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Crowd-averse cyber-physical systems: the paradigm of robust mean-field games

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***Abstract*—For a networked controlled system we illustrate the paradigm of robust mean-field games. This is a modeling framework at the interface of differential game theory, mathematical physics, and H_∞-optimal control that tries to capture the mutual influence between a crowd and its individuals. First, we establish a mean-field system for such games including the effects of adversarial disturbances. Second, we identify the optimal response of the individuals for a given population behavior. Third, we provide an analysis of equilibria and their stability.**

I. Introduction

Cyber-physical systems (CPSs) involve computation and physical processes with, possibly, humans in the loop. CPSs are required to maintain a "good" performance even in the presence of adversarial disturbances or cyber-attacks. A second issue is *concurrency* as physical processes are compositions of many parallel dynamics, in contrast to software processes, which are rooted in sequential steps. Thus the *need to bridge an inherently sequential semantics with an intrinsically concurrent physical world* [8]. In hybrid systems, a similar aspect yields to minimum attention control [9].

Robust mean-field games intersect CPSs in at least the following aspects: i) the game describes a large-scale distributed system where the players may represent the system components, ii) worst-case adversarial disturbances represent cyber-attacks on each single system component, iii) the mean-field term in the cost accounts for the congestion in the communication network. In addition, heuristics rather than cumbersome strategies on the part of the players are due to the limited computational capabilities of the humans in the loop. For the above reasons, we have identified in the "mean-field game theory" a suitable paradigmatic modeling framework.

Highlights of contributions. Each player evolves according to a linear stochastic differential equation (SDE) and minimizes a cost functional which includes a cross-coupling term. Such a term penalizes the use of the shared communication network when congested and therefore constitutes the coupling term between individuals and population. An adversarial disturbance with limited energy resources attacks each individual player in order to maximize the cost functional [18].

The contribution of this paper is three-fold: First, we establish a mean-field system for such a game including the effects of adversarial disturbances, which we call "robust" mean-field game. Second, we identify the optimal response of the individuals for a given population behavior. The latter is captured by the mean-field term. Third, we provide a detailed analysis of equilibria.

Related literature on mean-field games. Mean field games were formulated by Huang et al. in [11] and independently by Lasry and Lions in [14] and arise in several application domains (see [1, 6, 10, 11, 13, 16, 20]). The approach leads to a system of two partial differential equations (PDEs). The first PDE is the Hamilton-Jacobi-Bellman equation. The second PDE is the Fokker-Planck-Kolmogorov (FPK) equation which describes the density of the players [14, 19]. Explicit solutions exist for the linear-quadratic structure, see [2], while in general a variety of numerical solution schemes are available in the literature [1]. More recently, robustness and risk-sensitivity have been brought into the picture of mean-field games [4, 5, 19]. The first PDE is then the Hamilton-Jacobi-Isaacs (HJI) equation.

The paper is organized as follows. In Section II, we formulate the problem. In Section III, we illustrate the mean-field game. In Section IV, we study equilibria and stability. In Section V, we provide numerical studies. Finally, in Section VI, we provide conclusions.

Notation We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space. We let $\mathcal{B}$ be a finite-dimensional standard Brownian motion defined on this probability space. We define $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, its natural filtration augmented by all the $\mathbb{P}-$null sets (sets of measure-zero with the respect $\mathbb{P}$). We write ∂_x and ∂^2_{xx} to stand respectively for the first and second derivatives with respect to x. We denote by $\mathbb{R}_+$ the set of nonnegative reals. For any real $\xi \in \mathbb{R}$, $[\xi]_+$ denotes the positive part.

II. Crowd-aversion problem set-up

Consider a set of players $\mathcal{N} = \{1, \ldots, n\}$ and let $x_{j,0} \in \mathbb{R}_+$ be the initial state of generic player $j \in \mathcal{N}$, which is realized according to the probability distribution m_0. The state of player j at time t, denoted by $x_{j,t} \in \mathbb{R}_+$ evolves over a finite horizon $T > 0$ as:

$$dx_{j,t} = [\alpha x_{j,t} + \beta u_{j,t}]dt + \sigma\left[x_{j,t} d\mathcal{B}_{j,t} + \zeta_t dt\right], \qquad (1)$$

where $u_{j,t} \in \mathbb{R}_+$ is the control input, $\mathcal{B}_{j,t}$ is a standard Brownian motion, which is independent of the initial state

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$x_{j,0}$, independent across players and time, $\alpha, \sigma \in \mathbb{R}$ and $\beta < 0$ are parameters, ζ_t is an adversarial disturbance.

Let us denote the empirical measures of the states and of the controls at time t by $m_t = \frac{1}{n}\sum_{j=1}^{n} \delta_{x_{j,t}}$ and $z_t = \frac{1}{n}\sum_{j=1}^{n} \delta_{u_{j,t}}$, respectively, where δ is the Dirac measure. In addition, let $\bar{z}_t$ and $\bar{m}_t$ be the mean of the process z_t and m_t, respectively. Let us introduce the following cost functional with penalty on final state $g(\cdot)$, stage cost function $c(\cdot)$, and quadratic penalty on the unknown disturbance

$$\begin{aligned} J(x_{j,0}, u_j, \bar{z}, \zeta) = \mathbb{E}\Big(& g(x_{j,T}) \\ & + \int_0^T c(x_{j,t}, u_{j,t}, \bar{z}_t) dt \quad - \quad \gamma^2 \int_0^T |\zeta_t|^2 dt\Big). \end{aligned}$$

Players wish to stabilize their states to zero, and therefore we can take for the stage cost

$$c(x_{j,t}, u_{j,t}, \bar{z}_t, \zeta_t) = h(\bar{z}_t) u_{j,t} + \left[\frac{a}{2}(x_{j,t})^2 + \frac{b}{2}(u_{j,t})^2\right],$$

where $h(\bar{z}_t)$ is a measure of the "crowd", and thus $h(\bar{z}_t)u_{j,t}$ is a penalty on the control of the single player which is proportional to the crowd in the control loop for the whole system; $\frac{a}{2}(x_{j,t})^2$ where $a > 0$ is the cost of a nonnull state, and $\frac{b}{2}(u_{j,t})^2$ where $b > 0$ accounts for the control energy. The penalty on final state $g(x_{j,T}) = \phi(x_{j,T})^2$, for a given scalar $\phi > 0$, namely it is quadratic with minimum in zero thus penalizing non null states at the end of the horizon. We assume that the crowd is proportional to the magnitude of the average control, namely

$$\begin{aligned} h(\bar{z}_t) \quad &= k|\bar{z}_t| = k\left|\frac{1}{n}\sum_{j=1}^{n} u_{j,t}\right|, k \in \mathbb{R}_+. \\ &= k\left|\frac{1}{\beta}\frac{d}{dt}\bar{m}_t - \frac{\alpha}{\beta}\bar{m}_t - \frac{\sigma}{\beta}\zeta_t\right|. \end{aligned} \tag{2}$$

The last equality is obtained by introducing expectations in (1), by considering deterministic disturbance ζ_t, and by using indistinguishability, from which we can write:

$$\begin{aligned} [\mathbb{E}u_{j,t}] \quad &= \frac{1}{\beta}\left(\frac{d}{dt}[\mathbb{E}x_{j,t}]\right) - \frac{\alpha}{\beta}\left([\mathbb{E}x_{j,t}]\right) - \frac{\sigma}{\beta}\zeta_t \\ &= \frac{1}{\beta}\left(\frac{d}{dt}\int x m_t(dx)\right) - \frac{\alpha}{\beta}\left(\int x m_t(dx)\right) - \frac{\sigma}{\beta}\zeta_t. \end{aligned}$$

When $n \longrightarrow +\infty$, we have the following robust mean-field game problem [4, 5].

Problem 1. *(***Robust mean-field response problem***) Let $\mathcal{B}$ be a one-dimensional Brownian Motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the natural filtration generated by $\mathcal{B}$. Let x_0 be independent of $\mathcal{B}$ and with density $m_0(x)$. Consider the problem in $\mathbb{R}$ and $(0, T]$*

$$\left\{ \begin{array}{l} \qquad \inf_{\{u_t\}_t} \sup_{\{\zeta_t\}_t} J(x, u, \bar{z}, \zeta) \\ dx_t = [\alpha x_t + \beta u_t + \sigma\zeta_t]\, dt \quad + \quad \sigma x_t d\mathcal{B}_t. \end{array} \right.$$

Model (1) may represent a multi-tank system [12], where the state is the tank level, the control stabilizes the level to zero, while an adversary provides obstacles to this. Model (1) fits also to the case of a power grid, where the state is the rotor angle of each generator, the control operates in order to guarantee transient stability despite the volatility of wind or solar power sources [17]. A third example is given by cyber-physical economic systems; here (1) shares similarity with the Black and Scholes model [7] derived in the context of portfolio selection.

III. The resulting mean-field game

Let us denote by $v_t(x)$ the (upper) value of the robust optimization problem under worst-case disturbance starting from time t at state x. Let the Hamiltonian be given by

$$H(x, p, \bar{z}) = \inf_u \{c(x, u, \bar{z}) + p(\alpha x + \beta u)\},$$

where p is the co-state. The next result introduces the mean-field system for the case of crowd-averse CPSs and closed-loop control and disturbance.

Theorem 1. *The closed-loop robust mean-field game for the crowd-averse CPSs takes on the form:*

$$\left\{ \begin{array}{l} \partial_t v_t + \left[-\frac{1}{2b}\beta^2 + \left(\frac{\sigma}{2\gamma}\right)^2\right] |\partial_x v_t|^2 \\ + \left[-\frac{1}{2b}(2h(\bar{z}_t)\beta) + \alpha x_t\right] \partial_x v_t \qquad\qquad (HJI) \\ -\frac{1}{2b}h(\bar{z}_t)^2 + \frac{a}{2}(x_t)^2 + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} v_t = 0, \\ \; in\ \mathbb{R}_+ \times [0, T),\ v_T = \phi|x|^2,\ in\ \mathbb{R}_+ \\ \\ \partial_t m_t + \partial_x \Big[m_t\Big(\alpha x_t + \beta\frac{-h(\bar{z}_t) - \partial_x v_t \beta}{b} \\ + \frac{\sigma^2}{2\gamma^2}\partial_x v_t\Big)\Big] + \frac{\sigma^2}{2\gamma^2}\partial_x(m_t \partial_x v_t) \qquad (FPK) \\ -\frac{1}{2}\sigma^2 \partial^2_{xx}\left[x^2 m_t\right] = 0,\ in\ \mathbb{R}_+ \times [0, T), \\ m_0(x)\ given\ in\ \mathbb{R}_+, \\ \dot{\bar{m}}_t = \alpha\bar{m}_t + \beta\bar{z}^*_t + \sigma\bar{\zeta}^*_t,\ \bar{m}_0\ given, \end{array} \right. \tag{3}$$

where $\bar{z}^*_t := \int_{\mathbb{R}_+} u^*_t(x) m_t(x) dx$, $\bar{\zeta}^*_t := \int_{\mathbb{R}_+} \zeta^*_t(x) m_t(x) dx$ and the optimal closed-loop control and disturbance are

$$\left\{ \begin{array}{l} u^*_t(x) = \frac{-h(\bar{z}_t) - \partial_x v_t \beta}{b}, \\ \zeta^*_t(x) = \frac{\sigma}{2\gamma^2}\partial_x v_t(x). \end{array} \right. \tag{4}$$

Proof: Given in the appendix. ∎

The significance of the above result is that to find the optimal control input we need to solve the two coupled PDEs in (3) in v and m with given boundary conditions (second and fourth conditions). Any solution of the above system of equations is referred to as *worst-disturbance feedback mean-field equilibrium*. The difference with a mean-field equilibrium is that the first PDE is now a Hamilton-Jacobi-Isaacs (HJI) equation involving a *minimax* optimization and not a Hamilton-Jacobi-Bellman equation. Analogously to the mean-field equilibrium case, such a fixed point can be calculated iteratively solving the HJI equation for fixed m_t and by entering the optimal u^*_t and the worst-case disturbance ζ^*_t in the Fokker-Planck-Kolmogorov equation in (3), until a fixed point in v_t and m_t is reached.

Remark 1. *Let m_0 be absolutely continuous with a continuous density function with finite second moment. Since the running cost is convex in u, and concave in the disturbance ζ, one gets a convex-concave stage cost function that satisfies $\frac{c - \gamma^2\|\zeta\|^2}{\|u\|} \longrightarrow +\infty$ and $\frac{c - \gamma^2\|\zeta\|^2}{\|\zeta\|} \longrightarrow -\infty$, as $\|u\|, \|\zeta\|$ goes to infinity. The drift is linear and hence Lipschitz continuous because α, β, σ are bounded. We assume that the Fenchel transform of c is Lipschitz in (x, z). Finally, we assume that the function $p \longmapsto \frac{\sigma^2}{4\gamma^2}\|p\|^2 + H$ is strictly convex, differentiable and $\frac{\sigma^2}{4\gamma^2}\|p\|^2 + H$ is Lipschitz continuous. Under the above main assumptions, the existence of solution is established in Theorem 2.6 in [14].*

IV. A HEURISTIC APPROACH

In this section, we present a heuristic approach to approximate the mean-field equilibrium and provide performance bounds. The method was first developed in [3] and it is here adapted to the problem at hand. The heuristic approach reframes the problem in an extended state space involving both the state of the player and the average state distribution.

A. Extended state space

In the next assumption, we consider a lower bound on the rate of change of the mean $\bar{m}_t$. At the end of this section, we establish a specific value for such a lower bound.

Assumption 1. *Suppose there exists a $\theta > 0$ and a corresponding $\tilde{m}_t$ such that*

$$\begin{cases} \frac{d}{dt}\bar{m}_t \geq \frac{d}{dt}\tilde{m}_t = -\theta\tilde{m}_t, \quad \text{for all } t \in [0,T], \\ \bar{m}_0 = \tilde{m}_0. \end{cases} \tag{5}$$

In addition to this, let us also assume that $\bar{\zeta}_t = \delta\tilde{m}_t$.

From (2) and substituting $\bar{m}_t$ by the approximate dynamics $\frac{d}{dt}\tilde{m}_t = -\theta\tilde{m}_t$, $\bar{m}_0 = \tilde{m}_0$, we can rewrite

$$h(\bar{z}_t) = k\Big|\frac{-\theta - \alpha - \sigma\delta}{\beta}\tilde{m}_t\Big| := 2s\tilde{m}_t. \tag{6}$$

We can then approximate the problem at hand as follows:

$$\begin{aligned} \inf_{\{u_t\}_t} \sup_{\{\zeta_t\}_t} \int_0^T \Big[2s\tilde{m}_t u_t + \frac{q}{2}\tilde{m}_t^2 \\ + \Big(\tfrac{a}{2}x_t^2 + \tfrac{b}{2}u_t^2 - \gamma^2\zeta_t^2\Big)\Big]dt + g(x_T) \\ \text{s.t.} \begin{bmatrix} dx_t \\ d\tilde{m}_t \end{bmatrix} = \Big(\begin{bmatrix} \alpha & 0 \\ 0 & -\theta \end{bmatrix}\begin{bmatrix} x_t \\ \tilde{m}_t \end{bmatrix} + \begin{bmatrix} \beta \\ 0 \end{bmatrix} u_t \\ + \begin{bmatrix} \sigma \\ 0 \end{bmatrix}\zeta_t\Big)dt + \begin{bmatrix} \sigma x_t d\mathcal{B}_t \\ 0 \end{bmatrix}, \end{aligned}$$

where the term $\frac{q}{2}\tilde{m}_t^2$ is here introduced to guarantee convexity of the cost as formalized later in Assumption 2.

Reformulating the problem in terms of the expanded state and a new control expressed as:

$$X_t = \begin{bmatrix} x_t \\ \tilde{m}_t \end{bmatrix}, \quad \tilde{u}_t = u_t + \frac{2}{b}s\tilde{m}, \tag{7}$$

and by completing the square in the objective function we obtain the following linear quadratic problem:

$$\begin{cases} \inf_{\{\tilde{u}_t\}_t} \sup_{\{\zeta_t\}_t} \int_0^T \Big[\frac{1}{2}(X_t^T\tilde{Q}X_t + R\tilde{u}_t^2 - \Gamma\zeta_t^2)\Big]dt + g(x_T) \\ \qquad dX_t = (\tilde{A}X_t + B\tilde{u}_t + C\zeta_t)dt + Cx_t d\mathcal{B}_t, \end{cases}$$

where

$$\tilde{Q} = \begin{bmatrix} a & 0 \\ 0 & q - \frac{4}{b}s^2 \end{bmatrix}, \qquad R = b, \qquad \Gamma = 2\gamma^2,$$
$$\tilde{A} = \begin{bmatrix} \alpha & -\beta\frac{2}{b}s \\ 0 & -\theta \end{bmatrix}, \quad B = \begin{bmatrix} \beta \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} \sigma \\ 0 \end{bmatrix}.$$

Now the idea is to consider a new value function $\mathcal{V}_t(x, \tilde{m})$ (in compact form $\mathcal{V}_t(X)$) in the expanded state space, which satisfies

$$\begin{cases} \partial_t\mathcal{V}_t(X) + H(X, \partial_X\mathcal{V}_t(X)) \\ + \left(\frac{\sigma}{2\gamma}\right)^2 |\partial_x\mathcal{V}_t(X)|^2 + \frac{1}{2}\sigma^2x^2\partial^2_{xx}\mathcal{V}_t(X) = 0, \\ \mathcal{V}_T(X) = g(x). \end{cases} \tag{8}$$

Let us take for it the following quadratic expression:

$$\mathcal{V}(x, \bar{m}, t) = [x_t \quad \tilde{m}_t]\underbrace{\begin{bmatrix} P_{11,t} & P_{12,t} \\ P_{21,t} & P_{22,t} \end{bmatrix}}_{P_t}\begin{bmatrix} x_t \\ \tilde{m}_t \end{bmatrix},$$

where the matrix P_t, must be solution of the differential Riccati equation

$$\begin{aligned} \dot{P}_t + P_t\tilde{A} + \tilde{A}^TP_t - 2P_t(BR^{-1}B^T \\ -C\Gamma^{-1}C^T)P_t + \tfrac{\tilde{Q}}{2} + W = 0, \end{aligned} \tag{9}$$

and where

$$\begin{aligned} BR^{-1}B^T - C\Gamma^{-1}C^T &= \begin{bmatrix} \frac{1}{b}\beta^2 + \frac{1}{2\gamma^2}\sigma^2 & 0 \\ 0 & 0 \end{bmatrix}, \\ W &= \begin{bmatrix} \sigma^2P_{11} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{10}$$

Assumption 2. *Parameters q and s are such that*

$$\tilde{Q} = \begin{bmatrix} a & 0 \\ 0 & q - \frac{4}{b}s^2 \end{bmatrix} \geq 0. \tag{11}$$

Let P be solution of the above differential Riccati equation, then we know that the optimal value for control $\tilde{u}$ is of the form

$$\begin{aligned} \tilde{u}_t^*(X_t) &= -2R^{-1}B^TPX_t \\ &= -\tfrac{2}{b}[\beta \; 0]\begin{bmatrix} P_{11,t} & P_{12,t} \\ P_{21,t} & P_{22,t} \end{bmatrix}\begin{bmatrix} x_t \\ \tilde{m}_t \end{bmatrix} \\ &= -\tfrac{2\beta}{b}(P_{11,t}x_t + P_{12,t}\tilde{m}_t). \end{aligned} \tag{12}$$

From the above expression and from (7) it is immediate to derive the current optimal control

$$u_t^*(X_t) = -\tfrac{2}{b}[\beta P_{11,t}x_t + (\beta P_{12,t} + s)\tilde{m}_t], \tag{13}$$

and the worst disturbance is

$$\begin{aligned} \tilde{\zeta}_t^*(X_t) &= 2\Gamma^{-1}C^TPX_t \\ &= \tfrac{1}{\gamma^2}[\sigma \; 0]\begin{bmatrix} P_{11,t} & P_{12,t} \\ P_{21,t} & P_{22,t} \end{bmatrix}\begin{bmatrix} x_t \\ \tilde{m}_t \end{bmatrix} \\ &= \tfrac{1}{\gamma^2}\sigma(P_{11,t}x_t + P_{12,t}\tilde{m}_t). \end{aligned} \tag{14}$$

Note that if we take the average in (14) then the condition $\bar{\zeta}_t = \delta\tilde{m}_t$ in Assumption 1 is satisfied.

In addition, a possible value for θ can be obtained by taking $\tilde{m}_t = 0$ for all t in (13) and (14). By averaging we obtain $\bar{z}_t^*$ and $\bar{\zeta}_t^*$, which we can substitute in $\dot{\bar{m}}_t = \alpha\bar{m}_t + \beta\bar{z}_t^* + \sigma\bar{\zeta}_t^*$ to obtain the following expression

$$\theta = \Big[\alpha + (-\frac{2\beta^2}{b} + \frac{\sigma^2}{\gamma^2})P_{11,t}\Big].$$

Bounds for the proposed heuristics when $\sigma = 0$ can be obtained as follows. For the lower bound we take $\tilde{m}_t = 0$ for all t and solve the resulting linear quadratic problem. This yields the Riccati equation

$$\dot{\pi}_0 + 2\pi_0\alpha - \frac{2}{b}\pi_0^2\beta^2 + \frac{1}{2}a = 0. \tag{15}$$

For the upper bound, let us take $\tilde{m}_t = \bar{m}_0$ for all t, and consider the Taylor expansion $\pi = \pi_0 + \bar{m}_0\pi_1$. The Riccati equation takes the form

$$\begin{aligned} &\dot{\pi} + 2\pi\Big(\alpha + (-\tfrac{2}{b}\beta s)\bar{m}_0\Big) - \tfrac{2}{b}\pi^2\beta^2 + \tfrac{1}{2}\Big(a + (q - \tfrac{2}{b}s^2)\bar{m}_0\Big) \\ &= \tfrac{d}{dt}(\pi_0 + \bar{m}_0\pi_1) + 2(\pi_0 + \bar{m}_0\pi_1)\Big(\alpha + (-\tfrac{2}{b}\beta s)\bar{m}_0\Big) \\ &-\tfrac{2}{b}(\pi_0 + \bar{m}_0\pi_1)^2\beta^2 + \tfrac{1}{2}\Big(a + (q - \tfrac{2}{b}s^2)\bar{m}_0\Big) = 0. \end{aligned}$$

Neglecting higher order infinitesimals, from (15) and collecting all terms in π_1 we have

$$\dot{\pi}_1 + \pi_1(2\alpha - \tfrac{4}{b}\pi_0\beta^2) + 2\pi_0(-\beta\tfrac{2}{b}s) + \tfrac{1}{2}(q - \tfrac{2}{b}s^2) = 0.$$

Observing that for a sufficiently small $\bar{m}_0$, then $(q - \frac{2}{b}s^2)\bar{m}_0 \geq (q - \frac{2}{b}s^2)\bar{m}_0^2$, we can conclude

$$\pi_0 x^2 \leq v_t(x), \mathcal{V}_t(X) \leq \pi x^2.$$

Remark 2. *Let $\mathcal{P}_1$ be the set of Borel probability measures m on $\mathbb{R}$ with finite first order moment and let $\bar{m}$ be its mean. Also, let $\tilde{c}(x, \bar{m}) := 2s\bar{m}u + \frac{q}{2}\bar{m}^2 + \frac{a}{2}x^2$. It holds $\forall (x_1, m_1), (x_2, m_2) \in \mathbb{R} \times \mathcal{P}_1$*

$$|\tilde{c}(x_1, \bar{m}_1) - \tilde{c}(x_2, \bar{m}_2)| \leq C_0[|x_1 - x_2| + \mathbf{d}_1(m_1, m_2)],$$

where $\mathbf{d}_1$ is the Kantorovitch- Rubinstein distance. Then the solution to the game with infinite number of players, namely when $n \longrightarrow +\infty$, approximates the game with a finite number of players following the same approximation bounds established in [11, 14].

B. Asymptotic stability and mean-field equilibrium

Using the optimal control and worst-case disturbance (13)-(14) in the SDE (1) we obtain

$$dx_t = \alpha x_t + (-\tfrac{2\beta^2}{b} + \tfrac{\sigma^2}{\gamma^2})P_{11,t}x_t + [(-\tfrac{2\beta^2}{b} + \tfrac{\sigma^2}{\gamma^2})P_{12,t} \\ -\beta\tfrac{2}{b}s]\tilde{m}_t + \sigma x_t d\mathcal{B}_t,\ t \in (0,T],\ x_0 \in \mathbb{R}.$$

The above SDE is linear and time-varying. The corresponding stochastic process can be studied in the framework of stochastic stability theory [15].

To do this, let us take as Lyapunov function the quadratic function $V(x) = \Phi x^2$, then the stochastic derivative of $V(x)$ is obtained by applying the infinitesimal generator to $V(x)$ which yields $\mathcal{L}V(x) = [\sigma^2 + 2(\alpha - \frac{2\beta^2}{b} + \frac{\sigma^2}{\gamma^2})]\Phi x^2$.

Theorem 2 ([15]). *If $V(x) \geq 0$, $V(0) = 0$ and $\mathcal{L}V(x) \leq -\eta V(x)$ on $Q_\epsilon := \{x : V(x) \leq \epsilon\}$ for some $\eta > 0$ and for arbitrarily large ϵ, then the origin is asymptotically stable "with probability one", and*

$$P_{x_0}\{\sup_{T \leq t < +\infty} x_t^2 \geq \lambda\} \leq \frac{V(x_0)e^{-\psi T}}{\lambda}$$

for some $\psi > 0$.

From the above theorem, we have the following result, which establishes exponential stochastic stability of the mean-field equilibrium provided above.

Corollary IV.1. *If $[\sigma^2 + 2(\alpha - \frac{2\beta^2}{b} + \frac{\sigma^2}{\gamma^2})]\Phi < 0$ then $lim_{t\to\infty} x_t = 0$ almost surely and*

$$P_{x_0}\{\sup_{T \leq t < +\infty} x_t^2 \geq \lambda\} \leq \frac{V(x_0)e^{-\psi T}}{\lambda}$$

for some $\psi > 0$.

The interpretation of the above result is that the players stabilize their states to zero asymptotically, while predicting the evolution of congestion as formulated in Problem 1.

We can approximate the mean-field equilibrium, which is captured by the evolution of $\bar{m}_t$ over the horizon $(0, T]$, as

$$\tfrac{d}{dt}\bar{m}_t = \Big[\alpha + (-\tfrac{2\beta^2}{b} + \tfrac{\sigma^2}{\gamma^2})(P_{11,t} + P_{12,t}) - \beta\tfrac{2}{b}s\Big]\bar{m}_t \\ t \in (0,T],\ x_0 \in \mathbb{R}.$$

	$\sigma\,[10^{-2}]$	$std(m_0)$	b	Q
I	10^{-2}	$\{2, 5, 10\}$	25	5
II	$\{2, 4, 10\}$	1	25	5
III	1	5	$\{20, 25, 100\}$	$\{4, 5, 20\}$

TABLE I
VARYING SIMULATION PARAMETERS WITH DIFFERENT REGIMES.

Actually, we can derive a differential equation which represents a bound for the mean distribution:

$$\begin{cases} \bar{m}_t = \bar{m}_0 e^{\rho t} \\ \rho = \alpha + (-\frac{2\beta^2}{b} + \frac{\sigma^2}{\gamma^2})(P_{11,t} + P_{12,t}) - \beta\frac{2}{b}s. \end{cases}$$

The equation above corresponds to saying that the mean distribution converges exponentially to zero in absence of the stochastic disturbances (the Brownian motion), under the assumption that ρ is strictly negative.

V. NUMERICAL STUDIES

Consider a number of players $n = 10^3$ and a discretized set of states $\mathcal{X} = \{x_{min}, x_{min} + 1, \ldots, x_{max}\}$ where $x_{min} = 0$ and $x_{max} = 100$ (see parameters in in Table I). We set $\alpha = 0$ and $\beta = -1$ and consider the influence of ζ_t implicitly by increasing the coefficient Q used in the quadratic approximation of the value function $v_t = Qx^2$. The horizon length is $T = 40$. We assume m_0 to be Gaussian with mean $\bar{m}_0$ between 20 and 70 and standard deviation $std(m_0)$ between 1 and 10. We adopt the linear function

$$\hat{h}(\bar{m}) = \left(\frac{10^2 - \bar{m}}{10^2}\right)h_{min} + \left(1 - \frac{10^2 - \bar{m}}{10^2}\right)h_{max}. \tag{16}$$

The above function represents a linear approximation of $h(\bar{m}, .)$ with minimal value $h_{min} = 0$ when the mean distribution is minimal, $\bar{m} = x_{min}$, and maximal value $h_{max} = 10^2$ when the mean distribution is maximal, $\bar{m} = x_{max}$. Note that the heuristic method provides linear state feedback control and worst-case disturbance policies. Then the right-hand-side of (2) is linear in $\bar{m}$. Furthermore, we approximate $\partial_x v_t = Qx$ and thus we replace the optimal production in (4) by

$$u_t^* = \frac{-\hat{h}(\bar{m}_t) + 2Qx}{b}. \tag{17}$$

According to a first pattern, we have a constant decrease of $\bar{m}_t$ with time t, as well as of the standard deviation $std(\bar{m}_t)$. Figure 1, left, from top to bottom, shows m_t vs. x_t at different times. The initial distribution m_0 has mean $\bar{m}_0 = 70$ and standard deviation $std(m_0) = 1$ (top), $std(m_0) = 5$ (middle), $std(m_0) = 10$ (bottom). The graphics on the right column display the time plot $\bar{m}_t$ (solid line and y-axis labeling on the left) and the evolution of $std(m_t)$ (dashed line and y-axis labeling on the right). Note that, at approximately $t = 30$, $\bar{m}_t$ reaches zero while $std(m_t)$ drastically decreases to zero in less than 20 seconds, which means that all the players first reach consensus on their states and then drive their states to zero.

The second pattern shows the effects of the Brownian motion. Indeed, the standard deviation $std(m_t)$ as well as

Algorithm

Input: Set of parameters as in Table I.
Output: Distribution function m_t, mean $\bar{m}_t$
and standard deviation $std(m_t)$.

```
Initialize. Generate x_0 given m̄_0 and std(m_0)
for time t = 0, 1, ..., T − 1 do
    if t > 0, then compute m_t, m̄_t, and std(m_t)
    end if
    compute congestion term ĥ(m̄_t),
    for player i = 1, 2, ..., n do
        compute new state x_{t+1} by executing (1)
    end for
end for
STOP
```

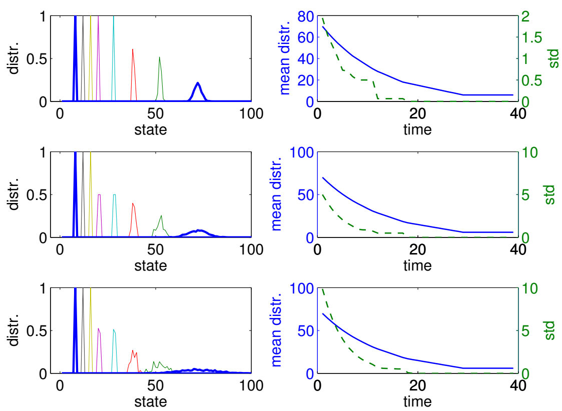

Fig. 1. First pattern: mean distribution $\bar{m}_t$ decreases and as well as standard deviation $std(m_t)$.

sparsity increase with time though the mean distribution $\bar{m}_t$ decreases to zero. Before $\bar{m}_t$ reaches zero the standard deviation $std(m_t)$ inverts the slope and drastically decays to zero. This is summarized in Figure 2, left. From top to bottom, the figure shows the distribution evolution m_t vs. the state x_t at different times. The coefficient weighing the Brownian motion increases from top to bottom, and in particular is $\sigma = 0.02$ (top), $\sigma = 0.04$ (middle), $\sigma = 0.1$ (bottom). The graphics on the right column display the time plot $\bar{m}_t$ (solid line and y-axis labeling on the left) and the evolution of the standard deviation $std(m_t)$ (dashed line and y-axis labeling on the right). Note that the standard deviation $std(m_t)$ first increases for $t \leq 8$ and then drastically decreases to zero.

The third pattern highlights the effects of a higher linear term $\frac{Qx}{b}$ in comparison with the constant term $\frac{\hat{h}(\bar{m}_t)}{b}$ in the control input expression (17). A higher value for Q can be linked back to the effects of the disturbance ζ_t. This is shown in Figure 3, left. From top to bottom, the figure displays the distribution evolution m_t vs. the state x_t at different times. The linear term $\frac{Qx}{b}$ dominates more and more in comparison with the constant term $\frac{\hat{h}(\bar{m}_t)}{b}$ from top

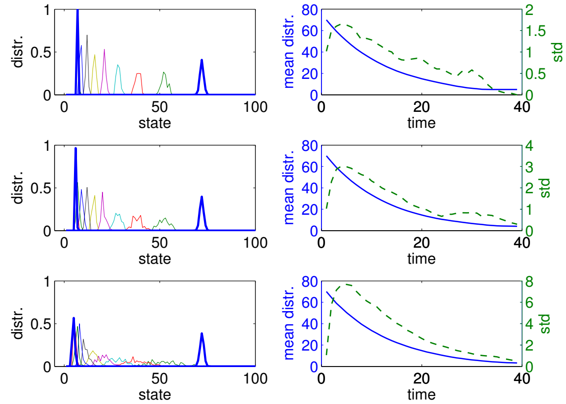

Fig. 2. Second pattern showing the effects of the Brownian motion: mean distribution $\bar{m}_t$ decreases and standard deviation $std(m_t)$ first increases and then decays drastically to zero.

to bottom. Actually $Q = 4$, $b = 20$ (top), $Q = 5$, $b = 25$ (middle), and $Q = 20$, $b = 100$ (bottom). Note that the ratio $\frac{Q}{b}$ is kept constant whereas $\frac{\hat{h}(.)}{b}$ is strongly decreasing from top to bottom. Apparently, the speed of convergence increases. This is clear from observing the graphics on the right column which display the time plot $\bar{m}_t$ (solid line and y-axis labeling on the left) and the evolution of the standard deviation $std(m_t)$ (dashed line and y-axis labeling on the right). Note that both the mean distribution $\bar{m}_t$ and the standard deviation $std(m_t)$ decrease monotonically to zero.

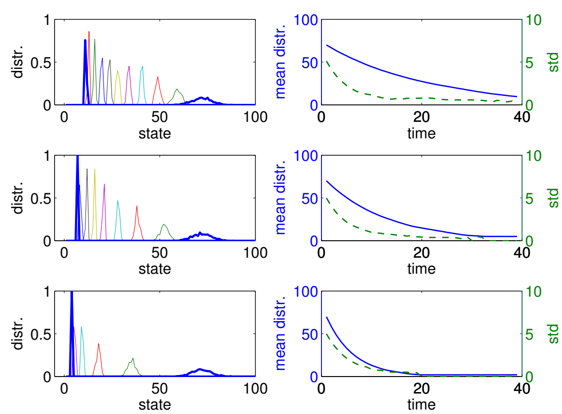

Fig. 3. Third pattern showing the effects of a higher control coefficient Q (associated with a stronger disturbance ζ_t): both the mean distribution $\bar{m}_t$ and the standard deviation $std(m_t)$ decrease monotonically.

VI. Concluding Remarks

We have illustrated robust mean-field games as a paradigm for CPSs. Future directions include the study of i) the connection with *risk-sensitive* optimal control problems; ii) the vector state case and infinite horizon (with discounted payoff and time-average payoff), iii) a cyber-physical economic market with some big players and many other small players.

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Appendix

Proof of Theorem 1. To obtain (4) let the Hamiltonian be:

$$H(x_t, \partial_x v_t, \bar{z}_t) = \inf_u \Big\{ h(\bar{z}_t) u_t + \Big[\frac{a}{2} x^2 + \frac{b}{2} u_t^2 \Big] + \partial_x v_t (\alpha x_t + \beta u_t) \Big\} = 0. \qquad (18)$$

After differentiation we obtain $bu_t + h(\bar{z}_t) + \partial_x v_t \beta = 0$ from which we have the expression of $u_t^*(x)$ in (4). To derive the expression of $\zeta_t^*(x)$, we need to solve $\sup_{\zeta_t} \Big\{ -\gamma^2 \zeta_t^2 + \partial_x v_t \sigma \zeta_t \Big\}$, from which, after differentiation and assuming concavity on ζ_t (see Remark 1), we have $-2\gamma^2 \zeta_t^* + \partial_x v_t \sigma = 0$ and therefore $\zeta_t^*(x) = \frac{\sigma}{2\gamma^2} \partial_x v_t$.

We now prove (3). From [4] the robust mean-field game for Problem 1 is given by

$$\left\{ \begin{array}{l} \partial_t v_t(x) + H(x, \partial_x v_t, \bar{z}_t) \\ + \left(\frac{\sigma}{2\gamma} \right)^2 |\partial_x v_t(x)|^2 + \frac{1}{2} \sigma^2 x^2 \partial_{xx}^2 v_t(x) = 0, \\ \text{in } \mathbb{R}_+ \times [0, T),\ v_T(x) = \phi x^2, \text{in } \mathbb{R}_+, \\ \ \partial_t m_t(x) + \partial_x \left(m_t(x) \partial_p H(x, \partial_x v_t, \bar{z}_t) \right) \\ + \frac{\sigma^2}{2\gamma^2} \partial_x (m_t(x) \partial_x v_t(x)) \\ - \frac{1}{2} \sigma^2 \partial_{xx}^2 \left[x^2 m_t(x) \right] = 0, \\ \text{in } \mathbb{R}_+ \times [0, T),\ m_0(x) \text{ given in } \mathbb{R}_+. \end{array} \right. \qquad (19)$$

First notice that the second and fourth equations are the boundary conditions and derive straightforwardly from Bellman equations and the evolution of the law of states.

To obtain the HJI, let us replace u appearing in the Hamiltonian (18) by its expression (4):

$$\begin{aligned} H(x_t, \partial_x v_t, \bar{z}_t) &= u_t^* [h(\bar{z}_t) + \partial_x v_t \beta] \\ &\quad + \frac{a}{2} (x_t)^2 + \frac{b}{2} (u_t^*)^2 + \partial_x v_t \alpha x_t \\ &= -\frac{1}{2b} \beta^2 |\partial_x v_t|^2 + \Big[-\frac{1}{2b} (2h(\bar{z}_t)\beta) \\ &\quad + \alpha x_t \Big] \partial_x v_t - \frac{1}{2b} h(\bar{z}_t)^2 + \frac{a}{2} (x_t)^2. \end{aligned}$$

Using the above expression of the Hamiltonian in the HJI equation in (19), we obtain the HJI in (3).

To prove the third equation, which is a PDE representing the FPK equation we simply substitute (4) into the FPK in (19).

Finally, the ordinary differential equation representing the time evolution for $\bar{m}_t$ is simply obtained from (1) by averaging over the state space and this concludes the proof.