} \|U\|_{L^2(\Omega'; L^2(0,T;\mathbb{L}^2))} \left(\mathbb{E}' \int_0^t \|u'_n(s) - u'(s)\|^4_{\mathbb{L}^4} \, \mathrm{d}s\right)^{\frac{1}{4}} t^{\frac{1}{4}} m(D)^{\frac{1}{4}} \to 0. \end{aligned}$$

Hence by Lemmata 6.15 and 6.16, we deduce that

$$\lim_{n\to\infty} {}_{L^2(\Omega';X^{-\beta})}\langle M'_n(t), U\rangle_{L^2(\Omega';X^\beta)} = {}_{L^2(\Omega';X^{-\beta})}\langle M'(t), U\rangle_{L^2(\Omega';X^\beta)}.$$

This concludes the proof of Lemma 7.1.

Before we can continue with the proof that $u'$ is the weak solution of equation (2.1), we need to establish that the processes $W'$ and $W'_n$ from Proposition 6.2 are Brownian Motions. This will be stated in Lemmata 7.2 and 7.3, which can be proved as in [9]. The proofs however will be omitted.

**Lemma 7.2.** *Suppose the $W'_n$ defined in $(\Omega', \mathcal{F}', \mathbb{P}')$ has the same distribution as the Brownian Motion $W$ defined in $(\Omega, \mathcal{F}, \mathbb{P})$ as in Proposition 6.2. Then $W'_n$ is also a Brownian Motion.*

**Lemma 7.3.** *The process $(W'(t))_{t\in[0,T]}$ is a real-valued Brownian Motion on $(\Omega', \mathcal{F}', \mathbb{P}')$ and if $0 \leq s < t \leq T$ then the increment $W'(t) - W'(s)$ is independent of the $\sigma$-algebra generated by $u'(r)$ and $W'(r)$ for $r \in [0, s]$.*

*Remark* 7.4. We will denote $\mathbb{F}'$ the filtration generated by $(u', W')$ and $\mathbb{F}'_n$ the filtration generated by $(u'_n, W'_n)$. Then by Lemma 7.3, $u'$ is progressively measurable with respect to $\mathbb{F}'$ and by Lemma 7.2, $u'_n$ is progressively measurable with respect to $\mathbb{F}'_n$.

## 7.2 Step 2

Let us summarize what we have achieved so far. We have got our process $M'$ and have shown $W'$ is a Wiener process. Next we will show a similar result as in equation (7.2) to prove $u'$ is a weak solution of the Equation (2.1). But before that we still need some preparation.

In what follows we assume that $\beta > \frac{1}{4}$. The following result is needed to prove Lemma 7.6.

**Proposition 7.5.** *If $h \in \mathbb{L}^\infty \cap \mathbb{W}^{1,3}$, then there exists $c_h > 0$: for every $u \in X^\beta$, $u \times h \in X^{-\beta}$ and*

$$\|u \times h\|_{X^{-\beta}} \leq c_h \|u\|_{X^{-\beta}} < \infty. \tag{7.4}$$

*Proof.* Let us fix $h \in \mathbb{L}^\infty \cap \mathbb{W}^{1,3}$. Then there exists $c > 0$ such that for every $z \in \mathbb{H}^1$

$$\begin{aligned} \|z \times h\|^2_{\mathbb{H}^1} &= \|\nabla(z \times h)\|^2_{\mathbb{L}^2} + \|z \times h\|^2_{\mathbb{L}^2} \leq 2(\|\nabla z \times h\|^2_{\mathbb{L}^2} + \|z \times \nabla h\|^2_{\mathbb{L}^2}) + \|z \times h\|^2_{\mathbb{L}^2} \\ &\leq 2(\|h\|^2_{\mathbb{L}^\infty}\|\nabla z\|^2_{\mathbb{L}^2} + \|\nabla h\|^2_{\mathbb{L}^3}\|z\|^2_{\mathbb{L}^6}) + \|h\|^2_{\mathbb{L}^\infty}\|z\|^2_{\mathbb{L}^2} \leq 2(\|h\|^2_{\mathbb{L}^\infty} + c^2\|\nabla h\|^2_{\mathbb{L}^3})\|z\|^2_{\mathbb{H}^1}. \end{aligned}$$

So the linear map $M_h : \mathbb{H}^1 \ni z \longmapsto z \times h \in \mathbb{H}^1$ is bounded. Since $M_h : \mathbb{L}^2 \to \mathbb{L}^2$ is also bounded and $X^\beta = [\mathbb{L}^2, \mathbb{H}^1]_\beta$, by the interpolation theorem we infer that $M_h : X^\beta \to X^\beta$ is bounded.

Next, let us fix $u \in \mathbb{L}^2 \subset X^{-\beta}$ and $z \in X^\beta$. Since $X^{-\beta}$ is equal to the dual space of $X^\beta$ we have

$$|\langle u \times h, z\rangle| = |\langle u, z \times h\rangle| \leq \|u\|_{X^{-\beta}}\|M_h(z)\|_{X^\beta} \leq c_h\|u\|_{X^{-\beta}}\|z\|_{X^\beta}.$$

By the density of $\mathbb{L}^2$ in $X^{-\beta}$ the above inequality holds for every $u \in X^{-\beta}$. In particular, for every $u \in X^{-\beta}$, $u \times h \in X^{-\beta}$ and inequality (7.4) holds. The proof is complete.

The proof of next Lemma is omitted because it is similar as part of the proof of Lemma 5.2 in Brzeźniak, Goldys and Jegaraj [9].

**Lemma 7.6.** *For each $m \in \mathbb{N}$, we define the partition $\left\{s_i^m := \frac{iT}{m},\ i = 0, \dots, m\right\}$ of $[0,T]$. Then for any $\varepsilon > 0$, there exists $m_0(\varepsilon) \in \mathbb{N}$ such that for all $m \geq m_0(\varepsilon)$, we have:*

*(i)*

$$\lim_{n\to\infty}\left(\mathbb{E}'\left[\left\|\sum_{j=1}^{N}\int_0^t\left(\pi_n(u_n'(s)\times h_j) - \sum_{i=0}^{m-1}\pi_n(u_n'(s_i^m)\times h_j)1_{(s_i^m, s_{i+1}^m]}(s)\right)\mathrm{d}W_{jn}'(s)\right\|_{X^{-\beta}}^2\right]\right)^{\frac{1}{2}} < \frac{\varepsilon}{2};$$

*(ii)*

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}'\Bigg[\Bigg\|&\sum_{i=0}^{m-1}\sum_{j=1}^{N}\pi_n(u_n'(s_i^m)\times h_j)(W_{jn}'(t\wedge s_{i+1}^m) - W_{jn}'(t\wedge s_i^m))\\ &-\sum_{i=0}^{m-1}\sum_{j=1}^{N}\pi_n(u'(s_i^m)\times h_j)(W_j'(t\wedge s_{i+1}^m) - W_j'(t\wedge s_i^m))\Bigg\|_{X^{-\beta}}^2\Bigg] = 0;\end{aligned}$$

*(iii)*

$$\lim_{n\to\infty}\left(\mathbb{E}'\left[\left\|\sum_{j=1}^{N}\int_0^t(\pi_n(u'(s)\times h_j) - \sum_{i=0}^{m-1}\pi_n(u'(s_i^m)\times h_j)1_{(s_i^m, s_{i+1}^m]}(s))\,\mathrm{d}W_j'(s)\right\|_{X^{-\beta}}^2\right]\right)^{\frac{1}{2}} < \frac{\varepsilon}{2};$$

*(iv)*

$$\lim_{n\to\infty}\mathbb{E}'\left[\left\|\sum_{j=1}^{N}\int_0^t(\pi_n(u'(s)\times h_j) - (u'(s)\times h_j))\,\mathrm{d}W_j'(s)\right\|_{X^{-\beta}}^2\right] = 0.$$

Now we are ready to state the Theorem which means that $u'$ is the weak solution of equation (2.1).

**Theorem 7.7.** *For each $t \in [0,T]$ we have $M'(t) = \sum_{j=1}^{N}\int_0^t(u'(s)\times h_j)\,\mathrm{d}W_j'(s)$.*

*Proof.* **Step 1:** We will show that

$$M_n'(t) = \sum_{j=1}^{N}\int_0^t \pi_n(u_n'(s)\times h_j)\,\mathrm{d}W_{jn}'(s) \tag{7.5}$$

$\mathbb{P}'$ almost surely for each $t \in [0,T]$ and $n \in \mathbb{N}$.

Let us fix that $t \in [0,T]$ and $n \in \mathbb{N}$. Let us also fix $m \in \mathbb{N}$ and define the partition $\left\{s_i^m := \frac{iT}{m}, i = 0, \dots, m\right\}$ of $[0,T]$. Let us recall that $(u_n', W_n')$ and $(u_n, W)$ have the same laws on the separable Banach space $C([0,T]; H_n)\times C([0,T];\mathbb{R}^N)$. Since the following map is continuous,

$$\begin{aligned}\Psi:\quad & C([0,T];H_n)\times C([0,T];\mathbb{R}^N) \to H_n\\ & (u_n, W) \longmapsto M_n(t) - \sum_{i=0}^{m-1}\sum_{j=1}^{N}\pi_n(u_n(s_i^m)\times h_j)(W_j(t\wedge s_{i+1}^m) - W_j(t\wedge s_i^m)),\end{aligned}$$

by invoking the Kuratowski Theorem we infer that the $\mathbb{L}^2$-valued random variables:

$$M_n(t) - \sum_{i=0}^{m-1} \sum_{j=1}^{N} \pi_n(u_n(s_i^m) \times h_j)(W_j(t \wedge s_{i+1}^m) - W_j(t \wedge s_i^m))$$

$$M_n'(t) - \sum_{j=0}^{m-1} \sum_{j=1}^{N} \pi_n(u_n'(s_i^m) \times h_j)(W_{jn}'(t \wedge s_{i+1}^m) - W_{jn}'(t \wedge s_j^m))$$

have the same laws. Let us denote $u_{n,m} := \sum_{i=0}^{m-1} u_n(s_i^m) 1_{[s_i^m, s_{i+1}^m)}$. By the Itô isometry, we have

$$\begin{aligned} & \left\| \sum_{i=0}^{m-1} \pi_n(u_n(s_i^m) \times h_j)(W_j(t \wedge s_{i+1}^m) - W_j(t \wedge s_i^m)) - \int_0^t \pi_n(u_n(s) \times h_j)\, \mathrm{d}W_j(s) \right\|^2_{L^2(\Omega;\mathbb{L}^2)} \qquad (7.6) \\ = \;& \mathbb{E} \left\| \int_0^t \Big( \pi_n(u_{n,m}(s) \times h_j) - \pi_n(u_n(s) \times h_j) \Big)\, \mathrm{d}W_j(s) \right\|^2_{\mathbb{L}^2} \le \|h_j\|^2_{\mathbb{L}^\infty} \mathbb{E} \int_0^t \|u_{n,m}(s) - u_n(s)\|^2_{\mathbb{L}^2}\, \mathrm{d}s. \end{aligned}$$

Since $u_n \in C([0,T]; H_n)$ $\mathbb{P}$-almost surely, we have

$$\lim_{m\to\infty} \int_0^t \|u_{n,m}(s) - u_n(s)\|^2_{\mathbb{L}^2}\, \mathrm{d}s = 0, \quad \mathbb{P} - a.s.. \qquad (7.7)$$

Moreover by equality (4.1), we infer that

$$\begin{aligned} \sup_m \mathbb{E} \left| \int_0^t \|u_{n,m}(s) - u_n(s)\|^2_{\mathbb{L}^2}\, \mathrm{d}s \right|^2 & \le \sup_m \mathbb{E} \left| \int_0^t \Big( 2\|u_{n,m}(s)\|^2_{\mathbb{L}^2} + 2\|u_n(s)\|^2_{\mathbb{L}^2} \Big)\, \mathrm{d}s \right|^2 \qquad (7.8) \\ & \le \; \mathbb{E} \left| 4\|u_0\|^2_{\mathbb{L}^2} T \right|^2 = 16 \|u_0\|^4_{\mathbb{L}^2} T^2 < \infty. \end{aligned}$$

By (7.8), we have $\int_0^t \|u_{n,m}(s) - u_n(s)\|^2_{\mathbb{L}^2}\, \mathrm{d}s$ is uniformly (with respect to $m$) integrable. Therefore by the uniform integrability and (7.7), we have

$$\lim_{m\to\infty} \mathbb{E} \int_0^t \|u_{n,m}(s) - u_n(s)\|^2_{\mathbb{L}^2}\, \mathrm{d}s = 0.$$

Then by above equality and (7.6), we have

$$\lim_{m\to\infty} \left\| \sum_{i=0}^{m-1} \pi_n(u_n(s_i^m) \times h_j)(W_j(t \wedge s_{i+1}^m) - W_j(t \wedge s_i^m)) - \int_0^t \pi_n(u_n(s) \times h_j)\, \mathrm{d}W_j(s) \right\|^2_{L^2(\Omega;\mathbb{L}^2)} = 0.$$

Similarly, because $u_n'$ satisfies the same conditions as $u_n$, we also get

$$\lim_{m\to\infty} \left\| \sum_{i=0}^{m-1} \pi_n(u_n'(s_i^m) \times h)(W_{jn}'(t \wedge s_{i+1}^m) - W_{jn}'(t \wedge s_i^m)) - \int_0^t \pi_n(u_n'(s) \times h_j)\, \mathrm{d}W_{jn}'(s) \right\|^2_{L^2(\Omega;\mathbb{L}^2)} = 0.$$

Hence, since the $L^2$ convergence implies the weak convergence, we infer that the random variables $M_n(t) - \sum_{j=1}^N \int_0^t \pi_n(u_n(s) \times h_j)\, \mathrm{d}W_j(s)$ and $M_n'(t) - \sum_{j=1}^N \int_0^t \pi_n(u_n'(s) \times h_j)\, \mathrm{d}W_{jn}'(s)$ have same laws. But $M_n(t) -$

$\sum_{j=1}^N \int_0^t \pi_n(u_n(s)\times h_j)\,\mathrm{d}W_j(s) = 0$ $\mathbb{P}$-almost surely, so (7.5) follows.

**Step 2:** From Lemma 7.6 and the Step 1, we infer that $M_n'(t)$ converges in $L^2(\Omega'; X^{-\beta})$ to $\sum_{j=1}^N \int_0^t (u'(s)\times h_j)\,\mathrm{d}W_j'(s)$ as $n\to\infty$. This completes the proof of Theorem 7.7.

Summarizing, it follows from Theorem 7.7 that for every $t\in[0,T]$ the following equation is satisfied in $L^2(\Omega'; X^{-\beta})$:

$$\begin{aligned} u'(t) = u_0 \quad &+ \quad \lambda_1 \int_0^t (u'\times(\Delta u' - \nabla\phi(u')))\,(s)\,\mathrm{d}s \\ &- \quad \lambda_2 \int_0^t u'(s)\times(u'\times(\Delta u' - \nabla\phi(u')))\,(s)\,\mathrm{d}s \\ &+ \quad \sum_{j=1}^N \int_0^t (u'(s)\times h_j)\circ\,\mathrm{d}W_j'(s). \end{aligned} \tag{7.9}$$

Hence by Definition 2.5, $u'$ is a weak solution of Equation (2.1).

## 8 Verification of the constraint condition

Now we will start to show some regularity of $u'$.

**Theorem 8.1.** *The process $u'$ from Proposition 6.2 satisfies:*

$$|u'(t,x)|_{\mathbb{R}^3} = 1, \quad \textit{for Lebesgue a.e. } (t,x)\in[0,T]\times D \textit{ and } \mathbb{P}' - a.s.. \tag{8.1}$$

To prove Theorem 8.1, we need to use [21, Theorem 1.2]. The proof similar to the proof of [9, property (2.11) ] and although we can add some missing details, the proof is omitted.

From Theorem 8.1 we can deduce the following result.

**Theorem 8.2.** *The process $u'$ from Proposition 6.2 satisfies: for every $t\in[0,T]$, in $L^2(\Omega';\mathbb{L}^2)$,*

$$\begin{aligned} u'(t) \quad = \quad & u_0 + \lambda_1 \int_0^t (u'\times(\Delta u' - \nabla\phi(u')))\,(s)\,\mathrm{d}s \\ & -\lambda_2 \int_0^t u'(s)\times(u'\times(\Delta u' - \nabla\phi(u')))\,(s)\,\mathrm{d}s \\ & +\sum_{j=1}^N \int_0^t (u'(s)\times h_j)\circ\,\mathrm{d}W_j'(s). \end{aligned} \tag{8.2}$$

*Proof.* It is enough to prove that the terms in equation (8.2) are in the space $L^2(\Omega';\mathbb{L}^2)$. For this aim let us note that by (6.12), Lemma 6.15 and (8.1),

$$\mathbb{E}'\left(\int_0^T \left\|(u'\times(\Delta u' - \nabla\phi(u')))\,(t)\right\|_{\mathbb{L}^2}^2\,\mathrm{d}t\right)^r < \infty, \quad r\geq 1. \tag{8.3}$$

$$\mathbb{E}' \int_0^T \left\| u'(t) \times (u' \times (\Delta u' - \nabla\phi(u')))(t) \right\|_{\mathbb{L}^2}^2 \, dt < \infty.$$

This completes the proof of Theorem 8.2.

**Theorem 8.3.** *The process $u'$ defined in Proposition 6.2 satisfies: for every $\alpha \in (0, \frac{1}{2})$,*

$$u' \in C^\alpha([0,T];\mathbb{L}^2), \qquad \mathbb{P}' - a.s.. \tag{8.4}$$

Proof of Theorem 8.3 follows from the Kolmogorov test, Jensen and Burkholder-Davis-Gundy inequalities, equation (7.9) and our estimates (8.3) and (6.6).

## A Some explanation

This Appendix aims to clarify the meaning of the process $\Lambda$ from Notation 6.11 and Lemma 6.10. And the explanation present here goes back to Visintin [24].

**Definition A.1.** Assume that $D \subset \mathbb{R}^d$, $d \leq 3$. Suppose that $M \in H^1(D)$. We say that $M \times \Delta M$ exists in the $L^2(D)$ sense (and write $M \times \Delta M \in L^2(D)$) iff there exists $B \in L^2(D)$ such that for every $u \in W^{1,3}(D)$,

$$\langle B, u\rangle_{\mathbb{L}^2} = \sum_{i=1}^{3} \langle D_i M, M \times D_i u\rangle_{\mathbb{L}^2}, \tag{A.1}$$

where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathbb{L}^2}$.

*Remark.* Since $H^1(D) \subset L^6(D)$ and $D_i u \in L^3(D)$, the integral on the RHS above is convergent.

*Remark.* If $M \in D(A)$, then $B = M \times \Delta M$ can be defined pointwise as an element of $L^2(D)$. Moreover by Proposition 3.3, (A.1) holds, so $M \times \Delta M$ in the sense of Definition A.1. The next result shows that this can happen also for less regular $M$.

**Proposition A.2.** *Suppose that $M_n \in H^1(D)$ so that $\Lambda_n := M_n \times \Delta M_n \in L^2(D)$ and*

$$|\Lambda_n|_{\mathbb{L}^2} \leq C.$$

*Suppose that*

$$|M_n|_{H^1} \leq C.$$

*Suppose that*

$$M_n \to M \text{ weakly in } H^1(D).$$

*Then $M \times \Delta M \in L^2(D)$.*

*Proof.* By the assumptions there exists a subsequence $(n_j)$ and $\Lambda \in L^2(D)$ such that for any $q < 6$ (in particular $q = 4$)

$$\begin{aligned}
&\Lambda_{n_j} \to \Lambda \text{ weakly in } L^2(D)\\
&M_{n_j} \to M \text{ strongly in } L^q(D)\\
&\nabla M_{n_j} \to \nabla M \text{ weakly in } L^2(D)
\end{aligned}$$

We will prove that $M \times \Delta M = \Lambda \in L^2$. Let us fix $u \in W^{1,4}(D)$.

First we will show that

$$\langle \Lambda, u\rangle = \sum_{i=1}^{3} \langle D_i M, M \times D_i u\rangle, \tag{A.2}$$

Since $\langle \Lambda_n, u\rangle = \sum_{i=1}^{3} \langle D_i M_n, M_n \times D_i u\rangle$ we have

$$\begin{aligned}
-\langle \Lambda_n, u\rangle \quad &+ \quad \sum_{i=1}^{3} \langle D_i M, M \times D_i u\rangle\\
&= -\sum_{i=1}^{3} \langle D_i M_n, M_n \times D_i u\rangle + \sum_{i=1}^{3} \langle D_i M, M \times D_i u\rangle\\
&= \sum_{i=1}^{3} \langle D_i M - D_i M_n, M \times D_i u\rangle + \sum_{i=1}^{3} \langle D_i M_n, M \times D_i - M_n \times D_i u\rangle\\
&= I_n + II_n
\end{aligned}$$

Since $M \times D_i u \in L^2$ and $D_i M - D_i M_n \to 0$ weakly in $L^2$ we infer that $I_n \to 0$. Moreover, by the Hölder inequality we have

$$|II_n| \quad \leq \quad \sum_{i=1}^{3} |D_i M_n|_{\mathbb{L}^2} |M - M_n|_{L^4} |D_i u|_{L^4} \to 0.$$

Thus, $\langle \Lambda_n, u\rangle \to \sum_{i=1}^{3} \langle D_i M, M \times D_i u\rangle$. On the other hand, $\langle \Lambda_n, u\rangle \to \langle \Lambda, u\rangle$, what concludes the proof of equality (A.2) for $u \in W^{1,4}(D)$.

Since both sides of equality (A.2) are continuous with respect to $W^{1,3}(D)$ norm of $u$ and the space $W^{1,4}(D)$ is dense in $W^{1,3}(D)$, the result follows.

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