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THE DECAY OF A WEAK LARGE-SCALE MAGNETIC FIELD IN TWO-DIMENSIONAL TURBULENCE

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ABSTRACT

We investigate the decay of a large-scale magnetic field in the context of incompressible, two-dimensional magnetohydrodynamic turbulence. It is well established that a very weak mean field, of strength significantly below equipartition value, induces a small-scale field strong enough to inhibit the process of turbulent magnetic diffusion. In light of ever-increasing computer power, we revisit this problem to investigate fluids and magnetic Reynolds numbers that were previously inaccessible. Furthermore, by exploiting the relation between the turbulent diffusion of the magnetic potential and that of the magnetic field, we are able to calculate the turbulent magnetic diffusivity extremely accurately through the imposition of a *uniform* mean magnetic field. We confirm the strong dependence of the turbulent diffusivity on the product of the magnetic Reynolds number and the energy of the large-scale magnetic field. We compare our findings with various theoretical descriptions of this process.

Key words: diffusion – magnetohydrodynamics (MHD) – turbulence

1. INTRODUCTION

In a seminal paper, Vainshtein & Rosner (1991) proposed a mechanism by which an extremely weak large-scale magnetic field could suppress turbulent magnetic diffusion. More specifically, they argued that even if the energy in the large-scale field were as small as the kinetic energy divided by the magnetic Reynolds number *Rm*, the ratio of Ohmic to advective timescales, then the associated small-scale field would be sufficiently strong to inhibit the turbulent diffusion process. Given that *Rm* is invariably immense in astrophysics, this raises the important consideration that extremely weak large-scale magnetic fields may have dynamically significant consequences. These ideas were substantiated for the case of two-dimensional magnetohydrodynamic (MHD) turbulence by the illuminating computations of Cattaneo & Vainshtein (1991) and Cattaneo (1994).

While two-dimensional MHD turbulence has a relatively constrained domain of application, further theoretical and numerical work undertaken in this direction by various authors (Kulsrud & Anderson 1992; Tao et al. 1993; Gruzinov & Diamond 1994; Cattaneo & Hughes 1996) demonstrated that similar principles to those underlying the suppression of magnetic diffusion in two dimensions may be used to explain the marked suppression of the mean-field dynamo α-effect in three dimensions. Such a suppression, sometimes referred to as the "catastrophic quenching" of the α-effect, presents a serious difficulty for the operation of any α-effect dynamo at high *Rm* in the non-linear regime.

Bearing in mind these wider implications of the conceptually simple problem of turbulent diffusion in two-dimensional MHD, here we revisit this problem in light of the tremendous increases in computational power that have taken place since the pioneering calculations of Cattaneo & Vainshtein (1991). Current multi-processor machines allow the simulation of two-dimensional MHD turbulence with a wide range of spatial scales and with reasonably high values of the fluid and magnetic Reynolds numbers.

One of the novel features of our paper is the employment of a non-standard technique to determine the turbulent magnetic diffusivity η_{T}. The obvious way to calculate η_{T} is via a measurement of the decay rate of the large-scale component of a zero-mean, large-scale magnetic field introduced into a turbulent hydrodynamic flow. Determining this decay rate accurately is, however, as we shall discuss, a procedure with inherent inaccuracies. In a different approach, provided that one can equate the turbulent diffusion of the magnetic field and the magnetic potential (which, as we shall see, can be justified in certain cases), we may use the result of Zeldovich (1957) to obtain

$$\frac{\langle b^2 \rangle}{\langle B \rangle^2} = \frac{\eta_{\mathrm{T}}}{\eta}, \tag{1}$$

where $\langle b^2 \rangle$ is twice the magnetic fluctuation energy, $\langle B \rangle$ is the strength of the mean (large-scale) magnetic field, and η is the laminar magnetic diffusivity. The angle brackets denote spatial averages over a scale that is large compared with that of the turbulent flow. As it stands, expression (1) is a prescription for η_{T} that is time-dependent; determination of a mean (time-independent) value can be achieved by additional time averaging. The beauty, and indeed the surprise, of expression (1) is that it holds for a *uniform* (and hence non-decaying) magnetic field. There is thus no finite interval of time over which measurements must be made; hence, in principle at least, results for η_{T} can be obtained to arbitrary accuracy.

The layout of the paper is as follows. Section 2 summarizes the results of previous work in this direction. Section 3 contains the mathematical formulation of the problem. In Section 4 we make a comparison between the two different ways of determining η_{T}, the *decaying field* and the *imposed field* methods, and explain in some detail the conditions under which the turbulent diffusion of the magnetic field and its magnetic potential can be considered to be equivalent—a necessary requirement for the imposed field method to work. The results and discussion are in Section 5, followed by the concluding remarks in Section 6.

2. PREVIOUS APPROACHES

For the investigation of large-scale astrophysical magnetic fields with energy weak compared with the turbulent kinetic energy of the flow, it is natural, at least as a first approximation, to regard the magnetic field as kinematic (i.e., passive), with no dynamical influence on the flow. The evolution of the field is then governed solely by the magnetic induction equation. For the specific case of a planar two-dimensional field, independent of the z-coordinate, the magnetic field can be expressed as $\boldsymbol{B} = \boldsymbol{\nabla} \times (A\hat{\boldsymbol{z}})$, where A is the magnetic potential. The induction equation then reduces to the heat equation, which describes the advection and diffusion of a passive scalar, i.e.,

$$\frac{\partial A}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} A = \eta \nabla^2 A. \tag{2}$$

Under the assumption that the flow is kinematic (i.e., the Lorentz force can be neglected), the turbulent diffusion of the magnetic potential therefore simply becomes that of a passive scalar, i.e.,

$$\eta_{\mathrm{T}} \sim u\ell, \tag{3}$$

where u is a characteristic flow speed and ℓ is a typical eddy scale.

However, the idea that a weak large-scale field has no dynamical consequences was challenged by Vainshtein & Rosner (1991) with a simple, yet powerful argument. Multiplying the induction equation (2) by A and averaging over the characteristic length scale of the mean field leads to the result

$$\frac{\partial \langle A^2 \rangle}{\partial t} = -2\eta \langle B^2 \rangle, \tag{4}$$

assuming that the surface term vanishes. In order to determine a relationship between the large-scale and fluctuating magnetic fields, two assumptions are made. The first is the basic premise of the problem, namely that there is a well-defined turbulent diffusion timescale,

$$\tau = L^2/\eta_{\mathrm{T}}, \tag{5}$$

where L is the (long) length scale characteristic of the system under consideration. The second is that the large-scale field is linked to the magnetic potential through the simple, though not entirely obvious, scaling

$$\langle A^2 \rangle^{1/2} \sim |\langle \boldsymbol{B} \rangle| \, L. \tag{6}$$

Under these assumptions, balancing the terms in Equation (4) leads to expression (1), on identifying $\langle B^2 \rangle$ with $\langle b^2 \rangle$ and on equating the turbulent diffusivities of the magnetic potential and the magnetic field. In Section 4.2 we shall re-derive (1) from a more formal, mean-field approach, which sheds further light on the scaling (6). The crucial property of (1), namely that it does not require the kinematic approximation, will also be evident in this approach.

Using (3), expression (1) can be written as

$$\langle B^2 \rangle \sim Rm \, \langle B \rangle^2, \tag{7}$$

thereby showing the increased energy in the magnetic fluctuations compared with that in the mean at high Rm. Vainshtein & Rosner (1991) argued that if the magnetic energy in the fluctuations becomes comparable with the kinetic energy, then the field can no longer be regarded as kinematic; in particular, its turbulent diffusion would be inhibited. Significantly, given the scaling (7), the transition to non-kinematic behavior occurs for very weak large-scale magnetic fields. Cattaneo & Vainshtein (1991) thus argued that turbulent diffusion would be inhibited if

$$M^2 \equiv \frac{\langle u^2 \rangle}{V_{\mathrm{A}}^2} \lesssim Rm, \tag{8}$$

where V_{A} is the Alfvén speed of the *large-scale* field; this leads to the following estimate for the timescale for magnetic diffusion,

$$\tau = \frac{L^2}{\eta}\left(\frac{1}{Rm} + \frac{1}{M^2 + 1}\right). \tag{9}$$

For very weak large-scale fields ($M^2 \gtrsim Rm$), the diffusion is kinematic, with turbulent diffusivity given by (3). By contrast, for equipartition large-scale fields (i.e., $M = O(1)$), the timescale is Ohmic. For $1 \lesssim M \lesssim Rm$, values that still represent weak large-scale fields, the small-scale field is of sufficient strength to inhibit the diffusive process.

More formally, one can seek expressions for the turbulent magnetic diffusivity through a standard mean-field approach. On neglecting molecular diffusion, the evolution for the mean magnetic potential $\langle A \rangle$ is given by

$$\frac{\partial \langle A \rangle}{\partial t} = -\boldsymbol{\nabla} \cdot \langle a\boldsymbol{u} \rangle, \tag{10}$$

where a denotes the fluctuating magnetic potential. From a standard Fickian assumption, the flux $\boldsymbol{\Gamma} = \langle a\boldsymbol{u} \rangle$ can be expressed as $\boldsymbol{\Gamma} = -\eta_{\mathrm{T}} \boldsymbol{\nabla} \langle A \rangle$ (see also Section 4.2). In general it is not straightforward to solve the fluctuation induction equation for a. However, under the assumption that the correlation time of the turbulence τ_{c} is much smaller than an eddy turnover time (the "short-sudden" approximation), a may be approximated by

$$a \approx -\tau_{\mathrm{c}} \boldsymbol{u} \cdot \boldsymbol{\nabla} \langle A \rangle, \tag{11}$$

leading to the expression

$$\eta_{\mathrm{T}} = \tau_{\mathrm{c}} \langle u^2 \rangle, \tag{12}$$

again equating η_{T} for A and $\boldsymbol{B}$. It is important to note that there is nothing intrinsically kinematic about the steps leading to expression (12). Subject to the short sudden approximation—and we shall return to this point—it is valid also in the dynamical regime. However, the non-linearities are contained in both τ_{c} and $\langle u^2 \rangle$ and so, without further information, the result as it stands cannot be readily applied.

In a quasi-linear approach, the flux $\boldsymbol{\Gamma}$ is expressed in terms of perturbations to some existing $\{a, \boldsymbol{u}\}$ state as

$$\boldsymbol{\Gamma} = -\langle \delta a \, \boldsymbol{u} \rangle - \langle a \, \delta \boldsymbol{u} \rangle. \tag{13}$$

Straightforward scalings for $\delta \boldsymbol{u}$ and δa from the momentum and induction equations, assuming the short-sudden approximation and the same correlation time in both, then lead to the result (Gruzinov & Diamond 1994, 1996)

$$\eta_{\mathrm{T}} = \tau_{\mathrm{c}} (\langle u^2 \rangle - \langle b^2 \rangle). \tag{14}$$

This is an appealing expression, representing the turbulent diffusion process as a competition between the fluctuating kinetic and magnetic energies; in the kinematic limit it reduces to (3). However, as pointed out by Proctor (2003), in his discussion of the related problem of α-quenching, one has to reconcile the results (12) and (14). In expression (12), the quantities τ_c and $\langle u^2 \rangle$ evolve with the flow and magnetic field. If they are known, or can be deduced from additional physical considerations, then expression (12), subject to the simplifications inherent in its derivation, tells the full story. By contrast, expression (14) comes from a small perturbation of an existing MHD state, where τ_c is characteristic of this state. Some caution should therefore be exercised in interpreting (14) as a formula for the quenching of the diffusivity as MHD turbulence evolves.

Combining (14) with the result (1), but again bearing in mind that the two expressions for $\langle b^2 \rangle$ are derived under different assumptions, leads to the suppression formula for the turbulent diffusivity

$$\eta_T = \frac{ul}{1 + Rm \langle B \rangle^2 / \langle u^2 \rangle}. \quad (15)$$

This is in agreement with the result (9) for the regime $M^2 \gtrsim Rm$. Equation (15) can also be derived from the more general EDQNM theory (Pouquet et al. 1976; Pouquet 1978) on the assumption of a single τ_c for the entire magnetic and velocity spectra (Diamond et al. 2005). This assumption is more likely to be valid when the field is relatively strong and dominated by Alfvén Waves.

3. MATHEMATICAL FORMULATION

3.1. The Governing Equations

We consider the two-dimensional flow of an electrically conducting fluid with density ρ, kinematic viscosity ν and magnetic diffusivity η (all constant). The fluid is maintained in a turbulent state via a homogeneous and isotropic excitation centered on a particular length scale.

The velocity and magnetic field are confined to the xy-plane and are dependent on x, y, and time t. It is thus convenient to express the velocity and magnetic field in terms of a stream function and flux function, respectively, i.e.,

$$\boldsymbol{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0 \right), \qquad \boldsymbol{B} = \left(\frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}, 0 \right). \quad (16)$$

The vorticity $\boldsymbol{\nabla} \times \boldsymbol{u} = \omega \hat{\boldsymbol{z}}$, where $\omega = -\nabla^2 \psi$.

The dynamics is then governed by two scalar equations: the vorticity equation with forcing and the (uncurled) magnetic induction equation. In dimensionless form these can be expressed as

$$\partial_t \omega = J(\Psi, \omega) + Q\, J(A, \nabla^2 A) + \kappa^{-1} \nabla^2 \omega + F(k_f, t_f) + S, \quad (17)$$

$$\partial_t A = J(\Psi, A) + \varepsilon^{-1} \nabla^2 A, \quad (18)$$

where the Jacobian $J(g, h) = \partial_x g\, \partial_y h - \partial_y g\, \partial_x h$. Time has been scaled with $T_0 = 1/\omega_0$, where ω_0 is some suitably chosen normalization of the vorticity. The parameters $\kappa = \omega_0 L^2/\nu$ and $\varepsilon = \omega_0 L^2/\eta$, where L is a representative length scale, are dimensionless measures of the inverse of viscosity and magnetic diffusivity; their ratio is the magnetic Prandtl number, $Pm = \varepsilon/\kappa$. The parameter Q, which may be regarded as the inverse of the square of the magnetic Mach number, is defined by $Q = (B_0^2/\mu_0 \rho)/u_0^2$, where $u_0 = \omega_0 L$, B_0 is some measure of the large-scale magnetic field strength, and μ_0 is the magnetic permeability. A distinction can be drawn in MHD between problems for which a magnetic field is imposed, such as the problem considered here or that of magnetoconvection, and those for which the strength of the field emerges through dynamical considerations, as in dynamo studies. For the former, one of the input parameters of the problem (Q in our case) measures the strength of the imposed field, whereas for the latter no such parameter is needed.

The term $F(k_f, t_f)$ denotes the forcing of a wave with wavevector k_f, the orientation of which changes randomly with a period t_f; the simulations described here adopt $k_f = 100$. In two-dimensional hydrodynamic turbulence, there is an inverse cascade of kinetic energy to large scales. Thus, in order to maintain a steady state, it is necessary to remove this energy via some sort of sink term, denoted by S. There are various prescriptions available for this procedure (see, for example, Danilov & Gurarie 2001); here we employ an inverse Laplacian ∇^{-2}.

In this paper, in order to compare different techniques for calculating the turbulent magnetic diffusivity we consider two different types of problem, outlined in detail in Section 4. In one, the initial magnetic field is of large scale and has zero spatial mean. In the other, a uniform magnetic field is imposed. The magnetic potential A in Equations (17) and (18) is thus the potential for the *entire* magnetic field (including, where appropriate, the imposed uniform component).

The velocity and magnetic field are assumed to be periodic over the computational domain, which, in dimensionless units, has length 2π.

Equations (17) and (18) are solved numerically, with input parameters κ, ε, Q, k_f, t_f, and $|F(k_f, t_f)|$, using a parallel pseudo-spectral code implementing the second order, Runge–Kutta time-stepping scheme described in Cox & Matthews (2002). The simulations with $Pm = 1$ were carried out on a collocation grid of 2048^2 points, while those with $Pm > 1$ employed 4096^2 points.

3.2. Quantifying Turbulence

In addition to the input parameters defined above, which provide a unique formulation of the problem, it is helpful to define a further set of output parameters in order to quantify the properties of the turbulent flows generated. As the characteristic length scale for the flow, we adopt the Taylor microscale (see Monin & Yaglom 1975)

$$\lambda_u \equiv \frac{2\pi}{k_u} \equiv \sqrt{\overline{u^2}/\overline{\omega^2}}, \quad (19)$$

where bars denote an average over space and time. A characteristic length scale for the magnetic field can be defined in a similar way, namely

$$\lambda_m \equiv \frac{2\pi}{k_m} \equiv \sqrt{\overline{a^2}/\overline{b^2}}. \quad (20)$$

Turbulent quantities related to κ, ε, and Q are defined by

$$Re \equiv \frac{\bar{u} \lambda_u}{\nu}, \quad (21)$$

$$Rm \equiv \frac{\bar{u} \lambda_u}{\eta}, \quad (22)$$

$$M^2 \equiv Q^{-1} \overline{u^2}, \quad (23)$$

$$q \equiv M^{-2}. \quad (24)$$

Table 1
Averages of Turbulent Quantities Measured Using the Imposed Field Method

Pm	*Q*	k_u	k_m	*Re*	*Rm*	*q*	η_T/η
1	0e+00	5.24e+01	1.75e+02	2.18e+02	2.18e+02	0.00e+00	1.26e+02
	1e–08	5.25e+01	1.73e+02	2.17e+02	2.17e+02	3.05e–07	1.21e+02
	1e–07	5.25e+01	1.74e+02	2.17e+02	2.17e+02	3.05e–06	1.21e+02
	1e–06	5.57e+01	1.74e+02	1.93e+02	1.93e+02	3.40e–05	1.02e+02
	1e–05	6.91e+01	1.69e+02	1.27e+02	1.27e+02	5.09e–04	4.78e+01
	5e–05	8.34e+01	1.61e+02	8.56e+01	8.56e+01	3.88e–03	2.12e+01
2	0e+00	5.22e+01	2.43e+02	2.19e+02	4.39e+02	0.00e+00	2.46e+02
	1e–08	5.23e+01	2.44e+02	2.18e+02	4.37e+02	3.03e–07	2.48e+02
	1e–07	5.29e+01	2.43e+02	2.13e+02	4.26e+02	3.10e–06	2.43e+02
	1e–06	5.81e+01	2.41e+02	1.79e+02	3.57e+02	3.66e–05	1.82e+02
	1e–05	7.53e+01	2.32e+02	1.07e+02	2.14e+02	6.06e–04	6.89e+01
4	0e+00	5.23e+01	3.44e+02	2.19e+02	8.74e+02	0.00e+00	5.02e+02
	1e–08	5.24e+01	3.44e+02	2.17e+02	8.67e+02	3.05e–07	5.05e+02
	1e–07	5.37e+01	3.43e+02	2.08e+02	8.32e+02	3.17e–06	4.64e+02
	1e–06	6.22e+01	3.38e+02	1.57e+02	6.28e+02	4.14e–05	2.93e+02
	1e–05	8.18e+01	3.14e+02	8.85e+01	3.54e+02	7.53e–04	9.98e+01
	1e–04	1.01e+02	2.55e+02	5.38e+01	2.15e+02	1.34e–02	2.65e+01

Note. For all simulations listed here: $\kappa = 10^4$, $k_f = 100$, $t_f = 0.2$ and $|F(k_f, t_f)| = 50$.

The values of the parameters for any particular run, both input and derived, are contained in Table 1. The characteristic Reynolds number *Re* and its magnetic counterpart *Rm* are much smaller than the domain-related quantities κ and ε since the flow is dominated by small-scale motions. The parameter q denotes the ratio of the energy of the large-scale magnetic field to the kinetic energy; as the simulations are all in the weak-field regime, this number is always very small.

4. DETERMINING THE TURBULENT DIFFUSIVITY

In this section we describe the two techniques that we have employed for measuring the turbulent diffusivity of the magnetic field, compare the ensuing results, and discuss the pros and cons of the two methods.

4.1. Decaying Large-scale Field

Here, once the hydrodynamic turbulence is established, we impose a weak large-scale magnetic field *of zero mean*; in our simulations this is given by the magnetic potential $A = \sin x$. As shown by Figure 1, the magnetic energy exhibits strong initial transient growth before decaying (Zeldovich's anti-dynamo theorem stipulates that it is impossible for the magnetic field to be maintained indefinitely). The turbulent diffusivity η_T can then be measured directly from the decay of the large-scale magnetic field by fitting a decaying exponential to the amplitude of the large-scale magnetic field.

Although this procedure is manifestly correct, it has a number of inherent inaccuracies. Simply stated, the problem boils down to the difficulties in accurately fitting a straight line to the plots in Figure 2, which, on log-linear axes, show several examples of the decay of the large-scale field. Figure 3 shows that the decay rate (i.e., a measure of the turbulent magnetic diffusivity) itself changes with time, thereby introducing an undesirable level of arbitrariness in the selection of the time interval used to estimate the decay coefficient. Initially, when the field is only large-scale and weak, the decay is kinematic; the rate of decay then decreases as the magnetic fluctuations

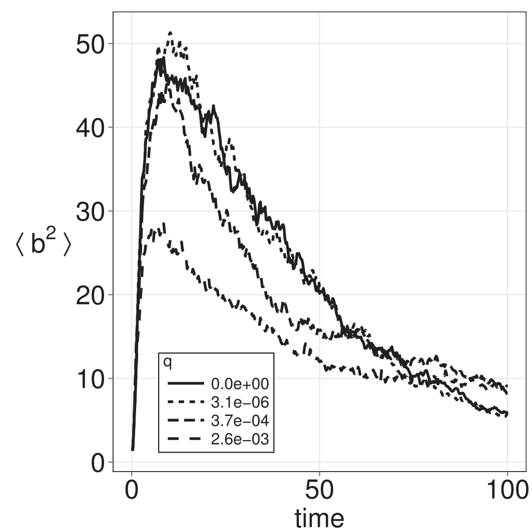

Figure 1. Mean square fluctuation of the magnetic field (i.e., twice the energy of the magnetic fluctuations) for decaying runs at different values of q and with $\varepsilon = 10^4$.

grow (Figure 1). For definiteness, we define a single representative decay coefficient to be the average of a set of values obtained by fitting at various times during the most dynamic phase of the decay. The arbitrariness is thus quantified by the standard deviation.

4.2. Imposed Uniform Field

Maybe somewhat surprisingly, it is also possible to estimate η_T through consideration of turbulence with an imposed *uniform* magnetic field (see Diamond et al. 2005). Suppose we decompose the magnetic potential into the sum of mean (large-scale) and fluctuating (small-scale) components, $A = \langle A \rangle + a$, where angle brackets again denote spatial averages. Averaging the induction equation (in dimensional

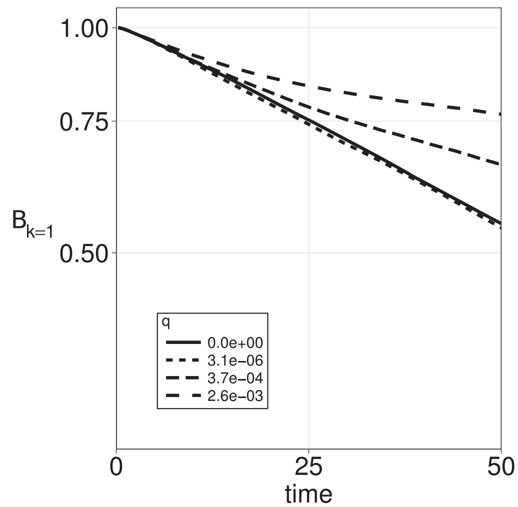

Figure 2. Decay of the large-scale magnetic field mode ($k = 1$) for several values of q.

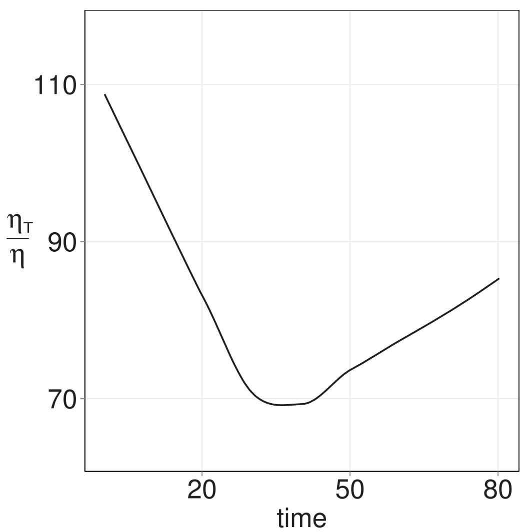

Figure 3. Evolution of the magnetic diffusivity η_T for a decaying field run with $Pm = 2$, $Q = 10^{-5}$.

form) over some intermediate spatial scale then gives

$$\partial_t \langle A \rangle + \boldsymbol{\nabla} \cdot \langle \boldsymbol{u} a \rangle = \eta \nabla^2 \langle A \rangle, \tag{25}$$

where the fluctuating component satisfies

$$\partial_t a + \boldsymbol{u} \cdot \boldsymbol{\nabla} a = -\boldsymbol{u} \cdot \boldsymbol{\nabla} \langle A \rangle + \eta \nabla^2 a. \tag{26}$$

Upon multiplying by a, averaging and using incompressibility, Equation (26) becomes

$$\frac{1}{2}\left(\frac{\partial}{\partial t}\langle a^2 \rangle + \langle \boldsymbol{\nabla} \cdot (\boldsymbol{u} a^2) \rangle\right) = -\langle a\boldsymbol{u} \rangle \cdot \boldsymbol{\nabla} \langle A \rangle + \eta \langle a \nabla^2 a \rangle, \tag{27}$$

which simplifies, on adopting periodic boundary conditions for the fluctuating potential a, and assuming a statistically stationary state, to

$$\eta \langle b^2 \rangle = -\langle a\boldsymbol{u} \rangle \cdot \boldsymbol{\nabla} \langle A \rangle, \tag{28}$$

where b is the strength of the fluctuating magnetic field. From a standard mean-field approach to expression (26), which relates a and $\langle A \rangle$, we may express the flux in terms of a Fickian diffusion as

$$\langle a\boldsymbol{u} \rangle = -\hat{\eta}_{\mathrm{T}} \boldsymbol{\nabla} \langle A \rangle, \tag{29}$$

where $\hat{\eta}_{\mathrm{T}}$ denotes the turbulent diffusion of $\langle A \rangle$. Combining (28) and (29) then gives

$$\langle b^2 \rangle = \frac{\hat{\eta}_{\mathrm{T}}}{\eta} (\boldsymbol{\nabla} \langle A \rangle)^2 = \frac{\hat{\eta}_{\mathrm{T}}}{\eta} \langle B \rangle^2. \tag{30}$$

Expression (30) is an appealing result, linking the energies of the small- and large-scale magnetic fields. It is, however, only useful for calculating the turbulent diffusion of the magnetic field if it can be shown that the turbulent diffusivities of A (i.e., $\hat{\eta}_{\mathrm{T}}$) and of B (i.e., η_{T}) amount to the same thing.

In order to address this question, it is instructive to revisit the analysis of Cattaneo et al. (1988), who explored some of the differences between the transport of scalar and vector fields in two-dimensional turbulence, paying particular attention to the turbulent advection of mean fields. Following Cattaneo et al. (1988), we express the evolution equation for $\langle A \rangle$ both as a standard scalar advection-diffusion equation, i.e.,

$$\frac{\partial \langle A \rangle}{\partial t} + \boldsymbol{\nabla} \cdot \langle \boldsymbol{u} a \rangle = \eta \nabla^2 \langle A \rangle, \tag{31}$$

and as the "uncurled" induction equation, i.e.,

$$\frac{\partial \langle A \rangle}{\partial t} - \langle \boldsymbol{u} \times \boldsymbol{b} \rangle_z = \eta \nabla^2 \langle A \rangle. \tag{32}$$

As in Moffatt (1983), the flux term in (31) may be expressed as

$$F_i \equiv \langle u_i a \rangle = -D_{ij} \frac{\partial \langle A \rangle}{\partial x_j} - E_{ijk} \frac{\partial^2 \langle A \rangle}{\partial x_j \partial x_k} + \cdots, \tag{33}$$

and the electromotive force in (32) as

$$\mathcal{E}_i \equiv \langle \boldsymbol{u} \times \boldsymbol{b} \rangle_i = \alpha_{ij} \langle B \rangle_j + \beta_{ijk} \frac{\partial \langle B \rangle_j}{\partial x_k} + \cdots. \tag{34}$$

Comparison of these two transport terms in Equations (31) and (32) thus gives

$$\frac{\partial}{\partial x_i}\left(D_{ij} \frac{\partial \langle A \rangle}{\partial x_j} + E_{ijk} \frac{\partial^2 \langle A \rangle}{\partial x_j \partial x_k} + \cdots\right) \equiv \tag{35}$$

$$\hat{z}_i \left(\epsilon_{jpq} \alpha_{ij} \frac{\partial \langle A \rangle}{\partial x_p} \hat{z}_q + \epsilon_{jpq} \beta_{ijk} \frac{\partial \langle A \rangle}{\partial x_k \partial x_p} \hat{z}_q + \cdots\right), \tag{36}$$

where $\hat{z}_i = \delta_{i3}$. Equating derivatives of $\langle A \rangle$ then leads to the expressions

$$\frac{\partial D_{il}}{\partial x_i} = \hat{z}_i \epsilon_{jlq} \alpha_{ij} \hat{z}_q, \tag{37}$$

$$D_{lm}^{(s)} + \frac{\partial E_{ilm}}{\partial x_i} = \hat{z}_i \epsilon_{jmq} \beta_{ijl} \hat{z}_q, \tag{38}$$

where $D_{lm}^{(s)}$ denotes the symmetric part of the tensor. Cattaneo et al. (1988) concentrated on Equation (37) and its consequences, in particular highlighting an important difference in the turbulent pumping velocity of two-dimensional scalar and vector fields. Here, in considering turbulent diffusion, our interest lies in Equation (38). The left-hand side describes the diffusion of the mean scalar potential $\langle A \rangle$, and the right-hand side describes the diffusion of a mean magnetic field (the "β-effect" of mean-field electrodynamics). Comparison of expressions (29) and (33) shows that $\hat{\eta}_{\mathrm{T}}$ is identified with D_{11}; hence (38) can be expressed as

$$\hat{\eta}_{\mathrm{T}} + \frac{\partial E_{111}}{\partial x} = \beta_{231} = \eta_{\mathrm{T}}. \tag{39}$$

Thus, in general, for inhomogeneous turbulence, η_{T} and $\hat{\eta}_{\mathrm{T}}$ will not be the same; however, for the case considered here of homogeneous turbulence (for which, by definition, gradients of any transport tensors will vanish), we can indeed identify η_{T} and $\hat{\eta}_{\mathrm{T}}$. Thus expression (30) becomes expression (1) for the turbulent diffusion of the magnetic field. We note that the scaling (6), which also leads to expression (1), may thus be interpreted as an assumption of homogenous turbulence.

In the derivation above, the details of the averaging procedure are not important, provided the average is over some spatial scale intermediate between that of the large-scale field and that of the small-scale turbulence. In order to make use of expression (1) to calculate η_{T} computationally, it is most convenient to adopt a uniform imposed field as the large-scale field; spatial averages are then taken over the computational domain. The uniform field component, which will of course remain constant in time, may be regarded as being formally of infinite lengthscale.

Expression (30), involving spatial averages, provides a means of calculating η_{T} at any particular instant. In a sufficiently large domain, this value will be essentially independent of time; however, the domain size needed for this to hold is truly immense, as discussed by Cattaneo & Hughes (2009). Thus, even with the large domains considered here, η_{T} defined by (30) varies with time, and so further temporal averaging is needed in order to improve its accuracy. This is illustrated by Figure 4, which shows the growth and subsequent saturation of the spatial averages of the magnetic and velocity fluctuations. The time averaging in the calculation of η_{T} is started once the system enters a stationary state, the precision of the result being determined by the length of the averaging interval.

In order to verify the correspondence between temporal and spatial averaging, we have considered the determination of η_{T} in a series of simulations with differing scale separation between that of the turbulent forcing and the size of the domain. Figure 5, which plots the cumulative time average of η_{T} versus time for three different scale separations, reveals two features. The first is that there is a well-defined time average of η_{T}, regardless of the domain size (although of course this must be sufficiently large to allow unconstrained turbulent dynamics), thus confirming that temporal and spatial averages may be exchanged. The second is that, as expected, the degree of fluctuation decreases with increasing domain size.

4.3. Comparison Between the Two Methods

Figure 6 shows a comparison between the two methods of calculating η_{T}, for a range of ε and q. The standard deviation, which quantifies the unavoidable inaccuracies associated with the derivation of a single η_{T} from a decaying large-scale field, is represented by the error bars. In principle, the results obtained from the experiments with an imposed uniform field

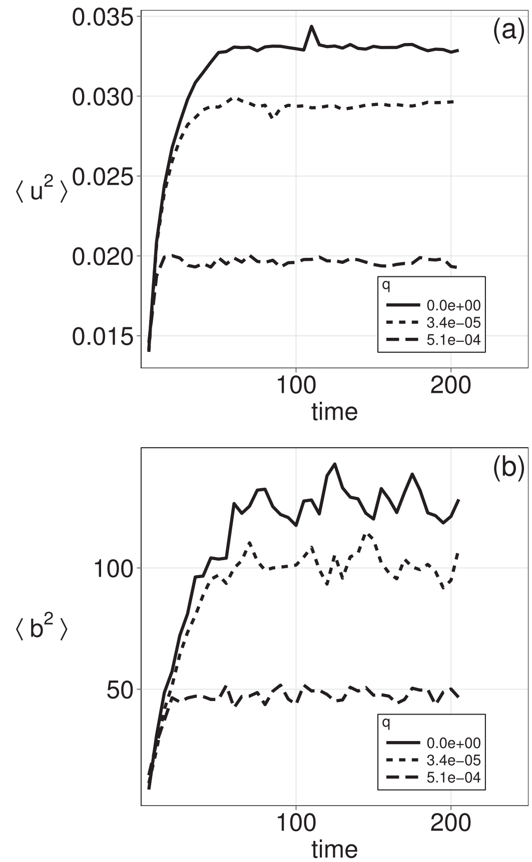

Figure 4. (a) Mean square of the velocity fluctuations as a function of time, for several values of q and with $\varepsilon = 10^4$. (b) Same as (a), but for the magnetic fluctuations.

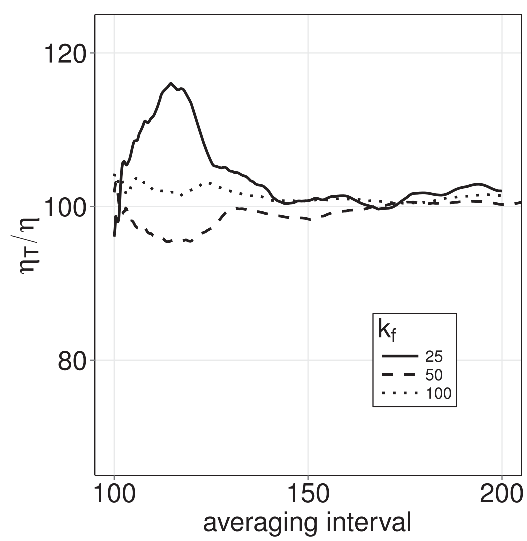

Figure 5. Cumulative time average of η_{T} for runs with the same physical setup (i.e., the same local Reynolds number), but with different scale separation. Scale separation is quantified using the forcing wavenumber k_{f}. (The case with the largest k_{f} is the $Pm = 1$, $Q = 10^{-6}$ entry in Table 1.)

also have a spread of values; this, however, turns out to be insignificant compared with that of the decaying field cases. The results match surprisingly well, given the different nature of the measurements and the forms of the initial large-scale components, further supporting the validity of the calculations with an imposed uniform field. At this point, we also note that

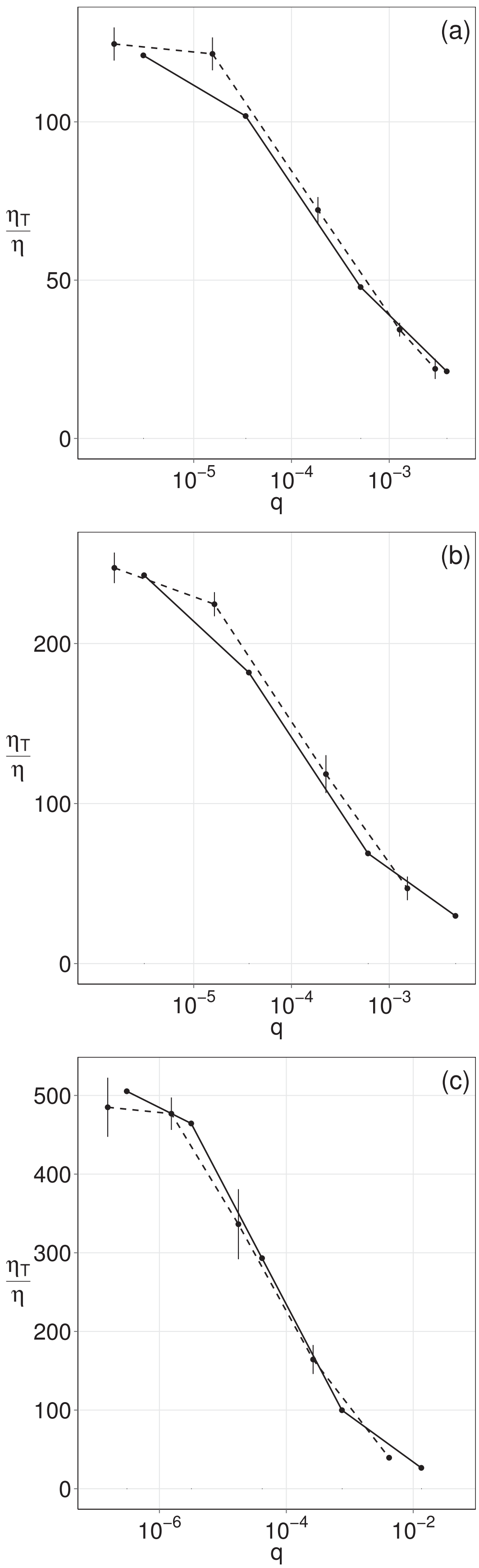

Figure 6. Comparison of η_{T} obtained from the decaying (dotted line) and the imposed field (solid line) runs for different ε: (a) $\varepsilon = 10^4$, (b) $\varepsilon = 2 \times 10^4$, (c) $\varepsilon = 4 \times 10^4$.

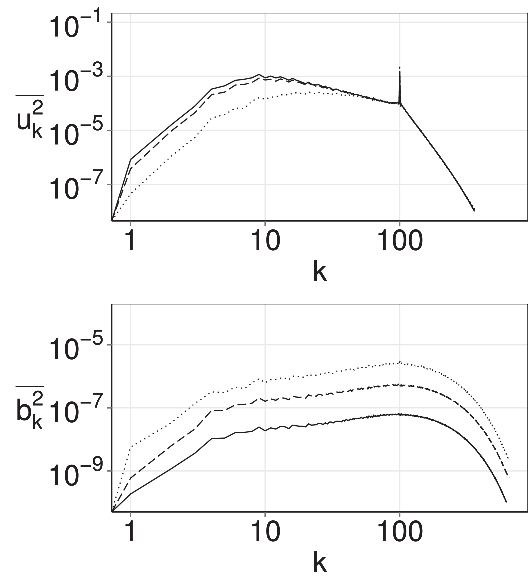

Figure 7. Time-averaged spectra of (a) u^2 and (b) b^2 for $Pm = 1$ and $Q = 10^{-7}$ (solid line), $Q = 10^{-6}$ (dashed line) and $Q = 10^{-5}$ (dotted line).

the imposed uniform field method has been successfully compared at moderate Reynolds numbers with the "Turbulent Ångström Method" and the "Method of Oscillatory Sines" proposed by Tobias & Cattaneo (2013). These methods may also be utilized to calculate the turbulent diffusivity to any required accuracy.

5. TURBULENT MAGNETIC DIFFUSION

5.1. Dependence on Rm, Pm and Q

Given the advantages of the imposed field method, we chose to employ it in order to determine the dependence of η_{T} on the various parameters of the problem: Q, Rm, and Pm. Computations of the governing equations (17)–(18) were performed for three different values of Pm, each for a set of several Q values. In all cases, the value of κ is the same; changing Pm is thus equivalent to changing ϵ, the magnetic Reynolds number defined by the dimension of the domain. The input parameter range for Q was chosen to span the transition between the purely kinematic regime ($Q = 0$) and the point where, although the large-scale field strength is several orders of magnitude below the equipartition value, η_{T} is significantly suppressed. The main output of the simulations is the mean square fluctuation of the magnetic field; this is the only observable required to calculate η_{T} from (1).

All other quantities of relevance are summarized in Table 1. From the wavenumber column, it is clear that the characteristic length scale of the flow decreases once the Lorentz force is non-negligible ($Q \neq 0$). This trend is reflected in the behavior of Rm, Re, and most importantly, η_{T}, and corresponds to the growing influence of the generated small-scale magnetic fluctuations on the turbulent motions.

Figure 7 shows the time-averaged spectra for (twice) the kinetic energy and magnetic energy, for $Pm = 1$ and three different values of the imposed field, represented by the parameter Q. The key point to note is that the characteristic inverse cascade of two-dimensional hydrodynamic turbulence is interrupted by an extremely weak large-scale field ($Q = 10^{-5}$). This is therefore an indication that very weak large-scale fields can have a significant dynamical effect. The interesting question of the physics underlying the transition between inverse and forward cascades in two-dimensional MHD turbulence has been addressed recently by Seshasayanan & Alexakis (2016).

As discussed above, the crucial idea under investigation here is the ability of even a very weak large-scale field to reduce the turbulent magnetic diffusivity significantly, as a consequence of the strong small-scale fluctuations. As can be seen in Figure 8 (solid lines), which shows the dependence of η_T on the strength of the large-scale magnetic field, this quenching of the turbulent diffusion is confirmed by our high-resolution simulations. The parameter q measures the ratio of the large-scale component of the magnetic energy to the total kinetic energy. For each of the three sets of runs distinguished by a different value of Pm in Figure 8 (left column), the turbulent diffusivity has decreased to about half of its kinematic value even for q as small as 10^{-4}, i.e., when q is of the order of the inverse large-scale magnetic Reynolds number. At $q = 10^{-2}$, η_T is reduced by an order of magnitude. Furthermore, as the right column of Figure 8 demonstrates, the disruptive effect of the generated small-scale field is significant even when its energy contribution is only a few percent of the total kinetic energy.

Although, as described above, it is feasible to explore a wide range of imposed field strengths at given, quite large, Reynolds numbers, exploring a significant range of variation in Rm at a fixed (high) value of Re remains extremely challenging. Figure 9 shows the results for η_T for a very modest range of Pm at the fixed value of $\kappa = 10^4$. The trend is as expected; however, determining the precise ε dependence accurately will require runs at phenomenally high resolution.

5.2. The Mechanism of Suppression

Expression (12) reveals the simple dependence of η_T on the amplitude and correlation time of the turbulence under the short-sudden approximation. Although this formula is not strictly appropriate for general MHD turbulence, which is neither particularly short nor sudden, it is nonetheless instructive to examine the dependence of η_T on τ_c and $\langle u^2 \rangle$. Cattaneo (1994) considered the Lagrangian correlation function for the case of two-dimensional MHD turbulence with an imposed field, and concluded that the suppression of turbulent diffusion resulted from the emergence of a memory time long compared with the turbulent turnover time, with very little change in the amplitude of the turbulence. In a different line of attack, Keating et al. (2008) considered the *cross-phase* $\mathcal{P}$, defined, for two-dimensional turbulence with an imposed magnetic field $\boldsymbol{B} = B_0 \hat{y}$, by

$$\mathcal{P} = \frac{u_x a}{\sqrt{\langle u_x^2 \rangle \langle a^2 \rangle}}. \tag{40}$$

They showed that the suppression of turbulent diffusion could be accounted for solely by a reduction in $\mathcal{P}$ as B_0 is increased, again with essentially no reduction in the turbulence intensity.

Here we address this issue by considering how the correlation length of the flow changes with the strength of the mean field. Figure 10 shows the variation in the structure of the velocity in the kinematic and weak-field regimes, with the magnetic field causing a shift toward smaller scales. To quantify this, and with the assumption that the correlation time scales with the turnover time of the flow, the correlation length may be calculated via the correlation function

$$\rho_u(\boldsymbol{x}, \boldsymbol{l}) = \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{l}). \tag{41}$$

We define the correlation length l_c to be the length scale at which the normalized average of ρ_u over the spatial domain is reduced by a factor e. Since the velocity field is approximately isotropic and homogeneous, l_c does not depend strongly on direction or position. Figures 11(a), (b) show plots of the rms velocity $\bar{u}$ and the correlation length l_c as functions of q. Although both the velocity and the correlation length drop off as the magnetic field strength is increased, the correlation length suffers a much stronger reduction; depending on Pm, by a factor of at least two, while the velocity decreases by only about 30%. This again demonstrates that the suppression of the turbulent magnetic diffusivity is associated predominantly with a change in the correlation properties of the flow, rather than with a decrease in the turbulent intensity. Figure 11(c) shows the correlation time calculated in two different ways: as the ratio $l_c/\bar{u}$ and as $\eta_T/\overline{u^2}$. Their functional form is in agreement; the mismatch in amplitude is not surprising, as there is some arbitrariness in determining l_c from the correlation function (41).

It is also possible to use l_c as a diagnostic tool by adopting it as the characteristic scale of turbulence, i.e., to calculate various turbulent quantities independently of those derived from Zeldovich's theorem, and then to compare them. We note that the local magnetic Reynolds number defined by $Rm_{l_c} = l_c\, u\, \eta^{-1}$ is in quantitatively better agreement with η_T/η than Rm based on the Taylor microscale; this can be seen from a comparison of Rm_{l_c} in Figure 12 with the values of η_T/η and Rm in Table 1.

5.3. Comparison with Theoretical Models

In order to compare the results of our simulations with the suppression formula (15), we calculate Rm_{l_c} as a ratio of the kinematic turbulent diffusivity (i.e., that of the $Q = 0$ case) to the molecular diffusivity. We then use Q from Table 1, together with Rm_{l_c}, to obtain η_T from (15). The left column of Figure 8 shows both the model and the measured η_T. It turns out that the model consistently predicts weaker suppression of η_T. The discrepancy is almost negligible near the kinematic state, as expected, but increases as the field becomes more dynamic (Figure 8). Furthermore, this effect is exhibited more strongly in the runs at higher ε.

Finally, it is of interest to examine the meaning of expression (14) and to see whether, despite the misgivings expressed in Section 2, it can be of practical use in the prediction of η_T suppression. Given that, subject to the short-sudden approximation, η_T is given by expression (12), with τ_c and $\langle u^2 \rangle$ evolving with the turbulence, it would seem that in expression (14), τ_c and $\langle u^2 \rangle$ must be interpreted as taking their *kinematic* values. From our numerical experiments we are able to calculate both η_T and $\tau_c(\langle u^2 \rangle - \langle b^2 \rangle)$ independently; with τ_c and $\langle u^2 \rangle$ taking their kinematic ($Q = 0$) values, the only dynamical element in the formula is $\langle b^2 \rangle$. Figure 13 plots the ratio of these two quantities, after an additional averaging in time, as a function of the imposed field strength. It can be seen that there is agreement only in the kinematic regime, and hence

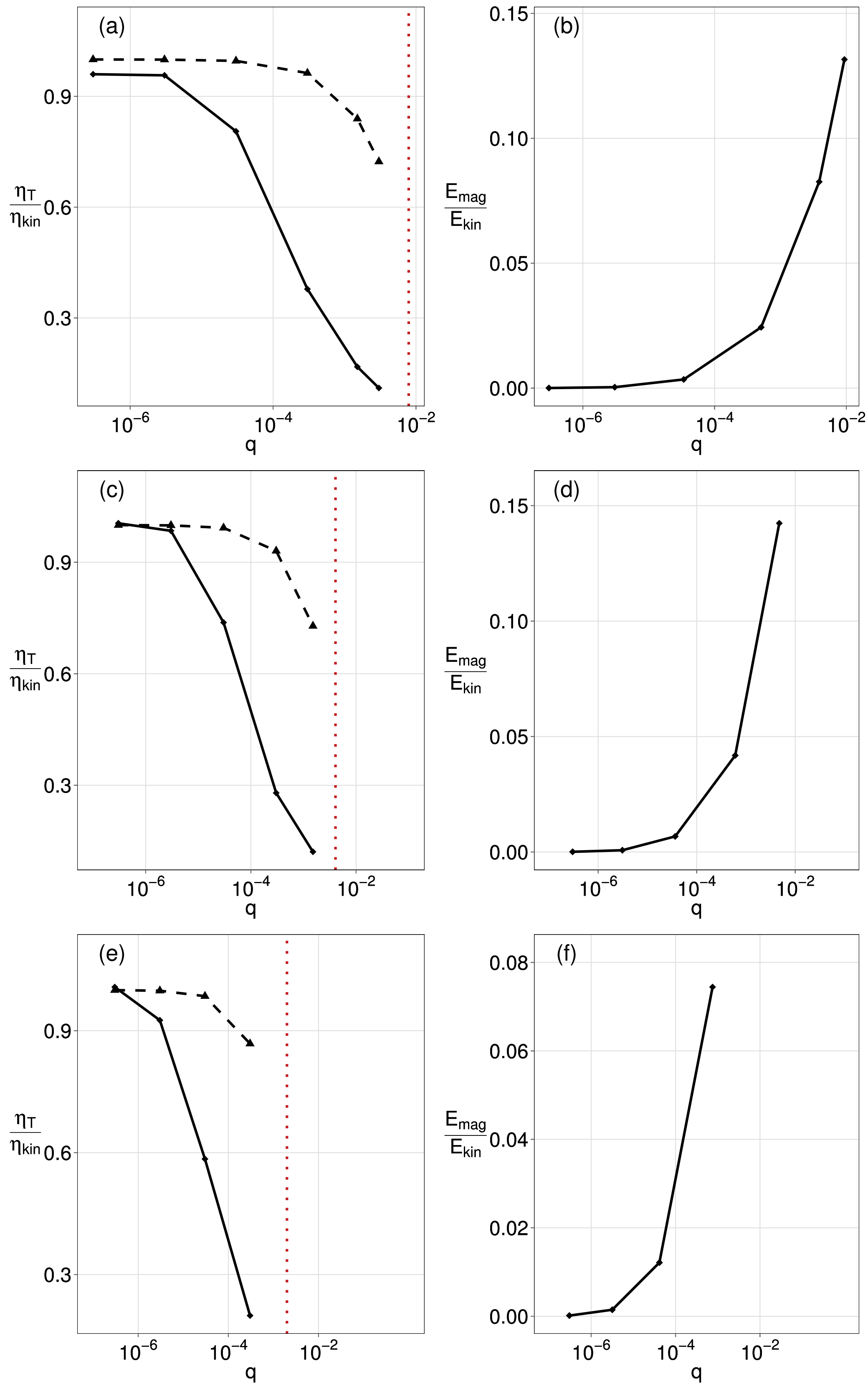

Figure 8. (a), (c), (e): Turbulent diffusivity, normalized by its kinematic value, as a function of q. (b), (d), (f): Ratio of magnetic to kinetic energy. For (a) and (b), $\varepsilon = 10^4$; for (c) and (d), $\varepsilon = 2 \times 10^4$; for (e) and (f), $\varepsilon = 4 \times 10^4$. Solid curves are obtained directly from the simulations, by measuring η_T. Dashed lines are calculated from the closure model equation (15). The critical q, given by $q_{cr} = Rm_{l_c}^{-1} = \eta/\eta_{T(kin)}$, is marked as the dotted line.

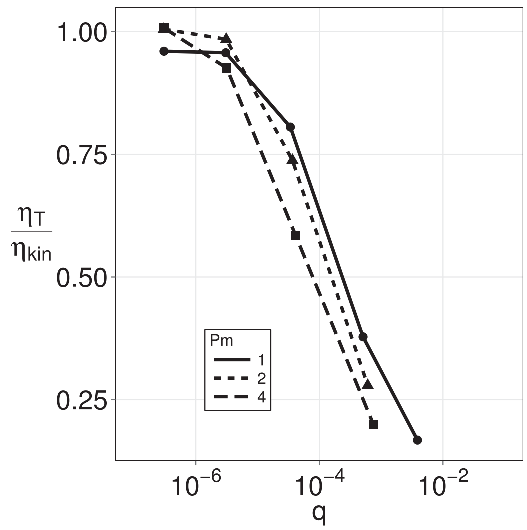

Figure 9. Turbulent diffusivity, normalized by its kinematic value, as a function of q for different values of Pm.

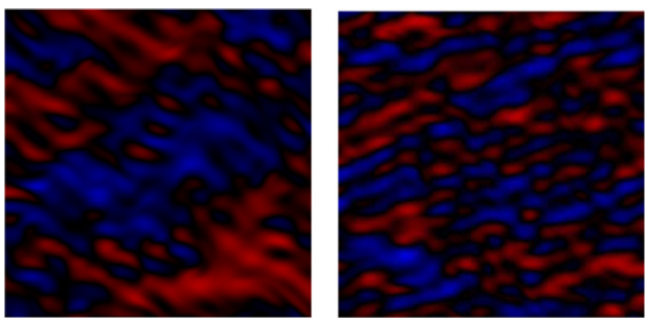

Figure 10. Sections (1/16 of the simulation area surface) of the x-component of the velocity for $Pm = 1$. Left: kinematic case; right: $Q = 10^{-5}$.

that expression (14) is not a particularly good description of the suppression process.

6. CONCLUSIONS

We have revisited the classical problem of the diffusion of magnetic field in two-dimensional MHD turbulence. Today's computational power allows two important properties of turbulent flows to be captured, particularly when attention is restricted to two-dimensional flows. One is the considerable separation of spatial scales between the size of the system and that of the energy-containing eddies; the other is the large (though not astronomically large) values of the Reynolds numbers ($O(10^4)$ on the system scale and $O(10^2)$ on the scale of the eddies). Also, rather than estimating the turbulent diffusivity η_T through the decay of a spatially varying large-scale magnetic field, we have instead considered the diffusion of the vector potential A for a uniform imposed gradient in A, i.e., a uniform magnetic field; although the turbulent diffusivities of $\boldsymbol{B}$ and A are, in general, not the same, they are equivalent for homogeneous turbulence. This method, which analyzes a stationary state of MHD turbulence, thus allows the calculation of η_T to arbitrary accuracy.

Our high-resolution simulations confirm the key finding of Vainshtein & Rosner (1991) and Cattaneo & Vainshtein (1991) that a very weak large-scale magnetic field has a strong

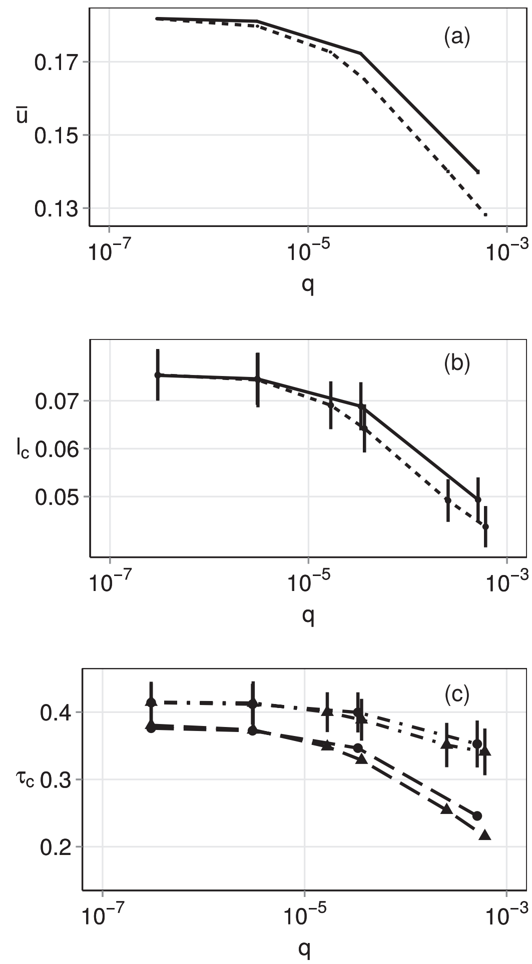

Figure 11. (a) rms velocity $\bar{u}$ as a function of q for $Pm = 1$ (solid line) and $Pm = 2$ (dashed line). (b) l_c as a function of q; the error bars represent the standard deviation resulting from the time averaging. (c) Correlation time calculated in two different ways. The dot–dashed curves show τ_c defined by $\tau_c = l_c/\bar{u}$ (circles denote $Pm = 1$, triangles $Pm = 2$). The dashed curves show τ_c defined by $\tau_c = \eta_T/\overline{u^2}$.

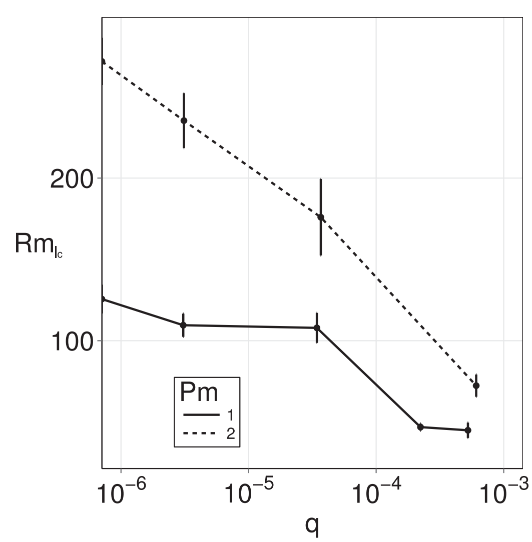

Figure 12. Local magnetic Reynolds number Rm_{l_c} as a function of q for $Pm = 1$ and $Pm = 2$. The error bars represent the standard deviation resulting from the time average of Rm_{l_c}.

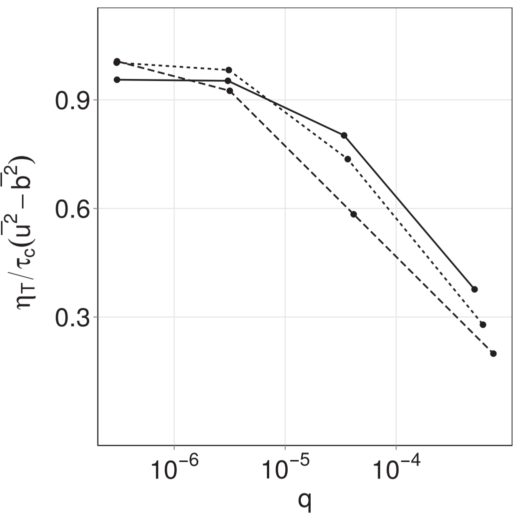

Figure 13. Comparison of η_T, derived from (1), with the expression $\tau_c(\overline{u^2} - \overline{b^2})$, with τ_c and $\overline{u^2}$ taking their kinematic values; the solid line denotes $Pm = 1$, the dotted line denotes $Pm = 2$, and the dashed line denotes $Pm = 4$.

dynamical influence, through the small scales generated, on the turbulent magnetic diffusivity. Indeed, as shown in Figure 8, the suppression of η_T appears to be even stronger than suggested by the model formula (15). We have explored the dependence of η_T on a range of large-scale field strengths B_0 at several fixed large values of the magnetic Reynolds number Rm; unfortunately, even with the high resolutions currently possible, exploring a wide range of Rm remains impracticable.

As discussed in Section 2, theoretical considerations ascribe the suppression of the turbulent diffusivity to changes in either the correlation time or the amplitude of the turbulence (or some combination of the two). Our findings back up those resulting from the Lagrangian analysis of Cattaneo (1994), and the cross-phase analysis of Keating et al. (2008), in that the reduction of the turbulent diffusivity arises primarily not from a marked reduction in the turbulence amplitude, but through a disruption of the correlation of the phase of the flow. This rather subtle modification of the small-scale features of the flow through the forces arising from a small-scale magnetic field that, ultimately, derives from a weak large-scale field, is of widespread astrophysical relevance. Here we have considered the rather simple case of the turbulent magnetic diffusivity in two-dimensional turbulence. As noted in Section 1, similar considerations apply also to the quenching of the mean-field dynamo α-effect. In a different context, Tobias et al. (2007) showed that jet formation in β-plane turbulence, which arises through the coherence of the small-scale Reynolds stresses, can be totally disrupted by the inclusion of an extremely weak large-scale field; as in the current study, the large-scale field leads to dynamical consequences even when its energy is $O(Rm^{-1})$ times the turbulent kinetic energy. In another area of astrophysical importance, rapidly rotating convective turbulence, the Reynolds stresses can lead to the formation of large-scale vortices (e.g., Favier et al. 2014; Guervilly et al. 2014). The flow may then act as a dynamo, as a consequence of which the strong small-scale magnetic field diminishes the coherence of the Reynolds stresses, and hence the driving for the large-scale vortices. Whether this signifies the demise of the large-scale vortices, or whether they can recover, depends critically on the value of Rm, which determines whether the small-scale field can be maintained (Guervilly et al. 2015). Although the physics is more complicated than that considered in this paper, in that the magnetic field is self-generated rather than imposed, the main underlying idea is again that a strong small-scale field can inhibit turbulent transport.

The set-up we have considered here is essentially the simplest model of MHD turbulence. Still within the context of two-dimensional turbulence, there are two interesting extensions that we are currently pursuing. The first is to consider inhomogeneous turbulence, which brings in the possibility of turbulent pumping of the magnetic field (the γ-effect of mean-field electrodynamics, sometimes referred to as "turbulent diamagnetism"); the important question is then to quantify any suppression of the pumping by a dynamic magnetic field (cf. Tao et al. 1998). The second is to explore the possible suppression of the turbulent transport of passive scalar concentration in MHD turbulence (cf. Diamond & Gruzinov 1997). This is of significance, for example, in considering the transport of light elements in magnetized stellar interiors.

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REFERENCES

Cattaneo, F. 1994, ApJ, 434, 200
Cattaneo, F., & Hughes, D. W. 1996, PhRvE, 54, 4532
Cattaneo, F., & Hughes, D. W. 2009, MNRAS, 395, L48
Cattaneo, F., Hughes, D. W., & Proctor, M. R. E. 1988, GApFD, 41, 335
Cattaneo, F., & Vainshtein, S. I. 1991, ApJL, 376, L21
Cox, S. M., & Matthews, P. C. 2002, JCoPh, 176, 430
Danilov, S., & Gurarie, D. 2001, PhRvE, 63, 061208/1
Diamond, P. H., & Gruzinov, A. V. 1997, PhRvL, 78, 3306
Diamond, P. H., Hughes, D. W., & Kim, E. 2005, in Fluid Dynamics and Dynamos in Astrophysics and Geophysics, ed. A. M. Soward et al. (Boca Raton, FL: CRC Press)
Favier, B., Silvers, L. J., & Proctor, M. R. E. 2014, PhFl, 26, 096605
Gruzinov, A. V., & Diamond, P. H. 1994, PhRvL, 72, 1651
Gruzinov, A. V., & Diamond, P. H. 1996, PhPl, 3, 1853
Guervilly, C., Hughes, D. W., & Jones, C. A. 2014, JFM, 758, 407
Guervilly, C., Hughes, D. W., & Jones, C. A. 2015, PhRvE, 91, 041001
Keating, S. R., Silvers, L. J., & Diamond, P. H. 2008, ApJL, 678, L137
Kulsrud, R., & Anderson, S. 1992, ApJ, 396, 606
Moffatt, H. K. 1983, RPPh, 46, 621
Monin, A. S., & Yaglom, A. M. 1975, Statistical Fluid Mechanics: Mechanics of Turbulence, Vol. II (Cambridge, MA: MIT Press)
Pouquet, A. 1978, JFM, 88, 1
Pouquet, A., Frisch, U., & Léorat, J. 1976, JFM, 77, 321
Proctor, M. R. E. 2003, in Stellar Astrophysical Fluid Dynamics, ed. M. J. Thompson, & J. Christensen-Dalsgaard (Cambridge: Cambridge Univ. Press)
Seshasayanan, K., & Alexakis, A. 2016, PhRvE, 93, 013104
Tao, L., Cattaneo, F., & Vainshtein, S. I. 1993, in Solar and Planetary Dynamos, ed. M. R. E. Proctor, P. C. Matthews, & A. M. Rucklidge (Cambridge: Cambridge Univ. Press)
Tao, L., Proctor, M. R. E., & Weiss, N. O. 1998, MNRAS, 300, 907
Tobias, S. M., & Cattaneo, F. 2013, JFM, 717, 347
Tobias, S. M., Diamond, P. H., & Hughes, D. W. 2007, ApJL, 667, L113
Vainshtein, S. I., & Rosner, R. 1991, ApJ, 376, 199
Zeldovich, Y. B. 1957, Sov. Phys. JETP, 4, 460