



This is a repository copy of *Characterizing asymptotically anti-de Sitter black holes with abundant stable gauge field hair*.

White Rose Research Online URL for this paper:
<http://eprints.whiterose.ac.uk/98101/>

Version: Accepted Version

Article:

Shepherd, B.L. and Winstanley, E. orcid.org/0000-0001-8964-8142 (2012) Characterizing asymptotically anti-de Sitter black holes with abundant stable gauge field hair. *Classical and Quantum Gravity*, 29. 155004. ISSN 0264-9381

<https://doi.org/10.1088/0264-9381/29/15/155004>

Reuse

Unless indicated otherwise, fulltext items are protected by copyright with all rights reserved. The copyright exception in section 29 of the Copyright, Designs and Patents Act 1988 allows the making of a single copy solely for the purpose of non-commercial research or private study within the limits of fair dealing. The publisher or other rights-holder may allow further reproduction and re-use of this version - refer to the White Rose Research Online record for this item. Where records identify the publisher as the copyright holder, users can verify any specific terms of use on the publisher's website.

Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.



eprints@whiterose.ac.uk
<https://eprints.whiterose.ac.uk/>

Characterizing asymptotically anti-de Sitter black holes with abundant stable gauge field hair

Ben L. Shepherd and Elizabeth Winstanley

Consortium for Fundamental Physics, School of Mathematics and Statistics,
The University of Sheffield, Hicks Building, Hounsfield Road,
Sheffield. S3 7RH United Kingdom

E-mail: E.Winstanley@sheffield.ac.uk

Abstract. In the light of the “no-hair” conjecture, we revisit stable black holes in $\mathfrak{su}(N)$ Einstein-Yang-Mills theory with a negative cosmological constant Λ . These black holes are endowed with copious amounts of gauge field hair, and we address the question of whether these black holes can be uniquely characterized by their mass and a set of global non-Abelian charges defined far from the black hole. For the $\mathfrak{su}(3)$ case, we present numerical evidence that stable black hole configurations are fixed by their mass and two non-Abelian charges. For general N , we argue that the mass and $N - 1$ non-Abelian charges are sufficient to characterize large stable black holes, in keeping with the spirit of the “no-hair” conjecture, at least in the limit of very large $|\Lambda|$ and for a subspace containing stable black holes (and possibly some unstable ones as well).

1. Introduction

According to the “no-hair” conjecture [1], equilibrium black holes are extraordinarily simple objects, characterized completely by their mass, angular momentum and charge. These three quantities, as well as being parameters in the Kerr-Newman metric, are also physical, global charges, which could, at least in principle, be measured far from the black hole. For stationary, asymptotically flat black holes in four-dimensional Einstein-Maxwell theory, the “no-hair” conjecture has been proved (see for example [2] for reviews). It is perhaps unsurprising that, if one of the above assumptions (asymptotically flat space-time, four space-time dimensions, Einstein-Maxwell theory) is relaxed, then black hole uniqueness no longer holds. Here we are particularly interested in matter theories other than Einstein-Maxwell, and we focus on the four-dimensional Einstein-Yang-Mills (EYM) model, on which there is now an extensive literature (see, for example, [3, 4] for reviews).

The original discovery of soliton [5] and black hole [6] solutions of $\mathfrak{su}(2)$ EYM theory in four-dimensional, asymptotically flat space, seemed, at least at first, to provide counter-examples to the “no-hair” conjecture. The black hole solutions have no global charges and are indistinguishable from the usual Schwarzschild black holes at infinity. However, both the black hole and soliton solutions are unstable [7], so while the “letter”

of the “no-hair” conjecture (which, in its original form, says nothing about stability) is violated, its “spirit” remains intact, because stable black holes do seem to be uniquely characterized by global charges. These results led Bizon to reformulate the “no-hair” conjecture as follows [8]:

Within a given matter model, a *stable* stationary black hole is uniquely determined by global charges.

For four-dimensional EYM theory in asymptotically flat space, this result holds at least for spherically symmetric black holes with a purely magnetic gauge field, as the unique stable solution in this case is the Schwarzschild black hole[‡]. For asymptotically flat, four-dimensional EYM black holes with both an electric and magnetic field, the situation is less clear, with the recent discovery [10] of stable dyonic black holes with a non-vanishing electric charge but zero magnetic charge in an extended EYM theory containing higher order terms in the field strength.

When a negative cosmological constant Λ is introduced into the model, so that the space-time is asymptotically anti-de Sitter (adS) rather than asymptotically flat, stable soliton [11] and black hole solutions [12] of $\mathfrak{su}(2)$ EYM theory exist. For $\mathfrak{su}(2)$ gauge group, the purely magnetic gauge field is described by a single function ω . Purely magnetic solutions have been proven to be stable [11, 12, 13], when the function ω has no zeros, provided that $|\Lambda|$ is sufficiently large. Unlike their asymptotically flat counterparts, these solutions do have a global magnetic charge.

Enlarging the gauge group to $\mathfrak{su}(N)$, the purely magnetic gauge field is now described by $N - 1$ functions ω_j . For any fixed N , the existence of stable, purely magnetic, soliton and black hole solutions (for which all the ω_j have no zeros) has been proven provided that $|\Lambda|$ is sufficiently large [14, 15, 16, 17]. These stable black holes have $N - 1$ independent gauge degrees of freedom and it turns out (see [16] and section 4) that they are described by $N + 1$ independent parameters, for gauge group $\mathfrak{su}(N)$. We conclude [14] that there is no limit to the amount of stable gauge field hair with which a black hole in adS can be endowed.

Our purpose in this paper is to revisit these $\mathfrak{su}(N)$ EYM black holes in the light of the “no-hair” conjecture as reformulated by Bizon. In particular, we investigate whether these black holes are uniquely characterized by global charges. After a brief review of the $\mathfrak{su}(N)$ EYM model and its black hole solutions in section 2, we proceed, in section 3, to construct global charges for the $\mathfrak{su}(N)$ gauge field. We follow a two-pronged approach to examine whether these charges uniquely characterize the black hole solutions, firstly, in section 4, performing numerical investigations of the solution space, and then, in section 6, giving an analytic argument that the black holes are uniquely characterized by global charges, at least for sufficiently large $|\Lambda|$. Our conclusions on the consequences of this result for the “no-hair” conjecture are presented in section 7.

[‡] It has been proven that all spherically symmetric, four-dimensional, black hole solutions of EYM theory in asymptotically flat space with a purely magnetic gauge field are unstable, for all gauge groups [9].

2. Hairy black holes in $\mathfrak{su}(N)$ Einstein-Yang-Mills theory in anti-de Sitter space

In this section we outline the salient features of $\mathfrak{su}(N)$ EYM theory in anti-de Sitter space, and the black hole solutions found numerically in [15], whose existence, for sufficiently large $|\Lambda|$, was proven in [16]. We also discuss the thermodynamic properties of the black holes, and use the boundary counter-term formalism [18] to compute their mass.

2.1. Ansatz and field equations

We consider four-dimensional $\mathfrak{su}(N)$ EYM theory with a negative cosmological constant, described by the following action, in suitable units:

$$S_{\text{EYM}} = \frac{1}{2} \int d^4x \sqrt{-g} [R - 2\Lambda - \text{Tr} F_{\mu\nu} F^{\mu\nu}], \quad (1)$$

where R is the Ricci scalar, Λ the cosmological constant and Tr denotes a Lie algebra trace. Throughout this paper, the metric has signature $(-, +, +, +)$ and we use units in which $4\pi G = 1 = c$. In addition, we fix the gauge coupling constant $g = 1$. In this article we focus on a negative cosmological constant, $\Lambda < 0$. Varying the action (1) gives the field equations

$$\begin{aligned} T_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}; \\ 0 &= D_\mu F_\nu{}^\mu = \nabla_\mu F_\nu{}^\mu + [A_\mu, F_\nu{}^\mu]; \end{aligned} \quad (2)$$

where the YM stress-energy tensor is

$$T_{\mu\nu} = F_{\mu\lambda}^a F_\nu{}^{\lambda\sigma} - \frac{1}{4} g_{\mu\nu} F_{\lambda\sigma}^a F^{a\lambda\sigma}, \quad (3)$$

with summation over the Lie-algebra index a understood, so that $T_{\mu\nu}$ involves a Lie-algebra trace. The Yang-Mills gauge field $F_{\mu\nu}$ is given in terms of the gauge potential A_μ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \quad (4)$$

In this paper we are interested in static, spherically symmetric black hole solutions of the field equations (2), and we write the metric in standard Schwarzschild-like coordinates as:

$$ds^2 = -\mu S^2 dt^2 + \mu^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (5)$$

where the metric functions μ and S depend on the radial co-ordinate r only. In the presence of a negative cosmological constant $\Lambda < 0$, it is convenient to write the metric function μ as

$$\mu(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}. \quad (6)$$

We emphasize that in this paper we are considering only spherically symmetric black holes and not topological black holes which have been found in the $\mathfrak{su}(2)$ case [19].

With a suitable choice of gauge, we take the purely magnetic $\mathfrak{su}(N)$ gauge potential to have the form [20]

$$A = \frac{1}{2} (C - C^H) d\theta - \frac{i}{2} [(C + C^H) \sin \theta + D \cos \theta] d\phi, \quad (7)$$

where C and D are $(N \times N)$ matrices and C^H is the Hermitian conjugate of C . The constant matrix D takes the form:

$$D = \text{Diag}(N - 1, N - 3, \dots, -N + 3, -N + 1), \quad (8)$$

and the matrix C is upper-triangular, with non-zero entries only immediately above the diagonal:

$$C_{j,j+1} = \omega_j(r), \quad (9)$$

for $j = 1, \dots, N - 1$. The gauge field is therefore described by the $N - 1$ functions $\omega_j(r)$. The derivation of the ansatz (7) uses the Yang-Mills equations and assumes that all the $\omega_j(r)$ are not identically zero (see, for example, [21] for the possibilities in asymptotically flat space if this assumption does not hold). We comment that our ansatz (7) is by no means the only possible choice in $\mathfrak{su}(N)$ EYM. Techniques for finding *all* spherically symmetric $\mathfrak{su}(N)$ gauge potentials can be found in [22], where all irreducible models are explicitly listed for $N \leq 6$.

With the ansatz (7), there are $N - 1$ non-trivial Yang-Mills equations for the $N - 1$ gauge field functions ω_j :

$$r^2 \mu \omega_j'' + \left(2m - 2r^3 p_\theta - \frac{2\Lambda r^3}{3} \right) \omega_j' + W_j \omega_j = 0 \quad (10)$$

for $j = 1, \dots, N - 1$, where a prime ' denotes d/dr ,

$$p_\theta = \frac{1}{4r^4} \sum_{j=1}^N \left[(\omega_j^2 - \omega_{j-1}^2 - N - 1 + 2j)^2 \right], \quad (11)$$

$$W_j = 1 - \omega_j^2 + \frac{1}{2} (\omega_{j-1}^2 + \omega_{j+1}^2), \quad (12)$$

and $\omega_0 = \omega_N = 0$. The Einstein equations take the form

$$m' = \mu G + r^2 p_\theta, \quad \frac{S'}{S} = \frac{2G}{r}, \quad (13)$$

where

$$G = \sum_{j=1}^{N-1} \omega_j'^2. \quad (14)$$

The field equations (10, 13) are invariant under the transformation

$$\omega_j(r) \rightarrow -\omega_j(r) \quad (15)$$

for each j independently, and also under the substitution:

$$j \rightarrow N - j. \quad (16)$$

2.2. Boundary conditions

Our primary interest in this paper is black hole solutions of the field equations (10, 13). However, we will need to consider solitons in section 6.2 to ensure that solitons cannot be mistaken for black holes by measuring global charges at infinity. The field equations are singular at the origin, at an event horizon $r = r_h$ if there is one, and at infinity $r \rightarrow \infty$. Boundary conditions therefore have to be specified in a neighbourhood of these singular points. Local existence of solutions of the field equations satisfying the boundary conditions outlined below is proven in [16].

2.2.1. Origin The boundary conditions at the origin are more complicated than near the event horizon or at infinity. The full form is derived in detail in [16] (following the analysis of [23] for the asymptotically flat case). Here we simply state the basic features which are needed for our analysis in section 6.2. Near the origin, the field variables have the following form:

$$\begin{aligned} m(r) &= m_3 r^3 + O(r^4); \\ S(r) &= S_0 + S_2 r^2 + O(r^3); \\ \omega_j(r) &= \pm [j(N-j)]^{\frac{1}{2}} + O(r^2). \end{aligned} \tag{17}$$

To fully specify the form of the gauge field in a neighbourhood of the origin, a complicated power series has to be developed, up to $O(r^N)$, the details of which can be found in [16] but which are not necessary for our purposes in this paper.

2.2.2. Event horizon For black hole solutions, we assume that there is a regular, non-extremal event horizon at $r = r_h$, where $\mu(r)$ has a single zero. This fixes the value of $m(r_h)$ to be

$$m(r_h) = \frac{r_h}{2} - \frac{\Lambda r_h^3}{6}. \tag{18}$$

We assume that the field variables $\omega_j(r)$, $m(r)$ and $S(r)$ have regular Taylor series expansions about $r = r_h$:

$$\begin{aligned} m(r) &= m(r_h) + m'(r_h)(r - r_h) + O(r - r_h)^2; \\ \omega_j(r) &= \omega_j(r_h) + \omega'_j(r_h)(r - r_h) + O(r - r_h)^2; \\ S(r) &= S(r_h) + S'(r_h)(r - r_h) + O(r - r_h). \end{aligned} \tag{19}$$

Setting $\mu(r_h) = 0$ in the Yang-Mills equations (10) fixes the derivatives of the gauge field functions at the horizon:

$$\omega'_j(r_h) = -\frac{W_j(r_h)\omega_j(r_h)}{2m(r_h) - 2r_h^3 p_\theta(r_h) - \frac{2\Lambda r_h^3}{3}}. \tag{20}$$

Therefore the expansions (19) are determined by the $N + 1$ quantities $\omega_j(r_h)$, r_h , $S(r_h)$ for fixed cosmological constant Λ . For the event horizon to be non-extremal, it must be the case that

$$2m'(r_h) = 2r_h^2 p_\theta(r_h) < 1 - \Lambda r_h^2, \tag{21}$$

which weakly constrains the possible values of the gauge field functions $\omega_j(r_h)$ at the event horizon. Since the field equations (10, 13) are invariant under the transformation (15), we may consider $\omega_j(r_h) > 0$ without loss of generality.

2.2.3. Infinity At infinity, we require that the metric (5) approaches adS, and therefore the field variables $\omega_j(r)$, $m(r)$ and $S(r)$ converge to constant values as $r \rightarrow \infty$. We assume that the field variables have regular Taylor series expansions in r^{-1} near infinity:

$$\begin{aligned} m(r) &= M + O(r^{-1}); & S(r) &= 1 + O(r^{-1}); \\ \omega_j(r) &= \omega_{j,\infty} + c_j r^{-1} + O(r^{-2}). \end{aligned} \quad (22)$$

We have included the $\mathcal{O}(r^{-1})$ terms in $\omega_j(r)$ as they are central to our analysis in section 4. If the space-time is asymptotically flat, with $\Lambda = 0$, then the values of $\omega_{j,\infty}$ are constrained to be

$$\omega_{j,\infty} = \pm \sqrt{j(N-j)}. \quad (23)$$

This condition means that the asymptotically flat black holes have no magnetic charge at infinity (see section 3 for the definition of magnetic charges). Therefore, at infinity, they are indistinguishable from Schwarzschild black holes. However, if the cosmological constant is negative, then there are no *a priori* constraints on the values of $\omega_{j,\infty}$. In general, therefore, the adS black holes will be magnetically charged. In section 3 we will construct appropriate non-Abelian charges.

2.3. Embedded solutions

The field equations (10, 13) are non-linear and coupled, but they do have two analytic, trivial solutions.

Schwarzschild-adS Setting

$$\omega_j(r) \equiv \pm \sqrt{j(N-j)} \quad (24)$$

for all j gives the Schwarzschild-adS black hole with

$$m(r) = M = \text{constant}. \quad (25)$$

Reissner-Nordström-adS Setting

$$\omega_j(r) \equiv 0 \quad (26)$$

for all j gives the Reissner-Nordström-adS black hole with metric function

$$\mu(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}, \quad (27)$$

where the magnetic charge Q (see section 3 for a definition of this quantity) is fixed by

$$Q^2 = \frac{1}{6}N(N+1)(N-1). \quad (28)$$

Only for this value of the magnetic charge is the Reissner-Nordström-adS black hole a solution of the field equations.

As well as these effectively Abelian embedded solutions, there is also a class of embedded $\mathfrak{su}(2)$ non-Abelian solutions, given by writing the $N - 1$ gauge field functions $\omega_j(r)$ in terms of a single function $\omega(r)$ as follows:

$$\omega_j(r) = \pm \sqrt{j(N-j)} \omega(r) \quad \forall j = 1, \dots, N-1. \quad (29)$$

It is shown in [16] that, with a suitable rescaling of the other field variables, the field equations (10, 13) reduce to the $\mathfrak{su}(2)$ field equations for the function ω . Therefore any $\mathfrak{su}(2)$, asymptotically adS, EYM black hole solution can be embedded into $\mathfrak{su}(N)$ EYM to give an asymptotically adS black hole.

2.4. Properties of the $\mathfrak{su}(N)$ solutions

The properties of soliton and black hole solutions of the field equations (10, 13) have already been studied in detail elsewhere [4, 14, 15, 16], therefore here we simply summarize the salient features required for our subsequent analysis.

For large $|\Lambda|$, numerical investigations [14, 15] find both soliton and black hole solutions for which all the gauge field functions $\omega_j(r)$ have no zeros. It has been proven [16] that, for fixed r_h and $\omega_j(r_h)$, black hole solutions for which all the $\omega_j(r)$ have no zeros exist for all sufficiently large $|\Lambda|$. In view of these results, the focus in numerical work [15] has been on properties of the phase space of solutions for fixed r_h and different values of Λ . In this work we are interested in the properties of the phase space for fixed Λ and varying horizon radius r_h . We follow the standard method for finding numerical solutions (for further details, see [15]). Using a shooting method, the field equations (10, 13) are integrated from close to the event horizon, out towards infinity.

We find that, for sufficiently large $|\Lambda|$, all numerical solutions for varying r_h are such that the gauge field functions $\omega_j(r)$ have no zeros. For example, in figure 1 we show the phase space of solutions for $\mathfrak{su}(2)$ black holes with $\Lambda = -10$ and varying r_h . Corresponding phase space studies for larger N with Λ fixed and r_h variable are more complex because of the number of parameters involved, but in each case investigated we find similar results. Phase space plots for $N > 2$ can only be produced by fixing some of the parameters, see, for example, figure 2.

At least some of these black hole solutions for which all ω_j have no zeros are stable [14, 17]. For $\mathfrak{su}(2)$, black hole solutions for which the single gauge field function $\omega(r)$ has no zeros are shown to be stable if $\omega(r) > 1/\sqrt{3}$ everywhere on and outside the event horizon [12]. For $N > 2$, stability under linear, spherically symmetric perturbations can be proven for sufficiently large $|\Lambda|$, for $\mathfrak{su}(N)$ solutions satisfying the inequalities [14, 17]:

$$\omega_j(r)^2 > 1 + \frac{1}{2} [\omega_{j+1}^2(r) + \omega_{j-1}^2(r)] \quad (30)$$

and which are sufficiently close to stable embedded $\mathfrak{su}(2)$ solutions. The inequalities (30) have to be satisfied for all values of $r \geq r_h$. In figure 2 we plot the region of phase space in the $\mathfrak{su}(3)$ case, with $\Lambda = -3$ and $r_h = 1$, for which the inequalities (30) are satisfied on the black hole event horizon and for which both $\omega_1(r)$ and $\omega_2(r)$ have no zeros. For

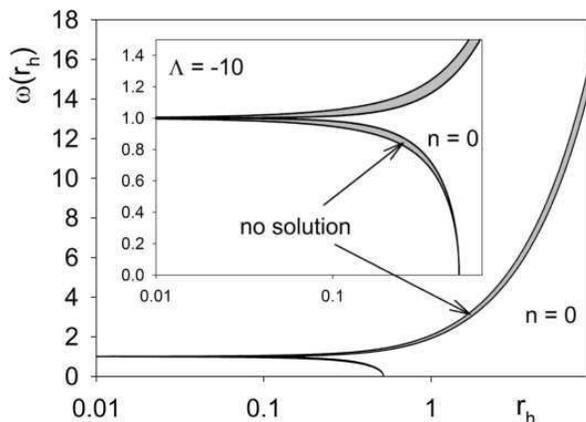


Figure 1. Phase space of $\mathfrak{su}(2)$ solutions with $\Lambda = -10$ and varying r_h . The insert is a blow-up for smaller values of r_h so that the structure can be seen. All the numerical solutions we find are such that $\omega(r)$ has no zeros. The light grey region corresponds to values of $\omega(r_h)$ such that the condition (21) is satisfied, but we do not find a numerical solution. Below the lower boundary of the upper grey region and above the upper boundary of the lower grey region, in the region labelled “ $n = 0$ ”, we find numerical solutions for which $\omega(r)$ has no zeros. Above the upper grey region and below the lower grey region, the condition (21) for a regular event horizon is not satisfied.

at least some of these solutions, the inequalities (30) are satisfied for all $r \geq r_h$ [14, 17]. However, the inequalities (30) are satisfied only in a comparatively small region of the phase space for which there are nodeless solutions (compare the size of the region in figure 2 with figure 2 in [16], where the entire phase space for $\mathfrak{su}(3)$ black holes with $\Lambda = -3$ and $r_h = 1$ is plotted). Furthermore, due to the algebraic complexity of the gravitational sector perturbation equations, stability for this sector can only be proven for sufficiently large $|\Lambda|$ and for $\mathfrak{su}(N)$ solutions sufficiently close to stable embedded $\mathfrak{su}(2)$ solutions, with, as is usually the case in this kind of proof, little indication of how large “sufficiently large” is, or how close “sufficiently close” is [17]. Our focus in this article is the characterization of stable $\mathfrak{su}(N)$ black holes. We therefore restrict attention to those black holes for which the inequalities (30) are satisfied, at least on the event horizon. It is possible that our results include some black holes which are in fact unstable.

2.5. Thermodynamics of $\mathfrak{su}(N)$ EYM black holes

While the “no-hair” conjecture as outlined in the introduction is primarily concerned with the characterization of black holes which are *classically* stable, it nonetheless makes sense to consider the thermodynamic stability of the black holes. The entropy of $\mathfrak{su}(N)$ EYM black holes is given, as usual, by one-quarter of the area of the event horizon, while their Hawking temperature T_H is

$$T_H = \frac{1}{4\pi r_h} (1 - 2m'(r_h) - \Lambda r_h^2) S(r_h), \quad (31)$$

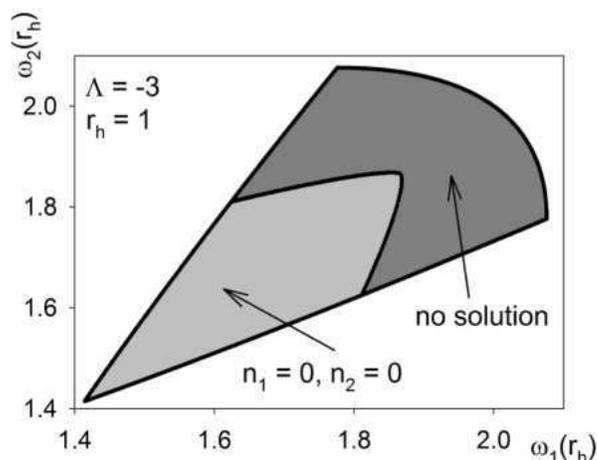


Figure 2. Phase space of $\mathfrak{su}(3)$ solutions with $\Lambda = -3$ and $r_h = 1$, with n_1 the number of zeros of the gauge field function $\omega_1(r)$ and n_2 the number of zeros of $\omega_2(r)$. We plot only the region in which the inequalities (30) are satisfied on the event horizon. The light grey region indicates solutions for which both the gauge field functions $\omega_1(r)$ and $\omega_2(r)$ have no zeros. The dark grey region corresponds to values of $\omega_1(r_h)$ and $\omega_2(r_h)$ for which the inequalities (30) are satisfied on the event horizon, together with the condition (21) for a regular event horizon, but for which we do not find solutions.

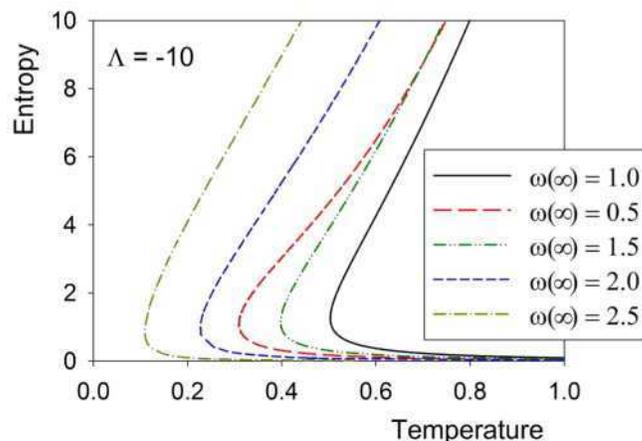


Figure 3. Entropy-temperature curves for $\mathfrak{su}(2)$ black holes with $\Lambda = -10$. For each curve, the value of the gauge field function $\omega(r)$ at infinity is fixed. The curve with $\omega(\infty) = 1$ corresponds to embedded Schwarzschild-adS black holes.

where we emphasize that $S(r_h)$ is a metric function (5), and not the entropy of the black hole. For $\mathfrak{su}(2)$ black holes, the thermodynamics has already been studied for $\Lambda = -3$ in [24]. Their results are similar to ours in figure 3 for the $\Lambda = -10$ case. In figure 3, we plot entropy as a function of Hawking temperature for fixed $\omega(\infty)$ (we will see in section 3 that this corresponds to fixing the magnetic charge of the black hole). For each value of $\omega(\infty)$, there are two branches of black hole solutions. Firstly, there is a branch with small entropy and negative specific heat, so that the black holes on this branch

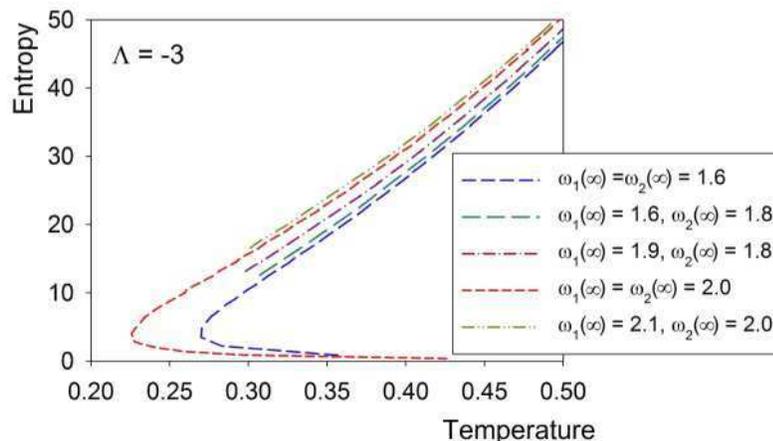


Figure 4. Entropy-temperature curves for $\mathfrak{su}(3)$ black holes with $\Lambda = -3$. For each curve, the values of the gauge field functions $\omega_1(r)$ and $\omega_2(r)$ at infinity are fixed. Curves with $\omega_1(\infty) = \omega_2(\infty)$ correspond to embedded $\mathfrak{su}(2)$ solutions.

are thermodynamically unstable. Secondly, there is an upper branch of black holes with larger entropy and positive specific heat, corresponding to thermodynamically stable black holes. This is the behaviour found by [24] when $\Lambda = -3$, and we found similar behaviour for other values of Λ . The curve in figure 3 with $\omega(\infty) = 1$ corresponds to embedded Schwarzschild-adS black holes. As predicted in [25] for generic hairy black holes in asymptotically flat space, we see from figure 3 that the non-Abelian $\mathfrak{su}(2)$ black holes have lower temperatures than the embedded Schwarzschild-adS black hole with the same entropy (and hence horizon area). Similar behaviour is observed for $\mathfrak{su}(3)$ black holes, as can be seen in figure 4. In figure 4, again anticipating the results of section 3, we fix the values of the two gauge field functions $\omega_1(r)$ and $\omega_2(r)$ at infinity, and then plot the curve of entropy as a function of temperature. If $\omega_1(\infty) = \omega_2(\infty)$, then we have embedded $\mathfrak{su}(2)$ solutions and the curves have a very similar shape to those in figure 3, with two branches of solutions. In this case we are interested in the upper branch of solutions which are thermodynamically stable. For $\omega_1(\infty) \neq \omega_2(\infty)$, we have genuinely $\mathfrak{su}(3)$ solutions, and in this case the curves appear to have just one branch of solutions, which have positive specific heat and so are thermodynamically stable. We are not able to say conclusively whether the fact that we have been unable to find a thermodynamically unstable branch of solutions is due to numerical difficulties or whether such a lower branch of solutions does not in fact exist.

2.6. Mass of $\mathfrak{su}(N)$ EYM black holes

The mass of the non-Abelian black holes is readily computed using the background counter-term formalism [18]. As is well known, the gravitational action for asymptotically adS space-times typically diverges as the boundary $\partial\mathcal{M}$ of the region (over which the Lagrangian is integrated) is taken to infinity. This divergence is removed by the addition of boundary counter-terms to the gravitational action, which do not alter

the bulk equations of motion. As we are working in four space-time dimensions, the boundary counter-terms which are sufficient to yield a finite bulk gravitational action are [18]:

$$I_{ct} = - \int_{\partial\mathcal{M}} d^3x \sqrt{-\gamma} \left(\frac{2}{\ell} + \frac{\ell}{2} \mathcal{R} \right), \quad (32)$$

where $\ell = \sqrt{-\frac{\Lambda}{3}}$ is the adS length and the boundary metric γ has Ricci scalar \mathcal{R} . The boundary stress tensor, resulting from the variation of the total gravitational action S_{grav} (which is the usual gravitational action plus the counter-terms (32)) is then [18]

$$T_{\mu\nu}^B = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{grav}}}{\delta \gamma^{\mu\nu}} = \frac{1}{2} \left(\Theta_{\mu\nu} - \Theta \gamma_{\mu\nu} - \frac{2}{\ell} \gamma_{\mu\nu} - \ell \mathcal{G}_{\mu\nu} \right), \quad (33)$$

where $\Theta_{\mu\nu}$ is the extrinsic curvature of the boundary (with trace Θ), and $\mathcal{G}_{\mu\nu}$ is the Einstein tensor of the boundary metric. Only the component T_{tt}^B is required to compute the mass of the black holes. To leading order, we find that

$$T_{tt}^B = \frac{M}{\ell r} \quad (34)$$

where $M = \lim_{r \rightarrow \infty} m(r)$ (22), so that the mass of the solutions is simply

$$\int_{\partial\mathcal{M}} \ell r T_{tt}^B d^2x = 4\pi M. \quad (35)$$

From here on we will therefore use the variable M to denote the ‘‘mass’’ of the black holes. Our results here are in complete agreement with those in [24], where it was found, for the $\mathfrak{su}(2)$ case, that the non-Abelian gauge fields only contribute to the boundary stress tensor at next-to-leading order, and therefore do not contribute directly to the mass of the solutions. Of course, they do contribute indirectly to the mass as they affect the value of M through the Einstein equations.

3. Defining charges for $\mathfrak{su}(N)$ EYM black holes

We now turn to the definition of global charges for $\mathfrak{su}(N)$ EYM black holes. Since the rank of the $\mathfrak{su}(N)$ Lie algebra is $N - 1$, we expect these black holes to carry $N - 1$ conserved charges, which, at least in principle, could be measured at infinity, far from the black hole. The definition of charges for general non-Abelian gauge fields is non-trivial because of the need for observable charges to be gauge-invariant, coupled with the fact that the gauge field $F_{\mu\nu}$ is not itself gauge-invariant in the non-Abelian case (unlike the situation for Abelian gauge fields). Methods for defining global, gauge-invariant, charges for non-Abelian gauge fields have been devised by a number of authors [26, 27, 28, 29]. While the quantization of non-Abelian charge has been studied [27], here we take a purely classical approach.

For an $\mathfrak{su}(2)$ gauge field, the single magnetic charge was defined in [30] as

$$Q = \frac{1}{4\pi} \int_{S_\infty} \sqrt{F_{\theta\phi}^a F_{\theta\phi}^a} d\theta d\phi, \quad (36)$$

where, under the square root, we have a Lie algebra trace over the $F_{\theta\phi}$ component of the gauge field, corresponding to a sum over the Lie algebra index a . The integral is taken over the sphere at infinity. The disadvantage of the formula (36) is that there is no natural generalization yielding $N - 1$ charges in the $\mathfrak{su}(N)$ case. An alternative $\mathfrak{su}(2)$ charge of the form

$$Q = 1 - \omega_\infty^2 \quad (37)$$

has been considered by a number of authors (see, for example, [24]), and has the advantage of readily extending to the larger gauge group. In particular, in this section we use the definitions of [28] (similar quantities were also defined in [29]) to construct magnetic charges. The approach of [28] has the advantage of yielding expressions which can be easily applied to our solutions.

Following [28], we define gauge invariant magnetic charges as follows. Let X be an element in \mathfrak{A} , the Cartan sub-algebra of the $\mathfrak{su}(N)$ Lie algebra. Then, for each X , a charge $Q(X)$ is defined by:

$$Q(X) = \frac{1}{4\pi} \sup_{g(r)} k \left(X, \int_{S_\infty} g^{-1} F g \right). \quad (38)$$

Here the integral is taken over the sphere at infinity, and the supremum is taken over all group elements $g(r)$ of the form

$$g(r) = \exp [f(r)\Sigma] \quad (39)$$

where $f(r)$ is a scalar function of r and Σ is a constant element in the Lie algebra. In (38), we have used $k(X, Y) = \text{Tr} \{ \text{ad } X \text{ ad } Y \}$ which is the Killing form, with $\text{ad } X$ denoting the adjoint representation of the Lie algebra element X . It is shown in [28] that the integrand in (38) takes its maximal value when $g^{-1} F g$ is in the Cartan sub-algebra \mathfrak{A} , and therefore we may restrict attention to those $g(r)$ for which this is the case.

Since we are integrating over a sphere at infinity, the integral in (38) simplifies to

$$Q(X) = \frac{1}{4\pi} k \left(X, \int_{S_\infty} g^{-1} F_{\theta\phi} g \, d\theta \, d\phi \right). \quad (40)$$

From the gauge potential ansatz (7), the required component of the field strength is found to be

$$F_{\theta\phi} = -\frac{i}{2} ([C, C^H] - D) \sin \theta, \quad (41)$$

where D is the constant matrix given in (8). From the form of the matrix C (9), it is straightforward to show that

$$[C, C^H] = \text{diag} \{ \omega_1^2, \omega_2^2 - \omega_1^2, \omega_3^2 - \omega_2^2, \dots, -\omega_{N-1}^2 \}. \quad (42)$$

We find that $F_{\theta\phi}$ (41) is an element of the Cartan sub-algebra \mathfrak{A} . To see this, we begin by defining the following generators of the Cartan sub-algebra [27]

$$H_k = -\frac{i}{\sqrt{2k(k+1)}} \text{diag} \left\{ \underbrace{1, 1, \dots, 1}_{k \text{ entries}}, -k, 0, 0, \dots, 0 \right\}. \quad (43)$$

Then $F_{\theta\phi}$ can be written in terms of the generators H_k as follows:

$$F_{\theta\phi} = \frac{\sin\theta}{2} \sum_{k=1}^{N-1} \sqrt{2k(k+1)} \left(\frac{\omega_k^2}{k} - \frac{\omega_{k+1}^2}{k+1} - 1 \right) H_k. \quad (44)$$

Since $F_{\theta\phi}$ is itself an element of the Cartan sub-algebra \mathfrak{A} , we can set $g(r) = e$, the identity element, in (40). For each element of the Cartan sub-algebra $X \in \mathfrak{A}$, equation (40) yields a charge $Q(X)$, however only $N-1$ of these will be independent because the rank of the $\mathfrak{su}(N)$ Lie algebra is $N-1$. We therefore need a consistent way of selecting an appropriate basis of charges Q_i , for $i = 1, \dots, N-1$.

First of all, we require our $\mathfrak{su}(N)$ charges (40) to reduce to (multiples of) the $\mathfrak{su}(2)$ charge (37) for embedded $\mathfrak{su}(2)$ solutions. Secondly, we fix an overall normalization constant by defining a total ‘‘effective’’ charge Q (reminiscent of the alternative $\mathfrak{su}(2)$ charge (36)), as follows. As $r \rightarrow \infty$, the metric function $m(r)$ takes the form

$$m(r) = M - \frac{Q^2}{2r} + O(r^{-2}), \quad (45)$$

with Q given by

$$Q^2 = \sum_{i=1}^{N-1} Q_i^2 = \frac{1}{2} \sum_{j=1}^N (\omega_{j,\infty}^2 - \omega_{j-1,\infty}^2 - N - 1 + 2j)^2. \quad (46)$$

Therefore Q plays the same role in the metric as the usual Abelian charge. For embedded Reissner-Nordström-adS black holes (26), the expression (46) reduces to

$$Q^2 = \frac{1}{6} N(N-1)(N+1). \quad (47)$$

With these two constraints, the natural basis of charges for $\mathfrak{su}(N)$ is

$$Q_j = \frac{\sqrt{j(j+1)}}{\sqrt{2}} \left[1 - \frac{\omega_{j,\infty}^2}{j} + \frac{\omega_{j+1,\infty}^2}{j+1} \right], \quad (48)$$

for $j = 1, \dots, N-1$. For the $\mathfrak{su}(2)$ case, there is just one charge $Q_1 = 1 - \omega(\infty)^2$, as required. For the $\mathfrak{su}(N)$ case and embedded $\mathfrak{su}(2)$ black holes with $\omega_j(r) = \sqrt{j(N-j)}\omega(r)$, the formula (48) reduces to

$$Q_j = \frac{\sqrt{j(j+1)}}{\sqrt{2}} [1 - \omega(\infty)^2] \quad (49)$$

so that all the $\mathfrak{su}(N)$ charges are proportional to the $\mathfrak{su}(2)$ charge, again as required. It is also straightforward to check that (46) holds.

From (48), it is clear that the charges are determined uniquely by the values of the gauge field functions ω_j at infinity. What is not so immediately apparent is that the converse is also true: the values of the gauge field functions at infinity can be determined (up to an overall irrelevant sign) from the charges Q_j (48). In particular, from (48), it can be shown that

$$\omega_{j,\infty}^2 = j(N-j) - j\sqrt{2} \sum_{k=j}^{N-1} \frac{Q_k}{\sqrt{k(k+1)}}. \quad (50)$$

4. Characterizing $\mathfrak{su}(N)$ EYM black holes for large $|\Lambda|$ - numerical work

Having defined a set of global charges for our $\mathfrak{su}(N)$ EYM black holes, we now address the question of whether these charges, together with the mass M defined in section 2.6 and the negative cosmological constant Λ , are sufficient to characterize stable $\mathfrak{su}(N)$ EYM black holes. We begin, in this section, with a numerical investigation, before turning, in section 6, to analytic arguments.

In [16], two local existence theorems are proved, for solutions in a neighbourhood of the event horizon and infinity. Near the event horizon, it is proven that the parameters Λ , r_h and $\omega_j(r_h)$ (19) determine the solutions in a neighbourhood of the event horizon, and, furthermore, that these local solutions are analytic in these parameters and the variable r . Therefore $N + 1$ parameters are required to completely specify the black hole solutions[§].

Near infinity, the situation is different. It is proven in [16] that $2N$ parameters are required to determine the local solutions in a neighbourhood of infinity, namely Λ , M , $\omega_{j,\infty}$ and c_j (22). Since we know that the black hole solutions form an $N + 1$ parameter family, it is clear that only $N + 1$ of these $2N$ parameters are independent, but the question is, which $N + 1$ parameters can we take to be a basis? Ideally we would like the $N - 1$ charges Q_j (48), defined in the previous section, together with the mass M and cosmological constant Λ , to be a suitable basis. From the analysis in the previous section, the charges Q_j are completely specified by the values of the gauge field functions $\omega_{j,\infty}$ at infinity, so equivalently we would like the constants c_j in (22) to be determined by Λ , M and $\omega_{j,\infty}$.

We begin our numerical investigations with the simplest case, namely $\mathfrak{su}(2)$ black holes. In figure 5 we plot c_1 against the mass M and charge Q_1 for stable $\mathfrak{su}(2)$ black hole solutions when $\Lambda = -10$ (that is, we consider only solutions for which the gauge field function $\omega(r)$ has no zeros and $\omega(r_h), \omega_\infty > 1$). It is clear from the surface in figure 5 that c_1 is a single-valued function of M and Q_1 . We find similar results for other values of Λ . For further evidence that M and Q_1 therefore characterize $\mathfrak{su}(2)$ black holes, in figure 6 we plot M as a function of Q_1 for various r_h , with $\Lambda = -10$. We have found no evidence that the (Q_1, M) curves for different r_h ever intersect. Therefore, for each value of (Q_1, M) for which there is a black hole solution, that solution is uniquely characterized by Q_1 and M .

For $\mathfrak{su}(3)$ and larger gauge groups, it is more difficult to see graphically whether the c_j are uniquely determined by M and the charges Q_j , simply because there are more variables to plot. However, all the numerical evidence we have been able to gather in the $\mathfrak{su}(3)$ case does indicate that the c_j are uniquely determined by M and Q_j . To illustrate this, we consider $\mathfrak{su}(3)$ black holes with $\Lambda = -3$, such that the inequalities

[§] Strictly speaking, there is one further parameter needed near the horizon, namely $S(r_h)$. However this parameter is fixed by the requirement that $S \rightarrow 1$ as $r \rightarrow \infty$ and it plays no further role in our analysis.

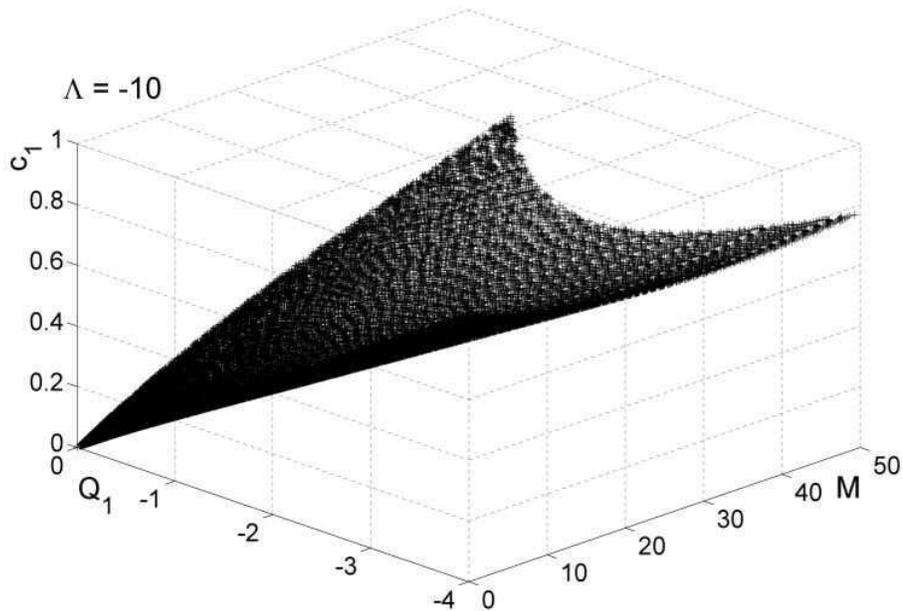


Figure 5. c_1 (22) as a function of mass M and charge Q_1 for stable $\mathfrak{su}(2)$ black hole solutions with $\Lambda = -10$ and varying r_h . Only solutions for which the gauge field function $\omega(r)$ has no zeros and $\omega(r_h), \omega_\infty > 1$ are plotted. From this surface it is evident that c_1 can be regarded as a single-valued function of M and Q_1 .

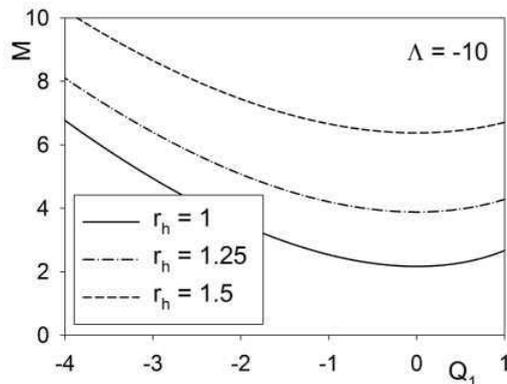


Figure 6. Plot of M against charge Q_1 for $\mathfrak{su}(2)$ black holes with $\Lambda = -10$ and varying r_h . There is no evidence that the curves for different r_h intersect, suggesting that M and Q_1 uniquely characterize the black holes.

(30) are satisfied \parallel . We would like to show that the c_1 and c_2 are determined by M , Q_1 and Q_2 , where the non-Abelian charges Q_1 and Q_2 are given by (48):

$$Q_1 = 1 - \omega_{1,\infty}^2 + \frac{1}{2}\omega_{2,\infty}^2, \quad Q_2 = \sqrt{3} \left(1 - \frac{1}{2}\omega_{2,\infty}^2 \right). \quad (51)$$

\parallel For ease of reference, we call these “potentially stable” black holes. We are not claiming to have shown that all black holes satisfying the inequalities (30) are stable. While the inequalities (30) are sufficient to guarantee stability in the sphaleronic perturbation sector, they do not guarantee stability in the gravitational sector [17].

To produce sensible plots, we first fix $M = 10 \pm 0.1$ (there has to be some range of values of M to obtain clear plots). We perform a scan over the black hole solutions, numerically integrating the field equations for a grid of values of r_h , $\omega_1(r_h)$, and $\omega_2(r_h)$. The results are shown in figures 7–8.

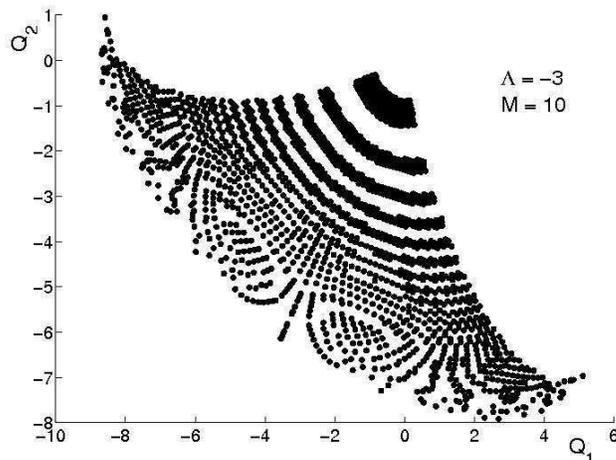


Figure 7. Scatter plot of non-Abelian charges Q_1 and Q_2 for potentially stable $\mathfrak{su}(3)$ black holes with $\Lambda = -3$ and mass $M = 10 \pm 0.1$. We perform a scan over the black hole solutions, and each data point represents a numerical black hole solution. The discreteness of the grid we use can be seen as the values of the charges become more negative. We find bands of charge values, although the bands seem to merge and the structure becomes less clear as the charges become more negative.

The values of Q_1 and Q_2 for these black holes are shown in figure 7. We observe that the values of the non-Abelian charges Q_1 and Q_2 have a band-like structure, with the bands curving round the origin. As the charges become more negative, the structure becomes less clear due to the finite grid on which we find our solutions. We found similar behaviour for other values of M .

Scatter plots of c_1 and c_2 as functions of Q_1 and Q_2 for $M = 10 \pm 0.1$ can be found in figure 8. It is difficult to see clearly from figure 8, but rotating the scatter plots on a computer screen indicates that both c_1 and c_2 are single-valued functions of Q_1 and Q_2 . Again, we find similar behaviour for other values of M . This provides some (albeit limited) numerical evidence for the parameters c_1 and c_2 being determined by the non-Abelian charges Q_1 and Q_2 .

As in the $\mathfrak{su}(2)$ case, further numerical evidence that the mass and non-Abelian charges characterize the black holes can be found by plotting mass M as a function of the charges. Firstly, in figure 9, we plot M as a function of the effective charge Q (46), to see if the quantities M and Q are sufficient to uniquely fix the black hole solution. In figure 9, we see that the surfaces corresponding to different values of the event horizon radius r_h overlap. For example, there are black holes with mass $M = 6.1$ and effective charge $Q = 5$ for $r_h = 1, 1.25$ and 1.5 . We therefore conclude that $\mathfrak{su}(3)$ black holes are

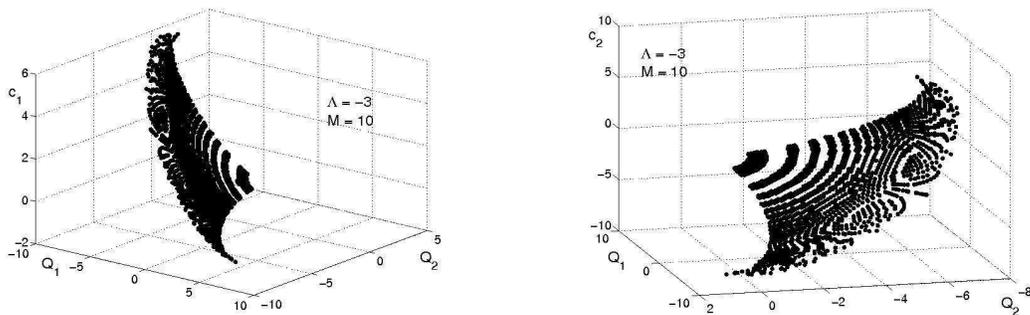


Figure 8. Scatter plot of the constants c_1 (left) and c_2 (right) against the non-Abelian charges Q_1 and Q_2 for the same black holes as in figure 7. It appears to be the case that c_1 and c_2 are single-valued functions of Q_1 and Q_2 .

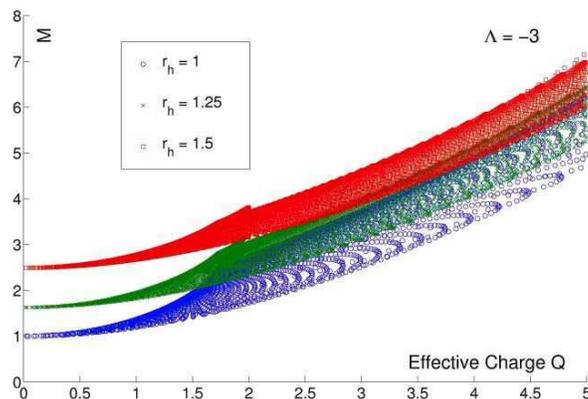


Figure 9. Scatter plot of mass M as a function of effective charge Q (46) for $\mathfrak{su}(3)$ black holes with $\Lambda = -3$ and event horizon radius $r_h = 1$ (black), 1.25 (blue), 1.5 (red). It can be seen that the surfaces for different r_h overlap, indicating that M and Q do not uniquely characterize the black hole solutions.

not uniquely specified by their mass M and effective charge Q .

On the other hand, if we plot M as a function of the non-Abelian charges Q_1 and Q_2 (51), a different structure can be seen in figure 10. The surfaces in figure 10 appear to foliate the parameter space, and we found no evidence of the surfaces for different r_h intersecting. This indicates that the parameters M , Q_1 and Q_2 do indeed uniquely characterize the black hole solutions.

5. Analyticity of M , Q_j and c_j as functions of horizon parameters

The numerical analysis of the previous section has indicated that the $N + 1$ parameters M , Λ and Q_j , for $j = 1, \dots, N - 1$, completely determine stable black hole solutions. In section 6, we will give an analytic argument to show that this assertion holds, at least for a large subspace of stable black holes. The essence of the argument is to consider

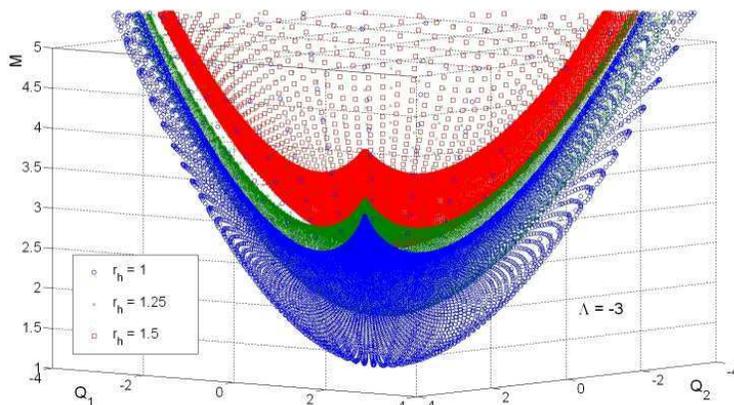


Figure 10. Scatter plot of mass M as a function of non-Abelian charges Q_1 and Q_2 (51) for the same black hole solutions as in figure 9. The different coloured surfaces are for different values of the event horizon radius r_h . There is no evidence that the surfaces for different r_h intersect, suggesting that M , Q_1 and Q_2 uniquely characterize the black holes.

the $N + 1$ parameters $(\Lambda, M, Q_1, \dots, Q_{N-1})$ as functions of the the $N + 1$ parameters $(\Lambda, r_h, \omega_1(r_h), \dots, \omega_{N-1}(r_h))$ which are known, from [16], to uniquely characterize black hole solutions of the field equations. If the map

$$(\Lambda, r_h, \omega_1(r_h), \dots, \omega_{N-1}(r_h)) \rightarrow (\Lambda, M, Q_1, \dots, Q_{N-1}) \quad (52)$$

is invertible, then we can deduce that $(\Lambda, M, Q_1, \dots, Q_{N-1})$ uniquely characterize the black hole solutions. A convenient way to show that the map (52) is invertible is to use the inverse function theorem, so that if we can show that the Jacobian J , given by

$$J = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \frac{\partial M}{\partial r_h} & \frac{\partial M}{\partial \omega_1(r_h)} & \dots & \frac{\partial M}{\partial \omega_{N-1}(r_h)} \\ 0 & \frac{\partial Q_1}{\partial r_h} & \frac{\partial Q_1}{\partial \omega_1(r_h)} & \dots & \frac{\partial Q_1}{\partial \omega_{N-1}(r_h)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\partial Q_{N-1}}{\partial r_h} & \frac{\partial Q_{N-1}}{\partial \omega_1(r_h)} & \dots & \frac{\partial Q_{N-1}}{\partial \omega_{N-1}(r_h)} \end{pmatrix}, \quad (53)$$

is non-vanishing, then the map (52) is invertible. However, this assumes that the map (52) is continuously differentiable. In fact, in this section we will be able to prove a stronger condition, namely that the quantities M , Q_j and c_j (22) are analytic functions of $(\Lambda, r_h, \omega_1(r_h), \dots, \omega_{N-1}(r_h))$.

We begin by writing the field equations (10, 13) in an alternative form, following Proposition 7 in [16]. New variables λ , ψ_k and ξ_k are defined by

$$\lambda = \frac{r}{r_h} \left(1 - \mu - \frac{\Lambda r^2}{3} \right), \quad \psi_j = \omega_j, \quad \xi_j = \frac{r^2}{r_h} \frac{d\omega_j}{dr}, \quad (54)$$

and then the field equations take the following form:

$$\begin{aligned} z \frac{d\lambda}{dz} &= z f_\lambda, & z \frac{d\psi_j}{dz} &= -z \xi_j, & z \frac{d\xi_j}{dz} &= z f_{\xi_j}, \\ z \frac{dS}{dz} &= z^4 f_S, & z \frac{d\Lambda}{dz} &= 0, & z \frac{dr_h}{dz} &= 0, \end{aligned} \quad (55)$$

where the new independent variable is $z = r_h/r$, so that $z \in [0, 1]$. The functions f_λ , f_{ξ_j} and f_S can be found in [16] (noting that in that reference, ξ_j is r_h times ξ_j as defined in (54)); their detailed form is not essential here, their key property being that they are analytic functions of z , Λ , r_h , λ , the ψ_j and the ξ_j , at least in a neighbourhood of $z = 0$. Proposition 7 of [16] then gives local existence of solutions of the differential equations (55), in a neighbourhood of $z = 0$, these solutions being analytic in M , Λ , r_h , $\omega_{j,\infty}$ and $\tilde{c}_j = c_j/r_h$.

The equations (55) are regular only on the interval $z \in [0, 1)$. In the proof of Proposition 6 in [16], equations of a similar form to (55), but with different independent and dependent variables, are derived and used to show the existence of local solutions in a neighbourhood of the event horizon, and analytic in Λ , r_h , and $\omega_j(r_h)$. These latter equations are regular on an interval equivalent to $z \in (0, 1]$. To show that M , $\omega_{j,\infty}$ and c_j are analytic functions of r_h , Λ and $\omega_j(r_h)$ we therefore have to match the solutions which exist in a neighbourhood of the event horizon with those that exist in a neighbourhood of infinity.

To do this, let $\tilde{z} = z - \frac{1}{2}$. Then the equations (55) can be written in terms of this new independent variable as

$$\begin{aligned} \tilde{z} \frac{d\lambda}{d\tilde{z}} &= \tilde{z} f_\lambda, & \tilde{z} \frac{d\psi_j}{d\tilde{z}} &= -\tilde{z} \xi_j, & \tilde{z} \frac{d\xi_j}{d\tilde{z}} &= \tilde{z} f_{\xi_j}, \\ \tilde{z} \frac{dS}{d\tilde{z}} &= \tilde{z} z^3 f_S, & \tilde{z} \frac{d\Lambda}{d\tilde{z}} &= 0, & \tilde{z} \frac{dr_h}{d\tilde{z}} &= 0. \end{aligned} \quad (56)$$

Therefore, by analogy with (55), we have a local existence theorem for solutions of the above differential equations, in a neighbourhood of $\tilde{z} = 0$ ($z = \frac{1}{2}$), and the solutions are analytic in r_h , Λ , and the values of λ , S , ψ_j and ξ_j at $\tilde{z} = 0$.

Since the equations (56) are obtained from (55) by a simple translation of the independent variable, the new equations (56) are regular at $\tilde{z} = -\frac{1}{2}$, which corresponds to $z = 0$. Therefore the values of the field variables at $z = 0$ are analytic in r_h , Λ and the values of λ , S , ψ_j and ξ_j at $\tilde{z} = 0$. In particular, the quantities M , Q_j and c_j will be analytic in r_h , Λ and the values of λ , S , ψ_j and ξ_j at $\tilde{z} = 0$.

Now suppose we have a space of black hole solutions of the original field equations (10, 13) which are regular everywhere between the event horizon and infinity. In particular, every member of this space of solutions is regular at $z = 0$ (infinity), $z = \frac{1}{2}$ and $z = 1$ (event horizon). From Proposition 6 in [16], these solutions are analytic in r_h , Λ , and $\omega_j(r_h)$ in the neighbourhood of the event horizon in which they exist. Therefore, the values of the field variables at $z = \frac{1}{2}$ ($\tilde{z} = 0$) are analytic in r_h , Λ and $\omega_j(r_h)$.

We now have the following situation: the values of the λ , S , ψ_j and ξ_j at $\tilde{z} = 0$ are analytic functions of r_h , Λ and $\omega_j(r_h)$. In addition, the quantities M , Q_j and c_j are

analytic in r_h , Λ and the values of λ , S , ψ_j and ξ_j at $\tilde{z} = 0$. Therefore we can conclude that M , Q_j and c_j are analytic in r_h , Λ and the $\omega_j(r_h)$. Furthermore, as a corollary, it must be the case that the Jacobian J (53) is also an analytic function of r_h , Λ and the $\omega_j(r_h)$.

6. Characterizing $\mathfrak{su}(N)$ EYM black holes for large $|\Lambda|$ - analytic work

Our purpose in this section is to argue that the charges Q_j (48), together with the mass M and cosmological constant Λ , are sufficient to uniquely characterize $\mathfrak{su}(N)$ hairy black holes, at least for some subset of those black holes which are linearly stable [17].

There are a number of difficulties in deriving such a result: firstly, general black hole solutions are known only numerically; secondly, while the local existence theorems in [16] near the event horizon and at infinity are valid for any values of the cosmological constant, event horizon radius and the other parameters in the theory, the theorems proving the existence of regular black hole solutions are valid only for sufficiently large $|\Lambda|$.

In other words, in [16] the existence of black hole solutions is shown for fixed r_h and $\omega_j(r_h)$, and sufficiently large $|\Lambda|$, where, as discussed in [16], how large “sufficiently large” is will likely depend on the values of the parameters r_h and $\omega_j(r_h)$. In the present analysis we wish to study the problem from a different perspective: we want to fix $|\Lambda|$ to be some (suitably large) value, and vary r_h and $\omega_j(r_h)$, and then show that, at least for some subset of the black hole solutions thus generated, the black holes can be uniquely characterized by their mass M and non-Abelian charges Q_j .

From our above discussion of the generalized “no-hair” conjecture, it is clear that we are only interested in characterizing stable hairy black hole solutions. Due to the symmetry (15) of the field equations, we may consider only $\omega_j(r_h) > 0$ without loss of generality. We therefore restrict attention to black holes for which the inequalities (30) are satisfied, both on and outside the event horizon. It is straightforward to show in this case that the functions $\omega_j(r)$ are monotonically increasing and that $\omega_j(r) > 0$ for all $r \geq r_h$. We also consider only those black holes which are thermodynamically stable (see section 2.5), which is equivalent to focussing attention on comparatively large black holes.

6.1. Characterizing black holes by M and Q_j

We begin our analysis with a more careful consideration of the space of black hole solutions for large $|\Lambda|$. There are two length scales in this problem: the event horizon radius r_h and $\ell = \sqrt{-3/\Lambda}$. We wish to study black hole solutions when ℓ is small, but allowing r_h to vary. To this end, it is helpful to introduce a dimensionless radial co-ordinate $x = r/r_h$, so that $x \in [1, \infty)$ irrespective of the value of the event horizon radius r_h . In terms of x the field equations (10, 13) take the form:

$$\frac{d\hat{m}}{dx} = \mu G + r_h^2 x^2 p_\theta;$$

$$\begin{aligned} \frac{1}{S} \frac{dS}{dx} &= \frac{2G}{x}; \\ 0 &= x^2 \mu \frac{d^2 \omega_j}{dx^2} + \left[2\hat{m} - 2r_h^2 x^3 p_\theta + \frac{2r_h^2 x^3}{\ell^2} \right] \frac{d\omega_j}{dx} + W_j \omega_j; \end{aligned} \quad (57)$$

where $\hat{m}(x) = m(r)/r_h$, the quantity W_j is given by (12) and equations (11, 14) now take the form

$$\begin{aligned} p_\theta &= \frac{1}{4x^4 r_h^4} \sum_{j=1}^N \left[(\omega_j^2 - \omega_{j-1}^2 - N - 1 + 2j)^2 \right]; \\ G &= \frac{1}{r_h^2} \sum_{j=1}^{N-1} \left(\frac{d\omega_j}{dx} \right)^2. \end{aligned} \quad (58)$$

At the event horizon, $x = 1$, we have

$$\hat{m}(1) = \frac{1}{2} \left(1 + \frac{r_h^2}{\ell^2} \right), \quad (59)$$

which becomes large as $\ell \rightarrow 0$. We define a further new variable $\tilde{m}(x)$ by

$$\hat{m}(x) = m_1 + \tilde{m}(x) \quad (60)$$

where $m_1 = \hat{m}(1)$ (59). In [16], it is shown that $\frac{d}{dx}(\tilde{m}\ell^2) \rightarrow 0$ as $\ell \rightarrow 0$ for fixed r_h , $\omega_j(r_h)$, but this does not necessarily mean that $\frac{d\tilde{m}}{dx}$ tends to zero in this limit. Indeed, we shall see below that this is not the case, as is borne out by Figures 5 and 6 in [16].

We now write the first and third field equations (57) in the form

$$\begin{aligned} \ell^2 \frac{d\tilde{m}}{dx} &= \frac{1}{r_h^2} \left[\ell^2 - \frac{2m_1 \ell^2}{x} - \frac{2\tilde{m}\ell^2}{x} + r_h^2 x^2 \right] \sum_{j=1}^{N-1} \left(\frac{d\omega_j}{dx} \right)^2 \\ &\quad + \frac{\ell^2}{4x^2 r_h^2} \sum_{j=1}^N \left[(\omega_j^2 - \omega_{j-1}^2 - N - 1 + 2j)^2 \right]; \\ 0 &= x^2 \left[\ell^2 - \frac{2m_1 \ell^2}{x} - \frac{2\tilde{m}\ell^2}{x} + r_h^2 x^2 \right] \frac{d^2 \omega_j}{dx^2} \\ &\quad + \left[2m_1 \ell^2 + 2\tilde{m}\ell^2 - 2r_h^2 x^3 \ell^2 p_\theta + 2r_h^2 x^3 \right] \frac{d\omega_j}{dx} + \ell^2 W_j \omega_j. \end{aligned} \quad (61)$$

The analysis leading to Proposition 11 in [16] essentially involves ignoring the terms $\tilde{m}\ell^2$ and $\ell^2 W_j \omega_j$ in the right-hand-sides of the above equations. It is clear that for any fixed values of r_h and $\omega_j(r_h)$, we may choose ℓ sufficiently small that these terms are negligible compared with the others in the equations. However, here we wish to vary r_h and $\omega_j(r_h)$, and therefore we need to carefully examine the magnitudes of all the quantities in the above equations for small ℓ . It is clear from equations (61) that there are subtleties if r_h is also small. For this reason, we consider only those black holes for which $r_h \gg \ell$, as these will be thermodynamically stable (see section 2.5).

For a regular event horizon at $x = 1$, we can vary r_h and $\omega_j(r_h)$ subject to the single constraint (21), which can be written as

$$\ell^2 \sum_{j=1}^N \left[(\omega_j^2(r_h) - \omega_{j-1}^2(r_h) - N - 1 + 2j)^2 \right] < 2r_h^2 \ell^2 + 6r_h^4. \quad (62)$$

Therefore, for each j , we have

$$\ell^2 (\omega_j^2(r_h) - \omega_{j-1}^2(r_h) - N - 1 + 2j)^2 < 2r_h^2 \ell^2 + 6r_h^4 \quad (63)$$

and, as a result, it must be the case that

$$\ell [\omega_j^2(r_h) - j(N - j)] < j [2r_h^2 \ell^2 + 6r_h^4]^{\frac{1}{2}}. \quad (64)$$

Our numerical results in section 2.4 indicate that we do not have regular black hole solutions for values of $\omega_j(r_h)$ close to the boundary of the region defined by (62), and therefore we do not need to consider all $\omega_j(r_h)$ such that (62) is satisfied. At the same time, we find that the region of parameter space for which we have regular black hole solutions which are uniquely determined by M and Q_j seems to grow as ℓ decreases. Therefore we would like to consider a region of the $\omega_j(r_h)$ parameter space which, for small ℓ , is smaller than the region defined by the inequality (62), but which nonetheless grows as ℓ decreases.

To this end, define new functions $q_j(x)$ by

$$\ell^2 [\omega_j^2(r) - j(N - j)]^2 = \ell^{2\sigma} q_j^2(x), \quad (65)$$

where $\sigma > 0$ is a presently unknown constant, and is the same for all j . We further assume that $q_j(x)$ is order one for small ℓ . Setting $\sigma = 1$ would correspond to an upper bound on ω_j which is fixed independent of ℓ , whilst setting $\sigma = 0$ corresponds to considering the whole of the region of parameter space satisfying (62). We therefore anticipate that $0 < \sigma < 1$ will be relevant for our analysis. In this case, there is an upper bound on ω_j^4 of the order of $\ell^{2\sigma-2} q_j^2$ which grows as ℓ decreases.

In the $\mathfrak{su}(2)$ case, it is shown in [12] that, for fixed r_h and $\omega(r_h)$, $\ell^{-1} \omega'(r) \rightarrow 0$ as $\ell \rightarrow 0$. This suggests the following definition of further new functions $\eta_j(x)$, again expected to be order one for small ℓ , such that

$$\ell^{-1} \frac{d\omega_j}{dx} = \ell^\kappa \eta_j(x) \quad (66)$$

for some $\kappa > 0$.

Let us now examine whether it is possible to find suitable values of σ and κ so as to obtain consistent approximate solutions of the field equations (61) when ℓ is small. First write the equations (61) in terms of the q_j and η_j :

$$\begin{aligned} \ell^2 \frac{d\tilde{m}}{dx} &= \frac{\ell^{2\kappa+2}}{r_h^2} \left[\ell^2 - \frac{2m_1 \ell^2}{x} - \frac{2\tilde{m} \ell^2}{x} + r_h^2 x^2 \right] \sum_{j=1}^{N-1} \eta_j^2(x) \\ &\quad + \frac{\ell^{2\sigma}}{4x^2 r_h^2} \sum_{j=1}^N [q_j(x) - q_{j-1}(x)]^2; \end{aligned} \quad (67)$$

$$\begin{aligned} 0 &= x^2 \ell^{\kappa+1} \left[\ell^2 - \frac{2m_1 \ell^2}{x} - \frac{2\tilde{m} \ell^2}{x} + r_h^2 x^2 \right] \frac{d\eta_j}{dx} \\ &\quad + \ell^{\kappa+1} [2m_1 \ell^2 + 2\tilde{m} \ell^2 - 2r_h^2 x^3 \ell^2 p_\theta + 2r_h^2 x^3] \eta_j(x) \\ &\quad + \frac{1}{2} \ell^{\sigma+1} [q_{j+1}(x) - 2q_j(x) + q_{j-1}(x)] \omega_j(x). \end{aligned} \quad (68)$$

Examining first (68), and bearing in mind the upper bound $\ell^{2\sigma-2}q_j^2$ on $\omega_j^4(x)$, for non-trivial solutions we require that the first two terms are of the same order in ℓ as the last term. This means that

$$\kappa = \frac{3\sigma}{2} - \frac{1}{2}. \quad (69)$$

Requiring that $\kappa > 0$ then implies that $\sigma > 1/3$, which is consistent with our assumptions. With this value of κ , we then have

$$2\kappa + 2 = 3\sigma + 1 > 2\sigma \quad (70)$$

for all $\sigma > 0$, which means that the first line of (67) is of subleading order in ℓ compared with the second. Furthermore, differentiating (65) gives

$$\frac{dq_j}{dx} \sim 2q_j\eta_j\ell^{1+\sigma} \quad (71)$$

for $\omega_j^4 \sim \ell^{2\sigma-2}q_j^2$ and κ given by (69). Therefore we may regard the functions $q_j(x)$ as constant to leading order in ℓ . Integrating (67) then gives, to leading order in ℓ ,

$$\begin{aligned} \ell^2\tilde{m}(x) &= \frac{\ell^{2\sigma}}{4r_h^2} \left(1 - \frac{1}{x}\right) \sum_{j=1}^N [q_j(1) - q_{j-1}(1)]^2 \\ &= \frac{\ell^2}{4r_h^2} \left(1 - \frac{1}{x}\right) \sum_{j=1}^N \left[(\omega_j^2(r_h) - \omega_{j-1}^2(r_h) - N - 1 + 2j)^2 \right], \end{aligned} \quad (72)$$

where we have used the initial condition $\tilde{m} = 0$ at the event horizon $x = 1$. Note that the answer (72) implies that the terms $\ell^2\tilde{m}$ in (68) are indeed small compared with $\ell^2m_1 = \mathcal{O}(1)$ for small ℓ and so can be ignored to first order. However, it is not necessarily the case that \tilde{m} itself is small. We have $\tilde{m} \sim \mathcal{O}(\ell^{2\sigma-2})$, so \tilde{m} will in fact diverge as $\ell \rightarrow 0$, albeit more slowly than m_1 (59).

To leading order, the Yang-Mills equations (68) become

$$\begin{aligned} 0 &= x^2 \left[r_h^2 x^2 - \frac{2m_1\ell^2}{x} \right] \frac{d\eta_j}{dx} + [2m_1\ell^2 + 2r_h^2 x^3] \eta_j(x) \\ &\quad + \frac{1}{2} [q_{j+1}(x) - 2q_j(x) + q_{j-1}(x)] q_j(x)^{\frac{1}{2}}, \end{aligned} \quad (73)$$

where we have ignored the term

$$2r_h^2 x^3 \ell^2 p_\theta = 2r_h^2 x^3 \ell^{2\sigma} \sum_{j=1}^N [q_j(x) - q_{j-1}(x)]^2, \quad (74)$$

and used the leading order approximation

$$\omega_j = \ell^{\frac{1}{2}(\sigma-1)} q_j^{\frac{1}{2}}. \quad (75)$$

To leading order in ℓ , we can treat the functions q_j as approximately constant, and in this case (73) can be integrated to give

$$\eta_j(x) = -\frac{1}{2r_h^2(x^2 + x + 1)} [q_{j+1}(1) - 2q_j(1) + q_{j-1}(1)] q_j(1)^{\frac{1}{2}}, \quad (76)$$

where we have chosen the arbitrary constant of integration to be such that $\eta(x)$ is finite at the event horizon $x = 1$. Restoring the original variables, it is straightforward to check that the solution (76) for η_j satisfies (20) at the event horizon.

Therefore we have obtained a consistent, approximate set of solutions of the field equations which are valid for all $r_h \gg \ell$ and all $\omega_j(r_h)$ such that

$$[\omega_j^2(r_h) - j(N - j)]^2 < \ell^{2\sigma-2} \quad (77)$$

for some $\sigma \in (\frac{1}{3}, 1)$.

At this stage we compare our approximate solutions $q_j \approx \text{constant}$, η_j (76) and \tilde{m} (72) with numerical solutions to test the validity of our approximations. To illustrate the behaviour, in figures 11–13, we consider $\mathfrak{su}(3)$ black holes with $r_h = 1$, $\omega_1(r_h) = 1$, $\omega_2(r_h) = 3$ and varying cosmological constant $\Lambda = -10^4$, -10^5 and -10^6 .

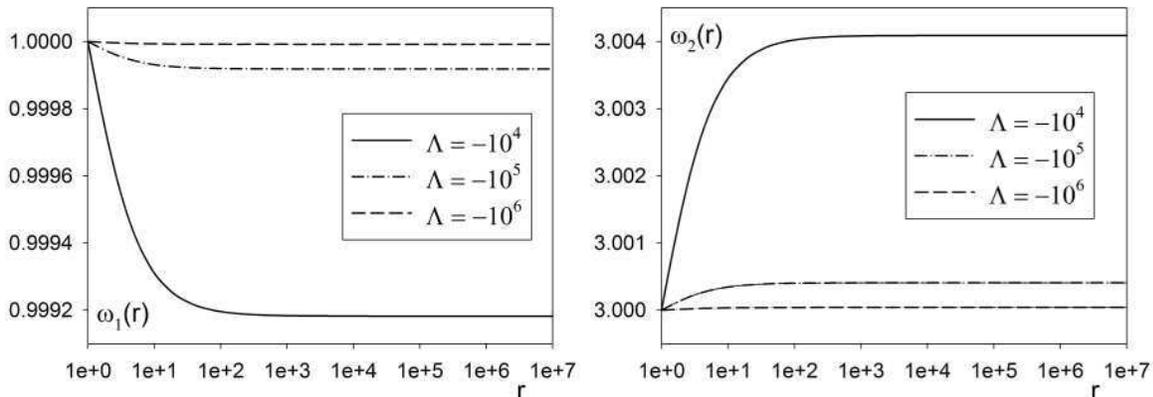


Figure 11. Gauge field functions $\omega_1(r)$ (left) and $\omega_2(r)$ (right) for $\mathfrak{su}(3)$ black holes with $r_h = 1$, $\omega_1(r_h) = 1$, $\omega_2(r_h) = 3$ and varying cosmological constant $\Lambda = -10^4$, -10^5 and -10^6 . The gauge field functions approach constants as $|\Lambda| \rightarrow \infty$.

Firstly, in figure 11 we plot the gauge field functions $\omega_1(r)$ and $\omega_2(r)$. It is clear that, as expected, the gauge field functions approach constants as $|\Lambda| \rightarrow \infty$.

To test the validity of the approximate solution for η_j (76), for each value of r we divide the numerically-generated value of $\omega'_j(r)$ by η_j given by (76). The results are plotted in figure 12, where we have also divided the answers by a constant (corresponding to the powers of ℓ in the definition (66)), so that all curves pass through ± 1 at the event horizon, to make comparisons easier. Since $\omega'_1(r) < 0$ and $\omega'_2(r) > 0$ for the particular black hole solutions we are considering, we have fixed the additional constant so that the curves for the $\omega'_1(r)$ approximation to pass through -1 at the event horizon, and those for the $\omega'_2(r)$ approximation pass through 1 at the event horizon. In figure 12, we see that the curves tend to ± 1 for all r as $|\Lambda| \rightarrow \infty$. This means that, in the large $|\Lambda|$ limit, the approximation η_j (76) becomes increasingly accurate.

To test the approximation \tilde{m} (72), we take the numerical values of $m(r) - m(r_h)$, divide by the approximation (72) and, as in figure 12, we also divide through by a

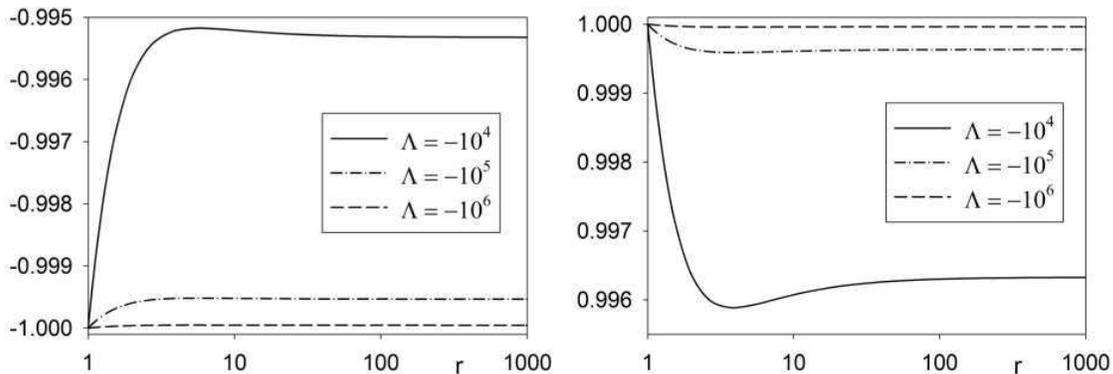


Figure 12. Test of the approximation (76) for $\omega'_1(r)$ (left) and $\omega'_2(r)$ (right). The numerical values of $\omega'_j(r)$ for the black hole solutions in figure 11 are divided by the approximate solution η_j (76) and a normalization constant, chosen so that the curves pass through ± 1 at the event horizon. The curves tend to ± 1 for all r as $|\Lambda|$ increases, indicating the validity of the approximation η_j (76).

constant (corresponding to powers of ℓ) so that all the curves pass through 1 as $r \rightarrow \infty$. We fix the normalization at $r \rightarrow \infty$ rather than at the event horizon because, as $r \rightarrow r_h$, both $m(r) - m(r_h)$ and \tilde{m} (72) vanish and numerical errors become an issue in dividing these two quantities. The results are plotted in figure 13. In figure 13 we see that the curves tend to 1 for all r as $|\Lambda| \rightarrow \infty$, showing that, in the large $|\Lambda|$ limit, the approximation \tilde{m} (72) becomes increasingly accurate.

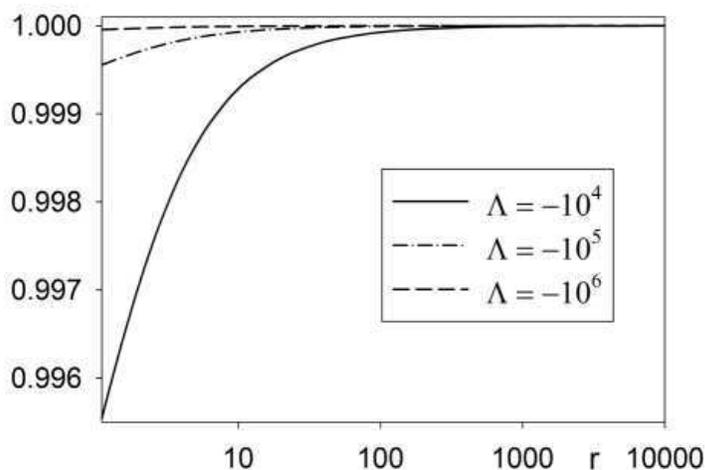


Figure 13. Test of the approximation (72) for $m(r) - m(r_h)$. The numerical values of $m(r) - m(r_h)$ for the black hole solutions in figure 11 are divided by the approximate solution for \tilde{m} (72), and a normalization constant, chosen so that the curves pass through 1 as $r \rightarrow \infty$. The curves tend to 1 for all r as $|\Lambda|$ increases, indicating the validity of the approximation for $m(r) - m(r_h)$ (72).

We emphasize an important point here. In our $\mathfrak{su}(3)$ numerical examples above, we fixed r_h , $\omega_1(r_h)$ and $\omega_2(r_h)$ and examined the solutions for varying values of the cosmological constant Λ . However, this is *not* the purpose of the approximation developed in this section. The approximation derived here is for *fixed* (but very large) Λ , and varying r_h (keeping $r_h \gg \ell$), $\omega_1(r_h)$ and $\omega_2(r_h)$ (subject to q_j (65) being of order one for small ℓ). We think of the approximation $\omega_j = \text{constant}$, η_j (76), \tilde{m} (72) as being the first term in an asymptotic series for the field variables, which is asymptotic for large $|\Lambda|$. This approximation is uniformly valid on a region of the parameter space in which $r_h \gg \ell$, and σ (65) is fixed between $\frac{1}{3}$ and 1. This does not cover the whole space of black hole solutions for this large, but fixed, value of $|\Lambda|$. However, this is not unexpected: the original existence theorems [12, 16] were only proved for $|\Lambda|$ “sufficiently large” and fixed r_h and ω_j . It is therefore unsurprising that we are only able to derive analytic approximations on some subspace of the set of black hole solutions. However, the space of black hole solutions that we are able to describe is large and increases in size as ℓ decreases.

Having derived the approximate solutions $\omega_j = \text{constant}$, η_j (76), \tilde{m} (72) for small ℓ , we now examine whether these approximate solutions are determined by their mass M and non-Abelian charges Q_i . For these approximate solutions, the gauge field functions ω_j are approximately constant, and therefore the charges (48) are given, to leading order in ℓ , in terms of the values of the gauge field functions on the event horizon:

$$Q_j = \frac{\sqrt{j(j+1)}}{\sqrt{2}} \left(1 - \frac{\omega_j(r_h)^2}{j} + \frac{\omega_{j+1}(r_h)^2}{j+1} \right). \quad (78)$$

In addition, the masses of the black holes are given, to leading order in ℓ , by

$$M = \frac{r_h}{2} - \frac{\Lambda r_h^3}{6} + \frac{1}{4r_h} \sum_{j=1}^N \left[(\omega_j^2(r_h) - \omega_{j-1}^2(r_h) - N - 1 + 2j)^2 \right]. \quad (79)$$

To show that the Jacobian J (53) does not vanish in this case, we require the partial derivatives

$$\begin{aligned} \frac{\partial M}{\partial r_h} &= \frac{1}{2} - \frac{\Lambda r_h^2}{2} - \frac{1}{4r_h^2} \sum_{j=1}^N \left[(\omega_j^2(r_h) - \omega_{j-1}^2(r_h) - N - 1 + 2j)^2 \right]; \\ \frac{\partial M}{\partial \omega_k(r_h)} &= -\frac{2}{r_h} W_k(r_h) \omega_k(r_h); \\ \frac{\partial Q_j}{\partial r_h} &= 0; \\ \frac{\partial Q_j}{\partial \omega_k(r_h)} &= \frac{\sqrt{j(j+1)}}{\sqrt{2}} \frac{2\omega_k(r_h)}{k} (-\delta_{j,k} + \delta_{j+1,k}). \end{aligned} \quad (80)$$

The fact that the Q_j do not depend on r_h means that the Jacobian J (53) is given by

$$J = \frac{\partial M}{\partial r_h} J_Q, \quad (81)$$

where J_Q is the Jacobian of the charges Q_j in terms of the $\omega_k(r_h)$. We observe that $\frac{\partial M}{\partial r_h} > 0$ follows immediately from (21), while J_Q must be non-zero by virtue of the fact

that the transformation (48) from the values of the ω_k to the charges Q_j is invertible. Therefore the Jacobian J (81) is non-zero. This means that the parameters Λ , M and Q_j uniquely specify the black hole solutions, at least in this approximation.

The approximation we have used in this section is exact in the limit $\ell \rightarrow 0$. We have therefore shown that the Jacobian J is non-zero in this limit. Using the fact that the Jacobian J is analytic in the parameters of the theory, as shown in section 5, it follows that $J \neq 0$ at least for sufficiently small ℓ . We therefore deduce that at least a subset of stable hairy $\mathfrak{su}(N)$ black holes, for sufficiently large $|\Lambda|$, are uniquely characterized by their mass, the (negative) cosmological constant Λ and a set of global conserved non-Abelian charges Q_j .

A further comment is in order. In previous sections, we have emphasized that Bizon's modified "no-hair" conjecture [8] applies only to stable black holes. In our numerical work in section 4, we restricted our attention to "potentially stable" black holes, namely those for which the inequalities (30), necessary (but not sufficient) for stability, are satisfied. However, in this section we have made no reference to these inequalities, which, with the definition (65), take the form

$$q_j(x) > \frac{1}{2} [q_{j+1}(x) + q_{j-1}(x)]. \quad (82)$$

Simply by restricting our attention to those black holes for which (82) hold at the event horizon $x = 1$, we trivially have that stable black holes are uniquely specified by Λ , M and the charges Q_j , for sufficiently small ℓ .

6.2. Characterizing solitons by M and Q_j

In the previous subsection we have shown that $\mathfrak{su}(N)$ black holes, at least for sufficiently large black holes in the presence of a sufficiently large $|\Lambda|$, are uniquely characterized by their mass M , the $\mathfrak{su}(N)$ charges Q_j and Λ . One remaining issue is whether it is possible for soliton solutions to have the same M and Q_j as black hole solutions for a particular Λ . In other words, is it possible to confuse black holes and solitons by measuring their mass M and non-Abelian charges Q_j ?

Our analysis mirrors that in the previous subsection. However, as in [16], we find that analytic work with soliton solutions is considerably more complicated than that for black hole solutions. The work in this section follows the approach and notation of [16] in dealing with the soliton solutions, and we refer the reader to that paper for more details, keeping the presentation in this section brief.

Our goal is to find an approximation for the soliton solutions which is valid for small ℓ . Since there is no event horizon for soliton solutions, we have just one length scale, namely ℓ , and we define a new dimensionless radial variable y by $y = r/\ell$. Our first task is to write the field equations (10, 13) in a form suitable for analysis. Following [16], we write the gauge field functions $\omega_j(r)$ as

$$\omega_j(r) = [j(N-j)]^{\frac{1}{2}} u_j(y) \quad (83)$$

and define a vector $\mathbf{u} = (u_1, \dots, u_{N-1})^T$. As in [16], we rewrite the vector \mathbf{u} as a sum over eigenvectors of the $(N-1) \times (N-1)$ matrix A whose entries are:

$$A_{i,j} = [j(N-j)]^{\frac{1}{2}} [2\delta_{i,j} - \delta_{i+1,j} - \delta_{i-1,j}], \quad (84)$$

where $\delta_{i,j}$ is the usual Kronecker delta. The form of \mathbf{u} in terms of eigenvectors of A reads (cf. (77) in [16]):

$$\mathbf{u}(y) = \mathbf{u}_0 + \sum_{k=2}^N \beta_k(y) y^k \ell^k \quad (85)$$

where $\mathbf{u}_0 = (1, 1, \dots, 1)^T$ and the β_k are vector functions which satisfy

$$A\beta_k = k(k-1)\beta_k. \quad (86)$$

Next we define scalar variables $\zeta_k(y)$ by (cf. (81) in [16])

$$\zeta_k(y) = \boldsymbol{\sigma}_k^T \beta_k(y), \quad (87)$$

where $\boldsymbol{\sigma}_k^T$ is the k -th left eigenvector of the matrix A .

We next define a rescaled metric variable $\hat{m}(y)$ as follows:

$$\hat{m}(y) = \frac{m(r)}{\ell}, \quad (88)$$

which satisfies the Einstein equation (13):

$$\frac{d\hat{m}}{dy} = \mu G + \frac{1}{2y} \tilde{P} \quad (89)$$

where G is defined in (14) and

$$\tilde{P}(y) = 2y^3 \ell^2 p_\theta, \quad (90)$$

with p_θ given by (11).

It is shown in [16] that the soliton solutions are determined, in a neighbourhood of the origin, by the N parameters Λ (or equivalently, ℓ) and $\zeta_k(0)$ [¶]. Unlike the black hole solutions, there are no a priori bounds on the values of $\zeta_k(0)$ for the existence of regular solutions. However, numerical analysis [15] shows that the region of parameter space for which soliton solutions in which all the gauge field functions ω_j have no zeros expands as ℓ decreases. It is argued in [31] that, for $\mathfrak{su}(2)$ solitons, the size of the parameter space expands as ℓ^{-1} as ℓ decreases. With this in mind, following (65), we define new variables $\alpha_k(y)$ by

$$\zeta_k(y) = \alpha_k(y) \ell^{\sigma_k - 1}, \quad (91)$$

where each σ_k is a constant, and we expect that $\sigma_k < 1$ so that the space of soliton solutions that we are considering expands as ℓ decreases ($\sigma_k = 0$ would correspond to the results for $\mathfrak{su}(2)$ solitons [31], but, as with the black holes, we expect to not be able to consider the entire solution space, but nonetheless a region of solutions which expands as ℓ decreases). Note that, unlike the black hole case (65), we allow for the

[¶] As with the black hole solutions, strictly speaking there is an additional parameter, namely $S(0)$, but we do not need to consider this further.

possibility of different values of σ_k for different ζ_k . We will assume that each $\alpha_k(y)$ is order one for all $y \in [0, \infty)$ and small ℓ .

The Yang-Mills equation (10) now becomes

$$0 = y^2 \mu \left[y^k \frac{d^2 \alpha_k}{dy^2} + 2ky^{k-1} \frac{d\alpha_k}{dy} + k(k-1)y^{k-2} \alpha_k \right] + \left[2\hat{m} + 2y^3 - \tilde{P} \right] \left[y^k \frac{d\alpha_k}{dy} + ky^{k-1} \alpha_k \right] + \frac{1}{\ell^{k+\sigma_k-1}} \boldsymbol{\sigma}_k^T \mathbf{W}, \quad (92)$$

where \tilde{P} is given by (90), and the vector \mathbf{W} is defined as $\mathbf{W} = (W_1, W_2, \dots, W_{N-1})^T$, with the W_j given by (12).

To determine the leading order (in ℓ , for small ℓ) behaviour of the Yang-Mills equation (92), we first consider the term $\boldsymbol{\sigma}_k^T \mathbf{W}$. It is shown in [16] that this takes the form

$$\boldsymbol{\sigma}_k^T \mathbf{W} = -k(k-1)y^k \ell^{k+\sigma_k-1} \alpha_k + \sum_{j=k+1}^Z \boldsymbol{\sigma}_k^T \boldsymbol{\tau}_j y^j \ell^j, \quad (93)$$

for some $Z \in \mathbb{N}$. The $\boldsymbol{\sigma}_k^T \boldsymbol{\tau}_j$ are rather complicated expressions which involve products of up to three of the ζ_k . Since each $\sigma_k < 1$, the leading order behaviour of $\boldsymbol{\sigma}_k^T \boldsymbol{\tau}_j$ will be from terms involving products of three ζ_k , which will be of order $\ell^{j-3+\sigma_a+\sigma_b+\sigma_c}$ for some a, b, c . These will be subleading compared to the first term in (93) if $k + \sigma_k - 1 < j - 3 + \sigma_a + \sigma_b + \sigma_c$, bearing in mind that $j > k$. This inequality is satisfied if $1 > \sigma_j > \frac{2}{3}$ for all j . Therefore, to find the leading order behaviour of the Yang-Mills equation (92), we need to keep only the first term in (93). This gives, to leading order in ℓ ,

$$0 = \mu \left[y^2 \frac{d^2 \alpha_k}{dy^2} + 2ky \frac{d\alpha_k}{dy} + k(k-1) \alpha_k \right] + \left[2\hat{m} + 2y^3 - \tilde{P} \right] \left[\frac{d\alpha_k}{dy} + ky^{-1} \alpha_k \right] - k(k-1) \alpha_k. \quad (94)$$

Further simplification of the equations (89, 94) requires analysis of the quantities G and \tilde{P} . First we write the vector functions $\boldsymbol{\beta}_k$ in terms of the scalar variables ζ_k (87) (with no summation implied):

$$\boldsymbol{\beta}_k(y) = \mathbf{v}_k \zeta_k(y) = \mathbf{v}_k \ell^{\sigma_k-1} \alpha_k(y), \quad (95)$$

where the $\mathbf{v}_k = (v_{k,1}, v_{k,2}, \dots, v_{k,N-1})$ are right-eigenvectors of the matrix A (84). Using the expressions (83, 85), we find

$$\omega_j = [j(N-j)]^{\frac{1}{2}} \left[1 + \sum_{k=2}^N v_{k,j} y^k \ell^{k+\sigma_k-1} \alpha_k(y) \right] \quad (96)$$

and therefore the leading order behaviour of G (14) is

$$G = \ell^{2\sigma_2} \Sigma_G \left[y^2 \frac{d\alpha_2}{dy} + 2y\alpha_2 \right]^2 + o(\ell^2), \quad (97)$$

where

$$\Sigma_G = \sum_{j=1}^{N-1} j(N-j) v_{2,j}^2. \quad (98)$$

From (90, 96), it can be seen that \tilde{P} is a complicated sum of terms involving products of at least two ζ_k , or, equivalently, at least two α_k . Products of the form $\alpha_{k_1}\alpha_{k_2}$ in \tilde{P} are multiplied by coefficients of order $\ell^{k_1+k_2+\sigma_{k_1}+\sigma_{k_2}-4}$ and therefore the leading order term of this form occurs when $k_1 = 2 = k_2$ and is of order $\ell^{2\sigma_2}$. Products of the form $\alpha_{k_1}\alpha_{k_2}\alpha_{k_3}$ are multiplied by coefficients of order $\ell^{k_1+k_2+k_3+\sigma_{k_1}+\sigma_{k_2}+\sigma_{k_3}-5}$ and are sub-leading compared with the $O(\ell^{2\sigma_2})$ term. Similarly, products of the form $\alpha_{k_1}\alpha_{k_2}\alpha_{k_3}\alpha_{k_4}$ are also sub-leading, and we deduce that the leading order behaviour of \tilde{P} is

$$\tilde{P} = 2\ell^{2\sigma_2}\Sigma_P\alpha_2^2y^3 + o(\ell), \quad (99)$$

where

$$\Sigma_P = \sum_{j=1}^{N-1} [j(N-j)v_{2,j} - (j-1)(N-j+1)v_{2,j-1}]^2. \quad (100)$$

Substituting for G and \tilde{P} in the Einstein equation (89), a consistent, non-trivial solution exists when

$$\hat{m} = \ell^{2\sigma_2}\chi(y), \quad (101)$$

with $\chi(y)$ satisfying, to leading order in ℓ , the differential equation

$$\frac{d\chi}{dy} = (1+y^2)\Sigma_G \left[y^2 \frac{d\alpha_2}{dy} + 2y\alpha_2 \right]^2 + \Sigma_P\alpha_2^2y^2. \quad (102)$$

The \hat{m} and \tilde{P} terms in (94) can therefore be ignored to leading order in ℓ provided $\sigma_2 > 0$, giving simplified Yang-Mills equations

$$0 = y(1+y^2) \frac{d^2\alpha_k}{dy^2} + 2[k+(k+1)y^2] \frac{d\alpha_k}{dy} + k(k+1)y\alpha_k. \quad (103)$$

The Yang-Mills equations (103) were also derived in [16] in the limit $\ell \rightarrow 0$, but here we have used a more subtle approximation (by including the σ_k in (91)). Equations (103) have the following solution regular at the origin [16]:

$$\alpha_k(y) = {}_2F_1 \left(\frac{1}{2}[k+1], \frac{k}{2}; k + \frac{1}{2}; -y^2 \right) \alpha_{k,0}, \quad (104)$$

where ${}_2F_1$ is a hypergeometric function and $\alpha_{k,0} = \alpha_k(0)$. The function (104) has magnitude bounded by $|\alpha_{k,0}|$ and, as $y \rightarrow \infty$, it tends monotonically to zero as $O(y^{-2k+4})$ for $k > 2$, and as $O(y^{-2})$ for $k = 2$. Then (102) can be integrated, to give $\chi(y)$, which is then a bounded function satisfying the required boundary conditions:

$$\chi(y) = O(y^3), \quad y \rightarrow 0; \quad \chi(y) = \chi_\infty + O(y^{-1}), \quad y \rightarrow \infty. \quad (105)$$

We therefore have a consistent set of solutions to the field equations, valid when $\ell \ll 1$ and providing $\frac{2}{3} < \sigma_k < 1$.

Returning to the original variables $m(r)$ and $\omega_j(r)$, we have $m(r) = O(\ell^{2\sigma_2+1})$ for all r , so that the mass of these soliton solutions is also $O(\ell^{2\sigma_2+1})$. On the other hand, the charges (48) can take on large values because

$$\omega_j = [j(N-j)]^{\frac{1}{2}} \left[1 + \sum_{k=2}^N v_{k,j} r^k \alpha_k \ell^{\sigma_k-1} \right] \quad (106)$$

and we have $\sigma_k < 1$. Therefore, for small ℓ , the soliton solutions all have negligibly small mass (this is confirmed by numerical calculations). This is in contrast to the black hole solutions considered in the previous subsection, which have non-negligible mass provided $r_h \gg \ell$. We therefore conclude that the soliton solutions cannot be mistaken for stable black hole solutions by measuring the mass and non-Abelian charges.

7. Conclusions

In this paper we have revisited stable furry black holes in $\mathfrak{su}(N)$ EYM with a negative cosmological constant Λ , examining the consequences for the “no-hair” conjecture. These black holes are “furry” because they have potentially unbounded amounts of stable gauge field hair, corresponding to $N - 1$ gauge degrees of freedom for $\mathfrak{su}(N)$. Bizon’s [8] reformulation of the “no-hair” conjecture states that, within this fixed matter model, stable black holes should be uniquely characterized by their mass and a set of global charges. Our purpose in this paper has been to investigate whether this modified “no-hair” conjecture holds for $\mathfrak{su}(N)$ EYM.

The space of black hole solutions of $\mathfrak{su}(N)$ EYM in anti-de Sitter space is extraordinarily rich. As well as purely magnetic, spherically symmetric, black hole solutions (the focus of our work in this paper), there are also dyonic spherically symmetric black holes with both electric and magnetic charges [11]; static axisymmetric solitons [32] and black holes [33] and rotating black holes [24], as well as a plethora of soliton solutions of each of these classes [24]. Furthermore, there are black holes with non-spherical event horizon topology [19]. These solutions are known only numerically, and, given the complexity of the field equations, deriving general analytic results for all these cases is challenging. For general N , it is known that for sufficiently large $|\Lambda|$, there exist purely magnetic, spherically symmetric, soliton and black hole solutions which are stable to spherically symmetric, linear perturbations [14, 16, 17].

We do not claim to have made a comprehensive study of all families in this zoo of solutions. Instead we have focussed on purely magnetic, static, spherically symmetric black holes. Within this simplified, restricted model, we have been able to present numerical and analytic evidence that black holes which are stable both thermodynamically and under linear, spherically symmetric perturbations, are uniquely determined by their mass M and a set of $N - 1$ non-Abelian global charges Q_j , constructed following [28]. Our analytic argument is based on the first term of an asymptotic series for the metric and gauge field functions, valid in the limit as the adS radius of curvature $\ell \rightarrow 0$. Furthermore, this series is only applicable for a subspace of

the full space of black hole solutions in this limit: in particular, we restricted attention to large black holes with event horizon radius $r_h \gg \ell$ and a (albeit large) subset of the parameter space of values of the gauge field functions on the event horizon.

With these limitations, we have provided evidence that Bizon’s modified “no-hair” conjecture holds for at least this subclass of furry black holes in $\mathfrak{su}(N)$ EYM with $\Lambda < 0$. Of course, it would be of great interest to extend these results to other families of black hole solutions within this matter model, particularly dyonic black holes and rotating black holes. However, we leave these problems for future work.

Acknowledgments

We thank Robert Bartnik for helpful discussions. EW thanks the Universities of Monash, Calgary and the Australian National University Canberra for hospitality while this work was completed. The work of BLS is supported by a studentship from EPSRC (UK). The work of EW is supported by the Lancaster-Manchester-Sheffield Consortium for Fundamental Physics under STFC grant ST/J000418/1, and by the European Cooperation in Science and Technology (COST) action MP0905 “Black Holes in a Violent Universe”.

References

- [1] Ruffini R and Wheeler J 1971 *Phys. Today* **24** 30–41
- [2] Chruściel P T 1994 *Contemp. Maths.* **170** 23–49
Heusler M 1996 *Black Hole Uniqueness Theorems* (Cambridge University Press)
Heusler M 1996 *Helv. Phys. Acta* **69** 501–28
Mazur P O 1987 *Proceedings of the 11th International Conference on General Relativity and Gravitation* 130–157
- [3] Volkov M S and Gal’tsov D V 1999 *Phys. Rept.* **319** 1–83
- [4] Winstanley E 2009 *Lect. Notes Phys.* **769** 49–87
- [5] Bartnik R and McKinnon J 1988 *Phys. Rev. Lett.* **61** 141–4
- [6] Bizon P 1990 *Phys. Rev. Lett.* **64** 2844–7
Kunzle H P and Masood-ul-Alam A K M 1990 *J. Math. Phys.* **31** 928
Volkov M S and Gal’tsov D V 1989 *JETP Lett* **50** 346
Volkov M S and Gal’tsov D V 1990 *Sov. J. Nucl. Phys.* **51** 747
- [7] Straumann N and Zhou Z 1990 *Phys. Lett. B* **237** 353–6
Straumann N and Zhou Z 1990 *Phys. Lett. B* **243** 33–5
Gal’tsov D V and Volkov M S 1992 *Phys. Lett. A* **162** 144–8
Volkov M S and Gal’tsov D V 1995 *Phys. Lett. B* **341** 279–85
Hod S 2008 *Phys. Lett. B* **661** 175–8
- [8] Bizon P 1994 *Acta Phys. Polon. B* **25** 877–898
- [9] Brodbeck O and Straumann N 1994 *Phys. Lett. B* **324** 309–314
Brodbeck O and Straumann N 1996 *J. Math. Phys.* **37** 1414–1433
- [10] Radu E and Tchrakian D H 2011 arXiv:1111.0418 [gr-qc]
- [11] Bjoraker J and Hosotani Y 2000 *Phys. Rev. Lett.* **84** 1853–6
Bjoraker J and Hosotani Y 2000 *Phys. Rev. D* **62** 043513
- [12] Winstanley E 1999 *Class. Quantum Grav.* **16** 1963–78
- [13] Sarbach O and Winstanley E 2001 *Class. Quantum Grav.* **18** 2125–46

- Winstanley E and Sarbach O 2002 *Class. Quantum Grav.* **19** 689–724
- [14] Baxter J E, Helbling M and Winstanley E 2008 *Phys. Rev. Lett.* **100** 011301
- [15] Baxter J E, Helbling M and Winstanley E 2007 *Phys. Rev. D* **76** 104017
- [16] Baxter J E and Winstanley E 2008 *Class. Quantum Grav.* **25** 245014
- [17] Baxter J E and Winstanley E, work in preparation
- [18] Balasubramanian V and Kraus P 1999 *Commun. Math. Phys.* **208** 413–428
- [19] van der Bij J and Radu E 2002 *Phys. Lett. B* **536** 107–13
- [20] Kunzle H P 1991 *Class. Quantum Grav.* **8** 2283–97
- [21] Gal'tsov D V and Volkov M S 1992 *Phys. Lett. B* **274** 173–8
Kleihaus B, Kunz J and Sood A 1995 *Phys. Lett. B* **354** 240–6
Kleihaus B, Kunz J and Sood A 1998 *Phys. Lett. B* **418** 284–93
Kleihaus B, Kunz J, Sood A and Wirschins M 1998 *Phys. Rev. D* **58** 084006
- [22] Bartnik R 1997 *J. Math. Phys.* **38** 3623–38
- [23] Kunzle H P 1994 *Commun. Math. Phys.* **162** 371–97
- [24] Mann R B, Radu E and Tehrastian D H 2006 *Phys. Rev. D* **74** 064015
- [25] Visser M 1992 *Phys. Rev. D* **46** 2445–2451
- [26] Chan H-M and Tsou S T 1980 *Phys. Lett. B* **95** 395
Creighton J D E and Mann R B 1995 *Phys. Rev. D* **52** 4569
Goddard P, Nuyts J and Olive D 1977 *Nucl. Phys. B* **125** 1
Kleihaus B, Kunz J, Sood A and Wirschins M 2002 *Phys. Rev. D* **65** 061502
Sudarsky D and Wald R M 1992 *Phys. Rev. D* **46** 1453
Tafel J and Trautman A 1983 *J. Math. Phys.* **24** 1087
- [27] Brandt R A and Neri F 1981 *Nucl. Phys. B* **186** 84
- [28] Chruściel P T and Kondraki W 1987 *Phys. Rev. D* **36** 1874–1881
- [29] Lai C, Oh C H and Soo C P 1987 *Phys. Rev. D* **36** 2532
- [30] Corichi A and Sudarsky D 2000 *Phys. Rev. D* **61** 101501
Kleihaus B, Kunz J and Navarro-Lerida F 2002 *Phys. Rev. D* **66** 104001
- [31] Breitenlohner P, Maison D and Lavrelashvili G V 2004 *Class. Quantum Grav.* **21** 1667–84
- [32] Radu E 2002 *Phys. Rev. D* **65** 044005
- [33] Radu E and Winstanley E 2004 *Phys. Rev. D* **70** 084023