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ON THE TORSION HOMOLOGY OF NON-ARITHMETIC
HYPERBOLIC TETRAHEDRAL GROUPS

MEHMET HALUK ŞENGÜN

Departament d'Àlgebra i Geometria, Facultat de Matemàtiques
Universitat de Barcelona, Gran Via de les Corts Catalanes 585
Barcelona, 08007, Spain
mehmet.sengun@uni-due.de

Numerical data concerning the growth of torsion in the first homology of congruence subgroups of non-arithmetic hyperbolic tetrahedral groups are collected. The data provide support the speculations of Bergeron and Venkatesh on the growth of torsion homology and the regulators for lattices in $\mathrm{SL}_2(\mathbb{C})$.

Keywords: growth of torsion homology; non-arithmetic lattices; hyperbolic three-manifolds

1. Introduction

In this note, I report on my computations related to the torsion in the first homology of certain non-arithmetic lattices in $\mathrm{SL}_2(\mathbb{C})$. The motivation for these computations is the recent paper of Bergeron and Venkatesh [2] on the size of the torsion in the homology of cocompact arithmetic lattices in semisimple Lie groups. For $\mathrm{SL}_2(\mathbb{C})$, they have the following result.

Let Γ be a lattice in $\mathrm{SL}_2(\mathbb{C})$. Let $\mathcal{O}$ be the ring obtained from the ring of integers of the invariant trace field of Γ by inverting the minimal set of of prime ideals which ensures that $\Gamma \subset \mathrm{SL}_2(\mathcal{O})$. A subgroup of Γ is called a congruence subgroup if it contains the kernel of the reduction map $\mathrm{SL}_2(\mathcal{O}) \to \mathrm{SL}_2(\mathcal{O}/\mathbb{I})$ modulo some ideal $\mathbb{I}$ of $\mathcal{O}$. Let $E_k(\mathcal{O})$ denote the space of homogeneous polynomials over $\mathcal{O}$ of degree k in two variables. There is a natural action of $\mathrm{SL}_2(\mathcal{O})$ on this space. Let

$$E_{k,\ell} := E_k(\mathcal{O}) \otimes_{\mathcal{O}} \overline{E_\ell(\mathcal{O})}$$

where the overline on the second factor means that the action of $\mathrm{SL}_2(\mathcal{O})$ is twisted by complex conjugation. Note that $E_{0,0}$ is of rank one with trivial action.

For a pair of non-negative integers k, ℓ, define

$$c_{k,\ell} := \frac{1}{6\pi} \times \frac{1}{2^3} \left\{ (k + \ell + 2)^3 - |k - \ell|^3 + 3(k + \ell + 2)(k - \ell)(k + \ell + 2 - |k - \ell|) \right\}.$$

Note that $c_{0,0} = \frac{1}{6\pi}$.

Theorem 1.1. *(Bergeron-Venkatesh) Let Γ be a cocompact arithmetic lattice in $SL_2(\mathbb{C})$. Let $\{\Gamma_n\}_{n \geq 1}$ be a decreasing tower of congruence subgroups of Γ such that $\bigcap_n \Gamma_n = \{1\}$. Let X_n denote the 3-fold $\Gamma_n \backslash \mathbb{H}$ where $\mathbb{H}$ is the hyperbolic 3-space. Then for nonnegative integers $k \neq \ell$, we have*

$$\lim_{n \to \infty} \frac{\log |H_1(\Gamma_n, E_{k,\ell})_{tor}|}{vol(X_n)} = c_{k,\ell},$$

where $E_{k,l}$ and $c_{k,\ell}$ are defined as above.

The goal of this note is to report my experiments on the asymptotic behavior of torsion for non-arithmetic lattices. More precisely, I computationally investigated the limit

$$\lim_{n \to \infty} \frac{\log |H_1(\Gamma_n, \mathbb{Z})_{tor}|}{vol(X_n)} \tag{1.1}$$

for families of congruence subgroups $\{\Gamma_n\}$ of non-arithmetic hyperbolic tetrahedral groups (defined below). Note that $\mathbb{Z} \simeq E_{0,0}(\mathbb{Z})$ in the notation of the theorem. The data I collected show that if one considers only the projective covers with vanishing first Betti number (that is, $\dim H_1(\Gamma_n, \mathbb{Q}) = 0$), then the limit (1) tends to $1/(6\pi)$. In the case of positive first Betti numbers, the ratios "log(torsion)/volume" get much smaller than $1/(6\pi)$. These observations support the general philosophy of Bergeron and Venkatesh as discussed in Sections 4 and 6. Moreover, the data show that it is extremely rare that residue degree one prime level projective covers of non-arithmetic hyperbolic 3-folds have positive first Betti number. This has been observed by Calegari and Dunfield in [5] in the case of twist-knot orbifolds.

2. Hyperbolic Tetrahedral Groups

A hyperbolic tetrahedral group is the index two subgroup consisting of orientation-preserving isometries in the discrete group generated by reflections in the faces of a hyperbolic tetrahedron whose dihedral angles are submultiples of π. Lannér [11] proved in 1950 that there are 32 such hyperbolic tetrahedra.

Consider a tetrahedron in the hyperbolic 3-space $\mathbb{H}$ with vertices A, B, C, D with its dihedral angles submultiples of π. If the dihedral angles along the edges AB, AC, BC, DC, DB, DA are π/λ_1, π/λ_2, π/λ_3, π/μ_1, π/μ_2, π/μ_3 respectively, then we denote the tetrahedron with $T(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3)$. A presentation for the tetrahedral group Γ associated to this tetrahedron is

$$\Gamma = \langle a, b, c, | \, a^{\mu_1} = b^{\mu_2} = c^{\lambda_3} = (ca)^{\lambda_2} = (cb^{-1})^{\lambda_1} = (ab)^{\mu_3} = 1 \rangle.$$

For my computations explicit realizations of these groups in $\mathrm{PSL}_2(\mathbb{C})$ are needed. Note that by Mostow's Rigidity Theorem, up to conjugation, such a realization will be unique. A general method to produce a realization is described to me by Grant Lakeland in [10].

It is well known that there are 7 non-arithmetic tetrahedral groups and only one of them is cocompact. Moreover, one of the non-cocompact ones is an index two subgroup of another. As nature of the experiment is insensitive to commensurability, I consider only the larger of these two tetrahedral groups. Thus I work with only 6 groups. Perhaps it should be noted that all the non-cocompact arithmetic tetrahedral groups are commensurable with one of the Bianchi groups $\mathrm{PSL}_2(\mathbb{Z}[\omega])$ with ω a 4th or 6th root of unity. These two Bianchi groups are covered with the computations in [13].

I will now go over each of the 6 groups, starting with the cocompact one. The volumes of the associated tetrahedra can be found in section 10.4 of the book [6] or the article [8]. Note that the volume of the 3-fold given by the tetrahedral group is twice the volume of the associated tetrahedron.

2.1. *Descriptions of the groups*

2.1.1. $H(1)$

Let $H(1)$ be the tetrahedral group attached to the Coxeter symbol
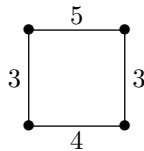
The tetrahedron associated to the Coxeter symbol can be described as $T(5, 3, 2; 4, 3, 2)$. The volume of the tetrahedron is $\simeq 0.3586534401$. A presentation can be given as (see [7])

$$H(1) = \langle a, b, c \mid a^3 = b^2 = c^5 = (bc^{-1})^3 = (ac^{-1})^2 = (ab)^4 = 1 \rangle.$$

where

$$a := \begin{pmatrix} \frac{2t^3+t^2+t+2}{5} & 1 \\ \frac{-t^3+t^2-2}{5} & \frac{-2t^3-t^2-t+3}{5} \end{pmatrix}, \quad b := \begin{pmatrix} \frac{-3t^3+t^2-4t+2}{5} & \beta \\ \alpha & \frac{3t^3-t^2+4t-2}{5} \end{pmatrix}, \quad c := \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$$

where t is a primitive 10-th root of unity and α is one of the two complex roots of the polynomial

$$x^4 + \frac{-6t^3+6t^2+8}{5}x^3 + \frac{-t^3+t^2-3}{5}x^2 + \frac{-4t^3+4t^2+2}{25}x + \frac{3t^3-3t^2+2}{25}.$$

Moreover,

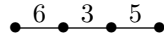$$\beta = (-20t^3+20t^2+35)\alpha^3 + (-50t^3+50t^2+80)\alpha^2 + (9t^3-9t^2-17)\alpha - 4t^3+4t^2+6.$$

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2.1.2. $H(2)$

Let $H(2)$ be the tetrahedral group attached to the Coxeter symbol

$$\bullet \!\!\overset{6}{\rule{1.5em}{0.4pt}}\!\! \bullet \!\!\overset{3}{\rule{1.5em}{0.4pt}}\!\! \bullet \!\!\overset{5}{\rule{1.5em}{0.4pt}}\!\! \bullet$$

The tetrahedron associated to the Coxeter symbol can be described as $T(5, 2, 2 \ ; 6, 2, 3)$. It has one ideal vertex. The volume of the tetrahedron is $\simeq$ 0.1715016613. A presentation is:

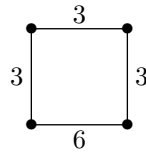$$H(2) = \langle a, b, c, | \, a^6 = b^2 = c^2 = (ca)^2 = (cb^{-1})^5 = (ab)^3 = 1\rangle,$$

where

$$a := \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad b := \begin{pmatrix} i & -(\frac{1+\sqrt{5}}{2})i \\ 0 & -i \end{pmatrix}, \quad c := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Here $\zeta = e^{i(\pi/6)}$.

2.1.3. $H(3)$

Let $H(3)$ be the tetrahedral group given by the Coxeter symbol

The tetrahedron associated to the Coxeter symbol can be described as $T(3, 3, 2 \ ; 6, 3, 2)$. It has two ideal vertices. The volume of the tetrahedron is $\simeq$ 0.3641071004. A presentation is

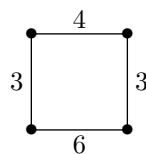$$H(3) = \langle a, b, c, | \, a^6 = b^3 = c^2 = (ca)^3 = (cb^{-1})^3 = (ab)^2 = 1\rangle,$$

where

$$a := \begin{pmatrix} \zeta & i \\ 0 & \zeta^{-1} \end{pmatrix}, \quad b := \begin{pmatrix} \zeta^2 & -i \\ 0 & \zeta^{-2} \end{pmatrix}, \quad c := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Here $\zeta = e^{i(\pi/6)}$.

2.1.4. $H(4)$

Let $H(4)$ be the tetrahedral group given by the Coxeter symbol

The tetrahedron associated to the Coxeter symbol can be described as $T(4, 3, 2 ; 6, 3, 2)$. It has two ideal vertices. The volume of the tetrahedron is $\simeq 0.5258402692$. A presentation is

$$H(4) = \langle a, b, c, \mid a^6 = b^3 = c^2 = (ca)^3 = (cb^{-1})^4 = (ab)^2 = 1 \rangle,$$
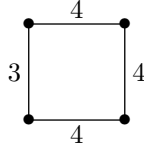
where

$$a := \begin{pmatrix} \zeta & i \\ 0 & \zeta^{-1} \end{pmatrix}, \quad b := \begin{pmatrix} \zeta^2 & -\sqrt{-2} \\ 0 & \zeta^{-2} \end{pmatrix}, \quad c := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Here $\zeta = e^{i(\pi/6)}$.

2.1.5. $H(5)$

Let $H(5)$ be the tetrahedral group given by the Coxeter symbol

The tetrahedron associated to the Coxeter symbol can be described as $T(3, 4, 2 ; 4, 4, 2)$. It has two ideal vertices. The volume of the tetrahedron is $\simeq 0.5562821156$. A presentation is

$$H(5) = \langle a, b, c, \mid a^4 = b^4 = c^2 = (ca)^4 = (cb^{-1})^3 = (ab)^2 = 1 \rangle,$$

where

$$a := \begin{pmatrix} \zeta & \sqrt{-2} \\ 0 & \zeta^{-1} \end{pmatrix}, \quad b := \begin{pmatrix} \zeta & -i \\ 0 & \zeta^{-1} \end{pmatrix}, \quad c := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Here $\zeta = e^{i(\pi/8)}$.

2.1.6. $H(6)$

Let $H(6)$ be the tetrahedral group given by the Coxeter symbol

The tetrahedron associated to the Coxeter symbol can be described as $T(5, 3, 2 ; 6, 3, 2)$. It has two ideal vertices. The volume of the tetrahedron is $\simeq 0.6729858045$. A presentation is

$$H(6) = \langle a, b, c, \mid a^6 = b^3 = c^2 = (ca)^3 = (cb^{-1})^5 = (ab)^2 = 1 \rangle,$$
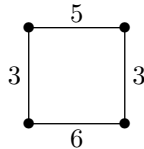
where

$$a := \begin{pmatrix} \zeta & i \\ 0 & \zeta^{-1} \end{pmatrix}, \quad b := \begin{pmatrix} \zeta^2 & -(\frac{1+\sqrt{5}}{2})i \\ 0 & \zeta^{-2} \end{pmatrix}, \quad c := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Here $\zeta = e^{i(\pi/6)}$.

3. Projective Covers

I will now describe the kind of subgroups that will be used for the experiment. Let Γ be a tetrahedral group as discussed above. Let K be the number field generated by the entries of the generators of Γ given in Section 2.1 and let $\mathcal{O}$ be the ring of integers of K. If S denotes the set of finite primes that appear in the denominators of the entries of the generators, then there is an inclusion

$$\Gamma \subset \mathrm{PSL}_2(\mathcal{O}_S)$$

where $\mathcal{O}_S$ is the localization of $\mathcal{O}$ by the primes in S. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ not in S and over the rational prime p. Let $\mathfrak{P}$ be the prime ideal of $\mathcal{O}_S$ corresponding to $\mathfrak{p}$. As the ideal is prime, the residue ring $\mathbb{F}$ is a finite field of characteristic p. We say that $\mathfrak{p}$ is of *residue degree a* if $\mathbb{F} \simeq \mathbb{F}_{p^a}$. Composing the above embedding with the reduction modulo $\mathfrak{P}$ map gives us a homomorphism

$$\phi: \ \Gamma \longrightarrow \mathrm{PSL}_2(\mathbb{F}).$$

By Strong Approximation Theorem (see Theorem 3.2. of [9]), this map is a surjection for almost all primes $\mathfrak{p}$. Let B denote the Borel subgroup of upper triangular elements in $\mathrm{PSL}_2(\mathbb{F})$. The elements in Γ which land in B under ϕ form a subgroup that we shall denote with $\Gamma_0(\mathfrak{p})$.

It is well known that the coset representatives of $\Gamma_0(\mathfrak{p})$ in Γ can be identified with the projective line $\mathbb{P}^1(\mathbb{F})$. With this in mind, we will call $\Gamma_0(\mathfrak{p})$ the *projective cover of level $\mathfrak{p}$* of Γ.

4. More Background for the Experiment

In this section I will *loosely* discuss some aspects of the work of Bergeron and Venkatesh. Any error is due solely to me.

In their recent paper [2], Bergeron and Venkatesh study the growth of torsion in the homology of cocompact arithmetic groups. Their strongest result is in the case where the ambient Lie group is $\mathrm{SL}_2(\mathbb{C})$. In this case one only needs to consider the first homology. Let us specialize our discussion to trivial coefficients (see [2] p.45). Let M be a hyperbolic 3-manifold of finite volume. The principal quantity of interest is the product

$$|H_1(M, \mathbb{Z})_{tor}| \cdot R(M)$$

where $R(M)$ is the regulator. When M is compact, the regulator $R(M)$ can be given explicitly as $\left| \left(\int_{\gamma_i} \omega_j \right)_{i,j} \right|^{-2}$ where $\omega_1, \ldots, \omega_n$ is an L^2-basis of harmonic 1-forms and $\gamma_1, \ldots, \gamma_n$ is a basis of $H_1(M, \mathbb{Z})$. In the non-compact case, harmonic 2-forms and $H_2(M, \mathbb{Z})$ also get involved in $R(M)$.

The proof of the main theorem of [2] suggests that given a collection of finite covers M_n of M that is locally converging to the hyperbolic 3-space $\mathbb{H}$ (in the sense of [1]), we should have

$$\frac{\log R(M_n)}{vol(M_n)} + \frac{\log |H_1(M_n, \mathbb{Z})_{tor}|}{vol(M_n)} \longrightarrow \frac{1}{6\pi}.$$

Note that $1/(6\pi)$ is the L^2-analytic torsion of $\mathbb{H}$.

Hence to study the growth of the torsion, we need to understand that of the regulator. For those covers M_n with the first Betti number (i.e. the dimension of $H_1(M_n, \mathbb{Q})$) equal to zero, the regulator is 1 and thus the ratio "log(torsion)/volume" should converge to $1/(6\pi)$ along such M_n. When the first Betti number is positive, the regulator comes into the play and one is faced with the problem of understanding the growth of the regulator. Extensive numerical data [13] show that in the case of arithmetic Kleinian groups, the ratio "log(torsion)/volume" converges to $1/(6\pi)$ regardless the first Betti numbers are zero or not. Bergeron and Venkatesh believe that this has to do with the existence of Hecke operators acting on the homology (see the discussion on p.45 of [2]). In fact, they conjecture that the ratio "log(regulator)/volume" goes to zero in the arithmetic case.

5. The Experiment

The method of my computations is essentially the same with the one I employed for the congruence subgroups of the Euclidean Bianchi groups in [13]. Namely, I compute the abelianization of the group instead of its first homology with trivial $\mathbb{Z}$-coefficients.

Let us fix one of our non-arithmetic tetrahedral groups above, call it Γ. Let K be the number field generated by the entries of the generators of Γ given in Section 2.1 and let $\mathcal{O}$ be the ring of integers of K. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ which does not divide the ideal generated by the denominators of the entries of the matrices. Then we can reduce the matrices modulo $\mathfrak{p}$ as the reductions of the denominators are invertible in the residue field $\mathbb{F} := \mathcal{O}/\mathfrak{p}$. The group $\Gamma_0(\mathfrak{p})$, defined in Section 3, has finite index in Γ and a set of coset representatives can be identified with the projective line $\mathbb{P}^1(\mathbb{F})$. The generators of Γ act as permutations on the set of coset representatives. The corresponding action on the projective line is the usual action of the matrices, after reducing the entries modulo $\mathfrak{p}$. After the presentation of Γ is given to a symbolic algebra program (I used Magma [3]), one can compute the action of the generator matrices on the projective line. The resulting permutations uniquely determine the subgroup $\Gamma_0(\mathfrak{p})$. Now computing the abelian quotient invariants is a standard functionality.

I have performed two main sets of computations for each of the six tetrahedral groups above. In the first set, I computed the ratios "log(torsion)/volume" for projective covers with prime level $\mathfrak{p}$ over a rational prime $400 < p$ with norm $N(\mathfrak{p}) \le 50000$. In this way, one captures only prime levels of residue degree one. In

Group	A	B	Group	A	B
$H(1)$	50000	400	$H(4)$	66000	257
$H(2)$	150000	400	$H(5)$	63000	257
$H(3)$	94200	307	$H(6)$	55000	233

Table 1. description of the ranges of the second computations

a second set, I aimed at prime levels with residue degree at least two. Table 1 gives the ranges A, B such that $N(\mathfrak{p}) \leq A$ and $p \leq B$ for this second set of computations for each group. In addition, I performed an extra set of computations for the groups $H(1)$ and $H(2)$ where I considered one prime level over each rational prime p which splits completely in the coefficient field attached to the given matrix realizations of these groups with $p \leq 90000$ and $p \leq 100000$ respectively.

6. Results of the Experiment and Observations

The complete data collected by the experiments are provided on my website [14]. Table 2 gives sample data for projective covers of prime level $\mathfrak{p}$ of the tetrahedral group $H(2)$. Here p is the rational prime that $\mathfrak{p}$ is over, $f(\mathfrak{p})$ is the residue degree of $\mathfrak{p}$, $N(\mathfrak{p})$ is the norm of $\mathfrak{p}$ (which is equal to $p^{f(\mathfrak{p})}$) and rank denotes the first Betti number of the associated 3-fold M.

It is interesting that %99.9 of the time the homology of a projective cover of residue degree one prime level has first Betti number equal to zero. For each of the six tetrahedral groups, I have computed approximately 5000 projective covers whose levels are prime with residue degree one. For each group, there were no more than 12 such covers with positive first Betti number, all of them with norm less than 200. Note that a similar paucity was observed by Calegari ad Dunfield [5] among the projective covers of twist knot orbifolds.

An immediate observation is that if we let Γ run only through the projective covers of G with first Betti number equal to *zero*, then the data strongly suggest that the limit (1) tends to $1/(6\pi) \simeq 0.053051647697298$. Such a growth was observed in [13] for the Euclidean Bianchi groups also, *but* regardless the first Betti number was zero or not. The convergence here seems to be faster than it was in [13].

We see that the behaviour is different in the cases of positive first Betti number. Here the ratios "log(torsion)/volume" get much smaller than $1/(6\pi)$. This suggests, under the general philosophy of Bergeron and Venkatesh, that the contribution of the regulator comes into play and decreases the contribution of the torsion. Remember that we are in the nonarithmetic case; there is no suitable infinite family of operators acting on the homology. This fact and the numerically observed contribution of the regulator are in favor of the validity of the speculations of Bergeron and Venkatesh on the general behaviour of the regulator.

Finally let me mention that I have been informed by the referee that upcoming paper [1] will show that the projective covers of Bianchi groups that I used in the

p	$f(\mathfrak{p})$	$N(\mathfrak{p})$	rank	log(torsion)/volume
149	2	22201	35	0.0022345997342923510246048917312...
157	2	24649	26	0.0012780079774939007014661120604...
163	2	26569	27	0.0011938873765947018146386110752...
179	2	32041	42	0.0028428945188512770764299178749...
179	2	32041	42	0.0027167589664083367808089498256...
191	2	36481	44	0.0028752360760017488846489518998...
191	2	36481	44	0.0029860204455915929982470840810...
193	2	37249	32	0.0009322818797127153757082245467...
197	2	38809	0	0.0538609569585085307702656919476
223	2	49729	37	0.0006339902340090512255204262310...
227	2	51529	0	0.0536582778982876893457516688742
233	2	54289	0	0.0533440524815412915499350425341
239	2	57121	56	0.0020485849346806113523095171371...
239	2	57121	56	0.0019778304903093237056534641587...
251	2	63001	58	0.0021391608853217776479337755163...
251	2	63001	58	0.0020750099794556671601351808966...
257	2	66049	0	0.0531408013299331295181799818203
269	2	72361	63	0.0010680977560234575793395952832...
277	2	76729	46	0.0004273784613047699177828682750...
281	2	78961	65	0.0011520638656404416378199109539...
283	2	80089	47	0.0004614725659989436045710261590...
307	2	94249	51	0.0003971774982145844853470545874...
311	2	96721	72	0.0016151951874606839952292550647...
311	2	96721	72	0.0016569812896026650641927489775...
313	2	97969	52	0.0003626214282744142379445363464...
317	2	100489	0	0.0530245496689115095907521079650
337	2	113569	56	0.0003166047979019520447065980395...
347	2	120409	0	0.0533616983192263459765735397830
353	2	124609	0	0.0533104721769799189934434296317
359	2	128881	84	0.0012712517097640717340213100102...
359	2	128881	84	0.0013026109016712169750346947981...
367	2	134689	61	0.0002556480925464262628635682232...
373	2	139129	62	0.0002626940006779962236741765516...
383	2	146689	0	0.0531817473569159738385919121240...

Table 2. sample data on growth of torsion for projective covers of H(2)

experiments of [13] form a collection that locally converges to the identity. While it is expected that the projective covers of non-arithmetic hyperbolic tetrahedral groups that I used for the experiments of this note also form a collection that locally

converges to the identity, I do not know how to show this.

Acknowledgments

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