Article:
Johnson, P. orcid.org/0000-0002-6472-3000 (2014) Equivariant GW Theory of Stacky Curves. Communications in Mathematical Physics, 327 (2). pp. 333-386. ISSN 0010-3616

https://doi.org/10.1007/s00220-014-2021-1

EQUIVARIANT GW THEORY OF STACKY CURVES

PAUL JOHNSON

Abstract. We extend Okounkov and Pandharipande's work on the equivariant Gromov-Witten theory of $\mathbb{P}^1$ to a class of stacky curves $\mathcal{X}$. Our main result uses virtual localization and the orbifold ELSV formula to express the tau function $\tau_{\mathcal{X}}$ as a vacuum expectation on a Fock space.

As corollaries, we prove the Decomposition Conjecture for these $\mathcal{X}$, and prove that $\tau_{\mathcal{X}}$ satisfies a version of the 2-Toda hierarchy. Coupled with degeneration techniques, the result should lead to treatment of general orbifold curves.

Contents

1. Introduction

In their trilogy [OP06b] [OP06a] [OP06c], Okounkov and Pandharipande completely determine the Gromov-Witten theory of curves. This paper is the beginning of a program to extend their results to stacky curves. More specifically, this paper is the stacky analog of the logical starting place of their program, [OP06a], which studies the equivariant Gromov-Witten theory of $\mathbb{P}^1$.

The main result of [OP06a] is an explicit formula for the $\mathbb{C}^*$-equivariant Gromov-Witten theory of $\mathbb{P}^1$ as a vacuum expectation on the infinite wedge. In addition to being an important tool for the rest of the trilogy, this description quickly implies that the equivariant Gromow-Witten theory of $\mathbb{P}^1$ satisfies a form of the 2-Toda hierarchy. In rough outline, the strategy of [OP06a] is to first use virtual localization and the ELSV formula to reduce everything to the combinatorics of the symmetric group. The resulting combinatorics are then handled using the infinite wedge, which also makes contact with integrable hierarchies.

We follow the structure and proofs of [OP06a] rather closely. Our main result is an explicit formula for the $\mathbb{C}^*$-equivariant Gromov-Witten theory of stacky toric $\mathbb{P}^1$ in terms of vacuum expectations on a certain Fock space. This is the natural analog of the main result [OP06a], and the proof follows the same method. The first step of the proof is essentially the same: the orbifold structure must be taken into account in the localization calculation, and in place of the ELSV formula there is the orbifold ELSV formula of [JPT11], but the general form is the same.

In the second step there is a larger novelty: for ineffective orbifolds, the infinite wedge is not a large enough Fock space. On an ineffective orbifold, also known as a gerbe, the generic point has a nontrivial stabilizer group K. In this case, instead of the symmetric group, the orbifold ELSV formula uses the wreath product K_n of S_n with K^n. To handle these combinatorics, we use a larger Fock space $\mathcal{Z}_K$, which has been studied in a context convenient for us by Wang and collaborators [FW01, QW07]. The Fock space $\mathcal{Z}_K$ plays the same role for the wreath product K_n as the infinite wedge plays for the symmetric group, and indeed $\mathcal{Z}_K$ can be viewed as a tensor product of $|K|$ copies of the infinite wedge. Utilizing this decomposition of $\mathcal{Z}_K$ as a tensor product, we can use the same proofs as [OP06a].

As a result of our formula, we again get a connection to the 2-Toda hierarchy. Furthermore, in the ineffective case we obtain a proof of the decomposition conjecture of [HHP$^+$07], which in our case expresses the τ function for ineffective orbifolds as a product of related τ functions for effective orbifolds after an appropriate change of variables.

In the rest of the introduction we introduce the notation needed (Section 1.1) to state our results more carefully (Section 1.2), then give an overview of the structure of the paper (Section 1.3) and a brief discussion of related work and future directions (Section 1.4).

1.1. **Basic setup.**

1.1.1. *Targets.* The effective orbifolds we consider are are denoted $\mathcal{C}_{r,s}$. The coarse moduli space of $\mathcal{C}_{r,s}$ is $\mathbb{P}^1$; however $\mathcal{C}_{r,s}$ has orbifold structure $\mathbb{Z}_r$ at 0 and $\mathbb{Z}_s$ at infinity.

Our general target $\mathcal{X}$ is a toric stack that is a gerbe over $\mathcal{C}_{r,s}$. In the one dimensional case, being toric is equivalent to $\mathcal{X}$ having all abelian isotropy groups. Denote the generic isotropy group by K, so that $\mathcal{X}$ is a K-gerbe over $\mathcal{C}_{r,s}$ with trivial band.

Denote the isotropy groups of $\mathcal{X}$ over 0 and ∞ as R and S, respectively. They are extensions of $\mathbb{Z}_r$ and $\mathbb{Z}_s$ by K, and so have canonical surjections to the corresponding cyclic groups. More explicit discussions of gerbes, the space $\mathcal{X}$ and the groups R and S are found in Section 2.

1.1.2. *Equivariant Chen-Ruan cohomology.* In orbifold Gromov-Witten theory, the state space is Chen-Ruan Cohomology. As a vector space, the Chen-Ruan cohomology $H^*_{CR,\mathbb{C}^*}(\mathcal{X})$ is the cohomology of the inertia stack $\mathcal{IX}$, which we think of as the space of constant loops $S^1 \to \mathcal{X}$. However, this is only as vector spaces – in general there is a grading shift and a different cup-product. For an introduction to orbifolds and Chen-Ruan cohomology, see [ALR07].

As we are doing $\mathbb{C}^*$-equivariant Gromov-Witten theory, we need to consider the $\mathbb{C}^*$-equivariant Chen-Ruan cohomology. Thus, our cohomology rings are modules over $H^*_{CR}(\mathrm{pt}) = \mathbb{C}[t]$. Let F be the fixed point set of the $\mathbb{C}^*$ action on $\mathcal{IX}$. Then Atiyah-Bott localization [AB84] implies that if we invert t, we have $H^*_{CR}(\mathcal{X}) \cong H^*_{CR}(F)$ as $\mathbb{C}[t, t^{-1}]$ modules. Throughout the rest of this paper we tacitly invert t and use this isomorphism.

Note that F consists of the constant loops that map to 0 or ∞, and so consists of $|R| + |S|$ isolated points, naturally labeled by elements of R and S. Use $\mathbf{0}_\rho$ and $\boldsymbol{\infty}_\sigma$ to denote the corresponding basis of $H^*_{CR}(\mathcal{X})$ as a $\mathbb{C}[t, t^{-1}]$ module.

The elements $\mathbf{0}_\rho$ have degree zero in $H^*(\mathcal{IX})$, but due to degree shifting have potentially nonzero degree $\iota(\rho)$ in $H^*_{CR}(\mathcal{X})$. Identifying the elements of $\mathbb{Z}_r$ with the rational numbers $k/r, 0 \leq k < 1$, we have $\iota(\rho) = \phi(\rho)$, where ϕ is the natural map $\phi : R \to \mathbb{Z}_r$.

We use $\mathfrak{r}$ and $\mathfrak{s}$ to denote tuples of elements of R and S, respectively. The number of elements in a tuple, i.e. its length, is denoted $\ell(\mathfrak{r})$, while the elements themselves are $\mathfrak{r}_i, 1 \leq i \leq \ell(\mathfrak{r})$. Finally, use

$$\iota(\mathfrak{r}) = \sum_{i=1}^{\ell(\mathfrak{r})} \iota(\mathfrak{r}_i).$$

1.1.3. *Moduli spaces.* Our main focus of study are the two moduli spaces $\overline{\mathcal{M}}_{g,\mathfrak{r},\mathfrak{s}}(\mathcal{X}, d)$ and $\overline{\mathcal{M}}_{g,\mathfrak{r}}(\mathcal{B}R)$, which we now describe.

In orbifold Gromov-Witten theory, the marked points get orbifold structure and are forced to map to certain twisted sectors. This gives additional discrete invariants for the moduli space of maps to the target space. Consider the moduli space $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,\mathfrak{r},\mathfrak{s}}(\mathcal{X}, d)$ of stable degree d maps from genus g nodal orbifold curves Σ to $\mathcal{X}$. The curve Σ has $\ell(\mathfrak{r}) + \ell(\mathfrak{s})$ marked points, where the ith marked point is pulling back the cohomology class $\mathbf{0}_{\mathfrak{r}_i}$ (or $\boldsymbol{\infty}_{\mathfrak{s}_i}$).

This space is generally singular, but it carries a virtual fundamental class of the expected dimension

$$\dim_{\mathbb{C}} [\overline{\mathcal{M}}_{g,\mathfrak{r},\mathfrak{s}}(\mathcal{X}, d)]^{\mathrm{vir}} = 2g - 2 + d(1/r + 1/s) + \ell(\mathfrak{r}) + \ell(\mathfrak{s}) - \iota(\mathfrak{r}) - \iota(\mathfrak{s}).$$

The $\mathbb{C}^*$ action on $\mathcal{X}$ induces a $\mathbb{C}^*$ action on $\overline{\mathcal{M}}$, and the virtual fundamental class becomes an equivariant class, so that we can pull back and integrate equivariant Chen-Ruan cohomology classes from $\mathcal{X}$. Applying virtual localization to $\overline{\mathcal{M}}$, the fixed point loci are products of simpler moduli spaces $\overline{\mathcal{M}}_{g,n}(\mathcal{B}R)$ and $\overline{\mathcal{M}}_{g,m}(\mathcal{B}S)$.

A point in $\overline{\mathcal{M}}_{g,n}(\mathcal{B}R)$ is given by a orbifold curve Σ together with a non-orbifold R-cover $\tilde{\Sigma}$. The monodromy at each marked point corresponds to a conjugacy class in R, and we use $\overline{\mathcal{M}}_{g,\mathfrak{r}}$ to denote the subset of $\overline{\mathcal{M}}_{g,n}(\mathcal{B}R)$ having monodromy $\mathfrak{r}_i$ around the ith marked point point.

A superscript bullet indicates disconnected source curves. For example, $\overline{\mathcal{M}}^{\bullet}_{g,\mathfrak{r}}$ denotes the same moduli space as above, but with Σ and $\tilde{\Sigma}$ allowed to be disconnected. The genus of a disconnected curve is defined by noting that the euler characteristic $\chi(\Sigma) = 2 - 2g$ for connected curves and is additive under disjoint union, and so we can use $\chi(\Sigma) = 2-2g$ to define the genus for disconnected curves.

1.1.4. *The τ function.* Over each of the moduli spaces we consider certain integrals packaged into generating functions – we describe this now.

The evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,\mathfrak{r},\mathfrak{s}}(\mathcal{X}, d) \to \mathcal{I}\mathcal{X}$ allow us to pull back Chen-Ruan cohomology classes.

In addition to pulled back classes, we also consider ψ classes, given by $\psi_i = c_1(\mathbb{L}_i)$, where the fiber of $\mathbb{L}_i$ over a curve Σ is the cotangent bundle at the ith marked point: $\mathbb{L}_i = T^*_{p_i}\Sigma$.

Define

$$\left\langle \prod_{i=1}^{\ell(\mathfrak{r})} \tau_{a_i}(\mathbf{0}_{\mathfrak{r}_i}) \prod_{j=1}^{\ell(\mathfrak{s})} \tau_{b_j}(\boldsymbol{\infty}_{\mathfrak{s}_j}) \right\rangle^{\mathcal{X}}_{g,d} = \int_{[\overline{\mathcal{M}}_{g,\mathfrak{r},\mathfrak{s}}(\mathcal{X},d)]^{\mathrm{vir}}} \prod_{i=1}^{\ell(\mathfrak{r})} \psi_i^{a_i} \mathrm{ev}_i^*(\mathbf{0}_{\mathfrak{r}_i}) \prod_{j=1}^{\ell(\mathfrak{s})} \psi_j^{b_j} \mathrm{ev}_j^*(\boldsymbol{\infty}_{\mathfrak{s}_j}).$$

The τ function is a generating function for all disconnected invariants of $\mathcal{X}$. It is a formal series in q and u, encoding the degree and the genus, respectively, as well as two infinite sets of variables $x = \{x_i(\rho) | 0 \leq i \in \mathbb{Z}, \rho \in R\}$ and $x^* = \{x^*_j(\sigma) | 0 \leq j \in \mathbb{Z}, \sigma \in S\}$, encoding the integrated classes.

The τ function is defined as:

$$\tau_{\mathcal{X}}(x, x^*, u) = \sum_{g\in\mathbb{Z}} \sum_{d\geq 0} u^{2g-2} q^d \left\langle \exp\left(\sum_{i,\rho} x_i(\rho)\tau_i(\mathbf{0}_\rho) + \sum_{j,\sigma} x^*_j(\sigma)\tau_j(\boldsymbol{\infty}_\sigma) \right) \right\rangle^{\bullet}_{g,d}$$

where in the sums i, j run from 0 to ∞, and ρ and σ run over R and S, respectively.

1.1.5. *The Hurwitz-Hodge function.* In calculating $\tau_{\mathcal{X}}$ through virtual localization, we are lead to consider certain *Hurwitz-Hodge integrals* over $\overline{\mathcal{M}}_{g,\mathfrak{r}}(\mathcal{B}R)$ that we now describe.

The Hodge bundle $\mathbb{E}$ over $\overline{\mathcal{M}}_{g,\mathfrak{r}}(\mathcal{B}R)$ has fiber $H^0(\tilde{\Sigma}, \omega_{\tilde{\Sigma}})$, where ω is the dualizing sheaf, which is the canonical sheaf if Σ is smooth. The R action on $\tilde{\Sigma}$ induces an R action on $\mathbb{E}$, and so $\mathbb{E}$ splits into subbundles $\mathbb{E}_V$, where $\mathbb{E}_V$ is the part of $\mathbb{E}$ where where R acts by a given irreducible representation V of R.

In the orbifold case, the orbifold structure at the marked point has an effect on this Chern class. If the ith marked point has a $\mathbb{Z}_{r_i}$ isotropy group, define $\overline{\psi}_i = r_i\psi_i$. The $\overline{\psi}$ classes ignore the effect of the orbifold structure.

The Hurwitz-Hodge classes are the Chern classes of the bundles $\mathbb{E}_V$, namely $\lambda_i^V = c_i(\mathbb{E}_V)$. Integrals of λ_i^V and $\overline{\psi}$ classes are called Hurwitz-Hodge integrals.

The representation of R of interest is $T_0\mathcal{X}$, which write as T for brevity. Package the Hurwitz-Hodge integrals that are linear in the λ_i^T into the following Hurwitz-Hodge generating function, which appears in the localization calculation:

$$H^\bullet_{\mathfrak{r}}(z_1, \ldots, z_n, u) = \sum_{g \in \mathbb{Z}} u^{2g-2} \int_{\overline{\mathcal{M}}^\bullet_{g,\mathfrak{r}}(\mathcal{B}R)} \prod_{i=1}^n \frac{z_i}{1 - z_i\overline{\psi}_i} \sum_{i=0}^m (-r)^i \lambda_i^T.$$

Here, the bullet in $\overline{\mathcal{M}}^\bullet$ means not only that we are allowing disconnected curves, but are actually allowing unstable curves. In the unstable case, there is an ad-hoc definition of the resulting integrals that is compatible with both localization and the orbifold ELSV formula. This convention allows for a uniform treatment of localization.

1.1.6. *Fock space formalism.* Our two theorems are expressions for the two generating functions $\tau_{\mathcal{X}}$ and $H^\bullet_{\mathfrak{r}}$ in terms of operators acting on a certain vector space $\mathcal{Z}_K$ called a Fock space. We give now just enough information about this Fock space to state our results – for more details, see Sections 4.3 and 4.4.

The Fock space $\mathcal{Z}_K$ occurs in the study of the representation theory of wreath products: $\mathcal{Z}_K$ is a graded vector space (the degree of an element is called its energy), and its nth graded piece has a natural basis indexed by the irreducible representations of $S_n \rtimes K^n$, the wreath product of S_n with K. From this, we see $\mathcal{Z}_K$ has an inner product $(\cdot,\cdot)$, and a preferred vector of energy zero, denoted $v_\emptyset$ and called the vacuum vector.

The Fock space $\mathcal{Z}_K$ is a representation of many important algebras. This is most easily seen by noting that another way of looking at $\mathcal{Z}_K$ is as a tensor product of n copies of the infinite wedge $\bigwedge^{\frac{\infty}{2}} V$. Since the infinite wedge is a projective representation of $\mathfrak{gl}_\infty$, $\mathcal{Z}_K$ has $|K|$ copies of $\mathfrak{gl}_\infty$ acting on it. If M is an operator in their product, we write $\langle M \rangle = (v_\emptyset, M v_\emptyset)$, called the vacuum expectation of M.

1.2. **Main Results.**

1.2.1. *Operator Expression for the Hurwitz-Hodge function.* Our first theorem is an expression for the Hurwitz-Hodge generating function as a vacuum expectation on $\mathcal{Z}_K$.

Theorem A.

$$H^\bullet_{\mathfrak{r}}\left(z_{\mathfrak{r}}, \frac{u}{r^{1/2}}\right) = (u|K|)^{-\ell(\mathfrak{r})} \left\langle \prod_{i=1}^{\ell(\mathfrak{r})} \mathcal{A}_{r_i}(z_i, u) \right\rangle.$$

The operators $\mathcal{A}_{r_i}(z_i, u)$ are defined in Equation (37) – they are complicated but explicit operators on the Fock space $\mathcal{Z}_K$.

The proof is presented in Section 5. The geometric content of the theorem is contained in the orbifold ELSV formula of [JPT11], and in some sense this theorem can be understood as an interpolation of this formula. In particular, the proof does not depend on the localization calculation of Section 3.

1.2.2. *Operator Expression.* Our main result is an expression for $\tau_{\mathcal{X}}$ as a vacuum expectation on $\mathcal{Z}_K$.

Theorem B.

$$\tau_{\mathcal{X}}(x, x^*, u) = \left\langle e^{\sum x_i(\mathfrak{r})\mathbf{A}_{\mathfrak{r}}[i]} e^{\frac{\alpha_r(0)}{u|R|}} q^{\widetilde{H}} e^{\frac{\alpha_{-s}(0)}{u|S|}} e^{\sum x_j^*(\mathfrak{s})\mathbf{A}^*_{\mathfrak{s}}[j]} \right\rangle.$$

Here the α are the bosonic operators acting on the Fock space, $\widetilde{H}$ is a twisted version of the energy operator (the twisting reflects the gerbe structure of $\mathcal{X}$), and the operators $\mathbf{A}$ are closely related to the operator $\mathcal{A}$ from our first theorem.

The proof of the main theorem is essentially combining the localization calculation of Section 3 with Theorem A. The main theorem is proved in the first part of Section 6. The remainder of this section explores several corollaries of the main theorem. We highlight two of these now.

1.2.3. *Decomposition Conjecture.* The first application of our main theorem, carried out in Section 6.2, is to verify the Decomposition Conjecture of [HHP$^+$07] for our $\mathcal{X}$. For more on the decomposition conjecture, see [TT].

Given any G-gerbe $\mathcal{X}$ over an effective orbifold $\mathcal{B}$, the Decomposition Conjecture describes how to construct a disconnected and effective cover $\mathcal{Y}$ of $\mathcal{B}$ and a flat gerbe c on $\mathcal{Y}$. It then asserts that conformal field theories on $\mathcal{X}$ are equivalent to those on $\mathcal{Y}$ twisted by c. This has several mathematical implications, in particular that the Gromov-Witten theory of $\mathcal{X}$ is equivalent to the Gromov-Witten theory of $\mathcal{Y}$ twisted by c.

In our case (and whenever $\mathcal{X}$ has trivial band) the space $\mathcal{Y}$ is $|K|$ copies of $\mathcal{C}_{r,s}$, indexed by representations γ of K. Furthermore, in our case the only effect of the twisting is a rescaling of variables. Thus informally we have:

Corollary C: Decomposition. *After an appropriate change of variables, we have:*

$$\tau_{\mathcal{X}} = \prod_{\gamma \in K^*} \tau_{\mathcal{C}^{\gamma}_{r,s}}.$$

The proof follows from our operator expression for $\tau_{\mathcal{X}}$, essentially since $\mathcal{Z}_K$ is a tensor product copies of $|K|$ copies of the infinite wedge.

1.2.4. *The 2-Toda Hierarchy.* The main corollary of our main theorem, derived in Section 6.5, is that our τ functions satisfy integrable hierarchies.

Corollary D: Integrability. *The tau function $\tau_{\mathcal{C}_{r,s}}$ is a τ function for the 2Toda hierarchy.*

Using a different approach [MT11], Milanov and Tseng have already obtained this result. Note that the exact integrable hierarchy that τ satisfies is some reduction of the 2-Toda hierarchy, as is already the case with non-orbifold $\mathbb{P}^1$. However, the relationship between this reduction and the expression in the infinite wedge is not yet clear.

In the noneffective case, since $\tau_{\mathcal{X}}$ a product of the $\tau_{\mathcal{C}^{\gamma}_{r,s}}$, we see that $\tau_{\mathcal{X}}$ satisfies $|K|$ commuting copies of the 2-Toda hierarchy. The Fock space $\mathcal{Z}_K$ was used in a similar way in [QW07], and this result and the decomposition conjecture were part of our motivation for bringing $\mathcal{Z}_K$ into the picture.

1.3. **Overview of the paper.** In this section, we give a quick overview of the body of the paper, section by section.

Section 2: Basics on orbifolds and gerbes. This section collects results on toric orbifold curves $\mathcal{X}$. All of our gerbes can be constructed as root stacks, which we review in Section 2.1. In Section 2.2 we use the root stack construction to give explicit cocycle descriptions of the isotropy groups R and S.

In Section 2.3 we prove two lemmas describing how the orbifold and gerbe structures of $\mathcal{X}$ interact with the orbifold structure of source curves Σ mapping into $\mathcal{X}$. Lemma 1 describes the effective isotropy, while Lemma 2 addresses the gerby isotropy. While the behavior of the effective isotropy is locally determined and well known, the gerby isotropy is a global phenomenon. For gerbes over $\mathbb{P}^1$ this global behavior was used in [CC09] to study Hurwitz-Hodge integrals. It is slightly more complicated, and we believe new, to describe what happens for gerbes over $\mathcal{C}_{r,s}$.

Section 3: Localization. In Section 3 we carry out the virtual localization calculation that determines $\tau_{\mathcal{X}}$ from $H^{\bullet}_{\mathfrak{r}}$.

The results of [GP99] say that Atiyah-Bott localization holds for virtual fundamental classes. Thus, to perform our equivariant integrals over $\overline{\mathcal{M}}_{g,\mathfrak{r},\mathfrak{s}}(\mathcal{X},\beta)$, it is enough to determine the fixed point locus F and its virtual normal bundle inside $\overline{\mathcal{M}}$.

In general, this calculation is standard, and essentially we use the same techniques as [GP99] and [OP06a]. However, localization in the context of orbifolds and gerbes is less standard, and so we provide a thorough description of the calculation.

Section 4: Orbifold ELSV formula and Fock spaces. Section 4 introduces the remaining tools in the proof: the orbifold ELSV formula and the Fock space $\mathcal{Z}_K$.

The orbifold ELSV formula expresses $H^{\bullet}_{\mathfrak{r}}$ in terms of Hurwitz numbers, which count covers of $\mathbb{P}^1$ having specified ramification. Considering the monodromy of the cover reduces Hurwitz theory to group theory. The resulting group theory question has a convenient expression in terms of representation theory, as was known already to Frobenius.

In the effective case, the pertinent groups are the symmetric groups, and the pertinent Hurwitz numbers count covers that have profile μ over $\mathbf{0}$, profile $(r, r, \ldots, r, r)$ over ∞, and simple ramification over the appropriate number of other points. In the noneffective case, the symmetric group is replaced by the wreath product.

The representation theory needed to calculate Hurwitz numbers is conveniently encoded in the Fock spaces. In Section 4.3 we introduce the infinite wedge $\bigwedge^{\frac{\infty}{2}} V$ and the operators on it, and explain how to express Hurwitz numbers as a vacuum expectation. Section 4.4 then extends this to wreath Hurwitz numbers and $\mathcal{Z}_K$.

Section 5: Interpolating orbifold ELSV. Section 5 is devoted to proving our Theorem A, which can be understood as follows.

The orbifold ELSV formula is only valid when the z_i are integers satisfying certain congruence relations, essentially because the Hurwitz numbers make sense only in this situation. However, for our application we require a formula valid for any z_i.

It is clear from the definition that the coefficients of u in $H^{\bullet}_{\mathfrak{r}}(z, u)$ are rational functions of z, andhence are determined by the infinite set of values provided by the orbifold ELSV formula. The expression for Hurwitz numbers in terms of Fock space, after some manipulation, allows for an interpolation to non-integer values. Theorem A says that this interpolation agrees with $H^{\bullet}_{\mathfrak{r}}(z, u)$.

The proof has two steps. First, we show that when z_i are integers of the right form, the complicated $\mathcal{A}$ operators simplify to the operators appearing in the description of Hurwitz numbers. We then show that their vacuum expectation is, after a combinatorial factor, rational.

The proof of the second step follows from a technical commutator calculation involving hypergeometric functions. We have not included this proof: it is lengthy, but once the correct definition of the $\mathcal{A}$ operators is found there is no substantive difference from the proof in [OP06a] for the non-orbifold setting. The complete proof may be found in Chapter VII of the author's thesis [Joh].

Section 6: The main theorem and applications. The main theorem is proved in section 6.1. The hard work has already been done; all that remains to do is to wire together Theorem A and the localization calculation. The remainder of Section 6 derives several consequences of Theorem B.

First, we prove the decomposition conjecture in Section 6.2. With the decomposition conjecture in hand, we can work with the effective case for the rest of the paper.

Before proving the full force of Corollary D, we derive a few smaller corollaries by hand, proving the equivariant divisor and string equations in Section 6.3, and deriving the lowest equation of the 2-Toda hierarchy by hand in Section 6.4.

The proof of Corollary D is contained in Section 6.5. The proof consists in showing that the $\mathbf{A}$ operators can be conjugated to the standard bosonic operators α_k. This conjugation gives a change of variables between the Gromov-Witten times and the 2-Toda times. The proof for the change of variables is rather technical and depends again on some hypergeometric function calculations. Again, we omit this as they are nearly identical to those found in [OP06a] and send the reader wanting full details to [Joh].

1.4. **Related work and future directions.** Gromov-Witten theory of orbifold curves has been studied before, largely from the angle of integrable systems and in particular Frobenius manifolds. Milanov and Tseng have studied $\mathcal{C}_{r,s}$ in both the equivariant [MT11] and nonequivariant [MT08] settings, in particular proving Corollary D. Rossi has approached them from the point of view of Symplectic Field Theory [Ros10] – see also [Ros08] for an approach to the Gromov-Witten theory of smooth curves from this perspective.

One benefit to our approach is that it handles the ineffective case. A longer term benefit is that our approach, coupled with degeneration techniques, should lead to a further extension of Okounkov and Pandharipande's program, yielding connections to Hurwitz theory, modular forms in case the target has Euler characteristic zero, and the Virasoro constraints. The symplectic field theory technique is quite similar to degeneration, and can obtain some of these results; it would be interesting to carefully compare the two approaches.

1.5. **Acknowledgements.** This paper is an adaptation of my thesis, [Joh], and would not have been possible without the guidance and encouragement of my advisor, Yongbin Ruan. I also greatly benefitted from conversations about the related work [JPT11] with my coauthors Rahul Pandharipande and Hsian-Hua Tseng. The author was supported in part by NSF grants DMS-0602191 and DMS-0902754.

2. Basics on orbifolds and gerbes

2.1. **Root construction.** In this section we collect some facts on the root construction, which produces $\mathbb{Z}_n$-gerbes from line bundles. It is a result of [FMN] that all toric stacks can be constructed by root constructions. For curves this result follows from the Meyer-Vietoris theorem.

We briefly recall three points of view on the root construction in terms of geometry, cohomology, and category theory. We also recall how iterating the root construction can produces general abelian groups K.

2.1.1. *Geometry.* Geometrically, the construction of a $\mathbb{Z}_n$ gerbe on $\mathcal{C}_{r,s}$ from a line bundle L is easily described. The total space of the line bundle has a natural $\mathbb{C}^*$ action, which is just the usual $\mathbb{C}^*$ action on each fiber. Removing the zero section of the total space of L gives the associated $\mathbb{C}^*$ bundle for L.

Taking the quotient by the natural $\mathbb{C}^*$ action gives $\mathcal{C}_{r,s}$ back. If instead we take the quotient by $\mathbb{C}^*$ acting by the nth power of the standard action, then the generic point has a $\mathbb{Z}_n$ stabilizer. The result is thus a $\mathbb{Z}_n$ gerbe over $\mathcal{C}_{r,s}$, which we denote by $\mathcal{C}_{r,s}^{(L,n)}$.

2.1.2. *Cohomology.* For an abelian group K, banded K-gerbes on $\mathcal{X}$ are classified by $H^2(X,K)$. In case $K=\mathbb{Z}_n$, we have the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}_n \to 0.$$

Part of the corresponding long exact sequence in cohomology is:

$$H^2(\mathcal{C}_{r,s},\mathbb{Z}) \xrightarrow{\cdot n} H^2(\mathcal{C}_{r,s},\mathbb{Z}) \xrightarrow{g} H^2(\mathcal{C}_{r,s},\mathbb{Z}_n) \to H^3(\mathcal{C}_{r,s},\mathbb{Z}). \tag{1}$$

Since $H^2(\mathcal{C}_{r,s},\mathbb{Z})$ classifies topological line bundles over $\mathcal{C}_{r,s}$ and $H^2(\mathcal{C}_{r,s},\mathbb{Z}_n)$ classifies $\mathbb{Z}_n$ gerbes with trivial band, the map g gives a way to construct a banded $\mathbb{Z}_n$ gerbe out of a line bundle. This is the cohomological description of the root construction.

It is easily seen using Meyer-Vietoris that $H^3(\mathcal{C}_{r,s},\mathbb{Z})=0$, and so every banded $\mathbb{Z}_n$ gerbe can be constructed this way.

2.1.3. *Universal property.* The most powerful view of the root construction, and the one that explains its name, is the categorical viewpoint. Namely, we construct $\mathcal{Y}^{L,n}$ by giving L an nth root.

Let $f:\mathcal{Y}^{L,n}\to\mathcal{Y}$ be the forgetful map. Then, $\mathcal{Y}^{L,n}$ "comes with" a line bundle M and an isomorphism $\varphi: M^n \to f^*L$, and this line bundle M is the only thing that distinguishes $\mathcal{Y}^{L,n}$ from $\mathcal{Y}$.

We understand $\mathcal{Y}^{L,n}$ if we understand all maps to it. Define a map to $\mathcal{Y}^{L,n}$ to be a map to $\mathcal{Y}$, together with a line bundle that's an nth root of the pullback of L. More precisely, a map $g:\mathcal{Z}\to\mathcal{Y}^{L,n}$ consists of a triple (h,N,φ), with h a map from $\mathcal{Z}$ to $\mathcal{Y}$, N a line bundle on $\mathcal{Z}$, and φ an isomorphism from N^n to $h^*(L)$.

2.1.4. *Iteration.* For K a more complicated abelian group, K-gerbes are easily constructed using a fibered product of the root construction.

Explicitly, with $K=\oplus_{j=1}^m\mathbb{Z}_{n_j}$, let $L_1,\dots,L_m$ be line bundles on $\mathcal{C}_{r,s}$. Then form a K-gerbe by taking the fibered product of the root constructions for each of these line bundles. We denote this by:

$$\mathcal{C}_{r,s}^{(L_j,n_j)} = \mathcal{C}_{r,s}^{(L_1,n_1)} \times_{\mathcal{C}_{r,s}} \cdots \times_{\mathcal{C}_{r,s}} \mathcal{C}_{r,s}^{(L_m,n_m)}.$$

2.2. **Cocycle description of isotropy groups.** We now use the categorical viewpoint of the root construction to give explicit cocycle descriptions of the isotropy groups R and S of $\mathcal{X}=\mathcal{C}_{r,s}^{(L_j,n_j)}$ over 0 and ∞. We do this by constructing faithful representations of R and S.

2.2.1. *A faithful representation.* We now construct a faithful representation V of R. The argument is easily adapted to give a faithful representation of S.

First, note that L_j is a representation of $\mathbb{Z}_r$. Let the action of $1_r \in \mathbb{Z}_r$ on L_j by $e^{2\pi i k_j/r}$.

Let M_j be the n_jth root of L_j. Then $M_j|_0$ is a one dimensional representation of R.

Another one dimensional representation of R is $T_0\mathcal{X}$. We now show that

$$V = \bigoplus_{j=1}^{m} M_j|_0 \oplus T_0\mathcal{X}$$

is a faithful representation of R, which identifies R as a subgroup of $(S^1)^{m+1}$.

To do this, note that although we do not yet know R, we know a generating set for R: a generating set for K together with any element that maps to $1_r \in \mathbb{Z}_r$.

From the fiber product construction, it is clear that each $\mathbb{Z}_{n_j}$ acts trivially on M_k for $k \neq j$, but nontrivially on M_j. In particular, R has an element g_j that generates $\mathbb{Z}_{n_j}$, acts on M_j by multiplication by $e^{2\pi i/n_j}$, and acts trivially on M_k for $k \neq j$.

Now, consider $T_0\mathcal{X}$. It is clear that K acts trivially on $T_0\mathcal{X}$, but any element mapping to $1 \in \mathbb{Z}_r$ acts as $e^{2\pi i/r}$. Hence, we see V is in fact faithful.

Identifying S^1 with $\mathbb{R}/\mathbb{Z}$m the element $g_j, 1 \leq j \leq m$ becomes $(0,\ldots,0,\frac{1}{n_j},0,\ldots,0)$.

2.2.2. *The cocycle description.* To construct a cocycle description of R, we need to choose a lifting of $1_r \in \mathbb{Z}_r$ to R. The representation V gives us one way of making this choice.

Since M_j is an n_jth root of L_j, we see that any preimage of 1_r in R must act by an n_jth root of $e^{2\pi i k_j/r}$. Now g_j acts on M_j by $e^{2\pi i/n_j}$ and is in K. Adding some g_j if necessary, we can choose a preimage of g_0 of 1_r that acts on M_j by $e^{2\pi i k_j/(r n_j)}$ for all j.

In the identification of R as a subgroup of $(S^1)^{m+1}$, this amounts to taking $g_0 = (\frac{k_j}{n_j r},\ldots,\frac{k_m}{n_m r},\frac{1}{r})$. Thus, writing down the multiplication in terms of the $g_i, 0 \leq i \leq m$ then gives us a K 2-cocycle β on $\mathbb{Z}_r$ that describes R as a possibly nontrivial extension of $\mathbb{Z}_r$ by K:

$$\beta(a,b) = \begin{cases} (\frac{k_1}{n_1},\ldots,\frac{k_m}{n_m}) & a+b \geq 1 \\ 0 & a+b < 1 \end{cases} . \tag{2}$$

The data of β is contained in the element $(\frac{k_1}{n_1},\ldots,\frac{k_m}{n_m}) \in K$. We denote this element by $\mathbb{k}_0$ for R, and the similar element of K describing the group S we denote by $\mathbb{k}_\infty$. As the arguments for R and S are often analogous, we sometimes just give the argument for R and drop the subscript and write $\mathbb{k}$. When we wish to use the cocycle description of elements of R, we write them as $r_i = (a_i, k_i) \in \mathbb{Z}_r \times_\beta K = R$.

2.2.3. *Inverses.* We now describe inverses in the cocycle description. Some care is required, since typically $-r_i \neq (-a_i, -k_i)$. Introduce

$$\delta_r(x) = \begin{cases} 0 & x \neq 0 \mod r \\ 1 & x = 0 \mod r \end{cases}$$

and

$$\delta_r^\vee(x) = \begin{cases} 1 & x \neq 0 \mod r \\ 0 & x = 0 \mod r \end{cases}$$

so that

$$\delta_r(x) + \delta_r^\vee(x) = 1.$$

Then it follows that

$$-(a,k) = (-a, -k - \delta_r^\vee(a)\mathbb{k}). \tag{3}$$

2.3. **Two lemmas about maps to $\mathcal{X}$.** In this section we prove two lemmas that together give us a complete understanding about how the orbifold structure of $\mathcal{X}$ affects maps from curves into $\mathcal{X}$. Due to localization, we only need to consider orbifold maps that give the map $z \mapsto z^d$ from $\mathbb{P}^1 \to \mathbb{P}^1$ on the underlying topological spaces. The results of this section describe the behavior of the orbifold structure of such maps. Again, due to localization, the source curve has orbifold structure only over 0 and ∞.

2.3.1. *The effective case.* The behavior of the effective orbifold structure is completely determined locally by the degree of the map.

Lemma 1. *Let $\mathbb{Z}_r$ and $\mathbb{Z}_n$ act on $\mathbb{C}$ as their standard embeddings in $\mathbb{C}^*$, and let $f : \mathbb{C}/\mathbb{Z}_r \to \mathbb{C}/\mathbb{Z}_n$ be a representable map of orbifolds which on coarse moduli spaces gives the map $z \mapsto z^d$. Then $r = n/\gcd(d,n)$ and the standard generator of $\mathbb{Z}_r$ maps to $d \in \mathbb{Z}_n$.*

Proof. The map $f : \mathbb{C}/\mathbb{Z}_r \to \mathbb{C}/\mathbb{Z}_n$ must lift to an equivariant map g from $\mathbb{C} \to \mathbb{C}$ which covers f:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{g} & \mathbb{C} \\ {\scriptstyle z^r}\downarrow & & {\scriptstyle z^n}\downarrow \\ \mathbb{C}/\mathbb{Z}_r & \xrightarrow{f} & \mathbb{C}/\mathbb{Z}_n \end{array}$$

Since f is of the form $z \mapsto z^d$, then we must have $g(z) = z^a$, and commutativity gives $an = rd$. Then there is some k with $a = kd/\gcd(d,n)$ and $r = kn/\gcd(d,n)$. We need to show that $k = 1$.

Suppose the generator $1 \in \mathbb{Z}_r$ maps to $l \in \mathbb{Z}_n$, then since g is equivariant, we have

$$e^{2\pi i a/r} z^a = g(e^{2\pi i/r} z) = e^{2\pi i l/r} g(z) = e^{2\pi i l/r} z^a$$

Since f is representable, the map on isotropy groups must be injective, and so we must have that $e^{2\pi i a/r}$ has order r, that is, a and r are relatively prime - which forces $k = 1$.

Finally, we see that the generator of $\mathbb{Z}_r$ maps to $a/r = d/n \in \mathbb{Z}_d$. □

2.3.2. *The gerby case.* In contrast to the effective part, the image of a degree d map in the ineffective part of the isotropy is completely unconstrained locally. Instead, there is a global monodromy constraint.

We only need to consider maps from $\mathbb{P}^1$ with two orbifold points, mapping to zero and infinity. The key point is that if the degree of the map and the orbifold behavior of one of the points is fixed, the orbifold behavior at the other marked point uniquely determined by the gerbe structure. We prove this first in the case of the $\mathbb{Z}_n$ gerbe coming from a line bundle L; the general case then follows from the fibered product construction.

The proof uses the categorical properties of the nth root construction and the existence and basic properties of the orbifold Chern-Weil class $cw_1(L) \in H^2(\mathcal{Y}, \mathbb{Q})$ of an orbifold line bundle L on $\mathcal{Y}$, (see, for instance, [CR04]).

Lemma 2. *Suppose that $\mathcal{C}$ is an orbifold that is topologically a $\mathbb{P}^1$ with orbifold structure only over 0 and ∞, $\mathcal{X} = \mathcal{C}_{r,s}^{(L,n)}$, and $f : \mathcal{C} \to \mathcal{X}$ is a representable, $\mathbb{C}^*$ fixed map of degree d. Suppose further that $1_r \in \mathbb{Z}_r$ acts on L_0 by a/r and that $1_s \in \mathbb{Z}_s$ acts on L_∞ by b/s. From this, we have $cw_1(L) = \ell + a/r + b/s$ for some $\ell \in \mathbb{Z}$.*

Then if the generator of the isotropy group of 0 in $\mathcal{C}$ maps to $(d, u) \in R$, and the generator of the isotropy over ∞ maps to $(d, v) \in S$, we have that

$$d\ell + \left\lfloor \frac{d}{r} \right\rfloor a + \left\lfloor \frac{d}{s} \right\rfloor b = u + v \mod n.$$

Note that since u and v are in $\mathbb{Z}_n$, Lemma 2 determines one from the other.

Proof. By Lemma 1, the image of the isotropy group in the effective parts of the isotropy groups are indeed as given, and so we must show that the ineffective parts of the isotropy satisfy the above relation.

By construction, on $\mathcal{X}$ the line bundle L has an nth root M . We have

$$cw_1(f^*(M)) = \frac{d}{n} cw_1(L) = \frac{d\ell}{n} + \frac{d}{r}\frac{a}{n} + \frac{d}{s}\frac{b}{n}. \tag{4}$$

On the other hand, we know that the fractional part of $cw_1(f^*(M))$ is determined by the behavior of the isotropy groups on M, which are known: $1 \in \mathbb{Z}_n$ acts as $1/n$, and $1/r \in \mathbb{Z}_r, 1/s \in \mathbb{Z}_s$ act by $a/(nr), b/(ns)$, respectively.

Thus, the generator of the isotropy group at 0 on $\mathcal{C}$ acts on $f^*(M)$ by $\left\langle \frac{d}{r} \right\rangle \frac{a}{n} + \frac{u}{n}$, while the generator of the isotropy group at ∞ acts on $f^*(M)$ by $\left\langle \frac{d}{s} \right\rangle \frac{b}{n} + \frac{v}{n}$.

Subtracting these contributions from the total Chern-Weil class of $f^*(M)$ in (4), we see that the contribution from 0 can be viewed as

$$\frac{d}{r}\frac{a}{n} - \left\langle \frac{d}{r} \right\rangle \frac{a}{n} - \frac{u}{n} = \left\lfloor \frac{d}{r} \right\rfloor \frac{a}{n} - \frac{u}{n}$$

and a similar equation holds for the contribution from zero. Thus we see that

$$\frac{d\ell}{n} + \frac{a}{n}\left\lfloor \frac{d}{r} \right\rfloor + \frac{b}{n}\left\lfloor \frac{d}{s} \right\rfloor - \frac{u}{n} - \frac{v}{n}$$

must be an integer, which is the desired result.

□

For the fibered product case, with $\mathcal{X}$ a K gerbe over $\mathcal{C}_{r,s}$, u and v are elements of K. The result of our lemma is an equation that holds in each $\mathbb{Z}_{n_i}$, with a, b, ℓ replaced by a_i, b_i, ℓ_i. The a_i and b_i package together to $\mathbb{k}_0$ and $\mathbb{k}_\infty$, respectively, and we package the ℓ_i as $\mathbb{L}$, so that we have:

$$u + v = d\mathbb{L} + \left\lfloor \frac{d}{r} \right\rfloor \mathbb{k}_0 + \left\lfloor \frac{d}{s} \right\rfloor \mathbb{k}_\infty \tag{5}$$

as an equation in K.

Note that this monodromy condition seemingly depends upon which line bundles L_i we pick to construct $\mathcal{X}$, and not just the gerbe $\mathcal{X}$ itself. This is because different line bundles produce different cocycles for the group extensions R and S. The change in Equation 5 produced by choosing a different line bundle is exactly countered by the different cocycle description of R.

3. Localization

3.1. **Generating Functions.**

3.1.1. *Hurwitz-Hodge generating function.* Localization expresses the Gromov-Witten invariants of $\mathcal{X}$ in terms of Hurwitz-Hodge integrals. These integrals are encoded in the generating function

$$H^{0,\circ}_{g,\mathfrak{r}}(z_1,\ldots,z_n) = \int_{\overline{\mathcal{M}}_{g,\mathfrak{r}}(\mathcal{B}R)} \prod_{i=1}^{\ell(\mathfrak{r})} \frac{z_i}{1-z_i\overline{\psi}_i} \sum_{i=0}^{\infty}(-r)^i\lambda_i^{T_0}.$$

Similarly, $H^{\infty,\circ}_{g,\mathfrak{s}}$ encodes the Hodge integrals that occur at ∞. The superscript $\circ$ denotes the connected theory; a superscript $\bullet$ means we allow disconnected curves. If neither a $\circ$ nor $\bullet$ superscript is present, the generating function is assumed to be connected. We frequently omit the 0 or ∞ subscript if it is unimportant.

3.1.2. *Unstable Contributions.* In the cases where $\overline{\mathcal{M}}_{g,\mathfrak{r}}(\mathcal{B}R)$ is unstable, we set the value of H°_g by hand. We give the unstable definitions for $H^{0,\circ}_g$; those for $H^{\infty,\circ}_g$ are obtained by replacing $|R|$ by $|S|$.

First, if $\sum_{i=1}^n r_i \neq 0 \in R$, the monodromy condition is not met, and we set $H^\circ_{g,\mathfrak{r}} = 0$. The remaining contributions are:

$$H^0_{0,id}(z) = \frac{1}{|R|z}, \quad H^0_{0,(r_1,-r_1)}(z_1,z_2) = \frac{z_1z_2}{|R|(z_1+z_2)}. \tag{6}$$

These definitions are ad-hoc, but they satisfy two consistency checks: they agree with the localization procedure and the orbifold ELSV formula.

Note that in the stable cases $H^\circ_{g,\mathfrak{r}}$ is a polynomial, while in the unstable cases it is only a rational function. This fact allows us to remove the unwanted unstable cases later.

3.1.3. *All genus generating functions.* Assemble the H_g into an all genus generating function, with the variable u indexing the genus:

$$H^\circ_{\mathfrak{r}}(z_1,\ldots,z_n,u) = \sum_{g\geq 0} u^{2g-2} H^\circ_{g,\mathfrak{r}}(z_1,\ldots,z_n).$$

In the disconnected all genus generating function we must allow negative genus:

$$H^\bullet_{\mathfrak{r}}(z_{\mathfrak{r}},u) = \sum_{g\in\mathbb{Z}} u^{2g-2} H^\bullet_{g,\mathfrak{r}}(z_{\mathfrak{r}}).$$

Although the sum is over arbitrary negative powers of u, for any fixed number of marked points there are only finitely many negative powers of u: only genus 0 connected curves contribute negative powers of u it must be genus 0, and the contribution of genus 0 curves vanish unless there are marked points.

Because of the unstable contributions, $H^\bullet_{\mathfrak{r}}(z_{\mathfrak{r}})$ is a rational function, with simple poles occurring at $z_i = 0$ for those i with $r_i = 0$ and at $z_i + z_j = 0$ when $r_i = -r_j$.

3.1.4. *n-point functions.* We denote by $G^\circ_{g,d,\mathfrak{r},\mathfrak{s}}(z_\mathfrak{r}, w_\mathfrak{s})$ the $\ell(\mathfrak{r})+\ell(\mathfrak{s})$-point function of genus g, degree d equivariant Gromow-Witten invariants of $\mathcal{X}$:

$$G^\circ_{g,d,\mathfrak{r},\mathfrak{s}}(z_\mathfrak{r}, w_\mathfrak{s}) = \int_{[\overline{\mathcal{M}}_{g,\mathfrak{r}+\mathfrak{s}}(\mathcal{X},d)]^{\mathrm{vir}}_{\mathbb{C}^*}} \prod_{i=1}^{\ell(\mathfrak{r})} \frac{z_i \mathrm{ev}_i^*(\mathbf{0}_{r_i})}{1 - z_i\overline{\psi}_i} \prod_{j=1}^{\ell(\mathfrak{s})} \frac{w_j \mathrm{ev}_j^*(\boldsymbol{\infty}_{s_j})}{1 - w_j\overline{\psi}_j}.$$

When $d = 0$ and the moduli space is unstable, we make the following conventions, which are compatible with the localization procedure and the unstable contributions defined earlier. All unstable 0-point functions are set to 0:

$$G^\circ_{0,0}() = G^\circ_{1,0}() = 0.$$

Any unstable 1 or 2 point functions that is empty by monodromy considerations are set to zero, i.e. if $\sum r_i$ or $\sum s_j$ are nonzero. The remaining 1 and 2-point functions are defined as follows:

$$G^\circ_{0,\{0\}}(z_1) = \frac{1}{|R|z_1}, \quad G^\circ_{0,\{0\}}(w_1) = \frac{1}{|S|w_1} \tag{7}$$

$$G^\circ_{0,0,r_1,-r_1}(z_1,z_2) = \frac{tz_1z_2}{|R|(z_1+z_2)}, \quad G^\circ_{0,0,s_1,-s_1}(w_1,w_2) = \frac{tw_1w_2}{|S|(w_1+w_2)}$$

$$G^\circ_{0,0,\{0\},\{0\}}(z_1,w_1) = 0.$$

Define $G^\circ_{d,\mathfrak{r},\mathfrak{s}}(z_\mathfrak{r}, w_\mathfrak{s}, u)$ to take into account all genus invariants:

$$G^\circ_{d,\mathfrak{r},\mathfrak{s}}(z_\mathfrak{r}, w_\mathfrak{s}, u) = \sum_{g\geq 0} u^{2g-2} G^\circ_{g,d,\mathfrak{r},\mathfrak{s}}(z_\mathfrak{r}, w_\mathfrak{s}).$$

Similarly, we denote the disconnected functions by $G^\bullet_{d,\mathfrak{r},\mathfrak{s}}(z_\mathfrak{r}, w_\mathfrak{s}, u)$.

$$G^\bullet_{d,\mathfrak{r},\mathfrak{s}}(z_\mathfrak{r}, w_\mathfrak{s}, u) = \sum_{P\in \mathrm{Part}_d(\mathfrak{r},\mathfrak{s})} \frac{1}{\mathrm{Aut}(P)} \prod_{i=1}^{\ell(P)} G^\circ_{d_i}(z_{P_i}, w_{P'_i}, u).$$

Here an element $P \in \mathrm{Part}_d(\mathfrak{r},\mathfrak{s})$ is a set of triples (d_i, P_i, P'_i) such that the d_i form a partition of d, where some parts could be 0, and the P_i, P'_i form partitions of $\mathfrak{r},\mathfrak{s}$, respectively, where some parts are allowed to be empty. Since the unstable zero point, zero degree functions are defined to vanish only a finite number of partitions have nonzero contribution to any given term.

3.2. **Fixed point loci.** In this section we describe the fixed point loci of the $\mathbb{C}^*$ action on $\overline{\mathcal{M}}_{g,\mathfrak{r},\mathfrak{s}}(\mathcal{X},d)$. For connected curves, the fixed point loci are denote $\overline{\mathcal{M}}_\Gamma$, where Γ ranges over certain labeled graphs. Summing over connected graphs of a given type is complicated. It is easier to work with disconnected curves, which give a sum over K-labeled partitions.

3.2.1. *Localization graphs.* We now describe how graphs correspond to fixed point loci. Our notation for graphs is as follows. The sets of edges e and vertices v of Γ are denoted $E(\Gamma)$ and $V(\Gamma)$, respectively. An incident edge-vertex pair is called a *flag* of Γ, and is denoted F, with $F(\Gamma)$ being the set of all flags.

Consider a stable map $f : C \to \mathcal{X}$ fixed under the induced $\mathbb{C}^*$ action on $\overline{\mathcal{M}}_{g,\mathfrak{r},\mathfrak{s}}(\mathcal{X},d)$. Any marked point, node, or contracted component must map to a fixed point of $\mathcal{X}$, namely 0 or ∞. Furthermore, any ramification points of a noncontracted component must lie over 0 or ∞ as well. Thus, any noncontracted component can be ramified over at most two points, and so, on the level of coarse curves the only possible noncontracted component allowed is the standard degree

d map $\mathbb{P}^1 \to \mathbb{P}^1, z \mapsto z^d$, which we call *edge* maps. The edges of a graph Γ are in bijection with the noncontracted components.

If we have two contracted components connected directly by a node, we may smooth that node and remain a fixed map. We call a maximal set of contracted components connected by nodes a *vertex* map. We always act as though there is a contracted component at either end of every edge graph – if there is not, we conventionally "destabilize" our curve and add an unstable component contracted to that component. Destabilization is discussed in the next subsection, 3.2.2

In contrast to nodes between contracted components, a node between an edge and a vertex map cannot be smoothed without leaving the fixed point locus. Such a node corresponds to an incident edge-vertex pair, and hence of a flag F of Γ.

3.2.2. *Destabilization.* As a first example of destabilization, consider the case where two edge maps joining directly in a node, with no contracted component between them. In this situation, destabilization means we act as though there were an unstable, contracted genus 0 curve two marked points in between the two edge maps. We then use our established conventions (6) for Hodge integrals over unstable curves in the localization procedure.

The advantage of the destabilization procedure is that it allows for uniform treatment of the localization process, rather than needing to deal with all the subcases. Of course, it is necessary to check that this convention is consistent with the localization procedure; this is easily done, or see Section 3.6 of [Joh].

An example illustrating essentially all the possibilities of destabilization is a genus 0 degree 3 stable map with one marked point to $\mathcal{C}_{2,3}$, consisting of two $\mathbb{P}^1$ components, one mapping with degree 2 to the target, joined by a node mapping to ∞ to the other component, which has degree 1. For this to happen, we see that our two components must be joined with a node with $\mathbb{Z}_3$ isotropy, and the point mapping to 0 with degree 1 must have $\mathbb{Z}_2$ isotropy – this is the one marked point. The destabilization consists of a chain of 5 $\mathbb{P}^1$ components, joined with nodes. The first and last components are contracted to 0, and the middle component is contracted to ∞. Thus, we consider the fixed point set of this graph to be $\overline{\mathcal{M}}_{0,2}(\mathcal{B}\mathbb{Z}_2, 1/2, 1/2) \times \overline{\mathcal{M}}_{0,1}(\mathcal{B}\mathbb{Z}_2, 0) \times \overline{\mathcal{M}}_{0,2}(\mathcal{B}\mathbb{Z}_3, 1/3, 2/3)$. We evaluate tautological classes on these unstable moduli spaces using Equation 6.

3.2.3. *Graph labelings.* Each edge $e \in E(\Gamma)$ carries a labeling of the degree $d(e) \in \mathbb{Z}_{\geq 1}$ of the map from $\mathbb{P}^1 \to \mathbb{P}^1$ that the edge represents. By Lemma 1, the degree also determines the behavior of the effective orbifold structure at each flag, and so for $\mathcal{C}_{r,s}$, this is the only labeling needed.

However, in the ineffective case, further labeling is necessary to determine how the edge interacts with the gerbe structure. By Lemma 2, to do so it is enough to specify a single element $k(e) \in K$. Define $k(e)$ so that

$$\left(d(e), -k(e) + \left\lfloor \frac{d(e)}{s} \right\rfloor \Bbbk_\infty \right) \in S \tag{8}$$

is the monodromy on the *edge side* of the node over ∞.

Given an edge e we use $\rho(e)$ and $\sigma(e)$ to denote the orbifold structure on the *vertex* sides of the nodes over both 0 and ∞. We now present formulas for $\rho(e)$ and $\sigma(e)$ in terms of $k(e)$.

By the balancing condition, $\sigma(e)$ is the inverse of the monodromy on the edge side of the node over ∞, which is given by Equation (8). Recalling Equation (3)

$$-(a,k) = (-a, -k - \delta_r^\vee(a))$$

and using

$$\left\lfloor \frac{a}{r} \right\rfloor + \left\lfloor \frac{-a}{r} \right\rfloor = -\delta_r^\vee(a), \tag{9}$$

it follows that

$$\sigma(e) = \left(-d(e), k(e) + \left\lfloor \frac{-d(e)}{s} \right\rfloor \mathbb{k}_\infty \right). \tag{10}$$

The factor of $\left\lfloor \frac{\pm d(e)}{s} \right\rfloor \mathbb{k}_\infty$ appears awkward, but it is a convenient, symmetric way to account for Equation (3) for $-(a,k)$. Additionally, this choice of $k(e)$ is convenient later in our operator description of Gromov-Witten theory.

To determine $\rho(e)$, we use Lemma 2. In particular, for a degree d map with monodromy $(d(e), -k(e) + \left\lfloor \frac{d(e)}{s} \mathbb{k}_\infty \right\rfloor)$ on the edge side at infinity, the monodromy on the edge side at 0 must be $(d(e), k(e) + \left\lfloor \frac{d(e)}{r} \right\rfloor \mathbb{k}_0 + d(e)\mathbb{L})$ from 5.

Since $\rho(e)$ is the monodromy on the *vertex side* over 0, using the balancing condition and arithmetic as in our deduction of Equation (10) gives

$$\rho(e) = \left(-d(e), -k(e) - d(e)\mathbb{L} + \left\lfloor \frac{-d(e)}{r} \right\rfloor \mathbb{k}_0 \right). \tag{11}$$

If F is a flag on e, we write $\rho(F)$ or $\sigma(F)$ to denote $\rho(e)$ or $\sigma(e)$.

3.2.4. *Vertex Labelings.* Each vertex $v \in V(\Gamma)$ carries the labeling of which fixed point it mapped to - we write v_0 or v_∞ when we want to indicate that a vertex is mapped to zero or infinity. The genus of the contracted curve is denoted $g(v)$.

In addition to these labelings, each vertex needs labelings for the marked points and the nodes connected to it.

The marked points contained on a contracted curve mapping to zero are a subset of the set $\mathfrak{r}$ of all marked points mapping to zero. We denote this subset by $\mathfrak{r}(v_0)$, or, for a vertex mapping to ∞, by $\mathfrak{s}(v_\infty)$.

The nodes on a vertex curve are in bijection with the edges e incident to the vertex v. We write $e(v)$ for the number of edges incident to v. Choosing a labeling for them as $e_1, \ldots, e_{e(v)}$, we use d_i to denote the degree of e_i.

For each edge e incident to a vertex v, the vertex curve has a marked point that is half of the corresponding node. The orbifold structure of this marked point is determined by the labelings of the edge e as discussed above. We write $\rho(v_0)$ or $\sigma(v_\infty)$ to denote the tuples $\rho(e_i)$ or $\sigma(e_i)$, where i ranges over all adjacent edges.

3.2.5. *Topology of connected fixed point sets.* To determine the topology of the the fixed point locus $\overline{\mathcal{M}}_\Gamma$, we note that the only deformations allowed while staying within the fixed point locus are deforming the vertex curves. Thus, each vertex v contributes a moduli space of stable maps to R or S, which we denote $\overline{\mathcal{M}}_v$. The discussion of vertex labelings makes clear that

$$\overline{\mathcal{M}}_{v_0} = \overline{\mathcal{M}}_{g(v_0),\mathfrak{r}(v_0)+\rho(v_0)}(\mathcal{B}R)$$

and similarly for a vertex over infinity.

Thus, the fixed point locus corresponding to a labeled graph Γ is the product of these moduli spaces over all vertices:

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in V(\gamma)} \overline{\mathcal{M}}_v.$$

However, there are extra automorphism factors coming from global symmetry of the graph and from the orbifold structure, which we now describe.

3.2.6. *Automorphisms and gluing factors.* To write $\overline{\mathcal{M}}_\Gamma$ as a product of $\overline{\mathcal{M}}_{v_0}$, we assumed that all vertices were distinguishable. However, this need not be the case. Let $\mathrm{Aut}(\Gamma)$ denote the of automorphisms of Γ that preserve all labelings of Γ. Then $\mathrm{Aut}(\Gamma)$ acts naturally on the product of $\overline{\mathcal{M}}_v$, and the moduli space is really the quotient.

Furthermore, each edge curve contributes some automorphisms. To begin with, we have the usual automorphism group $\mathcal{Z}_d$ of a degree d map obtained by rotating by dth roots of unity. Additionally, in the presence of a gerbe, each edge has an additional $|K|$ worth of automorphisms, see, e.g., [CC09].

Finally, there are subtle automorphism type factors coming from gluing nodes together. As presented in [AGV08], maps from a nodal curve C with components C_1 and C_2 glue along the *rigidified* inertia stack

$$\hom(C, \mathcal{X}) = \hom(C_1, \mathcal{X}) \amalg_{\overline{\mathcal{I}}(\mathcal{X})} \hom(C_2, \mathcal{X}).$$

and *not* over the usual inertia stack $\mathcal{I}$.

Recall that point (x, g) of the inertia stack has isotropy group $G_{(x,g)}$ isomorphic to $C(g)$, the centralizer of g in G_x. In the rigidified inertia stack, these isotropy groups are replaced by $C(g)/\langle g \rangle$. In our case, this is the group $R/\langle \rho(e) \rangle$.

The result of these gluing factors is that for each flag F over 0, the fundamental class of the actual fixed point set differs from that of the product of the $\overline{\mathcal{M}}_v$ by a factor of $|R|/|\rho(F)|$, and similarly for flags over ∞.

A lower level, geometric explanation of this factor is as follows. Consider the node corresponding to a flag F over 0. Let C_1 and C_2 be the two component curves meeting at the node, and let $\widetilde{C_i}$ be orbifold charts of C_i in neighborhoods of the orbifold node in question. Then the fibers of $\widetilde{C_1}, \widetilde{C_2}$ over the node are each isomorphic as R-sets to $R/\rho(e)$. Gluing the map into a map of nodal curves is equivalent to giving a R-equivariant isomorphism of these two fibers, and there are clearly $|R|/|\rho(e)|$ distinct such isomorphisms.

Taking all of these factors into account, on the level of virtual fundamental classes, we have:

$$\left[\overline{\mathcal{M}}_\Gamma\right] = \frac{1}{\mathrm{Aut}(\Gamma)} \prod_{e \in E(\Gamma)} \frac{1}{|K| d(e)} \frac{|R|}{|\rho(e)|} \frac{|S|}{|\sigma(e)|} \prod_{v \in v(\Gamma)} \left[\overline{\mathcal{M}}_v\right]. \tag{12}$$

3.2.7. *Disconnected Curves.* Some additional notation is needed for disconnected curves.

For any map, the set of $d(e)$ and $k(e)$ together form a K-weighted partition of d, which wedenote $\overline{\mu} = \{(\mu_i, k_i)\}$. We write $\rho(\overline{\mu})$ and $\sigma(\overline{\mu})$ to denote the set of all $\rho(e_i)$ and $\sigma(e_i)$, and $\rho(\overline{\mu}_i)$ to denote $\rho(e_i)$. We write g_0 and g_∞ for the genus of the (disconnected) curves over 0 and ∞.

3.3. **The virtual normal bundle.** We now determine the virtual normal bundle of each $\overline{\mathcal{M}}_\Gamma$.

3.3.1. *Virtual normal Bundle.* Assume for now that the moduli space $\overline{\mathcal{M}}$ is smooth. Consider a point $f \in \overline{\mathcal{M}}_\Gamma \subset \overline{\mathcal{M}}$ in some fixed point locus $\overline{\mathcal{M}}_\Gamma$. Then we have the splitting

$$T_f\overline{\mathcal{M}} = T_f\overline{\mathcal{M}}_\Gamma \oplus N_f\overline{\mathcal{M}}_\Gamma$$

of the tangent bundle of $\overline{\mathcal{M}}$ into the tangent bundle of $\overline{\mathcal{M}}_\Gamma$ and the normal bundle of $\overline{\mathcal{M}}_\Gamma$. This splitting is determined by the $\mathbb{C}^*$ action on $T_f\overline{\mathcal{M}}$: $T_f\overline{\mathcal{M}}_\Gamma$ is the 0-eigenspaces of the action, and the nonzero, or *moving*, eigenspaces make up $N_f\overline{\mathcal{M}}_\gamma$.

Though $\overline{\mathcal{M}}$ in general is not smooth, the same arguments hold if we work with the virtual tangent bundle.

Intuitively, the virtual tangent bundle should be the linear space of deformations of the map, minus the linear space of obstructions of the map. In addition, we subtract off first order automorphisms of the map/curve (which may also have first order obstructions we need to add back). In this rough view, sections of $f^*(T\mathcal{X})$ correspond to ways of deforming f, and its first cohomology is first order obstructions. Similarly, sections of TC correspond to first order automorphisms, and its cohomology to obstructions of these.

The precise version of this argument (see [GP99]) gives the exact sequence:

$$\begin{aligned} 0 \to & H^0(C, TC) \to H^0(C, f^*(T\mathcal{X})) \to \mathcal{T}^1 \to \\ \to & H^1(C, TC) \to H^1(C, f^*(T\mathcal{X})) \to \mathcal{T}^2 \to 0, \end{aligned}$$

where $\mathcal{T}^1 - \mathcal{T}^2$ is the virtual tangent bundle to $\overline{\mathcal{M}}$ in K-theory.

Using superscript m to denote the moving part, we find that the reciprocal of the Euler class of the normal bundle:

$$\frac{1}{e(N)} = \frac{e(H^0(C,TC)^m)}{e(H^1(C,TC)^m)} \frac{e(H^1(C, f^*(T\mathcal{X}))^m)}{e(H^0(C, f^*(T\mathcal{X}))^m)} \tag{13}$$

We now compute each of the four equivariant Euler classes appearing in Equation (13). The terms corresponding to TC are computed directly in the next two sections, 3.3.2 and 3.3.3, while the terms containg $f^*T\mathcal{X}$ are computed using the normalization long exact sequence in Section 3.4.

In the next two sections, we determine only the contributions of vertices over 0. The contributions coming of vertices over ∞ have the same form with t replaced by $-t$, r by s, and R by S.

3.3.2. $H^0(C, TC)$*: Automorphisms.* The term $H^0(C, TC)$ parameterizes infinitesimal automorphisms of the source curve. Conventionally, all vertex components are stable, and hence have no infinitesimal automorphisms.

Each edge curve is topologically $\mathbb{P}^1$, and by our destabilization convention, each edge curve has a node over 0 and ∞ that must be fixed by the automorphisms, and so we see that each $H^0(C_e, TC_e)$ is one dimensional. It is spanned by any section s of TC_e vanishing at both 0 and ∞, and hence s vanishes simply at each of 0 and ∞.

We calculate the $\mathbb{C}^*$ action on this section s using Chern-Weil theory. Choosing a connection ∇ on $T\mathbb{P}^1$, locally the derivative ∇s is a section of $TC_e \otimes T^*C_e$. Furthermore, since s vanishes simply, ∇s does not at 0. Hence we may identify $H^0(C_e, TC_e)$ with $TC_e \otimes T^*C_e$. Since the $\mathbb{C}^*$ actions on $T\mathbb{P}^1$ and $T^*\mathbb{P}^1$ have opposite weights and group actions, $T_0C_e \otimes T_0^*C_e$ has a weight 0 $\mathbb{C}^*$ action and contributes to the tangent bundle rather than the normal bundle. Hence, $e(H^0(C_e, TC_e)^m) = 1$.

Note that in fact $H^0(C, TC)$ can contain directions with nontrivial weights: for instance, in an edge curve e without nodes or marked points over 0, sections s do not have to vanish at 0, and so there are more automorphisms in these cases. The contributions from these extra automorphisms are contained in the unstable terms of our Hurwtiz-Hodge generating function $H(z)$.

3.3.3. *$H^1(C, TC)$: Node smoothings.* The term $H^1(C, TC)$ parameterizes infinitesimal smoothings of the nodes in the source curve. By our graph conventions there is a node for every flag, and these are the only nodes that contribute to the normal bundle. The node n between C_e and C_v contributes $T_nC_e \otimes T_nC_v$. Note that even if we have a twisted node, this space has trivial group action, since all nodes are balanced.

As T_nC_v is on the contracted component, it has trivial $\mathbb{C}^*$ action. Hence the Euler class of this bundle is the negative of the ψ class on $\overline{\mathcal{M}}_v$:

$$e(T_nC_e) = -\psi = -\frac{1}{|\rho(e)|}\overline{\psi}.$$

On the other hand, T_nC_e is topologically trivial on $\overline{\mathcal{M}}_\Gamma$, but has nontrivial $\mathbb{C}^*$ action.

Since $T_0\mathcal{X}$ has $\mathbb{C}^*$ weight 1, and the map has degree d and is equivariant, we see that $\mathbb{C}^*$ acts on T_nC_e with weight $1/d$. In addition, the Euler class receives a factor of $1/|\rho(e)|$ from the orbifold structure, and so the euler class of T_nC_e on $\overline{\mathcal{M}}_\Gamma$ is $1/(d|\rho(e)|)$.

Thus a node n attached to an edge of degree d, with isotropy mapping to $\rho(e)$ at 0 contributes to $e(H^1(C_e, TC_e))$ by:

$$\frac{t}{|\rho(e)|d} - \frac{1}{|\rho(e)|}\overline{\psi} = \frac{t}{|\rho(e)|d}(1 - d\overline{\psi}/t).$$

Putting together the contributions of all node smoothing terms at a given vertex v_0, we see the contribution to $\frac{1}{e(N_\Gamma)}$ is:

$$t^{-e(v)} \prod_{i=1}^{e(v_0)} \left(|\rho(e_i)| \frac{d_i}{1 - d_i\overline{\psi}_i/t} \right). \tag{14}$$

3.4. **Normalization exact sequence.** We calculate $H^0(C, f^*(T\mathcal{X}))$ and $H^1(C, f^*(T\mathcal{X}))$ together by using the normalization long exact sequence.

We have the short exact sequence

$$0 \to \mathcal{O}_C \to \bigoplus_{e\in \mathrm{E}(\Gamma)} \mathcal{O}_{C_e} \bigoplus_{v \in \mathrm{V}(\Gamma)} \mathcal{O}_{C_v} \to \bigoplus_{F\in \mathrm{F}(\Gamma)} \mathcal{O}_F \to 0$$

Tensoring the above sequence by $\xi = f^*(T\mathcal{X})$ and taking cohomology gives

$$\begin{aligned} 0 \quad \to \quad & H^0(C,\xi) \to \bigoplus_{e\in \mathrm{E}(\Gamma)} H^0(C_e,\xi) \bigoplus_{v\in \mathrm{V}(\Gamma)} H^0(C_v,\xi) \to \bigoplus_{F\in \mathrm{F}(\Gamma)} H^0(C_f,\xi) \to \\ \to \quad & H^1(C,\xi) \to \bigoplus_{e\in \mathrm{E}(\Gamma)} H^1(C_e,\xi) \bigoplus_{v\in \mathrm{V}(\Gamma)} H^1(C_v,\xi) \to \bigoplus_{F\in \mathrm{F}(\Gamma)} H^1(C_f,\xi) \to 0. \end{aligned}$$

We now exam the terms of this sequence in detail.

3.4.1. *Flags.* As the C_F are not curves but nodes, they are zero dimensional and so $H^1(C_F,\xi)=0$.

To calculate $H^0(C_F,\xi)$, first note that it is zero unless the action of the isotropy group of the node on ξ_0 is trivial. The R representation ξ_0 is the pullback of the standard representation of $\mathbb{Z}_r$, and so we see that ξ_0 is the trivial representation of R if and only if the image of $\rho(F)$ in $\mathbb{Z}_r$ is zero. But the image of $\rho(F)$ in $\mathbb{Z}_r$ is $d(F) \mod r$.

Finally, as a $\mathbb{C}^*$ representation, $H^0(C_F,\xi)$ is the equivalent to $T_0\mathcal{X}$, and so its Euler class is t/r.

Thus, the flag contribution from each vertex v_0 is

$$\left(\frac{t}{r}\right)^{(\#d_i=0 \mod r)}. \tag{15}$$

3.4.2. *Edges.* We compute the contribution of $H^i(C_e,\xi)$ by using the isomorphism with the cohomology of the desingularization $H^i(|C_e|,|\xi|)$. If C_e is an edge of degree d, then $\xi=f^*T\mathcal{X}$ has degree $cw_1(\xi)=d\cdot cw_1(\mathcal{X})=d(1/r+1/s)$.

The curve C_e has isotropy $\mathbb{Z}_{|\rho(e)|}$ at 0, and the generator acts on $f^*(T_0\mathcal{X})$ by its image in $\mathbb{Z}_r$, which is $d \mod r$. The generator acts by $1/|\rho(e)|$ on the tangent bundle and $\left\langle\frac{d}{r}\right\rangle$ on $f^*T\mathcal{X}$. Recalling the discussion of the desingularization, we see that the $\mathbb{C}^*$ weight of $|f^*T\mathcal{X}|$ is that of $f^*T\mathcal{X}\otimes(T^*C_e)^a$, where $a=|\rho(e)|\left\langle\frac{d}{r}\right\rangle$. Thus, the $\mathbb{C}^*$ weight of $|f^*T\mathcal{X}|$ at 0 is:

$$\frac{1}{r}-|\rho(e)|\left\langle\frac{d}{r}\right\rangle\frac{1}{|\rho(e)|d}=\frac{d}{dr}-\frac{d \mod r}{dr}=\frac{1}{d}\left\lfloor\frac{d}{r}\right\rfloor.$$

Similarly, we see that the degree of $|f^*T\mathcal{X}|$ is

$$\frac{d}{r}+\frac{d}{s}-\frac{d \mod r}{r}-\frac{d \mod s}{s}=\left\lfloor\frac{d}{r}\right\rfloor+\left\lfloor\frac{d}{s}\right\rfloor.$$

As the degree is nonnegative, $H^1(C_e,f^*T\mathcal{X})=0$, while $H^0(C_e,f^*T\mathcal{X})$ has dimension $\left\lfloor\frac{d}{r}\right\rfloor+\left\lfloor\frac{d}{s}\right\rfloor+1$. Any eigensection of the desingularization $|f^*T\mathcal{X}|$ vanishes only at 0 and ∞, and so the eigensections are given by sections that vanish to order k at 0 and order $\left\lfloor\frac{d}{r}\right\rfloor+\left\lfloor\frac{d}{s}\right\rfloor-k$ at ∞, for $0\le k\le\left\lfloor\frac{d}{r}\right\rfloor+\left\lfloor\frac{d}{s}\right\rfloor$.

To determine the weight of a section vanishing k times at 0, note that the kth derivative is locally a section of $|f^*T\mathcal{X}|\otimes\omega^k_{|C_e|}$, and that this section is nonzero at 0. Now, the weight of $\omega_{|C_e|}$ at 0 is $-1/d$, and so we see that the eigensection vanishing to order k at 0 has $\mathbb{C}^*$ weight $\left\langle\frac{d}{r}\right\rangle\frac{1}{d}-\frac{k}{d}$. Hence, as k varies there are sections of every weight that's a multiple of $1/d$ from $-\left\lfloor\frac{d}{s}\right\rfloor/d$ to $\left\lfloor\frac{d}{r}\right\rfloor/d$. One section has zero weight, and so contributes to the virtual tangent bundle rather than the virtual normal bundle.

We split this edge contribution between the zero and infinity by associating the positive weighted sections with 0 and the negatively weighted sections with ∞. With this convention, as well as the convention that contributions over ∞ have t replaced by $-t$, we see that the contribution coming from a degree d edge attached to 0 is:

$$\frac{d^{\lfloor\frac{d}{r}\rfloor}t^{-\lfloor\frac{d}{r}\rfloor}}{\lfloor\frac{d}{r}\rfloor!}. \tag{16}$$

3.4.3. *Vertices.* We now consider the terms $H^i(C_v, f^*T_0\mathcal{X})$. For a vertex over 0, we have $f : C_v \to \mathcal{B}R$, and so f is equivalent to a principal R bundle $\widetilde{C}_v$ over C_v. The bundle $f^*T_0\mathcal{X}$ on C_v corresponds to a topological trivial bundle on $\widetilde{C}_v$, but with a potentially nontrivial lift of the R action. The group $H^i(C_v, f^*T_0\mathcal{X})$ is isomorphic to the R invariant part of $H^i(\widetilde{C}_v, \mathcal{O}) \otimes T_0\mathcal{X}$.

The dimension of $H^0(\widetilde{C}_v, \mathcal{O})$ is the number of components of the R-cover $\widetilde{C}$. Letting $H \subset R$ be the monodromy group of the cover, $\widetilde{C}$ has $|R|/|H|$ components, and as an R representation $H^0(\widetilde{C}_v, \mathcal{O})$ is the regular representation of R/H.

Thus, $H^0(C_v, f^*T_0\mathcal{X})$ is one dimensional if the R action on $T_0\mathcal{X}$ factors through R/H, and zero otherwise. Equivalently, it is one dimensional if $H \subset K$, and zero otherwise.

Define δ_K to be 1 if $H \subset K$, and 0 otherwise. Then, since $T_0\mathcal{X}$ has $\mathbb{C}^*$ weight $1/r$, we have:

$$e(H^0(C_v, f^*T_0\mathcal{X})) = \left(\frac{t}{r}\right)^{\delta_K}. \tag{17}$$

To calculate $H^1(C_v, f^*T_0\mathcal{X})$, we apply Serre duality to see that:

$$\begin{aligned}(H^1(\widetilde{C}_v, \mathcal{O}) \otimes T_0\mathcal{X})^R &= (H^0(\widetilde{C}_v, \omega)^\vee \otimes T_0)^R \\ &= (\mathbb{E}^\vee)_{T_0^*} \\ &= \mathbb{E}^\vee_{T_0}.\end{aligned}$$

In addition to the topological structure of the bundle, $T_0\mathcal{X}$ has a $\mathbb{C}^*$ action with weight $1/r$.

Altogether, the equivariant Euler class of $H^1(C_v, f^*T_0\mathcal{X})$ is

$$\left(\frac{t}{r}\right)^m - \left(\frac{t}{r}\right)^{m-1} \lambda_1^{T_0} + \cdots \pm \lambda_m^{T_0} = \left(\frac{t}{r}\right)^m \sum_{i=0}^m \left(-\frac{r}{t}\right)^i \lambda_i^{T_0}. \tag{18}$$

Here we are using $m = \dim(\mathbb{E}_{T_0}) = g - 1 + \iota(\mathfrak{r}(v) + \rho(v)) + \delta_K$.

The δ_K term here exactly cancels the contribution of $H^0(C_v, \xi)$, and we cancel them here and in future occurences.

3.4.4. *Total Contribution.* We now combine contributions from Equations (12), (14),(15),(16) and (18) to find the total contribution of a vertex lying over 0 to $\frac{1}{e(N_\Gamma)}$.

First, note the $1/|\rho(e)|$ and $1/|\sigma(e)|$ factors appearing in (12) cancel the similar terms appearing in (14). Combining those terms and simplifying using

$$\rho(v) + \# d_i = 0 \mod r = e(v) - \sum_{i=1}^{e(v)} \left\langle \frac{d_i}{r} \right\rangle$$

and

$$\left\langle \frac{d_i}{r} \right\rangle + \left\lfloor \frac{d_i}{r} \right\rfloor = \frac{d_i}{r}$$

we obtain

$$\frac{t^{g-1+\iota(\mathfrak{r}(v)) - \sum d_i/r}}{r^{g-1+\iota(\mathfrak{r}(v)) - \sum \left\langle \frac{d_i}{r} \right\rangle + e(v)}} \prod_{i=1}^{e(v)} \left(\frac{d_i^{\left\lfloor \frac{d_i}{r} \right\rfloor}}{\left\lfloor \frac{d_i}{r} \right\rfloor !} \frac{d_i}{1 - d_i \overline{\psi}_i / t} \right) \sum (-r/t)^i \lambda_i^{T_0} \tag{19}$$

as the total contribution of a vertex v_0.

As always, the contribution from a vertex v_∞ is completely analogous,with t replaced by $-t$, and r and R replaced by s and S.

3.5. **Global localization contributions.** We now apply the localization calculations to express the Gromov-Witten generating function $G^\bullet$ in terms of the Hurwitz-Hodge generating functions $H^\bullet$. We've established the contribution to the virtual normal bundle from each vertex appearing in a localization graph. We now investigate the effect of localization on the integrands appearing in $G^\bullet_{\mathfrak{r},\mathfrak{s}}$.

3.5.1. *Integrands.* The integrand over the point 0 is exactly

$$\prod_{i=1}^{\ell(\mathfrak{r})} \frac{z_i \mathrm{ev}_i^*(\mathbf{0}_{r_i})}{1 - z_i\overline{\psi}_i}.$$

Let $\varphi : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,\mathfrak{r},\mathfrak{s}}(\mathcal{X}, d)$ be the inclusion. We have $\varphi^*(\overline{\psi}) = \overline{\psi}$; consider $\varphi^*(\mathbf{0}_{r_i})$. For r_i not belonging to K, the corresponding component of the $\mathcal{I}\mathcal{X}$ is zero dimensional, and we have $\varphi^*(r_i) = r_i$. However, for $r_i \in K$, the component of the twisted sector is one dimensional, and we have $\varphi^*(r_i) = tr_i$. Thus, localizing gives us a factor of

$$t^{\#(r_i \in K)},$$

and otherwise, considering both the integrand and the virtual normal bundle, the integral appearing is

$$\int_{\overline{\mathcal{M}}^\bullet_{g_0,\mathfrak{r}+\rho(\overline{\mu})}(\mathcal{B}R)} \prod_{i=1}^{\ell(\mathfrak{r})} \frac{z_i}{1 - z_i\overline{\psi}_i} \prod_{j=1}^{\ell(\overline{\mu})} \frac{\mu_i}{1 - \mu_i\overline{\psi}/t} \sum_{\ell=0}^{\infty} (-r/t)^\ell \lambda_\ell^{T_0}.$$

3.5.2. *Rescaling.* After some rescalings, we can express this in terms of $H^{0,\bullet}_{g,0}$. Namely, replacing $\overline{\psi}$ with $t\overline{\psi}$ and λ_i with $t^i\lambda_i$ multiplies the integral by t to the dimension of $\overline{\mathcal{M}}_{g_0,\mathfrak{r}+\rho(\overline{\mu})}(\mathcal{B}R)$, which is $3g_0 - 3 + \ell(\mathfrak{r}) + \ell(\overline{\mu})$. After this, the z terms appear as

$$\frac{z_i}{1 - tz_i\overline{\psi}_i},$$

and so we must multiply the integrand by $t^{\ell(\mathfrak{r})}$. Canceling part of this term with the factor of $t^{\#(r_i\in K)}$ appearing from localization, we see that the contribution can be written as

$$t^{-(3g_0 - 3 + \ell(\mathfrak{r}) + \ell(\overline{\mu}) + \#(r_i \notin K))} H^{0,\bullet}_{g_0,\mathfrak{r}+\rho(\overline{\mu})}(tz_\mathfrak{r}, \mu),$$

with the analogous statement for the integrals appearing over ∞.

Combining this with the other factors appearing in (19), as well as the factors of $|R|$ and $|S|$ appearing from the node gluing and automorphism in equation (12), we can write the total vertex contribution over 0 as

$$|K|^{\ell(\mu)} \frac{t^{2-2g_0+\iota(\mathfrak{r}(v)) - |\mu|/r - \#(r_i \notin K) - \ell(\overline{\mu}) - \ell(\mathfrak{r})}}{r^{g-1+\iota(\mathfrak{r}(v)) - \sum \left\langle \frac{\mu_i}{r} \right\rangle}} \left(\prod_{i=1}^{\ell(\overline{\mu})} \frac{\mu_i^{\left\lfloor \frac{\mu_i}{r} \right\rfloor}}{\left\lfloor \frac{\mu_i}{r} \right\rfloor !} \right) H^{0,\bullet}_{g_0,\mathfrak{r}+\rho(\overline{\mu})}(tz_\mathfrak{r}, \mu) \tag{20}$$

with similar contribution over ∞. To obtain the global contribution, we must combine these with the remaining global gluing and automorphism factor of

$$\frac{1}{\mathrm{Aut}(\Gamma)} \prod_{e \in E(\Gamma)} \frac{1}{|K| d(e)}.$$

This differs from equation (12) because we have canceled the factor of $|\sigma(e)||\rho(e)|$ in the previous section, as well as the contributions of $|R|$ and $|S|$ appearing just previously. Additionally, working with automorphisms of the weighted partition $\overline{\mu}$ correctly accounts for the weight shift:

$$\frac{1}{\mathfrak{z}(\overline{\mu})},$$

where

$$\mathfrak{z}(\overline{\mu}) = \mathrm{Aut}(\overline{\mu}) \prod_{i=1}^{\ell(\mu)} |K|\mu_i.$$

For each component of a graph, the corresponding genus is the sum of the genera of all the vertices, plus the number of loops in the graph, which can be calculated by $e - v + 1$. Thus the total genus of a connected graph is:

$$g(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1 + \sum_{v \in V(\Gamma)} g(v) = |E(\Gamma)| + 1 + \sum_{v \in V(\Gamma)} (g(v) - 1).$$

Working with our disconnected curves and partitions, it is more convenient to use the euler characteristic, which is additive under disjoin union:

$$2g(\Gamma) - 2 = (2g_0 - 2) + (2g_\infty - 2) + 2\ell(\overline{\mu}).$$

We can now express the disconnected $n+m$ point function $G^\bullet_{d,\mathfrak{r},\mathfrak{s}}$ in terms of the functions H

Define

(21)

$$\mathbf{J}_\mathfrak{r}(z_\mathfrak{r}, \overline{\mu}, u, t) = \frac{r^{\sum \left\langle \frac{\mu_i}{r} \right\rangle - \iota(\mathfrak{r})} (|K|u/t)^{\ell(\mu)}}{t^{|\mu|/r + \#(r_i \notin K) + \ell(\mathfrak{r}) - \iota(\mathfrak{r})}} \left(\prod_{i=1}^{\ell(\overline{\mu})} \frac{\mu_i^{\left\lfloor \frac{\mu_i}{r} \right\rfloor}}{\left\lfloor \frac{\mu_i}{r} \right\rfloor !} \right) H^{0,\bullet}_{\mathfrak{r}+\rho(\overline{\mu})} \left(\mu, t z_\mathfrak{r}, \frac{u}{t r^{1/2}} \right)$$

And for $\mathbf{J}_\mathfrak{s}$ we replace r with s and ρ with σ, but otherwise keep things the same. Then we have:

$$G^\bullet_{d,\mathfrak{r},\mathfrak{s}}(z_\mathfrak{r}, w_\mathfrak{s}, u) = \sum_{|\overline{\mu}|=d} \frac{1}{\mathfrak{z}(\overline{\mu})} \mathbf{J}_\mathfrak{r}(z_\mathfrak{r}, \overline{\mu}, u, t) \mathbf{J}_\mathfrak{s}(z_\mathfrak{s}, \overline{\mu}, u, -t). \tag{22}$$

4. Orbifold ELSV formula and Fock spaces

The previous sections have reduced the study of the Gromov-Witten invariants of $\mathcal{X}$ to studying Hurwitz-Hodge intergrals. That is the geometric portion of this paper; we now turn to algebra.

The transition from geometry to algebra is made by using the orbifold ELSV formula of [JPT11], which expresses Hurwitz-Hodge integrals in terms of certain wreath Hurwitz numbers.

We first recall the definition of the wreath Hurwitz numbers and the statement of the orbifold ELSV formula, and recast this formula in a form more convenient to us.

Wreath Hurwitz numbers can be calculated in terms of representation theory. In the second part of this section we explain how this representation theory is encoded in the Fock space $\mathcal{Z}_K$, though first we warm up with the infinite wedge $\bigwedge^{\frac{\infty}{2}} V$.

4.1. **Wreath Hurwitz Numbers.** To state the orbifold ELSV formula, we first introduce the notion of wreath Hurwitz numbers.

4.1.1. *Hurwitz Numbers and Representation Theory.* Hurwitz numbers count the number of maps between curves with prescribed monodromy in a given group G. That is, we pick a curve C, distinct points $p_1, \ldots, p_n \in C$, and conjugacy classes $(g_1), \ldots, (g_n)$ of G. The Hurwitz number then counts the G-covers of $C \setminus \{p_1, \ldots, p_n\}$ with monodromy (g_i) around p_i.

For C a genus g curve, we denote this number $\mathrm{Cov}_g((g_1), \ldots, (g_n))$. We weight each cover by the inverse of the size of its automorphism group, and do not require that the cover be connected.

The number $\mathrm{Cov}_g((g_1), \ldots, (g_n))$ has the following expression as a sum over irreducible representations, which Okounkov traces back at least as far as Burnside [Oko00]:

$$\mathrm{Cov}((g_1), \ldots, (g_n)) = \sum_R \left(\frac{\dim R}{|G|}\right)^{2-2g} \prod_{i=1}^n \frac{|(g_i)|\chi^R(g_i)}{\dim R}. \tag{23}$$

Detailed expositions of Equation (23) can be found in many places, for instance [LZ04] and [Rot].

4.1.2. *Wreath Products.* Use G_d to denote the wreath product of G^d with S_d:

$$G_d = G^d \wr S_d = \{(k, \sigma) | k = (k_1, \ldots, k_d) \in G^d, \sigma \in S_d\},$$
$$(g, \sigma)(g', \sigma') = (g\sigma(g'), \sigma\sigma').$$

4.1.3. *Labeled Partitions.* The conjugacy classes and representations of G_d are each indexed by labeled partitions. We set our notation for these now.

An over line distinguishes labeled partitions ($\overline{\mu}$) from unlabeled partitions (μ).

A labeled partition $\overline{\mu}$ of d consists of an unordered multi-set of pairs (μ_i, s), where each s is an element of some labeling set S, and the μ_i form a usual partition μ of d, called the underlying partition. The notation $|\overline{\mu}|$ and $\ell(\overline{\mu})$ refers to the size and length of the underlying partition μ.

For us, the labeling set is always one of two sets of cardinality equal to the class number of G – this is the number of conjugacy classes of G, or equivalently the number of irreducible representations of G. We use G^* to denote the set of irreducible representations of G and G_* to denote the set of conjugacy classes of G.

Partitions labeled with conjugacy classes of G index the conjugacy classes of G_d via a generalization of the usual cycle type for the symmetric group. We call such labeled partitions a cycle type. We use the symbol $\overline{\mu}$ or $\overline{\nu}$ to denote partitions labeled by conjugacy classes of G.

Partitions labeled with irreducible representations of G index the representations of G_d. We use $\overline{\lambda}$ to denote a partition labeled by representations of G.

Given an S-labeled partition $\overline{\mu}$, we can form S separate partitions $\overline{\mu}^s$, for $s \in S$, by taking only those parts of $\overline{\mu}$ labeled by s.

4.1.4. *Notation.* We use $\mathfrak{z}(\overline{\mu})$ to denote the size of the centralizer of an element of cycle type $\overline{\mu}$. We use ζ_c to denote the size of the centralizer of an element in the conjugacy class c of G. When G is abelian, we have

$$\mathfrak{z}(\overline{\mu}) = |G|^{\ell(\mu)} |\mathrm{Aut}(\overline{\mu})|,$$

where an automorphism of a labeled partition must preserve the labels.

For a cycle type $\overline{\mu}$ we denote the corresponding element in $\mathcal{Z}G_d$, the center of the group algebra of G_d, as $C_{\overline{\mu}}$. For $c \in G_*$, we denote by $T_c \in \mathcal{Z}G_d$ the element corresponding to the cycle type $(2_c) = \{(2, c), (1, \mathrm{Id}), \ldots (1, \mathrm{Id})\}$. Of particular interest is the class T_0, corresponding to the case where $c = \mathrm{Id}$.

4.1.5. *Wreath Hurwitz numbers.* The disconnected double wreath Hurwitz number, $\mathrm{Hur}^{\bullet}_{g,G}(\overline{\mu}, \overline{\nu})$, is the count of covers of $\mathbb{P}^1$, with arbitrary prescribed monodromies $\overline{\mu}$ and $\overline{\nu}$ over 0 and ∞, and monodromy T_0 around $b = 2g - 2 + \ell(\overline{\mu}) + \ell(\overline{\nu})$ other distinct points.

The connected version of wreath Hurwitz numbers does not require that the entire cover is connected, but only that a quotient of the cover is connected, see [JPT11]. Since we work primarily with the disconnected Hurwitz numbers, we will not touch further on this point.

Equation (23) simplifies in the case of double wreath Hurwitz numbers.

Define the central character $f_{\overline{\mu}}(\overline{\lambda})$ by

$$f_{\overline{\mu}}(\overline{\lambda}) = \frac{|C_{\overline{\mu}}|\chi^{\overline{\lambda}}(\overline{\mu})}{\dim \overline{\lambda}}.$$

We use $\overline{f}_{\overline{\mu}}(\overline{\lambda})$ to denote the complex conjugate of $f_{\overline{\mu}}(\overline{\lambda})$.

Since

$$\frac{\dim \overline{\lambda}}{|G_d|} f_{\overline{\mu}}(\overline{\lambda}) = \frac{|C_{\overline{\mu}}|}{|G_d|}\chi^{\overline{\lambda}}(\overline{\mu}) = \frac{1}{\mathfrak{z}(\overline{\mu})}\chi^{\overline{\lambda}}(\overline{\mu}),$$

in the case of double Wreath Hurwitz numbers Equation 23 becomes

$$\mathrm{Hur}^{\bullet}_{g,G}(\overline{\mu}, \overline{\nu}) = \frac{1}{\mathfrak{z}(\overline{\mu})}\frac{1}{\mathfrak{z}(\overline{\nu})}\sum_{\overline{\lambda}\vdash d}\chi^{\overline{\lambda}}(\overline{\mu})\chi^{\overline{\lambda}}(\overline{\nu}) f_T(\overline{\lambda})^b. \tag{24}$$

4.2. **Orbifold ELSV formula.** We now review the orbifold ELSV formula of [JPT11].

4.2.1. Recall that any irreducible representation ϕ of a finite abelian group R is pulled back from the standard representation U of $\mathbb{Z}_r$ as the group of units:

$$0 \to K \to R \xrightarrow{\phi} \mathbb{Z}_r \to 0.$$

Choose a preimage $x \in R$ of $1 \in \mathbb{Z}_r$, and define $\Bbbk = rx \in K$, and define $\overline{r}_{\Bbbk}$ to be the weighted partition

$$\overline{r}_{\Bbbk} = \{\underbrace{(r, -\Bbbk), \ldots (r, -\Bbbk)}_{d/r \text{ times}}\}.$$

Since K is abelian a K_*-weighted partition is really just a K weighted partition. For $\overline{\mu} = \{(\mu_i, k_i^{\mu})\}$, define an ℓ-tuple of elements of R by

$$-\overline{\mu} = \{k_1^{\mu} - \mu_1 x, \ldots, k_{\ell}^{\mu} - \mu_{\ell} x\}.$$

Note that while the parts of $\overline{\mu}$ are unordered, an ordering is chosen for $-\overline{\mu}$.

4.2.2. *The Formula.* With the notation established, Theorem 3 in [JPT11] gives the following formula for certain K_d Hurwitz numbers in terms of Hurwitz-Hodge integrals:

$$\begin{aligned}
\mathrm{Hur}^{\circ}_{g,K}(\overline{r}_{\Bbbk},\overline{\mu}) &= \frac{b!}{\mathrm{Aut}(\overline{\mu})} r^{1-g+\sum\left\langle\frac{\mu_i}{r}\right\rangle} \left(\prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\left\lfloor\frac{\mu_i}{r}\right\rfloor}}{\left\lfloor\frac{\mu_i}{r}\right\rfloor !}\right) \int_{\overline{\mathcal{M}}_{g,-\overline{\mu}}(\mathcal{B}R)} \frac{\sum_{i=0}^{\infty}(-r)^i\lambda_i^{\phi}}{\prod_{j=1}^{\ell(\mu)}(1-\mu_j\overline{\psi}_j)} \\
&= \frac{|K|^{\ell(\mu)}b!}{\mathfrak{z}(\overline{\mu})} r^{1-g+\sum\left\langle\frac{\mu_i}{r}\right\rangle} \left(\prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\left\lfloor\frac{\mu_i}{r}\right\rfloor}}{\left\lfloor\frac{\mu_i}{r}\right\rfloor !}\right) H^{0,\circ}_{g,-\overline{\mu}}(\mu). \qquad (25)
\end{aligned}$$

4.2.3. *Disconnected and Unstable.* For the representation theoretic calculation of the wreath Hurwitz numbers, it is more convenient to use disconnected invariants.

While the Hurwitz number side makes sense for all values of $\ell(\mu)$ and g, the Hurwitz-Hodge integral side is unstable when $g=0$ and $\ell(\mu)$ is one or two. However, in these cases it is easily checked by hand that our definition of the unstable terms, and Equation (6), agrees with resulting Hurwitz numbers.

4.2.4. *Orbifold ELSV restated.* We now restate the orbifold ELSV formula in a way more convenient for our purposes.

Suppose that $\left\langle\frac{-\mu_i}{r}\right\rangle = \frac{a_i}{r}, 0 \le a_i < r$. Then we have that

$$\begin{aligned}
-\overline{\mu}_i &= k_i^{\mu} - \mu_i x \\
&= k_i^{\mu} + \left(\left\lfloor\frac{-\mu_i}{r}\right\rfloor r + a_i\right) x \\
&= k_i^{\mu} + \left\lfloor\frac{-\mu_i}{r}\right\rfloor \Bbbk + a_i x,
\end{aligned}$$

giving the monodromy conditions $-\overline{\mu}$ in terms of the cocycle description of $R = \mathbb{Z}_r \times_\beta K$.

For

$$\mathfrak{r} = (r_1,\ldots,r_{\ell(\mathfrak{r})}); \qquad r_i = (a_i,k_i) \in \mathbb{Z}_r \times_\beta K = R$$

we introduce the K-weighted partition

$$\overline{\mu}_i^{\mathfrak{r}} = \left(\mu_i, k_i - \left\lfloor\frac{-\mu_i}{r}\right\rfloor \Bbbk\right).$$

Shifting the monodromy conditions of both sides of Equation (25) by $\left\lfloor\frac{-\mu_i}{r}\right\rfloor$, summing over genus, and passing to disconnected invariants, we see that for μ a partition with $-\mu_i \mod r = a_i$, we have:

$$H^{0,\bullet}_{\mathfrak{r}}(\mu,u) = \sum_g \left(ur^{1/2}\right)^{2g-2} \frac{r^{-\sum\left\langle\frac{\mu_i}{r}\right\rangle}}{|K|^{\ell(\mu)}} \frac{\mathfrak{z}(\overline{\mu}^{\mathfrak{r}})}{b!} \left(\prod_{i=1}^{\ell(\mu)} \frac{\left\lfloor\frac{\mu_i}{r}\right\rfloor !}{\mu_i^{\left\lfloor\frac{\mu_i}{r}\right\rfloor}}\right) \mathrm{Hur}^{\bullet}_{g,K}(\overline{r}_{\Bbbk},\overline{\mu}^{\mathfrak{r}}). \qquad (26)$$

4.3. **Infinite Wedge.** The character theory used to compute Hurwitz numbers is conveniently encoded as vacuum expectations on certain Fock spaces. In the case of an effective orbifold, the pertinent groups are the symmetric group, and the Fock space is the infinite wedge. For ineffective orbifolds, a larger Fock space is required that we introduce in the next section. This larger Fock space is a tensor product of copies of the infinite wedge.

This section gives a brief introduction to the infinite wedge. We do not give proofs, or even full definitions, but just set notation and collect needed facts. We follow the notation of Okounkov and Pandharipande. There are many introductions to the infinite wedge – we recommend [KR87] and [MJD00].

4.3.1. Let V be the vector space with basis labeled by the half-integers. We underline the label to denote the corresponding basis vector:

$$V = \bigoplus_{i \in \mathbb{Z}} \underline{i + \frac{1}{2}}.$$

The infinite wedge $\bigwedge^{\frac{\infty}{2}} V$ is the span of vectors of the form $\underline{i_1} \wedge \underline{i_2} \wedge \dots$ with $i_k \in \mathbb{Z} + \frac{1}{2}$ a decreasing series of half integers so that $i_k + k + 1/2$ is constant for $k >> 0$.

4.3.2. *The projective* $\mathfrak{gl}_\infty$ *action.* We use $\mathfrak{gl}_\infty$ to denote the lie algebra of operators with finitely many nonzero diagonals; that is, operators of the form

$$\left\{ \sum_{i,j} a_{i,j} E_{i,j} \middle| a_{i,j} = 0 | i - j | >> 0 \right\}.$$

If one attempts to have $\mathfrak{gl}_\infty$ act on the infinite wedge following the Leibniz rule, infinite sums occur. We can fix this by changing the action of the diagonal elements of the infinite wedge, but at the cost of having the infinite wedge be only a projective representation of $\mathfrak{gl}_\infty$.

The new action of the diagonal elements can be summarized as

$$E_{kk} \cdot \underline{i_1} \wedge \underline{i_2} \wedge \dots = \begin{cases} \underline{i_1} \wedge \underline{i_2} \wedge \dots & k > 0, k \in \{i_n\} \\ -\underline{i_1} \wedge \underline{i_2} \wedge \dots & k < 0, k \notin \{i_n\} \\ 0 & \text{otherwise} \end{cases}.$$

4.3.3. *Operators on the infinite wedge.* The *charge operator* C is the operator

$$C = \sum_{k \in \mathbb{Z}+1/2} E_{k,k}.$$

Vectors of $\bigwedge^{\frac{\infty}{2}} V$ that are eigenvectors of C with eigenvalue x are said to have charge x. Similarly, if L is an operator on $\bigwedge^{\frac{\infty}{2}} V$ with $[C, L] = x$ then L is said to have charge x.

The kernel of C consists of those vectors of $\bigwedge^{\frac{\infty}{2}} V$ with charge 0, and is denoted $\bigwedge_0^{\frac{\infty}{2}} V$. Note that charge 0 operators preserve $\bigwedge_0^{\frac{\infty}{2}} V$, and since the operators in $\mathfrak{gl}_\infty$ have charge 0, $\bigwedge_0^{\frac{\infty}{2}} V$ is a representation of $\mathfrak{gl}_\infty$.

The *energy operator* H is defined by

$$H = \sum_{k \in \mathbb{Z}+\frac{1}{2}} k E_{k,k}.$$

Vectors of $\bigwedge^{\frac{\infty}{2}} V$ that are eigenvectors of H with eigenvalue h are said to have energy h, and operators with $[H, L] = h$ are also said to have energy h.

For $0 \neq k \in \mathbb{Z}$, define the operator

$$\alpha_k = \sum_{i \in \mathbb{Z}+\frac{1}{2}} E_{i-k,i}.$$

The operators α_k form a Bosonic Heisenberg algebra – that is, they satisfy the following commutation relation:

$$[\alpha_n, \alpha_m] = n\delta_{n,-m}.$$

4.3.4. The subspace $\bigwedge_0^{\frac{\infty}{2}} V$ has a natural basis v_λ labeled by partitions λ:

$$v_\lambda = \underline{\lambda_1 - \frac{1}{2}} \wedge \underline{\lambda_2 - \frac{3}{2}} \wedge \underline{\lambda_3 - \frac{5}{2}} \wedge \dots.$$

We give $\bigwedge_0^{\frac{\infty}{2}} V$ an inner product by making the basis $\{v_\lambda\}$ orthonormal.

It is easily seen that the v_λ form an eigenbasis of $\bigwedge_0 V$ for H, with $Hv_\lambda = |\lambda| v_\lambda$.

The vacuum vector $|0\rangle$ corresponds to the zero partition. The vacuum expectation $\langle A \rangle$ of an operator A on $\bigwedge^{\frac{\infty}{2}} V$ is defined by the inner product

$$\langle A \rangle = (A\,|0\rangle\,, |0\rangle).$$

4.3.5. *Character Theory.* The infinite wedge is useful for encoding the character theory of the symmetric group.

Another basis of $\bigwedge_0^{\frac{\infty}{2}} V$ indexed by partitions μ is

$$\alpha_{-\mu}\,|0\rangle = \prod_{i=1}^{\ell(\mu)} \alpha_{-\mu_i}\,|0\rangle\,.$$

From the Murnaghan-Nakayama rule, it follows that the change of basis matrix between the v_λ and $\alpha_{-\mu}$ bases is the character table of the symmetric group:

$$\alpha_{-\mu}\,|0\rangle = \sum_{|\lambda|=|\mu|} \chi^\lambda(\mu) v_\lambda. \tag{27}$$

Furthermore, let T be the conjugacy class of a transposition, and define

$$\mathcal{F}_2 = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \frac{k^2}{2} E_{k,k}.$$

Then it follows from a formula of Frobenius that

$$\mathcal{F}_2 v_\lambda = f_T(\lambda) v_\lambda. \tag{28}$$

4.4. **Wreath Product Fock spaces.** We now introduce a larger Fock space $\mathcal{Z}_G$ that is useful in computing the representation theory of G_d. Our knowledge of $\mathcal{Z}_G$ comes from the work of Weiqiang Wang and collaborators; in particular, we make use of a formula from [FW01] and our intial motivation for applying $\mathcal{Z}_G$ to Gromov-Witten theory came from [QW07].

4.4.1. *Definition.* There are more subtle aspects to $\mathcal{Z}_G$, but for our applications it is enough to understand $\mathcal{Z}_G$ as the tensor product of $|G^*|$ copies of $\bigwedge^{\frac{\infty}{2}} V$, indexed by the representations of G. We denote vectors in the copy of $\bigwedge^{\frac{\infty}{2}} V$ labeled by γ by using a superscript γ.

We mostly work in the subspace of $\mathcal{Z}_G$ given by the tensor product of $|K^*|$ copes of $\bigwedge_0^{\frac{\infty}{2}} V$ – that is, the subspace where each component has charge zero. Denote this subspace $\mathcal{R}_G$, as there is a natural identification

$$\mathcal{R}_G = \bigoplus_d R(G_d)$$

of $\mathcal{R}_G$ with the direct sum of the representation rings of the wreath products G_d.

Give $\mathcal{R}_G$ an inner product in the following way: for a G^* labeled partition $\overline{\lambda}$ we introduce the vector

$$v_{\overline{\lambda}} = \bigotimes_{\gamma \in G^*} v_{\overline{\lambda}^\gamma}.$$

The $v_{\overline{\lambda}}$ form a basis, and we declare it to be orthonormal.

The vacuum vector $|0\rangle \in \mathcal{R}_G$ is the tensor product of the vacuum vector living in each copy of $\bigwedge^{\frac{\infty}{2}} V$.

4.4.2. *Operators on $\mathcal{Z}_G$.* For M an operator on $\bigwedge^{\frac{\infty}{2}} V$, and $\gamma \in G^*$, define an operator M^γ on $\mathcal{Z}_G$ by

$$M^\gamma = \mathrm{Id} \otimes \cdots \mathrm{Id} \otimes M \otimes \mathrm{Id} \cdots \mathrm{Id}$$

where the M occurs on the component labeled γ.

This notation allows us to transfer the operators C, H and α to operators C^γ, H^γ and α_n^γ on $\mathcal{Z}_G$.

The operators α_n^γ satisfy the commutation relations

$$[\alpha_n^\gamma, \alpha_m^{\gamma'}] = n\delta_{\gamma,\gamma'}\delta_{n,-m}.$$

There are also α operators indexed by conjugacy classes in G. For $c \in G_*$, define

$$\alpha_n^c = \alpha_n(c) = \sum_{\gamma \in G^*} \gamma(c^{-1})\alpha_n^\gamma. \tag{29}$$

The operators α_n^c and α_n^γ span the same space of operators, and so Equation (29) can be inverted using character theory:

$$\alpha_n^\gamma = \sum_{c \in G_*} \frac{1}{\zeta_c}\gamma(c)\alpha_n^c. \tag{30}$$

This observation and character theory tell us that

$$[\alpha_n^c, \alpha_m^{c'}] = n\zeta_c\delta_{c,c'}\delta_{n,-m}.$$

4.4.3. *Bases.* In this section we describe three distinguished bases of $\mathcal{R}_G$ and their relationship, following Remark 3.1 of [QW07].

Define the vector $\alpha_{-\overline{\mu}}|0\rangle$ by

$$\alpha_{-\overline{\mu}}|0\rangle = \prod_{i=1}^{\ell(\overline{\mu})} \alpha_{-\mu_i}^{c_i}|0\rangle.$$

The $\alpha_{-\overline{\mu}}|0\rangle$ form a basis of $\mathcal{R}_G$, and as an extension of Equation (27) expressing the change of basis between $\alpha_{-\mu}|0\rangle$ and v_λ, we have

$$\alpha_{-\overline{\mu}}|0\rangle = \sum_{|\overline{\lambda}|=|\overline{\mu}|} \chi^{\overline{\lambda}}(\overline{\mu})v_{\overline{\lambda}}. \tag{31}$$

Thus, the two bases $v_{\overline{\lambda}}$ and $\alpha_{-\overline{\mu}}|0\rangle$ encoded the representation theory of G_d.

The operators α_{-n}^γ give rise to a third basis for $\mathcal{R}_G$. For $\overline{\xi}$ a G^* labeled partition, define

$$\alpha_{-\overline{\xi}}|0\rangle = \prod_{i=1}^{\ell(\overline{\xi})} \alpha_{-\xi_i}^{\gamma_i}|0\rangle.$$

The basis $\alpha_{-\overline{\xi}}|0\rangle$ factors the representation theory of G_d into the representation theory of S_d and the representation theory of G: changing from the $\alpha_{-\overline{\mu}}$ basis to the $\alpha_{-\overline{\xi}}$ basis uses the representation theory of G, while the changing from the $\alpha_{-\overline{\xi}}$ basis to the $v_{\overline{\lambda}}$ basis uses the representation theory of S_d.

In more detail: to change basis between $\alpha_{-\overline{\xi}}|0\rangle$ and $\alpha_{-\overline{\mu}}|0\rangle$, one repeatedly applies either Equation (29) or (30); these Equations are just the character table of G, and the repetition gives some combinatorial complication.

The change of basis between $\alpha_{-\overline{\xi}}|0\rangle$ and $v_{\overline{\lambda}}$ is the tensor product of $|G_*|$ copies of the character table of S_d.

The third basis of $\mathcal{R}_G$ is essential to our proof of the decomposition conjecture. It was utilized in a very similar manner in [QW07].

4.4.4. *Multiplication by generalized transpositions.* Equation (28), which says that the operator $\mathcal{F}_2$ encodes multiplication by a transposition, is extended by Frenkel and Wang in [FW01] to wreath products. We recall their formula now.

For $c \in G_*$, define

$$\mathcal{F}_2^c = \sum_{\gamma \in G^*} \frac{|G|^2}{(\dim \gamma)^2} \frac{\gamma(c)}{\zeta_c} \mathcal{F}_2^\gamma.$$

Theorem (3) of [FW01] states that:

$$\mathcal{F}_2^c v_{\overline{\lambda}} = \chi^{\overline{\lambda}}(T_c) v_{\overline{\lambda}}. \tag{32}$$

4.4.5. *Hurwitz Numbers on Fock Space.* The connection between $\mathcal{R}_G$ and representation theory of G_d together with Equation (24) for wreath Hurwitz numbers in terms of representation theory leads to the following formula for wreath double Hurwitz numbers:

Lemma 3.

$$\mathrm{Hur}^\bullet_{g,G}(\overline{\mu},\overline{\nu}) = \frac{1}{\mathfrak{z}(\overline{\mu})}\frac{1}{\mathfrak{z}(\overline{\nu})} \left\langle \prod_{i=1}^{\ell(\mu)} \alpha_{\mu_i}(c_i^\mu) \left(\mathcal{F}_2^0\right)^b \prod_{j=1}^{\ell(\nu)} \alpha_{-\nu_j}(c_j^\nu) \right\rangle.$$

Lemma (3) is the wreath Hurwitz number analog of Okounkov's expression [Oko00] for double Hurwitz numbers in terms of the infinite wedge.

In [Oko00], Okounkov uses this expression to prove that double Hurwitz numbers satisfy the 2-Toda hierarchy. The decomposition conjecture leads us to expectthat wreath double Hurwitz numbers should satisfy commuting copies of the 2-Toda hierarchy. This follows from Lemma 3 in an argument parallel to that of [Oko00]. The result is straightforward and we do not present the details here, but it may be found for the abelian case in [Joh], and the general case in [ZZ12].

4.4.6. *Orbifold ELSV in terms of infinite wedge.* We end this section by combining the orbifold ELSV formula with our expression for wreath Hurwitz numbers on a Fock space.

Thus, we are specializing to $G = K$ is abelian. In this case some formulas simplify. In particular, the definition of $\mathcal{F}_2^c$ in Equation 32 for $c = 0$ is the identity becomes

$$\mathcal{F}_2^0 = |K| \sum_\gamma \mathcal{F}_2^\gamma. \tag{33}$$

To keep our formulas compact, for $m \in \mathbb{Z}$ and $k \in K$ we introduce the operator $\widetilde{\alpha}_m^k$ defined by:

$$\widetilde{\alpha}_m^k = \alpha_m(k - \left\lfloor \frac{m}{r} \right\rfloor \mathbb{k}). \tag{34}$$

Then by Equation (3) we have:

$$\begin{aligned}\mathrm{Hur}^{\bullet}_{g,K}(\overline{r}_{\Bbbk},\overline{\mu}^{\mathfrak{r}}) &= \frac{1}{|K|^{d/r}r^{d/r}(d/r)!}\frac{1}{\mathfrak{z}(\overline{\mu}^{\mathfrak{r}})}\left\langle \alpha_r(-\Bbbk)^{d/r}\left(\mathcal{F}_2^0\right)^b\prod_{i=1}^{\ell(\mu)}\widetilde{\alpha}^{k_i}_{-\mu_i}\right\rangle \\ &= \frac{1}{\mathfrak{z}(\overline{\mu}^{\mathfrak{r}})}\left\langle e^{\frac{\alpha_r(-\Bbbk)}{|R|}}\left(\mathcal{F}_2^0\right)^b\prod_{i=1}^{\ell(\mu)}\widetilde{\alpha}^{k_i}_{-\mu_i}\right\rangle.\end{aligned}$$

Though replacing a single power with an exponential seems to add other terms to the sum, these terms vanish as the vacuum expectation of an operator with nonzero energy is zero.

Substituting this into Equation (26) with u replaced with $u/r^{1/2}$, we obtain, for a partition μ with $-\mu_i \mod r = a_i$:

(35)

$$H^{0,\bullet}_{\mathfrak{r}}\left(\mu,\frac{u}{r^{1/2}}\right) = u^{-|\mu|/r-\ell(\mu)}\frac{r^{-\sum\left\langle\frac{\mu_i}{r}\right\rangle}}{|K|^{\ell(\mu)}}\left(\prod_{i=1}^{\ell(\mu)}\frac{\lfloor\frac{\mu_i}{r}\rfloor!}{\mu_i^{\lfloor\frac{\mu_i}{r}\rfloor}}\right)\left\langle e^{\frac{\alpha_r(-\Bbbk)}{|R|}}e^{u\mathcal{F}_2^0}\prod_{i=1}^{\ell(\mu)}\widetilde{\alpha}^{k_i}_{-\mu_i}\right\rangle.$$

5. Interpolating orbifold ELSV

In this section we prove Theorem A, which interpolates the orbifold ELSV formula to obtain an expression for the Hurwitz-Hodge function $H^{\bullet}_{\mathfrak{r}}$ as a vacuum operator expression.

Section 5.1 introduces the main operators $\mathcal{E}$ and $\mathcal{A}$ and states Theorem B. The remaining two sections carry out the proof. Section 5.2 carries out Step 1, which is to show that for z satisfying the congruence conditions of the previous section, the $\mathcal{A}$ operators simplify drastically. In this case, Theorem A reduces to the orbifold ELSV formula.

Step 2, covered in Section 5.3, shows that the operator expression converges for all values of z in a certain region, and is actually the expansion of a rational function there. Theorem A then follows, as both sides are rational functions, and by Step 1 they agree on a Zariski dense set, and therefore must be equal.

5.1. **Statement of the Theorem.**

5.1.1. *Basic Notation.* Before we introduce the key operators, we recall some preliminary notation. Recall the *Pochhammer symbol*:

$$(x+1)_n = \frac{(x+n)!}{x!} = \begin{cases}(x+1)(x+2)\cdots(x+n) & n\geq 0\\ (x(x-1)\cdots(x+n+1))^{-1} & n\leq 0\end{cases}.$$

From the definition, $(x+1)_n$ vanishes for $-n\leq x\leq -1$ an integer, and $1/(x+1)_n$ vanishes for $0\leq x\leq -(n+1)$ an integer.

We also use the notation

$$\varsigma(z) = e^{\frac{z}{2}} - e^{\frac{-z}{2}},$$

and

$$\mathcal{S}(z) = \frac{\varsigma(z)}{z} = \frac{\sinh(z/2)}{z/2}.$$

5.1.2. *The $\mathcal{E}$ operators.* In their work on the Gromov-Witten theory of curves, Okounkov and Pandharipande make extensive use of operators $\mathcal{E}_r$ for $r \in \mathbb{Z}$. Similarly, define:

$$\mathcal{E}_r^\gamma(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{r}{2})} E_{k-r,k}^\gamma + \frac{\delta_{r,0}}{\varsigma(z)}.$$

Note that while the definition of $\mathcal{E}^{(i)}(z)$ given in Section 4.2 of [QW07] is quite similar to our $\mathcal{E}_0^\gamma(z)$, they are not quite the same.

The operator $\mathcal{E}_r^\gamma(z)$ has energy $-r$, and specializes to the standard bosonic operator α_r^γ on $\mathcal{Z}_G$:

$$\mathcal{E}_r^\gamma(0) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} E_{k-r,k}^\gamma = \alpha_r^\gamma, \quad r \neq 0.$$

Additionally, we have $\mathcal{E}_r^\gamma(z)^* = \mathcal{E}_{-r}^\gamma(z)$, and

$$[\mathcal{E}_r^\gamma(z), \mathcal{E}_s^\gamma(w)] = \varsigma\left(\det\begin{bmatrix} r & z \\ s & w \end{bmatrix}\right)\mathcal{E}_{r+s}^\gamma(z+w). \tag{36}$$

For $\gamma \neq \gamma'$, we of course have

$$[\mathcal{E}_r^\gamma(z), \mathcal{E}_s^{\gamma'}(w)] = 0.$$

5.1.3. *The $\mathcal{A}$ operators.* The $\mathcal{A}$ operators have a complicated definition; the motivation is that this makes Theorem A hold. The reader desperate for motivation beforehand is invited to peek at the statement of Lemma 5.

For $0 \leq a \leq r-1$ and $\gamma \in K^*$, define

$$\mathcal{A}_{a/r}^\gamma(z,u) = \frac{z(\gamma(-\Bbbk)r)^{a/r}}{z+a}\mathcal{S}(|R|uz)^{\frac{z+a}{r}} \sum_{i=-\infty}^{\infty} \frac{z^i \mathcal{S}(|R|uz)^i}{(1+\frac{z+a}{r})_i}\mathcal{E}_{ir+a}^\gamma\big(|K|uz\big). \tag{37}$$

The operators $\mathcal{A}_{a,r}^\gamma(z,u)$ play the role of the operators $\mathcal{A}(x,y)$ in [OP06a], although we have made a few minor changes. First, we use z, u as the two variables, where [OP06a] has z, zu. Additionally, we have conjugated their operators, as we now discuss.

In case R is the trivial group, there is only one such operator, $\mathcal{A}_{0/1}^0$, which simplifies to

$$\mathcal{A}_{0/1}^0(z,u) = \mathcal{S}(uz)^z \sum_{i=-\infty}^{\infty} \frac{u^{-i}\varsigma(uz)^i}{(1+z)_i}\mathcal{E}_i^0(uz).$$

This is the same as the operator $\mathcal{A}(z,uz)$ from [OP06a] except for the factor of u^{-i}. As $\mathcal{E}_i$ has energy i, this change amounts to conjugating by the operator u^H – that is,

$$\mathcal{A}_{0/1}^0 = u^H \mathcal{A}(z,uz)u^{-H}.$$

Since H and its adjoint fix the vacuum, this does not affect operator expectations of products of the $\mathcal{A}$. This modification is rather natural – it appeared already in [OP06a] in the proof of Proposition 9.

From our perspective, the advantage of this modification is that it is natural from the point of view of the decomposition conjecture. The relevant properties are visible now: the only dependence of $\mathcal{A}$ on the choice of K and R appear in a global factor of $\gamma(\Bbbk)$ and in the factors of $|K|$. Due to our modification, this second dependence can be absorbed by rescaling u (recall that $|R| = r|K|$).

The operators that appear in the ELSV formula are linear combinations of the $\mathcal{A}^\gamma$ operators. For $\mathfrak{r} = (a,k) \in \mathbb{Z}_r \times_\beta K = R$, define:

$$\mathcal{A}_{\mathfrak{r}} = \sum_{\gamma \in K^*} \gamma(-k)\mathcal{A}^\gamma_{a/r}. \tag{38}$$

5.1.4. *The Theorem.* Our main theorem is that the Hurwitz-Hodge generating function is a vacuum expectation of a product of the operators $\mathcal{A}_{\mathfrak{r}}$:

Theorem A.

$$H^\bullet_{\mathfrak{r}}(z_{\mathfrak{r}}, \frac{u}{r^{1/2}}) = (u|K|)^{-\ell(\mathfrak{r})} \left\langle \prod_{i=1}^{\ell(\mathfrak{r})} \mathcal{A}_{r_i}(z_i,u) \right\rangle. \tag{39}$$

The remainder of this section is devoted to the proof of Theorem A.

5.2. **Step 1: agreement with orbifold ELSV.** In this section we prove the following statement:

Lemma 4. *Let $r_i = (a_i,k_i) \in \mathbb{Z}_r \times_\beta K = R^n$ and let z_i be positive integers with $z_i \equiv -a_i \mod r$. Then Theorem A (Equation 5.3.1) holds.*

The proof is straightforward. We unravel the definition of the $\mathcal{A}$ operators until it is plain that, given our restrictions on z and $\mathfrak{r}$, the operator expression in Equation 5.3.1 reduces to Equation 35.

5.2.1. We begin by recalling Equation (35):

$$H^{0,\bullet}_{\mathfrak{r}}\left(\mu, \frac{u}{r^{1/2}}\right) = u^{-|\mu|/r-\ell(\mu)} \frac{r^{-\sum\langle\frac{\mu_i}{r}\rangle}}{|K|^{\ell(\mu)}} \left(\prod_{i=1}^{\ell(\mu)} \frac{\lfloor\frac{\mu_i}{r}\rfloor!}{\mu_i^{\lfloor\frac{\mu_i}{r}\rfloor}}\right) \left\langle e^{\frac{\alpha_r(-\Bbbk)}{|R|}} e^{u\mathcal{F}^0_2} \prod_{i=1}^{\ell(\mu)} \widetilde{\alpha}^{k_i}_{-\mu_i} \right\rangle$$

for $\mu_i = -a_i \mod r$.

Since $\alpha_r(-\Bbbk)$ and $\mathcal{F}^0_2$ both annihilate the vacuum, the vacuum expectation above (ignoring the prefactors) is equivalent to

$$\left\langle \prod_{i=1}^{\ell(\mu)} e^{\frac{\alpha_r(-\Bbbk)}{|R|}} e^{u\mathcal{F}^0_2} \widetilde{\alpha}^{k_i}_{-\mu_i} e^{-u\mathcal{F}^0_2} e^{\frac{-\alpha_r(-\Bbbk)}{|R|}} \right\rangle.$$

5.2.2. It is useful to change our point of view so that the identification of $\mathcal{R}_K$ with $\bigotimes \bigwedge_0^{\frac{\infty}{2}} V^\gamma$ is more visible. By Equation (29) we have

$$\alpha_r(-\Bbbk) = \sum_{\gamma \in K^*} \gamma(\Bbbk)\alpha^\gamma_r.$$

Similarly, expanding $\widetilde{\alpha}^{k_i}_{-\mu_i}$ by its definition (34) we find that

$$\widetilde{\alpha}^{k_i}_{-\mu_i} = \sum_{\gamma\in K^*} \gamma(-k_i)\gamma(\Bbbk)^{\lfloor\frac{-\mu_i}{r}\rfloor} \alpha^\gamma_{-\mu_i}.$$

Using Equation (33) to expand $\mathcal{F}^0_2$, this gives

$$e^{\frac{\alpha_r(-\Bbbk)}{|R|}} e^{u\mathcal{F}^0_2} \widetilde{\alpha}^{k_i}_{-\mu_i} e^{-u\mathcal{F}^0_2} e^{\frac{-\alpha_r(-\Bbbk)}{|R|}}$$
$$= \sum_{\gamma\in K^*} \gamma(-k_i)\gamma(\Bbbk)^{\lfloor\frac{-\mu_i}{r}\rfloor} \left(e^{\frac{\alpha^\gamma_r}{|R|}} e^{u|K|\mathcal{F}^\gamma_2} \alpha^\gamma_{-\mu_i} e^{-u|K|\mathcal{F}^\gamma_2} e^{\frac{-\gamma(\Bbbk)\alpha^\gamma_r}{|R|}} \right).$$

5.2.3. Since both H^γ and $H^{\gamma*}$ annihilate the vacuum, we can further conjugate each operator by $c_\gamma^{H^\gamma}$, for any constants c_γ, and not change the vacuum expectation. This conjugation multiplies operators of energy E by c_γ^E.

Conjugate each operator by

$$\prod_{\gamma\in K^*} (u\gamma(\Bbbk))^{\frac{H^\gamma}{r}}.$$

This leaves $\mathcal{F}_2^\gamma$ fixed, rescales α_r^γ by $(u\gamma(\Bbbk))^{-1}$, and rescales $\alpha_{-\mu_i}^\gamma$ by $(u\gamma(\Bbbk))^{\mu_i/r}$. Thus, we see that it changes

$$e^{\gamma(\Bbbk)\frac{\alpha_r^\gamma}{|R|}} e^{u|K|\mathcal{F}_2^\gamma} \alpha_{-\mu_i}^\gamma e^{-u|K|\mathcal{F}_2^\gamma} e^{\frac{-\gamma(\Bbbk)\alpha_r^\gamma}{|R|}}$$

to

$$(u\gamma(\Bbbk))^{\mu_i/r} e^{\frac{\alpha_r^\gamma}{u|R|}} e^{u|K|\mathcal{F}_2^\gamma} \alpha_{-\mu_i}^\gamma e^{-u|K|\mathcal{F}_2^\gamma} e^{-\frac{\alpha_r^\gamma}{u|R|}}.$$

Applying this conjugation to Equation (35), we can combine and simplify powers of u and $\gamma(\Bbbk)$ using

$$\frac{m}{r} + \left\lfloor \frac{-m}{r} \right\rfloor = \left\langle \frac{m}{r} \right\rangle - \delta_r^\vee(m) = -\left\langle \frac{-m}{r} \right\rangle,$$

and see that the proof of Lemma 4 reduces to the following lemma.

Lemma 5. *For $z = -a \mod r$, we have*

$$\mathcal{A}_{a/r}^\gamma(z,u) = r^{-\left\langle \frac{z}{r}\right\rangle} \gamma(-\Bbbk)^{a/r} \frac{\left\lfloor \frac{z}{r} \right\rfloor !}{z^{\left\lfloor \frac{z}{r}\right\rfloor}} \left(e^{\frac{\alpha_r^\gamma}{u|R|}} e^{u|K|\mathcal{F}_2^\gamma} \alpha_{-z}^\gamma e^{-u|K|\mathcal{F}_2^\gamma} e^{-\frac{\alpha_r^\gamma}{u|R|}} \right).$$

5.2.4. *Proof of Lemma 5.* We begin by investigating the term in parentheses.

Equation (2.14) of [OP06a] states that

$$e^{u\mathcal{F}_2}\alpha_{-m}e^{-u\mathcal{F}_2} = \mathcal{E}_{-m}(um),$$

and so

$$e^{u|K|\mathcal{F}_2^\gamma} \alpha_{-z}^\gamma e^{-u|K|\mathcal{F}_2^\gamma} = \mathcal{E}_{-z}^\gamma(u|K|z).$$

Now we consider the effect of the $e^{\alpha_r^\gamma/(u|R|)}$ terms. Since

$$[\alpha_r, \mathcal{E}_{-m}(w)] = \varsigma(rw)\mathcal{E}_{-m+r}(w),$$

we see that

$$\begin{aligned}
e^{\frac{\alpha_r^\gamma}{u|R|}} \mathcal{E}_{-m}^\gamma(w) e^{\frac{-\alpha_r^\gamma}{u|R|}} &= \sum_{i=0}^\infty \frac{1}{i!}\left[\left(\frac{\alpha_r^\gamma}{u|R|}\right)^i, \mathcal{E}_{-m}^\gamma(w)\right] e^{\frac{-\alpha_r^\gamma}{u|R|}} + \mathcal{E}_{-m}^\gamma(w) \\
&= \sum_{0\le j\le i} \left(\frac{1}{u|R|}\right)^j \frac{1}{i!}\binom{i}{j} \underbrace{[\alpha_r^\gamma, [\ldots, [\alpha_r^\gamma}_{j \text{ times}}, \mathcal{E}_{-m}^\gamma(w)]]\ldots] \left(\frac{\alpha_r^\gamma}{u|R|}\right)^{i-j} e^{\frac{-\alpha_r^\gamma}{u|R|}} \\
&= \sum_{j=0}^\infty \frac{1}{j!}\left(\frac{\varsigma(rw)}{u|R|}\right)^j \mathcal{E}_{-m+jr}^\gamma(w).
\end{aligned}$$

When $m = z$ and $w = u|K|z$, we see that $\frac{\varsigma(rw)}{u|R|} = z\mathcal{S}(u|R|z)$. Writing $-z = a - (\frac{z+a}{r})r$, we set $b = \frac{z+a}{r}$, and $h = j - b$, so that the sum becomes:

$$(40) \qquad \sum_{h=-b}^{\infty} \frac{1}{(h+b)!} \left(z\mathcal{S}(u|R|z)\right)^{h+b} \mathcal{E}^{\gamma}_{a-br+jr}(u|K|z)$$
$$= \frac{1}{b!} \left(z\mathcal{S}(u|R|z)\right)^{b} \sum_{h=-b}^{\infty} \left(z\mathcal{S}(u|R|z)\right)^{h} \frac{b!}{(h+b)!} \mathcal{E}^{\gamma}_{a+hr}(u|K|z).$$

Since

$$\frac{b!}{(h+b)!} = \frac{1}{(1+b)_h} = \frac{1}{(1+\frac{z+a}{r})_h}$$

vanishes if $b \in \mathbb{Z}, b \leq -(h+1)$, extending the sum to all $h \in \mathbb{Z}$ does not change the value when $z = -a \mod r$. Using $b = (z+a)/r = \left\lfloor \frac{z}{r} \right\rfloor + \delta^{\vee}_r(a)$, we can rewrite the prefactor in Equation (40) as

$$\left(\frac{rz}{z+a}\right)^{\delta^{\vee}_r(a)} \frac{z^{\left\lfloor \frac{z}{r} \right\rfloor}}{\left\lfloor \frac{z}{r} \right\rfloor !} \mathcal{S}(u|R|z)^{\frac{z+a}{r}}.$$

Substituting this in and simplifying proves Lemma 5, and hence Lemma 4 as well.

5.3. **Step 2: convergence and rationality.** We now complete Step 2 of the proof, by showing the vacuum expectation of the $\mathcal{A}$ operators in Theorem A converge, and that each coefficient of a given power of u converges to a rational function.

The proof depends on several technical results that are the analogs of the appendices of [OP06a]. The proofs given in [OP06a] carry over to our situation with almost no change, and so omit them, though the interested reader may find them written out in full in [Joh].

5.3.1. *Convergence.* The first step is see for what values of $z_{\mathfrak{r}}$ the left hand side is well defined. When $z_i \neq -a_i \mod r$, the sum in the definition of $\mathcal{A}^{\gamma}_{a_i/r}$ is infinite in both dimensions, and so the energy of the operators $\mathcal{A}_{r_i}(z, u)$ is not bounded on either side, and so there is no immediate reason to suppose that Equation makes sense except as a formal power series.

In fact, the right hand side of Equation has nice convergence properties. In particular, define $\Omega \subset \mathbb{C}^n$ by

$$\Omega = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \middle| \forall k, |z_k| > \sum_{i=1}^{k-1} |z_i| \right\}.$$

The operators $\mathcal{A}^{\gamma}_{a/r}$ have poles at negative integers, but away from these, we have

Lemma 6. *Let K be a compact set,*

$$K \subset \Omega \cap \{z_i \neq -1, -2, \ldots, i = 1, \ldots, n\}.$$

Then for all $\gamma \in K^*, 0 \leq a_i < r$, and μ, λ partitions, the series

$$\left\langle \mathcal{A}^{\gamma}_{a_1/r}(z_1, u) \cdots \mathcal{A}^{\gamma}_{a_n/r}(z_n, u) v_\mu, v_\lambda \right\rangle$$

converges uniformly and absolutely for all sufficiently small $u \neq 0$.

This is the analog of Proposition 3 in Section 2.3 of [OP06a], and we omit the proof.

5.3.2. *Laurent Series.* As a consequence of Proposition 6, we see that the vacuum expectation

$$\left\langle \mathcal{A}^{\gamma}_{a_1/r}(z_1,u)\cdots\mathcal{A}^{\gamma}_{a_n/r}(z_n,u)\right\rangle$$

is an analytic function of $(z_1,\ldots,z_n,u)$ in a neighborhood of the origin intersect $\Omega\times\mathbb{C}^*$. Hence, we may expand it as a convergent Laurent series.

It is important for us to understand the terms appearing with negative exponents. To that end, for any ring U, we denote by $U((z))$ the ring of formal Laurent series with coefficients in U and degree bounded below:

$$U((z))=\Big\{\sum_{i\in\mathbb{Z}}u_iz^i\Big|u_i\in U, u_i=0\forall i<<0\Big\}.$$

Then we have

Lemma 7.

$$\left\langle \mathcal{A}^{\gamma}_{a_1/r}(z_1,u)\cdots\mathcal{A}^{\gamma}_{a_n/r}(z_n,u)\right\rangle\in\mathbb{C}[u^{\pm1}]((z_n))((z_{n-1}))\cdots((z_1)).$$

Note that this *does not* say that power of z_n appearing in the right hand side is bounded below. Rather, if we fix arbitrary $p_1,\ldots,p_{n-1}$, then the powers of z_n appearing as the coefficient of $z_1^{p_1}\cdots z_{n-1}^{p_{n-1}}$ are bounded below. However, as the p_i go to infinity, the powers of z_n can go to negative infinity.

Proof. The key point is that the power of z appearing in the energy m part of $\mathcal{A}^{\gamma}_{a/r}$ is bounded below by some function $f(m)$ that satisfies:

$$\lim_{m\to\infty}f(m)=\infty.$$

Being bounded below by $f(m)$ implies that the vacuum expectation of $\mathcal{A}^{\gamma}_{a/r}$ applied to any vector in the infinite wedge has exponent of z bounded below. Knowing that $f(m)$ goes to infinity implies that for any fixed k, the coefficient of z^k in $\mathcal{A}^{\gamma}_{a/r}$ can only raise the energy a bounded amount. To see this, observe that the energy m part has coefficients in $z^{f(m)}\mathbb{C}[z]$, and for m sufficiently high this does not contain z^k.

So it is enough to establish the key point. First note that the energy m part of $\mathcal{A}^{\gamma}_{a/r}$ is the term of Definition (37) containing $\mathcal{E}^{\gamma}_m$.

Next, note the expansion of $\mathcal{E}^{\gamma}_m$ contains only positive powers of z, except for the z^{-1} term appearing in $\mathcal{E}_0$.

Finally, the powers of z appearing in the coefficient of $\mathcal{E}^{\gamma}_m$ is bounded below by $\left\lfloor\frac{m}{r}\right\rfloor$. Indeed, the prefactors of $z/(z+a)$ and $\mathcal{S}$ appearing in the Definition (37) contribute only positive powers of z, the ς^i factor of the coefficient of $\mathcal{E}_{ir+a}$ has leading term z^i, and the factor $\frac{1}{(1+(z+a)/r)_i}$ also contributes only positive terms. □

5.3.3. *Commutators of the $\mathcal{A}$.* The rationality result follows from a commutator formula for the $\mathcal{A}$ operators, which we now motivate and state.

Let us first consider the function $H^{\bullet}_{\mathfrak{r}}(z_{\mathfrak{r}},u)$, and its coefficients of u, the rational functions $H^{\bullet}_{g,\mathfrak{r}}$. On the domain Ω, we can expand the function $H^{\bullet}_{g,\mathfrak{r}}$ as a Laurent series. Note that permuting the z_i changes the definition of the domain Ω, and hence the Laurent expansion.

In particular, the function $\frac{1}{z+w}$, which appears in $H^\bullet_{g,\mathfrak{r}}$ as part of the unstable contributions (Equation (6)), can be expanded as a geometric series in two different ways, depending on which of $|z|$ and $|w|$ is bigger:

$$\frac{1}{z+w} = \frac{1}{w} - \frac{z}{w^2} + \frac{z^2}{w^3} - \dots, \quad |z| < |w|$$

$$\frac{1}{z+w} = \frac{1}{z} - \frac{w}{z^2} + \frac{w^2}{z^3} - \dots, \quad |z| > |w|.$$

Taking the difference of these expansions gives the formal series

$$\delta(z,-w) = \frac{1}{w}\sum_{n\in\mathbb{Z}}\left(-\frac{z}{w}\right)^n,$$

which converges nowhere. However, this series acts as a formal delta function at $z=-w$ because it satisfies

$$(z+w)\delta(z,-w) = 0. \tag{41}$$

Let us investigate what changing this choice of Laurent expansion does to $H^\bullet_{\mathfrak{r}}$. To that end, let us always expand the variables out in the order they appear, and let $\tilde{\mathfrak{r}}$ and $\hat{\mathfrak{r}}$ be obtained from $\mathfrak{r}$ by adding some s and t, in the opposite orders:

$$\tilde{\mathfrak{r}} = \{r_1,\dots,r_n,s,t\} \quad \hat{\mathfrak{r}} = \{r_1,\dots,r_n,t,s\}.$$

Since the two point unstable contribution $\frac{z_iz_j}{|R|(z_i+z_j)}$ occurs in genus 0 and only when $r_i = r_{i+1}^{-1}$, we see that switching the order of (x,s) and (y,t) only matters when $t=-s$, when we have

$$\begin{aligned} H^\bullet_{\tilde{\mathfrak{r}}}(z_{\mathfrak{r}},x,y,\frac{u}{r^{1/2}}) - H^\bullet_{\hat{\mathfrak{r}}}(z_{\mathfrak{r}},y,x,\frac{u}{r^{1/2}}) &= \left(\frac{u}{r^{1/2}}\right)^{-2}\delta_{s,-t}\frac{xy}{|R|}\delta(x,-y)H^\bullet_{\mathfrak{r}}(z_{\mathfrak{r}},\frac{u}{r^{1/2}}) \\ &= \delta_{s,-t}\frac{xy}{u^2|K|}\delta(x,-y)H^\bullet_{\mathfrak{r}}(z_{\mathfrak{r}},\frac{u}{r^{1/2}}). \end{aligned}$$

Comparing this with Lemma 4 and taking note of the prefactor of $(u|K|)^{-\ell(\mathfrak{r})}$ suggests the following formula for the commutators of the $\mathcal{A}_{r_i}(z,u)$:

$$[\mathcal{A}_{r_1}(z,u),\mathcal{A}_{r_2}(w,u)] = \delta_{r_1,-r_2}|K|zw\delta(z,-w).$$

This formula is a corollary of the following commutator formula for the $\mathcal{A}^\gamma_{a/r}(z,u)$, which we make further use of later:

Lemma 8.

$$[\mathcal{A}^\gamma_{a/r}(z,u),\mathcal{A}^{\gamma'}_{b/r}(w,u)] = \delta_{\gamma,\gamma'}\delta_r(a+b)\gamma(\Bbbk)^{-(a+b)/r}zw\delta(z,-w).$$

The proof of Lemma 8 is only a minor modification of Theorem 1 from [OP06a] (proven in Section 5 of that paper). We omit it, but note again that the proof is written in full detail in [Joh].

We now show that the predicted formula for the commutators of $\mathcal{A}_r$ follows from Lemma 8.

Corollary 9.

$$[\mathcal{A}_{r_1}(z,u),\mathcal{A}_{r_2}(w,u)] = \delta_{r_1,-r_2}|K|zw\delta(z,-w).$$

Proof. Let $r_1 = (a, k_1)$ and $r_2 = (b, k_2)$ be elements of R, where we have written our elements using our description of $R = \mathbb{Z}_r \times_\beta K$. Then, expanding $\mathcal{A}_{r_1}, \mathcal{A}_{r_2}$ by Definition (38) and using Lemma 8, we have:

$$\begin{aligned}[][\mathcal{A}_{r_1}(z,u), \mathcal{A}_{r_2}(w,u)] &= \sum_{\gamma,\gamma' \in K^*} \gamma(-k_1)\gamma'(-k_2)[\mathcal{A}^{\gamma}_{a/r}(z,u), \mathcal{A}^{\gamma'}_{b/r}(w,u)] \\ &= \delta_{a,b^\vee} zw\delta(z,-w) \sum_{\gamma \in K^*} \gamma(-k_1 - k_2 - \delta_r^\vee(a)\mathbb{k})\end{aligned}$$

By character orthogonality, the sum is zero if $k_1 + k_2 \neq -\delta_r^\vee(a)\mathbb{k}$, and $|K|$ otherwise. From the definition of $R = \mathbb{Z}_r \times_\beta K$, this combines with $\delta_{a,b^\vee}$ to give $|K|\delta_{r_1,-r_2}$. □

5.3.4. *Rationality.* We now show that the the vacuum expectation of our products of $\mathcal{A}$ are rational. The point is that if Theorem A is true, we explicitly understand the denominators that occur: they are the factors of $1/z_i$ and $1/(z_i + z_j)$ coming from the unstable contributions. Once these contributions are taken into account, the remaining terms are actually *polynomial*.

Lemma 10. *The series:*

$$\left(\prod_{\substack{i<j \\ \mathfrak{r}_i = -\mathfrak{r}_j}} (z_i + z_j)\right) \langle \mathcal{A}_{\mathfrak{r}_1}(z_1,u) \cdots \mathcal{A}_{\mathfrak{r}_n}(z_n,u)\rangle$$

is independent of the ordering of the $(z_i, \mathfrak{r}_i)$*, and is an element of*

$$\prod_{\{i|\rho_i = 0\}} z_i^{-1}\mathbb{C}[u^{\pm 1}][[z_1, \ldots, z_n]].$$

Proof. That the series is independent of the ordering follows from Corollary 9 and Equation (41). Because the series is independent of ordering, to show that it is a power series except for a factor of z_i^{-1} for i with $\mathfrak{r}_i = 0$, it is enough to do so for z_1. However, this follows from the proof of Lemma 7 and Equation (38) expanding $\mathcal{A}_{\mathfrak{r}_i}$ in terms of $\mathcal{A}^{\gamma}_{a/r}$. □

Lemma 11. *The coefficients of powers of* u *in the right hand side of Lemma 4,*

$$[u^m]\langle \mathcal{A}_{\mathfrak{r}_1}(z_1,u) \cdots \mathcal{A}_{\mathfrak{r}_n}(z_n,u)\rangle, m \in \mathbb{Z}$$

are rational functions in the z_i*, with at most simple poles along the divisors* $z_i + z_j = 0$ *for* i, j *with* $\mathfrak{r}_i + \mathfrak{r}_j = 0$*, and divisors* $z_i = 0$ *with* $\mathfrak{r}_i = 0$*.*

Proof. From Lemma 10 and the fact that expanding $1/(z_i + z_j)$ on Ω only introduces negative powers of z_n, we see that it is enough to show that the coefficient of z_n is bounded from above.

We accomplish this by closely considering the powers of u. First, we pair any occurrence of z_n^ℓ for ℓ positive with a factor of $u^{\ell/2}$. Then we show that the remaining powers of u appearing have degree bounded below.

We consider the expansion of the $\mathcal{A}$ in terms of the $\mathcal{E}$, and hence terms of the form

$$\langle \mathcal{E}_{k_1}(u|K|z_1) \cdots \mathcal{E}_{k_n}(u|K|z_n)\rangle.$$

These terms vanish unless $\sum k_i = 0$ and $k_n \leq 0$.

In what follows, we work closely with Equation 37, the definition of $\mathcal{A}^{\gamma}_{1/r}$. Note first that $\mathcal{E}_k$ appears only if $k = a + ri$. Thus, we see that if $a_n = 0$, we have $i \leq 0$, while if $a_n \neq 0$ we have $i \leq -1$. In either case, the pole at $z = -a$ occurring in the prefactor of $\mathcal{A}$ is canceled, and the vacuum expectation depends on z_n only through terms of the form

$$(ur)^{a/r}\mathcal{S}(|R|uz)^{\frac{z+a}{r}} \tag{42}$$

from the prefactor,

$$e^{xu|K|z_n} \tag{43}$$

from the definition of $\mathcal{E}$, and

$$\frac{z_n}{r}\left(\frac{z_n+a_n}{r}-1\right)\cdots\left(\frac{z_n+a_n}{r}+i+1\right)\times(z_n\mathcal{S}(|R|uz_n))^i \tag{44}$$

from the coefficient of $\mathcal{E}_{a_n+ir}$, where the first term in the product is z_n/r instead of $(z_n + a_n)/r$ because we have multiplied it by the prefactor $z_n/(z_n + a_n)$.

Now, it is clear that in term (43), z_n^m occurs with coefficient u^m. There is a less obvious grouping for the terms of the form (42) - rewriting $\mathcal{S}$ as $e^{\ln \mathcal{S}}$, and using the Taylor expansion for $\ln(1+x)$, we see that the term z_n^{ℓ} occurs with a coefficient of u^p, with $p \geq \ell/2$. Finally, to handle the z_n appearing in (44), observe that the first product is a polynomial in z_n of degree $-i$, and so we can pair it with the z_n^i appearing, to get all negative powers of z_n, except for those paired with u. We have thus shown that all positive appearances of z_n occur with a positive power of u as well. Furthermore, the only u appearing as a negative power are those coming from the constant term of $\mathcal{E}_0$, and so we are done. □

6. The main theorem and applications

We now prove Theorem B and derive several corollaries.

6.1. Proof of the main theorem.

The proof of Theorem B consists of plugging the interpolated orbifold ELSV formula (Theorem A) into the localization calculation of Section 3 and simplifying. We work mostly with the n-point functions $G^{\bullet}_{d,\mathfrak{r},\mathfrak{s}}(z_{\mathfrak{r}}, w_{\mathfrak{s}}, u)$, and turn to $\tau_{\mathcal{X}}$ only at the end.

6.1.1. *Plugging in.* Recall that the culmination of our localization calculation was Equation (22):

$$G^{\bullet}_{d,\mathfrak{r},\mathfrak{s}}(z_{\mathfrak{r}}, w_{\mathfrak{s}}, u) = \sum_{|\overline{\mu}|=d} \frac{1}{\mathfrak{z}(\overline{\mu})}\mathbf{J}_{\mathfrak{r}}(z_{\mathfrak{r}}, \overline{\mu}, u, t)\mathbf{J}_{\mathfrak{s}}(z_{\mathfrak{s}}, \overline{\mu}, u, -t).$$

Combining the definition of $\mathbf{J}$, Equation (21), with Theorem A for H gives:

$$\mathbf{J}_{\mathfrak{r}}(z_{\mathfrak{r}}, \overline{\mu}, u, t) = \frac{r^{\sum\langle\frac{\mu_i}{r}\rangle-\iota(\mathfrak{r})}(|K|u/t)^{\ell(\mu)}}{t^{|\mu|/r+\#(r_i\notin K)+\ell(\mathfrak{r})-\iota(\mathfrak{r})}}\left(\prod_{i=1}^{\ell(\overline{\mu})}\frac{\mu_i^{\lfloor\frac{\mu_i}{r}\rfloor}}{\lfloor\frac{\mu_i}{r}\rfloor!}\right)H^{0,\bullet}_{\mathfrak{r}+\rho(\overline{\mu})}\left(\mu, tz_{\mathfrak{r}}, \frac{u}{tr^{1/2}}\right)$$

$$= \frac{r^{\sum\langle\frac{\mu_i}{r}\rangle-\iota(\mathfrak{r})}(|K|u/t)^{-\ell(\mathfrak{r})}}{t^{|\mu|/r+\#(r_i\notin K)+\ell(\mathfrak{r})-\iota(\mathfrak{r})}}\left(\prod_{i=1}^{\ell(\overline{\mu})}\frac{\mu_i^{\lfloor\frac{\mu_i}{r}\rfloor}}{\lfloor\frac{\mu_i}{r}\rfloor!}\right)\left\langle\prod_{i=1}^{\ell(\mathfrak{r})}\mathcal{A}_{\mathfrak{r}_i}\left(tz_i, \frac{u}{t}\right)\prod_{j=1}^{\ell(\overline{\mu})}\mathcal{A}_{\rho(\overline{\mu}_j)}\left(\mu_j, \frac{u}{t}\right)\right\rangle.$$

6.1.2. *From two vacuum expectations to one.* The overall localization calculation is quadratic in the $\mathbf{J}$ function, but we would like to express the GW invariants as a single vacuum expectation. To that end, define the operator $\mathbf{P}_\emptyset$ to be projection onto the vacuum vector. Then, taking the adjoint of the operator definition of $\mathbf{J}_{\mathfrak{s}}(z_{\mathfrak{s}}, \overline{\mu}, u, -t)$, we can write $G^\bullet$ as a single vacuum expectation as follows:

$$G^\bullet_{d,\mathfrak{r},\mathfrak{s}}(z_{\mathfrak{r}}, w_{\mathfrak{s}}, u) = \sum_{|\overline{\mu}|=d} \frac{1}{\mathfrak{z}(\overline{\mu})} \times \tag{45}$$

$$\frac{r^{\sum\left\langle\frac{\mu_i}{r}\right\rangle - \iota(\mathfrak{r})}\left(\frac{|K|u}{t}\right)^{-\ell(\mathfrak{r})}}{t^{|\mu|/r + \#(r_i\notin K) + \ell(\mathfrak{r}) - \iota(\mathfrak{r})}} \left(\prod_{i=1}^{\ell(\overline{\mu})} \frac{\mu_i^{\left\lfloor\frac{\mu_i}{r}\right\rfloor}}{\left\lfloor\frac{\mu_i}{r}\right\rfloor!}\right) \frac{s^{\sum\left\langle\frac{\mu_i}{s}\right\rangle - \iota(\mathfrak{s})}\left(\frac{|K|u}{-t}\right)^{-\ell(\mathfrak{s})}}{(-t)^{|\mu|/s + \#(s_i\notin K) + \ell(\mathfrak{s}) - \iota(\mathfrak{s})}} \left(\prod_{i=1}^{\ell(\overline{\mu})} \frac{\mu_i^{\left\lfloor\frac{\mu_i}{s}\right\rfloor}}{\left\lfloor\frac{\mu_i}{s}\right\rfloor!}\right)$$

$$\left\langle \prod \mathcal{A}_{\mathfrak{r}_i}\left(tz_i, \frac{u}{t}\right) \prod_{j=1}^{\ell(\overline{\mu})} \mathcal{A}_{\rho(\overline{\mu}_j)}\left(\mu_j, \frac{u}{t}\right) \mathbf{P}_\emptyset \prod_{j=1}^{\ell(\overline{\mu})} \mathcal{A}^*_{\sigma(\overline{\mu}_j)}\left(\mu_j, \frac{u}{-t}\right) \prod \mathcal{A}^*_{\mathfrak{s}_i}\left(-tw_i, -\frac{u}{t}\right) \right\rangle.$$

6.1.3. *Definitions.* We introduce some definitions to simplify Equation (45). We modify the $\mathcal{A}_{\mathfrak{r}}$ operators by including any of the prefactors pertaining to $\mathfrak{r}$ or $\mathfrak{s}$, and we gather the remaining $\mathcal{A}_\mu$ operators and all prefactors containing to μ into an operator we call $\mathbf{Q}_d$.

Collecting every factor relating to $\mathfrak{r}_i$ together, we have:

$$\begin{aligned}\frac{1}{t}\left(\frac{t}{r}\right)^{a/r} & \frac{t^{\delta_r(a)}}{|K|u}\mathcal{A}^\gamma_{a/r}(tz, u/t) \\ &= \frac{(t\gamma(-\mathbb{k}))^{a/r}}{|K|u}\frac{t^{\delta_r(a)}z}{(tz+a)}\mathcal{S}(|R|uz)^{\frac{tz+a}{r}} \sum_{i=-\infty}^{\infty} \frac{(tz\mathcal{S}(|R|uz))^i}{(1+\frac{tz+a}{r})_i}\mathcal{E}^\gamma_{ir+a}(|K|uz).\end{aligned}$$

A similar statement is true when working with $\mathfrak{s}$, after replacing r and its variants with s and replacing t with $-t$.

Conjugating the To simplify, we conjugate $\mathcal{A}_{\mathfrak{r}}$ and $\mathcal{A}_{\mathfrak{s}}$ by $t^{H/r}$ and $(-t)^{H/S}$, respectively. As before, since $\mathcal{E}_n(z)$ has energy $-n$, this has the effect of rescaling $\mathcal{E}_n(z)$ by $(\pm t)^{-n/r}$. We denote the resulting rescaled operator by $\mathbf{A}$:

$$\mathbf{A}^\gamma_{a/r} = \frac{(\gamma(-\mathbb{k}))^{a/r}}{|K|u}\frac{t^{\delta_r(a)}z}{(tz+a)}\mathcal{S}(|R|uz)^{\frac{tz+a}{r}} \sum_{i=-\infty}^{\infty} \frac{(z\mathcal{S}(|R|uz))^i}{(1+\frac{tz+a}{r})_i}\mathcal{E}^\gamma_{ir+a}(|K|uz). \tag{46}$$

Furthermore, following Definition (38), define

$$\mathbf{A}_{r_i}(z) = \sum_{\gamma\in K^*} \gamma(-k_i)\mathbf{A}^\gamma_{a/r}(z).$$

Having dealt with the terms dealing with $\mathfrak{r}$, we now examine those dealing with $\overline{\mu}$. Collect all terms dealing with $\overline{\mu}$ into the operator $\mathbf{Q}_d$. That is, define:

$$\mathbf{Q}_d = \sum_{|\overline{\mu}|=d} \frac{1}{\mathfrak{z}(\overline{\mu})} r^{\sum\left\langle\frac{\mu_i}{r}\right\rangle} t^{-|\mu|/r} s^{\sum\left\langle\frac{\mu_i}{s}\right\rangle} (-t)^{-|\mu|/s} \tag{47}$$

$$\left(\prod_{i=1}^{\ell(\overline{\mu})} \frac{\mu_i^{\left\lfloor\frac{\mu_i}{r}\right\rfloor}}{\left\lfloor\frac{\mu_i}{r}\right\rfloor!}\right)\left(\prod_{i=1}^{\ell(\overline{\mu})} \frac{\mu_i^{\left\lfloor\frac{\mu_i}{s}\right\rfloor}}{\left\lfloor\frac{\mu_i}{s}\right\rfloor!}\right) \prod_{j=1}^{\ell(\overline{\mu})} \mathcal{A}_{\rho(\overline{\mu}_j)}\left(\mu_j, \frac{u}{t}\right) \mathbf{P}_\emptyset \prod_{j=1}^{\ell(\overline{\mu})} \mathcal{A}^*_{\sigma(\overline{\mu}_j)}\left(\mu_j, \frac{u}{-t}\right).$$

With these definitions, and remembering the additional conjugations we have made, we see that Equation (45) can be compactly written as:

$$G^{\bullet}_{d,\mathfrak{r},\mathfrak{s}}(z_{\mathfrak{r}}, w_{\mathfrak{s}}, u) = \left\langle \prod \mathbf{A}_{\mathfrak{r}_i}(z_i) t^{H/r} \mathbf{Q}_d (-t)^{H/s} \prod \mathbf{A}^*_{\mathfrak{s}_i}(w_i) \right\rangle. \tag{48}$$

6.1.4. *Simplifying* $\mathbf{Q}_d$. We continue now by simplifying the form of the operator $\mathbf{Q}_d$. The simplification has two steps. First, the operators $\mathcal{A}$ appearing in $\mathbf{Q}_d$ are all evaluated at integers μ_i, where they have a particularly simple form. Second, currently we have $\mathbf{Q}_d$ written in terms of a projection onto the vacuum, but it is better written in terms of projection onto the energy d subspace of the Fock space.

We work now with the operators appearing before the $\mathbf{P}_\emptyset$, and then explain what differs when dealing with terms after the $\mathbf{P}_\emptyset$.

By Lemma 5, we have

$$\mathcal{A}_{\rho(\overline{\mu}_j)}\left(\mu_j, \frac{u}{t}\right) = r^{-\langle \frac{\mu_j}{r} \rangle} \frac{\lfloor \frac{\mu_j}{r} \rfloor !}{\mu_j^{\lfloor \frac{\mu_j}{r} \rfloor}} \times$$
$$\sum_{\gamma \in K^*} \gamma(-\Bbbk_0)^{\left\langle \frac{-\mu_j}{r} \right\rangle} \gamma(-\rho(\overline{\mu_j})) e^{\frac{t\alpha_r^\gamma}{u|R|}} e^{\frac{u|K|}{t}\mathcal{F}_2^\gamma} \alpha^\gamma_{-\mu_j} e^{-\frac{u|K|}{t}\mathcal{F}_2^\gamma} e^{-\frac{t\alpha_r^\gamma}{u|R|}}.$$

The prefactors on the first line here cancel with those in Equation (47), leaving only the power of t from (47). Furthermore, recalling Equation (11),

$$\rho(\overline{\mu}_j) = \left(-d(\overline{\mu}_j), -k_j - \mu_j \mathbb{L} + \left\lfloor \frac{-\mu_j}{r} \right\rfloor \Bbbk_0 \right),$$

we see that the γ terms in appearing in $\mathcal{A}_{\rho(\overline{\mu}_j)}\left(\mu_j, \frac{u}{t}\right)$ is

$$\gamma(-\Bbbk_0)^{\left\langle \frac{-\mu_j}{r} \right\rangle} \gamma(-\rho(\overline{\mu_j})) = \gamma(k_j)\gamma(\mathbb{L})^{\mu_j} \gamma(-\Bbbk_0)^{-\mu_j/r}.$$

Recall further that all the exponentials were introduced by conjugating each α operator by them, and we can undo this conjugation now. First, since $\mathcal{F}_2^\eta$ and α_k^η commute with α_n^γ for $\gamma \neq \eta$, we can replace the $\mathcal{F}_2^\gamma$ and α_r^γ by the corresponding sums over all representations γ, and further simplify using

$$\sum_\gamma \alpha_r^\gamma = \alpha_r(0), \qquad |K| \sum_\gamma \mathcal{F}_2^\gamma = \mathcal{F}_2^0.$$

Then, the exponentials at the end of one $\mathcal{A}$ are exactly inverse to those at the beginning of the next $\mathcal{A}$, and so all the intermediate exponentials cancel, leaving just those at the beginning and end.

Finally, the last two exponentials fix the vacuum vector, and so they can be forgotten, leaving just the initial exponential terms

$$e^{\frac{t\alpha_r(0)}{u|R|}} e^{\frac{u}{t}\mathcal{F}_2^0}.$$

Together, we see that for a given $\overline{\mu}$, the product of the $\mathcal{A}$ terms appearing before the $\mathbf{P}_\emptyset$ along with the appropriate prefactors together give:

$$e^{\frac{t\alpha_r(0)}{u|R|}} e^{\frac{u}{t}\mathcal{F}_2^0} \prod_{j=1}^{\ell(\overline{\mu})} \left(\sum_{\gamma \in K^*} \gamma(k_j)\gamma(\mathbb{L})^{\mu_j} \gamma(\Bbbk_0)^{\frac{\mu_j}{r}} \alpha^\gamma_{-\mu_j} \right).$$

Working with the operators after the $\mathbf{P}_\emptyset$ is nearly identical. Since the inner produce is Hermitian, and $(\alpha_n^\gamma)^* = \alpha_{-n}^\gamma$, effectively we replace all $\gamma(k)$ terms by $\gamma(-k)$. The terms α_n^0 and $\mathcal{F}_2^0$ contain only $\gamma(0)$ terms, and so remain unchanged.

Recalling Equation 10, we see:

$$\sigma(\overline{\mu}_j) = \left(-d(\overline{\mu}_j), k_j + \left\lfloor \frac{-\overline{\mu}_j}{s} \right\rfloor \Bbbk_\infty \right),$$

and so the γ factor appearing in $\mathcal{A}^*_{\rho(\overline{\mu}_j)}\left(\mu_j, \frac{u}{t}\right)$ is

$$\gamma(-\Bbbk_\infty)^{\left\langle \frac{-\mu_j}{s} \right\rangle} \gamma(\sigma(\overline{\mu_j})) = \gamma(k_j)\gamma(\Bbbk_\infty)^{-\overline{\mu}_j/s}.$$

Putting together these arguments, define:

$$\widetilde{\mathbf{P}}_d = \sum_{|\overline{\mu}|=d} \frac{1}{\mathfrak{z}(\overline{\mu})} \prod_{j=1}^{\ell(\overline{\mu})} \left(\sum_{\gamma \in K^*} \gamma(k_j)\gamma(\mathbb{L})^{\mu_j}\gamma(\Bbbk_0)^{\frac{\mu_j}{r}} \alpha^\gamma_{-\mu_j} \right) \mathbf{P}_\emptyset \left(\prod_{j=1}^{\ell(\overline{\mu})} \left(\sum_{\gamma \in K^*} \gamma(k_j)\gamma(\Bbbk_\infty)^{\frac{\mu_j}{s}} \alpha^\gamma_{-\mu_j} \right) \right)^*,$$

so that, simplifying using $|\mu| = d$, we have:

$$\mathbf{Q}_d = t^{-d/r}(-t)^{-d/s} \exp\left(\frac{t\alpha_r(0)}{u|R|}\right) \exp\left(\frac{u}{t}\mathcal{F}_2^0\right) \widetilde{\mathbf{P}}_d \exp\left(\frac{u}{-t}\mathcal{F}_2^0\right) \exp\left(\frac{-t\alpha_{-s}(0)}{u|S|}\right).$$

6.1.5. *Gerbes and* $\widetilde{\mathbf{P}}_d$. We now examine the operator $\widetilde{\mathbf{P}}_d$, explaining how it is a twisted version of the operator $\mathbf{P}_d$ that projects on to the energy d eigenspace of $\mathcal{Z}_K$, and furthermore that this twisting is related to the twisting of the gerbe.

Observe that we can write the projection operator $\mathbf{P}_d$ as

$$\begin{aligned} \mathbf{P}_d &= \sum_{|\overline{\mu}|=d} \frac{1}{\mathfrak{z}(\overline{\mu})} \prod_{j=1}^{\ell(\mu)} \alpha_{-\mu_j}(-k_i) \mathbf{P}_\emptyset \prod_{j=1}^{\ell(\mu)} \alpha_{\mu_j}(k_i) \\ &= \sum_{|\overline{\mu}|=d} \frac{1}{\mathfrak{z}(\overline{\mu})} \prod_{j=1}^{\ell(\mu)} \left(\sum_{\gamma\in K^*} \gamma(k_j)\alpha^\gamma_{-\mu_j} \right) \mathbf{P}_\emptyset \prod_{j=1}^{\ell(\overline{\mu})} \left(\sum_{\gamma \in K^*} \gamma(-k_j)\alpha^\gamma_{\mu_j} \right). \end{aligned}$$

Apart from the factors of $\gamma(\mathbb{L}), \gamma(\Bbbk_0)$ and $\gamma(\Bbbk_\infty)$ appearing in $\widetilde{\mathbf{P}}_d$, the two operators are the same: if $\mathbb{L} = \Bbbk_0 = \Bbbk_\infty$=0, then $\widetilde{\mathbf{P}}_d = \mathbf{P}_d$. The situation where these are zero corresponds to when $\mathcal{X}$ is the trivial K gerbe, with trivial cocycle description. The twisting of $\widetilde{\mathbf{P}}_d$ reflects the twisting of the gerbe.

To understand the relationship between $\mathbf{P}_d$ and $\widetilde{\mathbf{P}}_d$ better, it is convenient to think in terms of the decomposition of $\mathcal{Z}_K = \bigotimes \bigwedge^{\frac{\infty}{2}} V^\gamma$. For the projection operator $\mathbf{P}_d$, we have:

$$\begin{aligned} \mathbf{P}_d &= \sum_{\sum d_\gamma = d} \bigotimes_{\gamma \in K^*} \mathbf{P}^\gamma_{d_\gamma} \\ &= \sum_{\sum d_\gamma = d} \bigotimes_{\gamma \in K^*} \left(\sum_{|\mu^\gamma| = d_\gamma} \prod \alpha^\gamma_{-\mu^\gamma_j} \mathbf{P}^\gamma_\emptyset \prod \alpha^\gamma_{\mu^\gamma_j} \right). \end{aligned}$$

Since the twisted projection operator $\widetilde{\mathbf{P}}_d$ differs from $\mathbf{P}_d$ by multiplying $\alpha^\gamma_{-\mu_j}$ by $\left(\gamma(\mathbb{L})(\gamma(\Bbbk_0)/t)^{1/r}\right)^{\mu_j}$, and similarly with the operators over infinity, we have:

$$\widetilde{\mathbf{P}}_d = \sum_{\sum d_\gamma = d} \bigotimes_{\gamma \in K^*} \gamma(\Bbbk_0)^{d_\gamma / r} \gamma(\mathbb{L})^{d_\gamma} \gamma(\Bbbk_\infty)^{d_\gamma/s} \mathbf{P}^\gamma_{d_\gamma}.$$

Since $\widetilde{\mathbf{P}}_d$ acts diagonally in the $v_{\overline{\lambda}}$ basis, and the operator $\mathcal{F}_2^0$ does as well, they commute. This allows us to simplify

$$\mathbf{Q}_d = t^{-d/r}(-t)^{-d/s}\exp\left(\frac{t\alpha_r(0)}{u|R|}\right)\widetilde{\mathbf{P}}_d\exp\left(\frac{-t\alpha_{-s}(0)}{u|S|}\right).$$

Finally, recall that $\mathbf{Q}_d$ appeared as $H^{t/r}\mathbf{Q}_dH^{-t/s}$. Moving the H operator past an $\alpha_r(0)$ rescales it by $1/t$, and $H\widetilde{\mathbf{P}}_d = d\widetilde{\mathbf{P}}_d = \widetilde{\mathbf{P}}_dH$, we see that Equation (48) simplifies to:

$$G^\bullet_{d,\mathfrak{r},\mathfrak{s}}(z_\mathfrak{r}, w_\mathfrak{s}, u) = \left\langle \prod \mathbf{A}_{\mathfrak{r}_i}(z_i)e^{\frac{\alpha_r(0)}{u|R|}}\widetilde{\mathbf{P}}_d e^{\frac{\alpha_{-s}(0)}{u|S|}}\prod \mathbf{A}^*_{\mathfrak{s}_i}(w_i)\right\rangle.$$

6.1.6. *Generating Functions.* Our operator formula so far has been specifically for degree d Gromov-Witten invariants; we now extend this to the τ function, proving Theorem B.

Define

$$\widetilde{H} = \sum_d d\widetilde{\mathbf{P}}_d$$

and defining

$$G^\bullet_{\mathfrak{r},\mathfrak{s}}(z_\mathfrak{r}, w_\mathfrak{s}, u, q) = \sum_d G^\bullet_{d,\mathfrak{r},\mathfrak{s}}(z_\mathfrak{r}, w_\mathfrak{s}, u)q^d,$$

we have that

$$G^\bullet_{\mathfrak{r},\mathfrak{s}}(z_\mathfrak{r}, w_\mathfrak{s}, u, q) = \left\langle \prod \mathbf{A}_{\mathfrak{r}_i}(z_i)e^{\frac{\alpha_r(0)}{u|R|}}q^{\widetilde{H}}e^{\frac{\alpha_{-s}(0)}{u|S|}}\prod \mathbf{A}^*_{\mathfrak{s}_i}(w_i)\right\rangle. \tag{49}$$

Recall that the definition of $G^\bullet_{\mathfrak{r},\mathfrak{s}}$ includes unstable contributions, and hence is not the true Gromov-Witten potential. We now remedy this, and express the τ function as a vacuum expectation.

The unstable contributions in the definition of $G^\bullet_{\mathfrak{r},\mathfrak{s}}$, as defined in Equation (7), result from the degree 0, genus 0, one and two point functions, and so all unstable contributions contain a z_i or w_i with a non-positive exponent. Thus, if we restrict our attention to only positive powers of the variables there are no unstable contributions and we recover the usual Gromow-Witten potential.

Define $\mathbf{A}_\mathfrak{r}[i] = [z^{i+1}]\mathbf{A}_\mathfrak{r}(z)$, i.e., the coefficient of z^{i+1} in $\mathbf{A}_\mathfrak{r}(z)$. Then, we have

$$\sum_{g\in\mathbb{Z}}\sum_{d\geq 0}u^{2g-2}q^d\left\langle\prod\tau_{k_i}(\mathbf{0}_{\mathfrak{r}_i})\prod\tau_{\ell_j}(\infty_{\mathfrak{s}_j})\right\rangle^\bullet_{g,d}$$
$$= \left\langle \prod \mathbf{A}_{\mathfrak{r}_i}[k_i]e^{\frac{\alpha_r(0)}{u|R|}}q^{\widetilde{H}}e^{\frac{\alpha_{-s}(0)}{u|S|}}\prod \mathbf{A}^*_{\mathfrak{s}_j}[\ell_j]\right\rangle.$$

Additionally, define

$$\tau(x, x^*, u) = \sum_{g\in\mathbb{Z}}\sum_{d\geq 0}u^{2g-2}q^d\left\langle\exp\left(\sum x_i(\mathfrak{r})\tau_i(\mathbf{0}_\mathfrak{r}) + \sum x^*_j(\mathfrak{s})\tau_j(\infty_\mathfrak{s})\right)\right\rangle^\bullet_{g,d}$$

Letting

$$\Phi_R(x) = \exp\left(\sum_{i\geq 0}\sum_{\rho\in R}x_i(\rho)\mathbf{A}_\rho[i]\right)$$

and

$$\Phi_S(x^*) = \exp\left(\sum_{i\geq 0}\sum_{\sigma\in S} x_i^*(\sigma)\mathbf{A}_\sigma^*[i]\right),$$

we have:

Theorem B.

$$\tau(x,x^*,u) = \left\langle \Phi_R(x)e^{\frac{\alpha_r(0)}{u|R|}} q^{\widetilde{H}} e^{\frac{\alpha_{-s}(0)}{u|S|}} \Phi_S(x^*)\right\rangle.$$

6.2. **Decomposition.** We now show how the decomposition conjecture follows from the main theorem.

The decomposition conjecture is stated in terms of twisted Gromov-Witten theory. In our case, this twisting winds up just being a rescaling of certain variables in the τ function, which we now summarize

(50)

Meaning	Variable	Scaling factor
$\mathbf{0}_{a/r}\psi^i$ insertion	$x_i(a/r)$	$\gamma(\mathbb{k}_0)^{-a/r}$
$\infty_{b/s}\psi^j$ insertion	$x_j^*(b/s)$	$\gamma(\mathbb{k}_\infty)^{-b/s}$
Degree variable	q	$\gamma(\mathbb{k}_0)^{1/r}\gamma(\mathbb{k}_\infty)^{1/s}\gamma(\mathbb{L})$
Genus variable	u	$1/\lvert K\rvert$

First, we have to define the change of variables. Recalling that

$$\mathbf{A}_{(a/r,k)}(z) = \sum_{\gamma\in K^*}\gamma(-k)\mathbf{A}_{a/r}^\gamma(z),$$

define

$$y_i(a/r,\gamma) = \sum_{k\in K}\gamma(-k)x_i(a/r,k),$$

so that

$$\sum_{k\in K} x_i(a/r,k)\mathbf{A}_{(a/r,k)}[i] = \sum_{\gamma\in K^*} y_i(a/r,\gamma)\mathbf{A}_{a/r}^\gamma(z)[i].$$

Further, for a representation γ of K, define

$$\Phi_R^\gamma(y) = \exp\left(\sum_{i\geq 0}\sum_{a/rin\mathbb{Z}_r} y_i(a/r,\gamma)\mathbf{A}_{a/r}^\gamma[i]\right)$$

and

$$\Phi_S^\gamma(y^*)\exp\left(\sum_{i\geq 0}\sum_{b/sin\mathbb{Z}_s} y_i(b/s,\gamma)\mathbf{A}_{b/s}^\gamma[i]\right)$$

Then the above gives

$$\Phi_R(x) = \prod_{\gamma\in K^*}\Phi_R^\gamma(y_\gamma)$$

and similarly for S, where y_γ consists of the set of variables $y_i(a/r,\gamma)$.

Then, expressed in the y variables, we have that

Corollary C: Decomposition.

$$\tau(y,y^*,u) = \left\langle \prod_{\gamma\in K^*}\Phi_R^\gamma(y_\gamma)\left(\sum_{\gamma\in K^*} e^{\frac{\alpha_r^\gamma}{u|R|}}\right) q^{\widetilde{H}}\left(\sum_{\gamma\in K^*} e^{\frac{\alpha_{-s}^\gamma}{u|S|}}\right)\prod_{\gamma\in K^*}\Phi_S^\gamma(y_\gamma^*)\right\rangle$$

$$= \prod_{\gamma \in K^*} \left\langle \Phi_R^\gamma(y_\gamma) e^{\frac{\alpha_r^\gamma}{u|R|}} \left(q\gamma(\mathbb{k}_0)^{\frac{1}{r}} \gamma(\mathbb{k}_\infty)^{\frac{1}{s}} \gamma(\mathbb{L}) \right)^{H_\gamma} e^{\frac{\alpha_{-s}^\gamma}{u|S|}} \Phi_S^\gamma(y_\gamma^*) \right\rangle$$

The factor of q on each has been multiplied by $\gamma(\mathbb{k}_0)^{1/r}$, $\gamma(\mathbb{k}_\infty)^{1/s}$, and $\gamma(\mathbb{L})$. The operator $\mathbf{A}^\gamma_{a/r}[i]$ differs from $\mathbf{A}_{a/r}[i]$ by a factor of $\gamma(-\mathbb{k}_0)^{a/r}$, and finally u has been multiplied by $|K|$.

Given Corollary C, for the rest of the section we work in the effective case. Let

$$\mathbf{M} = e^{\sum x_i(a/r)\mathbf{A}_{a/r}[i]} e^{\frac{\alpha_r}{ur}} q^H e^{\frac{\alpha_{-s}}{us}} e^{\sum x_j^*(b/s)\mathbf{A}^*_{b/s}[j]},$$

so that $\tau = \langle \mathbf{M} \rangle$. Then, to show that τ is a τ function of the 2-Toda hiearchy, we must show that we can conjugate $\mathbf{M}$ to the form $\Gamma_+(t) M \Gamma_-(s)$, for appropriate M. This conjugation gives a linear change of variables relating the x_i and x_i^* variables of equivariant Gromov-Witten theory to the standard t_i, s_i variables of the 2-Toda hierarchy.

6.3. **Equivariant string and divisor equations.** In this section we derive the equivariant divisor equation and, from that, the equivariant string equation. Since our generating function includes unstable contributions, the usual geometric proof requires modification. Instead, we follow [OP06a], and derive it from the operator formalism.

6.3.1. *Divisor equation.* The equivariant divisor equation describes the effect of point class insertions with no ψ classes. Suppose that $\mathfrak{r}$ is an n-tuple, and let $\tilde{\mathfrak{r}}$ be the $n+1$-tuple obtained by adding 0 in the first position. Then the equivariant divisor equation is:

Proposition 12.

$$[z_0^1] G^\bullet_{d,\tilde{\mathfrak{r}},\mathfrak{s}}(z_{\tilde{\mathfrak{r}}}, w_{\mathfrak{s}}, u) = \left(d - \frac{1}{24} + t \sum z_i \right) G^\bullet_{d,\mathfrak{r},\mathfrak{s}}(z_{\mathfrak{r}}, w_{\mathfrak{s}}, u).$$

Proof. Using Equation (49) for G_d, we see that:

$$[z_0^1] G^\bullet_{d,\tilde{\mathfrak{r}},\mathfrak{s}}(z_{\tilde{\mathfrak{r}}}, w_{\mathfrak{s}}, u, q) = \left\langle \mathbf{A}_0[0] \prod \mathbf{A}_{\mathfrak{r}_i}(z_i) e^{\frac{\alpha_r(0)}{ur}} \widetilde{\mathbf{P}}_d e^{\frac{\alpha_{-s}(0)}{us}} \prod \mathbf{A}^*_{\mathfrak{s}_i}(w_i) \right\rangle.$$

The general method of proof is to commute $\mathbf{A}_{0/r}[0]$ to the right. However, we first replace $\mathbf{A}_{0/r}[0]$ with another operator that does not change the vacuum expectation but has better commutation properties.

Recall that $\mathbf{A}_{0/r}[0]$ is the coefficient of z in $\mathbf{A}_0$. Together with Equation (46), this gives

$$\mathbf{A}_{0/r}(z) = \frac{1}{u} \mathcal{S}(ruz)^{\frac{tz}{r}} \sum_{i=-\infty}^{\infty} \frac{(z\mathcal{S}(ruz))^i}{(1+\frac{tz}{r})_i} \mathcal{E}_{ir}(uz).$$

Note that $\mathbf{A}_{0/r}[0]$ receives contributions only from the $i \leq 1$ terms of this sum.

We now examine the $i = 1$ and $i = 0$ terms. First, observe that

$$\mathcal{E}_r(uz) = \alpha_r + O(z)$$

and

$$\mathcal{E}_0(uz) = \frac{1}{u} z^{-1} + C + (H - \frac{1}{24}) uz + O(z^2).$$

As the adjoints of both C and H annihilate the vacuum, as does any operator with positive energy, we see that

$$\mathbf{A}_{0/r}[0] = \frac{1}{u}\alpha_r - \frac{1}{24} + \dots,$$

where the dots are terms whose adjoint annihilates the vacuum. In fact, we find it convenient to include H in our operator, which does not change the vacuum expectation, and so we have:

$$[z_0^1]G^{\bullet}_{d,\tilde{\mathfrak{r}},\mathfrak{s}}(z_{\tilde{\mathfrak{r}}}, w_{\mathfrak{s}}, u, q) = \left\langle \left(\frac{1}{u}\alpha_r + H - \frac{1}{24}\right) \prod \mathbf{A}_{\mathfrak{r}_i}(z_i) e^{\frac{\alpha_r(0)}{ur}} \widetilde{\mathbf{P}}_d e^{\frac{\alpha_{-s}(0)}{us}} \prod \mathbf{A}^*_{\mathfrak{s}_i}(w_i) \right\rangle.$$

We now investigate the commutator of $\frac{1}{u}\alpha_r + H - \frac{1}{24}$ with $\mathbf{A}_{a/r}(z)$.

First, since

$$[\alpha_r, \mathcal{E}_{a+ir}(uz)] = \varsigma(urz)\mathcal{E}_{a+(i+1)r}(uz),$$

we have

$$\left[\frac{1}{u}\alpha_r, \sum_{i=-\infty}^{\infty} \frac{(z\mathcal{S}(ruz))^i}{(1+\frac{tz+a}{r})_i} \mathcal{E}^{\gamma}_{ir+a}(uz)\right] = \sum_{i=-\infty}^{\infty} (a+ri+tz) \frac{(z\mathcal{S}(ruz))^i}{(1+\frac{tz+a}{r})_i} \mathcal{E}^{\gamma}_{ir+a}(uz).$$

The sum here is the one that appears in the definition of $\mathbf{A}_{a/r}(z)$. The equality follows from the identity $(1+x+y)(1+x)_y = (1+x)_{1+y}$ and a reindexing of the sum.

Observe that we also have

$$[H, \mathcal{E}_{a+ir}(uz)] = -(a+ir)\mathcal{E}_{a+ir}(uz).$$

Canceling this from part of the previous commutator calculation gives

$$\left[\frac{1}{u}\alpha_r + H, \mathbf{A}_{a/r}(z)\right] = tz\mathbf{A}_{a/r}(z),$$

and so commuting $\frac{1}{u}\alpha_r + H - \frac{1}{24}$ past the $\mathbf{A}_{a/r}(z)$ produces the $t\sum z_i$ factor appearing in Proposition 12.

Since $[H, \alpha_r] = -r\alpha_r$, we have

$$\left[\frac{1}{u}\alpha_r + H - \frac{1}{24}, e^{\frac{\alpha_r}{ur}}\right] = -\frac{1}{u} e^{\frac{\alpha_r}{ur}} \alpha_r,$$

and so

$$\left(\frac{1}{u}\alpha_r + H - \frac{1}{24}\right) e^{\frac{\alpha_r}{ur}} = e^{\frac{\alpha_r}{ur}} \left(H - \frac{1}{24}\right),$$

and together with $H\widetilde{\mathbf{P}}_d = d\widetilde{\mathbf{P}}_d$, this finishes the proof of Proposition 12. □

6.3.2. *String equation.* The equivariant string equation describes insertions of the identity class in equivariant cohomology with no ψ classes. However, due to localization, we can express this in terms of insertions of $\mathbf{0}(0,0)$ and $\boldsymbol{\infty}(0,0)$, as

$$1 = \frac{\mathbf{0}(0) - \boldsymbol{\infty}(0)}{t}.$$

The following differential operator, then, inserts $\tau_0(1)$:

$$\partial = \frac{1}{t}\left(\frac{\partial}{\partial y_0(0)} - \frac{\partial}{\partial y_0^*(0)}\right). \tag{51}$$

To obtain an explicit form for the lowest equation of the hierarchy, we use the string equation in the following form

Proposition 13.

$$\left\langle e^{\tau_0(1)} \prod \tau_{k_i}(\mathbf{0}(a_i/r)) \prod \tau_{\ell_j}(\infty(b_j/s)) \right\rangle_{g,d}^{\bullet}$$
$$= \left[\prod z_i(a_i/r)^{k_i+1} \prod w_j(b_j/s)^{\ell_j+1}\right] e^{\sum z_i(a_i/r)+\sum w_j(b_j/s)} G^{\bullet}_{g,d}(z,w,u).$$

6.4. **2-Toda: Explicit form of the lowest equation.** Further following [OP06a], we now derive an explicit form of the lowest equation of the 2-Toda hierarchy in the effective case.

6.4.1. *The lowest equation of the 2-Toda hierarchy.* As explained in [OP06a], one way of writing the lowest form of the 2-Toda hierarcy is

$$\left\langle T^{-1}MT\right\rangle \left\langle TMT^{-1}\right\rangle = \langle M\rangle \langle \alpha_1 M \alpha_{-1}\rangle - \langle \alpha_1 M\rangle \langle M\alpha_{-1}\rangle . \tag{52}$$

Here, T is the translation operator on $\bigwedge^{\frac{\infty}{2}} V$:

$$T \cdot \underline{i_1} \wedge \underline{i_2} \wedge \cdots = \underline{i_1+1} \wedge \underline{i_2+1} \wedge \cdots .$$

To derive the lowest equation of the hierarchy, we determine the effect of the insertions of the T and α operators on the τ function.

6.4.2. *The α_1 operator in terms of GW theory.* We begin with the α_1 operator. The key observation is that $\mathbf{A}_{1/r}[1]$ is closely realated to the operator α_1. Indeed, using the same reasoning as in the proof of the equivariant divisor equation (Proposition 12), we see that

$$[z]\mathbf{A}_{1/r}(z) = \frac{1}{u}\alpha_1 + \ldots,$$

where the ... are terms whose adjoint annihilates the vacuum, or possibly constant terms in case $r = 1$. We disregard the possibility of constant terms, as it is clear that adding or removing a constant from either α_1 or α_{-1} does not change the validity of Equation (52).

With this assumption, we have that

$$\frac{\partial}{\partial x_0(1/r)}\tau(x, x^*, u) = \langle(\frac{1}{u}\alpha_1)\mathbf{M}\rangle$$

and

$$\frac{\partial}{\partial x_0^*(1/s)}\tau(x, x^*, u) = \langle\mathbf{M}(\frac{1}{u}\alpha_{-1})\rangle.$$

Clearly this works for taking both derivatives as well, and so substituting into Equation (52) gives

$$\tau\frac{\partial^2}{\partial x_0(1/r)\partial x_0^*(1/s)}\tau - \frac{\partial}{\partial x_0^*(1/s)}\tau\frac{\partial}{\partial x_0(1/r)}\tau = \frac{1}{u^2}\langle T^{-1}\mathbf{M}T\rangle\langle T\mathbf{M}T^{-1}\rangle. \tag{53}$$

6.4.3. *Conjugation by T in terms of GW theory.* We describe how conjugating $\mathbf{M}$ by powers of T affects the τ function.

We first describe how conjugating by T affects the $\mathbf{A}$ operators. The key equation here is

$$T^{-1}\mathcal{E}_r(z)T = e^z\mathcal{E}_r(z),$$

which follows from the definition, with some care needed for $\mathcal{E}_0$. From this, it follows that

$$T^{-1}\mathbf{A}_{a/r}(z)T = e^{uz}\mathbf{A}_{a/r}(z),$$

and so conjugating by T multiplies each $\mathbf{A}_{\mathfrak{r}_i}(z_i)$ by $\exp(z_i u)$. By our form of the equivariant string equation, Proposition 13, this is equivalent to applying the operator $e^{u\partial}$.

We now investigate how conjugating by T effects the remaining terms. Since α_k and T commute, the exponential terms remain unchanged, and we only need to determine the effect on H:

$$T^{-n}HT^n = H + nC + \frac{n^2}{2}.$$

Since C commutes with the $\mathbf{A}$ and the α_k and annihilates the vacuum, its appearance has no effect, and we may replace $T^{-n}q^H T^n$ with $q^{n^2/2}q^H$. This gives

$$\left\langle T^{-n}\mathbf{M}T^n \right\rangle = q^{n^2/2} e^{nu\partial}\tau \tag{54}$$

and so

$$\left\langle T^{-1}\mathbf{M}T \right\rangle \left\langle T\mathbf{M}T^{-1} \right\rangle = q e^{u\partial}\tau e^{-u\partial}\tau. \tag{55}$$

Equations (53) and (55) and a calculation give us the following form for the lowest equation of the 2-Toda hierachy.

Proposition 14. *Suppose, $r, s > 1$. Then the τ function satisfies the following 2-Toda equation:*

$$\frac{\partial^2}{\partial x_0(1/r)\partial x_0^*(1/s)} \log \tau = \frac{q}{u^2}\frac{e^{u\partial}\tau e^{-u\partial}\tau}{\tau^2}.$$

6.5. **2-Toda: The full hierarchy.** We end by discussing that $\mathbf{M}$ can be conjugated to the required form, and show the resulting linear change of variables from the Gromov-Witten times to the standard 2-Toda times. We treat all our matrices as operators acting on V, not on $\bigwedge^{\frac{\infty}{2}} V$.

More particular, we want to show that there exists an upper triangular matrix W_r, called the *dressing matrix*, so that

$$W_r^{-1} \exp\left(\sum x_i(a/r)\mathbf{A}_{a/r}[i]\right) W_r = \Gamma_+(t).$$

This gives a linear change of variables between the 2-Toda time variables t and the Gromov-Witten variables $x_i(a/r)$. Taking the adjoint and replacing t with $-t$, we set

$$W_s^\dagger = W_s^* \mid_{t\mapsto -t},$$

so that

$$W_s^\dagger \exp\left(\sum x_i^*(a/r)\mathbf{A}_{a/r}^*[i]\right) \left(W_s^\dagger\right)^{-1} = \Gamma_-(v).$$

Then we have that

$$\langle \mathbf{M} \rangle = \left\langle W_r \Gamma_+(t) M \Gamma_-(v) W_s^\dagger \right\rangle,$$

with

$$M = W_r^{-1} e^{\frac{\alpha_r}{ur}} q^H e^{\frac{\alpha_{-s}}{us}} \left(W_s^\dagger\right)^{-1}.$$

When W_r and $W_s^\dagger$ are upper triangular, we have

$$W_r^* v_\emptyset = W_s^\dagger v_\emptyset = v_\emptyset.$$

When W_r and $W_s^\dagger$ are unitriangular, we have in addition

$$W_r^* T^n v_\emptyset = W_s^\dagger T^n v_\emptyset = T^n v_\emptyset, \tag{56}$$

which implies

$$\tau_n = \left\langle T^{-n}\mathbf{M}T^n\right\rangle = \left\langle T^{-n}W_r\Gamma_+(t)M\Gamma_-(v)W_s^\dagger T^n\right\rangle = \left\langle T^{-n}\Gamma_+(t)M\Gamma_-(v)T^n\right\rangle,$$

and hence that the τ_n were τ functions of the 2-Toda hierarchy.

Our W_r are upper triangular but not unitriangular. In this case τ_0 is unchanged, but the τ_n are multiplied by functions of q, u and t. This is not a problem, as the τ_n for $n \neq 0$ are not a-priori related Gromov-Witten theory. Rather, τ_0 is the original Gromov-Witten potential, and the τ_n are related to Gromov-Witten theory via (54): $\tau_n = q^{n^2/2}e^{nu\partial}\tau_0$.

For $k \geq 0$, the operators $\mathbf{A}_{a/r}[k]$ have the form

$$\mathbf{A}_{a/r}[k] = \begin{cases} \frac{1}{qu}\alpha_{a+kr} + \dots & a \neq 0 \\ \frac{1}{u}\alpha_{(k+1)r} + \dots & a = 0 \end{cases}$$

where the dots stand for terms of larger energy. Hence, there exists an upper triangular matrix W_r so that

$$W_r^{-1}\mathbf{A}_{1/r}[0]W_r = \alpha_1.$$

Note that W is not unique - we can multiply W_r by an element that commutes with α_1.

Since the $\mathbf{A}_{a/r}[k]$ commute by Theorem 8, and have the form above, if we define

$$\widetilde{\mathbf{A}}_{a/r}[k] = W\mathbf{A}_{a/r}[k]W^{-1}$$

then we have

$$\widetilde{\mathbf{A}}_{a/r}[k] = \sum_{\ell \leq k+1} c_{a,k,\ell}(u,t)\alpha_{a+\ell r}, \tag{57}$$

since the $\widetilde{\mathbf{A}}_{a/r}[k]$ commute with $\widetilde{\mathbf{A}}_{1/r}[0] = \alpha_1$.

Corollary D: Integrability. *After the change of variables, $\tau_{\mathcal{C}_{r,s}}$ is the tau function for the 2-Toda hierarchy.*

In fact, we have the following fact

Lemma 15. *The coefficients $c_{a,k,\ell}(u,t)$ are monomials in u, t.*

The proof of Lemma 15 is somewhat involved, and again requires some hypergeometric function identities. Since the proof is a straightforward adaptation of Section 4.4.5 of [OP06a] (the Lemma itself being an analog of Propositions 14 and 15 from [OP06a]), and the full proof is written out in [Joh], we do not give it here.

However, once this lemma is in hand, the change of variables follows easily. If the $c_{a,k,\ell}(u,t)$ are monomials, they are identical to their asymptotics as $u \to 0$, and so the full change of variables is equivalent to the change of variables in the $u \to 0$ limit. But by Equation (46), in the $u \to 0$ limit, we have:

$$\mathbf{A}_{a/r}(z) \sim \frac{1}{u}\frac{t^{\delta_r(0)}z}{(tz+a)}\sum_{i=-\infty}^{\infty}\frac{z^i}{(1+\frac{tz+a}{r})_i}\alpha_{ir+a}$$

and in the $u \to 0$ limit the operator W_r is diagonal, and we have, for $a \neq 0$:

$$\sum_{k\geq 0} z^{k+1}\widetilde{\mathbf{A}}_{a/r}[k] = \frac{1}{u}\sum_{n\geq 0}\frac{z^{n+1}}{\prod_{i=0}^{n}(i+\frac{tz+a}{r})}\alpha_{a+nr}$$

and for $a = 0$:

$$\sum_{k\geq 0} z^{k+1}\widetilde{\mathbf{A}}_{0/r}[k] = \frac{1}{u}\sum_{n\geq 1}\frac{z^n}{\prod_{i=1}^n (i+\frac{tz}{r})}\alpha_{nr}.$$

References

[AB84] M.F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.

[AGV08] Dan Abramovich, Tom Graber, and Angelo Vistoli. Gromov-Witten theory of Deligne-Mumford stacks. *Amer. J. Math.*, 130(5):1337–1398, 2008.

[ALR07] Alejandro Adem, Johann Leida, and Yongbin Ruan. *Orbifolds and stringy topology.* Cambridge University Press, 2007.

[CC09] Charles Cadman and Renzo Cavalieri. Gerby localization, $\mathbb{Z}_3$ Hodge integrals and the GW theory of $[\mathbf{C}^3/\mathbb{Z}_3]$. *Amer. J. Math.*, 131(4):1009–1046, 2009.

[CR04] Weimin Chen and Yongbin Ruan. A new cohomology theory of orbifold. *Comm. Math. Phys.*, 248(1):1–31, 2004.

[FMN] Barbara Fantechi, Etienne Mann, and Fabio Nironi. Smooth toric DM stacks. Preprint, arXiv:0708.1254.

[FW01] Igor Frenkel and Weiqiang Wang. Virasoro algebra and wreath product convolution. *J. Alg.*, 242:656–671, 2001.

[GP99] Tom Graber and Rahul Pandharipande. Localization of virtual classes. *Invent. Math.*, 135(2):487–518, 1999.

[HHP$^+$07] Simeon Hellerman, André Henriques, Tony Pantev, Eric Sharpe, and Matt Ando. Cluster decomposition, T-duality, and gerby CFTs. *Adv. Theor. Math. Phys.*, 11(5):751–818, 2007.

[Joh] Paul Johnson. Equivariant Gromov-Witten theory of one dimensional stacks. arXiv:0903.1068.

[JPT11] Paul Johnson, Rahul Pandharipande, and Hsian-Hua Tseng. Abelian Hurwitz-Hodge integrals. *Michigan Mathematical Journal*, 60(1), 2011.

[KR87] V. G. Kac and A. K. Raina. *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras.* World Scientific Publishing Co., Inc., 1987.

[LZ04] Sergei K. Lando and Alexander K. Zvonkin. *Graphs on surfaces and their applications.* Encyclopaedia of Mathematical Sciences. Spring-Verlag, Berlin, 2004.

[MJD00] T. Miwa, M. Jimbo, and E. Date. *Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras.* Cambirdge University Press, 2000.

[MT08] Todor Milanov and Hsian-Hua Tseng. The spaces of Laurent polynomials, p^1-orbifolds, and integrable hierarchies. *Journal fur die Reine und Angewandte Mathematik (Crelle's Journal)*, 622:189–235, 2008.

[MT11] Todor Milanov and Hsian-Hua Tseng. Equivariant orbifold structures on the projective line and integrable hierarchies. *Advances in Mathematics*, 226(1):641–672, 2011.

[Oko00] Andrei Okounkov. Toda equations for Hurwitz numbers. *Math. Res. Lett.*, 7(4):447–453, 2000.

[OP06a] Andrei Okounkov and Rahul Pandharipande. The equivariant Gromow-Witten theory of $\mathbb{P}^1$. *Ann. of Math. (2)*, 163(2):561–605, 2006.

[OP06b] Andrei Okounkov and Rahul Pandharipande. Gromov-Witten theory, Hurwitz theory, and completed cycles. *Ann. of Math. (2)*, 163(2):517–560, 2006.

[OP06c] Andrei Okounkov and Rahul Pandharipande. Virasoro constraints for target curves. *Invent. Math.*, 163(1):47–108, 2006.

[QW07] Zhenbo Qin and Weiqiang Wang. Hilbert schemes of points on the minimal resolution and soliton equations. In Y.-Z. Huang and K. Misra, editors, *Lie algebras, vertex operator algebras and their applications*, volume 442 of *Contemp. Math.*, pages 435–462, (2007).

[Ros08] Paolo Rossi. Gromov-Witten invariants of target curves via symplectic field theory. *Journal of Geometry and Physics*, 58(8):931–941, 2008.

[Ros10] Paolo Rossi. Gromov-Witten theory of orbicurves, the space of tri-polynomials and symplectic field theory of Seifert fibrations. *Math. Ann.*, 348(2):265–287, 2010.

[Rot] Mike Roth. Counting covers of an elliptic curve. www.mast.queens.ca/ mikeroth/notes/covers.pdf.

[TT] Hsian-Hua Tseng and Xiang Tang. Duality theorems of étale gerbes on orbifolds. arXiv:1004.1376.

[ZZ12] HanXiong Zhang and Jian Zhou. Wreath Hurwitz numbers, colored cut-and-join equations, and 2-Toda hierarchy. *Sci. China Math.*, 55(8):1627–1646, 2012.