Proceedings Paper:

Bagni, C., Askes, H. and Susmel, L. (2016) Gradient elasticity: a new tool for the multiaxial high-cycle fatigue assessment of notched components. In: Proceedings of the Department of Civil and Structural Engineering, University of Sheffield - The Annual Postgraduate Research Student Conference – 2016. The Annual Postgraduate Research Student Conference – 2016 Frattura ed Integrità Strutturale, pp. 21-26. ISBN: 9788895940656.

Gradient elasticity: a new tool for the multiaxial high-cycle fatigue assessment of notched components

C. Bagni, H. Askes, L. Susmel
Department of Civil and Structural Engineering, the University of Sheffield, Sheffield S1 3JD, United Kingdom
c.bagni@sheffield.ac.uk; h.askes@sheffield.ac.uk; l.susmel@sheffield.ac.uk

Primary Supervisor: H. Askes – e-mail: h.askes@sheffield.ac.uk
Secondary Supervisor: L. Susmel – e-mail: l.susmel@sheffield.ac.uk

ABSTRACT. In this paper, the accuracy of gradient elasticity in estimating the fatigue strength of engineering components, characterised by the presence of stress risers and subjected to multiaxial high-cycle fatigue loadings, is assessed. In particular, a new approach, based on the combination of the Ru-Aifantis theory of gradient elasticity and the Theory of Critical Distances (TCD), is proposed for the fatigue assessment of notched metallic components. The proposed methodology represents an important step forward respect to the state of the art, allowing an accurate fatigue assessment of engineering components, by post-processing the relevant gradient-enriched stresses directly on the surface of the component, with evident advantages from a practical point of view.

KEYWORDS. Gradient elasticity; Theory of Critical Distances; Multiaxial Fatigue; Notch; Length Scale

INTRODUCTION

Engineering components are often characterised by complex shapes, presenting different stress concentration features that strongly affect the fatigue behaviour of the materials being assessed. Therefore, an accurate stress analysis is essential to perform a reliable fatigue assessment of components containing stress risers.

Unfortunately, classical elasticity produces unphysically high stress values in the neighbourhood of stress concentration features, leading to non-accurate fatigue assessments. To overcome this problem and deliver accurate fatigue assessments, different theories have been proposed in the past. Amongst these, the Theory of Critical Distances, also known as TCD (for a complete overview see [1]), represents a very effective and accurate tool for the estimation of high-cycle fatigue strength of notched components. The TCD, originally proposed by Neuber [2] and Peterson [3] for the estimation of the high-cycle fatigue strength of notched components, is based on the assumption that a fatigue crack propagates when the range of the effective stress (function of a critical distance, considered a material constant) exceeds the plain fatigue limit of the material, for the relevant load ratio $R = \sigma_{min}/\sigma_{max}$.

Despite its undoubted accuracy and advantages, the use of the TCD for the fatigue assessment of mechanical components requires the knowledge of the failure location into the analysed body a priori, which is not a trivial task from a practical point of view.

However, a family of theories known as *gradient elasticity* has been shown to be able to remove the singularities from the stress fields or, more in general, to smooth the stresses, respectively, in the vicinity of sharp crack tips and stress concentrators such as notches, etc. (see for example [4–6]). This ability has the great advantage of allowing a more accurate estimation of the stresses even at the tip of the stress concentrators, i.e. on the surface of the components, avoiding the need to predetermine the failure location into the material. Similarly to the TCD, gradient elasticity assumes that the relevant stress fields in the neighbourhood of crack/notch tips have to be calculated by directly including an internal length parameter (representative of the underlying microstructure) into the constitutive relations.

In this paper, a new methodology based on the combination of gradient elasticity, in particular the Ru-Aifantis theory (for a comprehensive overview see [5]), and the TCD is proposed. The accuracy and reliability of the proposed methodology in estimating the multiaxial high-cycle fatigue strength of notched components has been validated by using a large number of experimental results taken from the literature.

GRADIENT ELASTICITY

Gradient elasticity represents a family of theories which allows to take into account the influence of the underlying microstructure of the material on the global behaviour of a component. This is possible by enriching the constitutive relations by means of high-order gradients, of relevant state variables, accompanied by intrinsic length parameters.

In this paper, the theory developed by Aifantis and co-workers in the early 90s [7–9], and recently implemented in a unified finite element methodology [6], is considered. This theory consists in enriching the constitutive relations with the Laplacian of the strains as follows:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - \ell^2\varepsilon_{kl,mm} \tag{1}$$

where σ_{ij} is the Cauchy's stress tensor, ε_{ij} is the infinitesimal strain tensor, C_{ijkl} is the elastic tensor and ℓ is the intrinsic length scale of the material. This leads to the following equilibrium equations:

$$C_{ijkl}u_{k,jl} - \ell^2 u_{k,jlmm} + b_i = 0 \tag{2}$$

where u_k are the displacements and b_i the body forces.

Eq. (2) represents a system of fourth-order partial differential equations (p.d.e.), whose implementation in a finite element framework is non-trivial due to the stringent continuity requirements [5]. However, Ru and Aifantis [9] proposed a theorem consisting in a factorisation of the derivatives, which allows the solution of Eq. (2) as two decoupled systems of second-order p.d.e., allowing a straightforward and effective C^0 finite element implementation [6].

The first step of the aforementioned theorem consists in the solution of the standard equations of classical elasticity

$$C_{ijkl}u^c_{k,jl} + b_i = 0 \tag{3}$$

where u^c_k are the classical (or local) displacements.

Using then as source term the local displacements u^c_k calculated from Eq. (3), it is possible to solve the following second system of p.d.e. in terms of stresses [4, 5, 10]

$$\sigma^g_{ij} - \ell^2\sigma^g_{ij,mm} = C_{ijkl}u^c_{k,l} \tag{4}$$

where σ^g_{ij} are the gradient-enriched (or non-local) stresses.

THEORY OF CRITICAL DISTANCES

As stated by Taylor [1], the TCD represents a family of methods, all characterised by two main common features: the use of linear elastic analysis and of a constant critical distance (characteristic of the material). All these methods assume that, under cyclic loading, the threshold condition for crack propagation is given by [11]:

$$\Delta\sigma_{eff} = \Delta\sigma_0 \tag{5}$$

where $\Delta\sigma_{eff}$ represents the range of the effective stress, which is function of the critical distance, L, while $\Delta\sigma_0$ is the plain fatigue limit range, related to the load ratio $R=\sigma_{min}/\sigma_{max}$.

The different approaches, followed to determine the range of the effective stress $\Delta\sigma_{eff}$, identify different versions of the TCD [11]. In particular, the Point Method (PM) represents the simplest approach and consists in applying the TCD as proposed by Peterson [3]. In this case the range of the effective stress $\Delta\sigma_{eff}$ is given by the following expression:

$$\Delta\sigma_{eff}=\Delta\sigma_y\left(\theta=0, r=\frac{L}{2}\right) \tag{6}$$

where r and θ represent the local polar coordinates in a reference system centred at the notch tip. Furthermore, the critical distance L is defined as follows [12]:

$$L=\frac{1}{\pi}\left(\frac{\Delta K_{th}}{\Delta K_0}\right)^2 \tag{7}$$

where ΔK_{th} is the threshold value of the stress intensity factor range.

The TCD as just described can be applied only to notched components subjected to in-service Mode I fatigue loading. In order to extend its use to multiaxial situations, this approach has to be applied along with an appropriate multiaxial fatigue damage model [12, 13]. According to a systematic validation based on a large number of experimental results [12, 14, 15], the highest accuracy is obtained by applying the PM together with a bi-parametrical critical plane approach known as Modified Wöhler Curve Method (MWCM).

THE MODIFIED WÖHLER CURVE METHOD

The MWCM belongs to the family of the so-called *critical plane approaches*, where the critical plane is defined as the material plane experiencing the maximum shear stress amplitude, τ_a. The MWCM consists in estimating the multiaxial fatigue strength in terms of τ_a, by conveniently modifying the fatigue curves through a coefficient able to take into account the degree of multiaxiality of the load history as well as the damaging effect of non-zero mean stresses normal to the critical plane [12]. This coefficient, called *critical plane stress ratio*, ρ_{eff}, is defined as [16]:

$$\rho_{eff}=\frac{m\cdot\sigma_{n,m}+\sigma_{n,a}}{\tau_a} \quad \text{for} \quad \rho_{eff}<\rho_{lim} \qquad \text{and} \qquad \rho_{eff}=\rho_{lim}=\frac{\tau_{A,\infty}}{2\,\tau_{A,\infty}-\sigma_{A,\infty}} \quad \text{for} \quad \rho_{eff}\geq\rho_{lim} \tag{8}$$

where m is the mean stress sensitivity index, while $\sigma_{n,m}$ and $\sigma_{n,a}$ are, respectively, the mean value and the amplitude of the stress normal to the critical plane.

According to the MWCM, denoting with $\sigma_{A,\infty}$ and $\tau_{A,\infty}$, respectively, the fully-reversed uniaxial and torsional endurance limits extrapolated at N_A cycles to failure, the following equivalent shear stress can be defined:

$$\tau_{eq}=\tau_a-\left(\frac{\sigma_{A,\infty}}{2}-\tau_{A,\infty}\right)\cdot\rho_{eff} \tag{9}$$

COMBINING GRADIENT ELASTICITY WITH THE TCD

One of the main unresolved problems of gradient elasticity theories is the identification of the length scale parameters. Although it is widely accepted that these parameters are somehow related to microstructural features of the material, a clear identification is still missing.

Inspired by the similarities shared by the Ru-Aifantis theory and the TCD, Susmel and co-workers [17] tried to formally relate the Ru-Aifantis model and the TCD, finding a practical and effective relation between the internal length scale, ℓ, and the critical distance, L, in the case of cracked components:

$$\ell \approx \frac{L}{2\sqrt{2}} \tag{10}$$

Considering that both ℓ and L are intrinsic material properties (hence their values are independent of the geometry of the problem), in this paper, Eq. (10) has been used to relate the two length parameters also in the case of notched components.

VALIDATION THROUGH EXPERIMENTAL DATA – MULTIAXIAL FATIGUE LOADING

The accuracy and reliability of the proposed design approach were assessed by post-processing a large number of experimental data taken from the literature. The considered results were generated by testing, under in-phase and out-of-phase multiaxial fatigue loadings [12, 14, 18], metallic specimens containing different geometrical features. Table 1 summarises the experimental results used to perform the validation.

The analysed specimens can be subdivided into two families: shafts with shoulder fillet (SSF) for which, following Gough's experimental findings [19], the crack initiation was considered to occur at the junction of the fillet with the central part of the specimen, and circumferentially notched cylindrical bars (CNB) under in-phase and out-of-phase fully-reversed loadings. In order to assess its accuracy in taking into account the mean stress effect in fatigue, the proposed approach was also used to estimate the high-cycle fatigue strength of En3B [18] and S65A [19] notched samples subjected to biaxial fatigue loading with superimposed static stresses.

To be consistent with gradient elasticity, when the stress field is characterised by a certain gradient (in this case both bending and torsion), also the plain fatigue limits should be recalculated according to gradient elasticity. For this purpose the gradient-affected plain endurance limits $\sigma^{g}_{A,\infty}$ and $\tau^{g}_{A,\infty}$ were determined by analysing the plain bars presented in [19, 20] with the proposed methodology, using the ℓ-values reported in Table 1, under both pure bending and pure torsion.

Material	Ref.	R	$\sigma^{g}_{A,\infty}$ [MPa]	$\tau^{g}_{A,\infty}$ [MPa]	m	L [mm]	ℓ [mm]	Specimen Type[(a)]	Load Type[(b)]	r_n [mm]
SAE 1045	[20]	-1	303.1	175.5	-	0.159	0.056	SSF	B-T	5
Ck 45	[21]	-1	303.1	175.5	-	0.159	0.056	SSF	B-T	5
S65A	[19]	-1	581.6	369.7	0.37	0.056	0.020	SSF	B-T	0.838
0.4% C Steel	[19]	-1	325.1	203.1	-	0.178	0.063	CNB	B-T	0.005
3% Ni Steel	[19]	-1	337.2	201.9	-	0.144	0.051	CNB	B-T	0.005
3/3.5% Ni Steel	[19]	-1	330.6	252.6	-	0.516	0.182	CNB	B-T	0.01
Cr-Va Steel	[22]	-1	423.9	255.2	-	0.101	0.036	CNB	B-T	0.011
3.5% NiCr Steel (normal impact)	[22]	-1	530.5	346.4	-	0.150	0.053	CNB	B-T	0.022
3.5% NiCr Steel (low impact)	[22]	-1	502.5	320.2	-	0.109	0.039	CNB	B-T	0.022
NiCrMo Steel (75-80 tons)	[22]	-1	586.6	339.1	-	0.106	0.037	CNB	B-T	0.031
En3B	[18]	-1	346	266.5	0.22	0.048	0.017	CNB	Ax-T	0.2÷4.0

[(a)]SSF=cylindrical Shaft with Shoulder Fillet; CNB=Circumferential Notch in cylindrical Bar
[(b)]Ax=Axial loading; B=Bending; T=Torsion

Table 1: Summary of the experimental results generated under multiaxial fatigue loading.

To assess the accuracy of the proposed methodology in estimating the multiaxial high-cycle fatigue strength of notched components the following error was defined:

$$E_\tau[\%] = \frac{\tau^g_{eq} - \tau^g_{A,\infty}}{\tau^g_{A,\infty}} \cdot 100 \tag{11}$$

where τ^g_{eq} is obtained from Eq. (9) by using the gradient-enriched endurance limits, $\sigma^g_{A,\infty}$ and $\tau^g_{A,\infty}$, as well as the gradient-enriched critical plane stress ratio, ρ^g_{eff}, determined by substituting into Eq. (8) τ^g_a, $\sigma^g_{n,m}$ and $\sigma^g_{n,a}$, obtained by post-processing, through software Multi-FEAST (www.multi-feast.com), the gradient-enriched stress history determined at the crack initiation point.

In the performed analyses, the boundary conditions were taken as homogeneous essential for the first step (Eq. (3)), in order to restore the symmetry of the problem, and homogeneous natural throughout for the second step (Eq. (4)).

Figure 1 confirms that the proposed approach produces estimations mainly ranging in $\pm 20\%$ error interval (this holding true also for out-of-phase loading and under non-zero mean stresses). Furthermore, from Figure 1 it is possible to observe that the proposed multiaxial approach leads to highly accurate estimations also for simple nominal uniaxial cases.

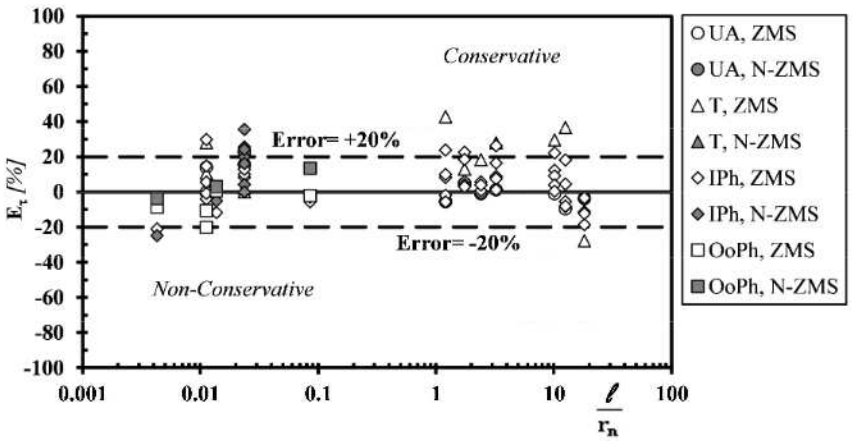

Figure 1: Accuracy of gradient elasticity applied along with the MWCM in estimating high-cycle fatigue strength of notched specimens subjected to uniaxial/multiaxial fatigue loading (UA=uniaxial loading; T=torsional loading; IPh=In-Phase loading; OoPh=Out-of-Phase loading; ZMS=Zero Mean Stress; N-ZMS=Non-Zero Mean Stress).

CONCLUSIONS

The present paper shows the effectiveness and accuracy of gradient elasticity and in particular of the Ru-Aifantis theory in estimating the fatigue strength of components presenting different types of stress concentration features and subjected to multiaxial high-cycle fatigue loadings. Thanks to the ability of gradient elasticity to smooth the stress field in the vicinity of stress risers, the proposed methodology allows the accurate estimation of the fatigue limit by considering the stress field directly at the assumed crack initiation point on the surface of the specimen. This leads to significant simplifications respect to many existing fatigue assessment approaches, which require the knowledge of the failure location into the assessed body a priori.

These results have important implications since, as soon as the length scale, ℓ, will be clearly identified, it will be possible to perform accurate fatigue assessments of notched components by simply using a gradient-enriched FE methodology, such as the one proposed in [6], without considering any linear elastic fracture mechanics (LEFM) concept. However, it has been shown that, in absence of a clear identification of the length scale, ℓ, the relation proposed in [17], based on well-known LEFM concepts, represents a good and practical approximation of the length scale.

Concluding, at the light of the results presented in the present paper, it is possible to state that the proposed approach can potentially become a powerful tool for the fatigue assessment and design of engineering components.

ACKNOWLEDGMENTS

The author gratefully acknowledges 3DS Dassault Systemes (www.3ds.com) for fully supporting the present research investigation.

REFERENCES

[1] Taylor, D., The Theory of Critical Distances: A New Perspective in Fracture Mechanics. Elsevier, Oxford, UK, (2007).
[2] Neuber, H., Theory of notch stresses: principles for exact calculation of strength with reference to structural form and material. Springer Verlag, Berlin, (1958).
[3] Peterson, R. E., Notch-sensitivity, in G. Sines, J. L. Waisman (Eds.), Metal Fatigue, McGraw Hill, New York, (1958) 293–306.
[4] Askes, H., Morata, I., Aifantis, E. C., Finite element analysis with staggered gradient elasticity. Comput. Struct., 86(2008) 1266–1279. doi: 10.1016/j.compstruc.2007.11.002.
[5] Askes, H., Aifantis, E. C., Gradient elasticity in statics and dynamics: An overview of formulations, length scale identification procedures, finite element implementations and new results. Int. J. Solids Struct., 48(13)(2011) 1962–1990. doi: 10.1016/j.ijsolstr.2011.03.006.
[6] Bagni, C., Askes, H., Unified finite element methodology for gradient elasticity. Comput. Struct., 160(2015) 100–110. doi: 10.1016/j.compstruc.2015.08.008.
[7] Aifantis, E. C., On the role of gradients in the localization of deformation and fracture. Int. J. Eng. Sci., 30(10)(1992) 1279–1299. doi: 10.1016/0020-7225(92)90141-3.
[8] Altan, S. B., Aifantis, E. C., On the structure of the mode III crack-tip in gradient elasticity. Scripta Metallurgica et Materialia, 26(1992) 319–324.
[9] Ru, C. Q., Aifantis, E. C., A simple approach to solve boundary-value problems in gradient elasticity. Acta Mechanica, 101(1993) 59–68. doi: 10.1007/BF01175597.
[10] Askes, H., Gitman, I., Non-singular stresses in gradient elasticity at bi-material interface with transverse crack, Int. J. Fract., 156(2009) 217–222.
[11] Taylor, D., Geometrical effects in fatigue: a unifying theoretical model, Int. J. Fatigue, 21(1999) 413–420. doi: 10.1016/S0142-1123(99)00007-9.
[12] Susmel, L., Multiaxial Notch Fatigue: from nominal to local stress-strain quantities. Woodhead & CRC, Cambridge, UK (2009), ISBN: 1 84569 582 8.
[13] Susmel, L., Taylor, D., Can the conventional High-Cycle Multiaxial Fatigue Criteria be re-interpreted in terms of the Theory of Critical Distances?, Structural Durability & Health Monitoring, 2(2)(2006) 91-108.
[14] Susmel, L., A unifying approach to estimate the high-cycle fatigue strength of notched components subjected to both uniaxial and multiaxial cyclic loadings, Fatigue Fract. Engng. Mater. Struct., 27(5)(2004) 391–411.
[15] Susmel, L., Taylor, D., Two methods for predicting the multiaxial fatigue limits of sharp notches, Fatigue Fract. Engng. Mater. Struct., 26(2003) 821-833.
[16] Susmel, L., Multiaxial Fatigue Limits and Material Sensitivity to Non-Zero Mean Stresses Normal to the Critical Planes, Fatigue Fract. Engng. Mater. Struct., 31(2008) 295-309.
[17] Susmel, L., Askes, H., Bennett, T., Taylor, D., Theory of Critical Distances versus Gradient Mechanics in modelling the transition from the short to long crack regime at the fatigue limit. Fatigue Fract. Engng. Mater. Struct., 36(2013) 861–869. doi: 10.1111/ffe.12066.
[18] Susmel, L., Taylor, D., The Modified Wöhler Curve Method applied along with the Theory of Critical Distances to estimate finite life of notched components subjected to complex multiaxial loading paths, Fatigue Fract. Engng. Mater. Struct., 31(12)(2008) 1047-1064.
[19] Gough, H. J., Engineering Steels under Combined Cyclic and Static Stresses. In: Proc. Inst. Mech. Engrs., 160(1949) 417-440.
[20] Kurath, P., Downing, S. D., Galliart, D. R., Summary of non-hardened notched shaft round robin program, in G. E. Leese and D. F. Socie (Eds.), Multiaxial Fatigue. Society of Automotive Engineers, AE-14(1989) 13–32.
[21] Sonsino, C. M., Fatigue Behaviour of Welded Components Under Complex Elasto-Plastic Multiaxial Deformation, LBF-Bericht, Nr. 6078 (1994).
[22] Taylor, D., Wang, G., The validation of some methods of notch fatigue analysis. Fatigue Fract. Engng. Mater. Struct., 23(2000) 387–394.