ON LIFTING AND MODULARITY OF REDUCIBLE RESIDUAL GALOIS REPRESENTATIONS OVER IMAGINARY QUADRATIC FIELDS

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ABSTRACT. In this paper we study deformations of mod p Galois representations τ (over an imaginary quadratic field F) of dimension 2 whose semi-simplification is the direct sum of two characters τ_1 and τ_2. As opposed to [BK13] we do not impose any restrictions on the dimension of the crystalline Selmer group $H^1_\Sigma(F, \mathrm{Hom}(\tau_2, \tau_1)) \subset \mathrm{Ext}^1(\tau_2, \tau_1)$. We establish that there exists a basis $\mathcal{B}$ of $H^1_\Sigma(F, \mathrm{Hom}(\tau_2, \tau_1))$ arising from automorphic representations over F (Theorem 8.1). Assuming among other things that the elements of $\mathcal{B}$ admit only finitely many crystalline characteristic 0 deformations we prove a modularity lifting theorem asserting that if τ itself is modular then so is its every crystalline characteristic zero deformation (Theorems 8.2 and 8.5).

1. INTRODUCTION

Let p be an odd prime. Let F be a number field, Σ a finite set of primes of F (containing all primes $\mathfrak{p}$ of F lying over p) and G_Σ the Galois group of the maximal extension of F unramified outside Σ. Let E be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $\mathbf{F}$. Let $\tau_1, \tau_2 : G_\Sigma \to \mathrm{GL}_{n_i}(\mathbf{F})$ be two absolutely irreducible non-isomorphic representations with $n_1 + n_2 = n$, which we assume lift uniquely to crystalline representations $\tilde{\tau}_i : G_\Sigma \to \mathrm{GL}_{n_i}(\mathcal{O})$.

The aim of this article is to study deformations of *non-semi-simple* continuous crystalline representations $\tau : G_\Sigma \to \mathrm{GL}_n(\mathbf{F})$ whose semi-simplification is $\tau_1 \oplus \tau_2$ in the case $n = 2$ and F is an imaginary quadratic field. We analyzed this deformation problem in [BK13] under the additional assumption that $H^1_\Sigma(F, \mathrm{Hom}(\tau_2, \tau_1))$ is one-dimensional (which is equivalent to saying that there exists only one such τ up to isomorphism). Here H^1_Σ denotes the subgroup of H^1 consisting of classes unramified outside Σ and crystalline at all $\mathfrak{p} \mid p$. In this paper we do not make any assumption on this dimension. Disposing of the "dim=1" assumption is more than a technicality as in the general case one can no longer expect to be able to identify the universal deformation ring with a Hecke algebra.

This question was studied by Skinner and Wiles for $n = 2$ and totally real fields F in the seminal paper [SW99]. In that paper the authors analyze primes $\mathfrak{q}$ of the (ordinary) universal deformation ring R_τ of τ and prove that they are 'pro-modular' in the sense that the trace of the deformation corresponding to $R_\tau \twoheadrightarrow R_\tau/\mathfrak{q}$ occurs in the Hecke algebra $\mathbf{T}$. In particular no direct identification of R_τ and $\mathbf{T}$ is made.

In this article we take a different approach and work with the reduced universal deformation ring R_τ^{red} and its ideal of reducibility. In the "dim=1"-case, the authors proved (as a consequence of an $R = T$-theorem - Theorem 9.14 in [BK13]) that R_τ^{red} is a finitely generated $\mathbf{Z}_p$-module. In contrast, if $\dim H_\Sigma^1(F, \mathrm{Hom}(\tau_2, \tau_1)) > 1$, while there are only finitely many automorphic representations whose associated Galois representations are deformations of τ, the ring R_τ^{red} may potentially be infinite over $\mathbf{Z}_p$ (Remark 2.15). This is a direct consequence of the existence of linearly independent cohomology classes inside the Selmer group which can be used to construct non-trivial lifts to $\mathrm{GL}_2(\mathbf{F}[[X]])$. The resulting (potentially large) characteristic p components of R^{red} do not arise from automorphic representations and in this paper we will ignore them by considering a certain torsion-free quotient R_τ^0 of R_τ^{red} instead of R_τ^{red} itself. It is however possible that by doing so we are excluding some characteristic p deformations whose traces may be modular in the sense of [CM09] (i.e. arise from torsion Betti cohomology classes).

On the other hand, as opposed to the situation studied in [SW99], over an imaginary quadratic field there are no reducible deformations to characteristic zero which in turn is a consequence of the finiteness of the Bloch-Kato Selmer group $H_\Sigma^1(F, \mathrm{Hom}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes \mathbf{Q}_p/\mathbf{Z}_p)$ (Lemma 2.19), where $\tilde{\tau}_1, \tilde{\tau}_2$ are (unique) lifts to characteristic zero of τ_1 and τ_2 respectively.

While each τ may possess non-modular reducible characteristic p deformations, the situation is complicated further by the fact that in general many τ's do not admit any modular deformations at all (this phenomenon does not arise in the "dim=1" case). Indeed, first note that two extensions in $\mathrm{Ext}_{G_\Sigma}^1(\tau_2, \tau_1)$ define isomorphic representation of G_Σ if and only if they are (non-zero) scalar multiples of each other. In particular, if $\dim_{\mathbf{F}} \mathrm{Ext}_{G_\Sigma}^1(\tau_2, \tau_1) = 1$, then there is a unique non-semi-simple representation of G_Σ with semi-simplification $\tau_1 \oplus \tau_2$. (Similarly, if $\dim_{\mathbf{F}} H_\Sigma^1(F, \mathrm{Hom}(\tau_2, \tau_1)) = 1$ then there exists a unique crystalline such representation.) However, if $\dim_{\mathbf{F}} \mathrm{Ext}_{G_\Sigma}^1(\tau_2, \tau_1) = m$, then there are $\frac{q^m - 1}{q - 1}$ non-isomorphic such representations where $q = \#\mathbf{F}$. This demonstrates that in general not all reducible representations τ can be modular (of a particular level and weight), as the number of such characteristic zero automorphic forms is fixed (in particular it is independent of making a residue field extension). Nevertheless, we are able to prove (see Corollary 4.8) that there exists an $\mathbf{F}$-basis $\mathcal{B}$ of $H_\Sigma^1(F, \mathrm{Hom}(\tau_2, \tau_1))$ arising from modular forms. For this we combine a congruence ideal bound for a Hecke algebra with the upper bound on the Selmer group of $\mathrm{Hom}(\tilde{\tau}_2, \tilde{\tau}_1)$ predicted by the Bloch-Kato conjectures.

Let $R_\tau^{\mathrm{tr},0}$ be the image in R_τ^0 of the subalgebra generated by traces of R_τ^{red} for τ arising from a modular form. As pointed out we can extend the set consisting of τ to a modular basis $\mathcal{B} := \{\tau^1 = \tau, \tau^2, \ldots, \tau^s\}$ of $H_\Sigma^1(F, \mathrm{Hom}(\tau_2, \tau_1))$. Our ultimate goal is to show that it is possible to identify $R_\tau^{\mathrm{tr},0}$ with the quotient $\mathbf{T}_\tau$ of a Hecke algebra $\mathbf{T}$. Here the quotient $\mathbf{T}_\tau$ corresponds to automorphic forms for which there exists a lattice in the associated Galois representation with respect to which the mod p reduction equals τ.

To prove our main modularity lifting theorem (Theorem 8.2) we work under the following two assumptions. On the one hand we assume that the modular basis $\mathcal{B}$ is unique in the sense that any other such consists of scalar multiples of the elements of $\mathcal{B}$. On the other hand we assume that all $\tau \in \mathcal{B}$ admit only finitely many characteristic zero deformations, which in particular implies that the quotient R_τ^0

we define is a finitely generated $\mathbf{Z}_p$-module. The first assumption can be replaced with the assumption that the Bloch-Kato Selmer group $H^1_\Sigma(F, \mathrm{Hom}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes \mathbf{Q}_p/\mathbf{Z}_p)$ is annihilated by p (Theorem 8.5). This second result is in a sense 'orthogonal' to the main results of [BK11] and [BK13], where the same Selmer group is assumed to be cyclic, but of arbitrary finite order.

Our approach relies on simultaneously considering all the deformation problems for representations τ^i ($i = 1, 2, \ldots, s$). As in [BK13] we first study "reducible" deformations via the quotients $R^{\mathrm{tr},0}_{\tau^i}/I^{\mathrm{tr},0}_{\tau^i}$ for the reducibility ideal $I^{\mathrm{tr},0}_{\tau^i}$ of the trace of the universal deformation into $\mathrm{GL}_2(R^0_\tau)$ as defined by Bellaïche and Chenevier. These ideals are the analogues of Eisenstein ideals J_{τ_i} on the Hecke algebra side. To relate $\# \prod_i R^{\mathrm{tr},0}_{\tau^i}/I^{\mathrm{tr},0}_{\tau^i}$ to the order of a Bloch-Kato Selmer group we make use of a lattice construction of Urban (Theorem 1.1 of [Urb01], see Theorem 4.1 in this paper). In fact it is a repeated application of Urban's theorem (on the Hecke side and on the deformation side) that allows us to prove a modularity lifting theorem. We show that when the upper bound on the Selmer group and the lower bound on the congruence ideal agree (which in many cases is a consequence of the Bloch-Kato conjecture), this implies that every reducible deformation which lifts to characteristic zero of every τ^i is modular (cf. section 6). It is here that we make use of the assumption on the 'uniqueness' of $\mathcal{B}$ to be able to use a result of Kenneth Kramer and the authors [BKK14] on the distribution of Eisenstein-type congruences among various residual isomorphism classes of Galois representations (cf. Section 5). Yet another application of Urban's Theorem allows us to prove the existence of a deformation to $\mathrm{GL}_2(R^{\mathrm{tr},0}_\tau)$ and as a consequence to identify $R^{\mathrm{tr},0}_\tau$ with R^0_τ (Theorem 6.2). Using the fact that the ideal of reducibility of R^0_τ is principal (Proposition 7.1) and applying the commutative algebra criterion (Theorem 4.1 in [BK13]) we are finally able to obtain an isomorphism $R^{\mathrm{red}}_\tau \cong \mathbf{T}_\tau$ and thus a modularity lifting theorem (Theorems 8.2 and 8.5).

Throughout the paper we work in a slightly greater generality than necessary for the imaginary quadratic case to stress that our results apply in a more general context if one assumes some standard conjectures. However, in section 8 we gather all the assumptions in the imaginary quadratic case as well as the statements of the main theorems (Theorems 8.1, 8.2 and 8.5) in this context. Hence the reader may refer directly to that section for the precise (self-contained) statements of the main results of the paper in that case.

2. Deformation rings

Let F be a number field and $p > 2$ a prime with $p \nmid \# \mathrm{Cl}_F$ and p unramified in $F/\mathbf{Q}$. Let Σ be a finite set of finite places of F containing all the places lying over p. Let G_Σ denote the Galois group $\mathrm{Gal}(F_\Sigma/F)$, where F_Σ is the maximal extension of F unramified outside Σ. For every prime $\mathfrak{q}$ of F we fix compatible embeddings $\overline{F} \hookrightarrow \overline{F}_\mathfrak{q} \hookrightarrow \mathbf{C}$ and write $D_\mathfrak{q}$ and $I_\mathfrak{q}$ for the corresponding decomposition and inertia subgroups of G_F (and also their images in G_Σ by a slight abuse of notation). Let

E be a (sufficiently large) finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $\mathbf{F}$. We fix a choice of a uniformizer ϖ.

2.1. **Deformations.** Denote the category of local complete Noetherian $\mathcal{O}$-algebras with residue field $\mathbf{F}$ by $\mathrm{LCN}(E)$. Let m be any positive integer. Suppose

$$r : G_\Sigma \to \mathrm{GL}_m(\mathbf{F})$$

is a continuous homomorphism.

We recall from [CHT08] p. 35 the definition of a *crystalline* representation: Let $\mathfrak{p} \mid p$ and A be a complete Noetherian $\mathbf{Z}_p$-algebra. A representation $\rho : D_{\mathfrak{p}} \to \mathrm{GL}_n(A)$ is crystalline if for each Artinian quotient A' of A, $\rho \otimes A'$ lies in the essential image of the Fontaine-Lafaille functor $\mathbf{G}$ (for its definition see e.g. [BK13] Section 5.2.1). We also call a continuous finite-dimensional G_Σ-representation V over $\mathbf{Q}_p$ (short) *crystalline* if, for all primes $\mathfrak{p} \mid p$, $\mathrm{Fil}^0 D = D$ and $\mathrm{Fil}^{p-1} D = (0)$ for the filtered vector space $D = (B_{\mathrm{crys}} \otimes_{\mathbf{Q}_p} V)^{D_\mathfrak{p}}$ defined by Fontaine (for details see [BK13] Section 5.2.1).

Following Mazur we call two representations $\tilde{r}_1, \tilde{r}_2 : G_\Sigma \to \mathrm{GL}_m(A)$ for $A \in \mathrm{LCN}(E)$ such that $r = \tilde{r}_1 = \tilde{r}_2 \pmod{\mathfrak{m}_A}$ *strictly equivalent* if there exists $M \in \ker(\mathrm{GL}_2(A) \to \mathrm{GL}_2(\mathbf{F}))$ such that $\tilde{r}_1 = M\tilde{r}_2 M^{-1}$. A (crystalline) $\mathcal{O}$-*deformation* of r is then a pair consisting of $A \in \mathrm{LCN}(E)$ and a strict equivalence class of continuous representations $\tilde{r} : G_\Sigma \to \mathrm{GL}_m(A)$ that are crystalline at the primes dividing p and such that $r = \tilde{r} \pmod{\mathfrak{m}_A}$, where $\mathfrak{m}_A$ is the maximal ideal of A. (So, in particular we do not impose on our lifts any conditions at primes in $\Sigma \setminus \Sigma_p$.) Later we assume that if $\mathfrak{q} \in \Sigma$, then $\#k_\mathfrak{q} \not\equiv 1 \pmod{p}$, which means that all deformations we consider will trivially be "Σ-minimal". As is customary we will denote a deformation by a single member of its strict equivalence class.

If r has a scalar centralizer then the deformation functor is representable by $R_r \in \mathrm{LCN}(E)$ since crystallinity is a deformation condition in the sense of [Maz97]. We denote the universal crystalline $\mathcal{O}$-deformation by $\rho_r : G_\Sigma \to \mathrm{GL}_m(R_r)$. Then for every $A \in \mathrm{LCN}(E)$ there is a one-to-one correspondence between the set of $\mathcal{O}$-algebra maps $R_r \to A$ and the set of crystalline deformations $\tilde{r} : G_\Sigma \to \mathrm{GL}_m(A)$ of r.

For $j \in \{1, 2\}$ let $\tau_j : G_\Sigma \to \mathrm{GL}_{n_j}(\mathbf{F})$ be an absolutely irreducible continuous representation. Assume that $\tau_1 \not\cong \tau_2$. Consider the set of isomorphism classes of n-dimensional residual (crystalline at all primes $\mathfrak{p} \mid p$) representations of the form:

$$(2.1) \qquad\qquad \tau = \begin{bmatrix} \tau_1 & * \\ & \tau_2 \end{bmatrix} : G_\Sigma \to \mathrm{GL}_n(\mathbf{F}),$$

which are non-semi-simple ($n = n_1 + n_2$).

From now on assume $p \nmid n!$.

Lemma 2.1. *Every representation τ of the form (2.1) has scalar centralizer.*

Proof. This is easy. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.2. **Pseudo-representations and pseudo-deformations.** We next recall the notion of a pseudo-representation (or pseudo-character) and pseudo-deformations (from [BC09] Section 1.2.1 and [Böc11] Definition 2.2.2).

Definition 2.2. Let A be a topological ring and R a topological A-algebra. A (continuous) A-valued pseudo-representation on R of dimension d, for some $d \in \mathbf{N}_{>0}$, is a continuous function $T : R \to A$ such that

(i) $T(1) = d$ and $d!$ is a non-zero divisor of A;

(ii) T is *central*, i.e. such that $T(xy) = T(yx)$ for all $x, y \in R$;

(iii) d is minimal such that $S_{d+1}(T)(x) = 0$, where, for every integer $N \geq 1$, $S_N(T) : R^N \to A$ is given by

$$S_N(T)(x) = \sum_{\sigma \in \mathcal{S}_N} \epsilon(\sigma) T^\sigma(x),$$

where for a cycle $\sigma = (j_1, \ldots j_m)$ we define $T^\sigma((x_1, \ldots x_{d+1})) = T(x_{j_1} \cdots x_{j_m})$, and for a general permutation σ with cycle decomposition $\prod_{i=1}^r \sigma_i$ we let $T^\sigma(x) = \prod_{i=1}^r T^{\sigma_i}(x)$.

In the case when $R = A[G_\Sigma]$ the pseudo-representation T is determined by its restriction to G_Σ (see [BC09] Section 1.2.1) and we will also call the restriction of T to G_Σ a pseudo-representation.

We note that if $\rho : A[G_\Sigma] \to M_n(A)$ is a morphism of A-algebras then $\operatorname{tr} \rho$ is a pseudo-representation of dimension n (see [BC09] Section 1.2.2).

According to [BC09] Section 1.2.1, if $T : R \to A$ is a pseudo-representation of dimension d and A' an A-algebra, then $T \otimes A' : R \otimes A' \to A'$ is again a pseudo-representation of dimension d.

Following [SW99] (see also [Böc11] Section 2.3) we define a pseudo-deformation of $\operatorname{tr} \tau_1 + \operatorname{tr} \tau_2$ to be a pair (T, A) consisting of $A \in \operatorname{LCN}(E)$ and a continuous pseudo-representation $T : G_\Sigma \to A$ such that $T = \operatorname{tr} \tau_1 + \operatorname{tr} \tau_2 \pmod{\mathfrak{m}_A}$, where $\mathfrak{m}_A$ is the maximal ideal of A.

By the sentence following [SW99] Lemma 2.10 (see also [Böc11] Proposition 2.3.1) there exists a universal pseudo-deformation ring $R^{\mathrm{ps}} \in \operatorname{LCN}(E)$ and we write $T^{\mathrm{ps}} : G_\Sigma \to R^{\mathrm{ps}}$ for the universal pseudo-deformation. For every $A \in \operatorname{LCN}(E)$ there is a one-to-one correspondence between the set of $\mathcal{O}$-algebra maps $R^{\mathrm{ps}} \to A$ and the set of pseudo-deformations $T : G_\Sigma \to A$ of $\operatorname{tr} \tau_1 + \operatorname{tr} \tau_2$. Any deformation of a representation τ as in (2.1) gives rise (via its trace) to a pseudo-deformation of $\operatorname{tr} \tau_1 + \operatorname{tr} \tau_2$, so there exists a unique $\mathcal{O}$-algebra map $R^{\mathrm{ps}} \to R_\tau$ such that the trace of the deformation equals the composition of T^{ps} with $R^{\mathrm{ps}} \to R_\tau$.

We write R_τ^{red} for the quotient of R_τ by its nilradical and ρ_τ^{red} for the corresponding universal deformation, i.e. the composite of ρ_τ with $R_\tau \twoheadrightarrow R_\tau^{\mathrm{red}}$. We further write $R_\tau^{\mathrm{tr}} \subset R_\tau^{\mathrm{red}}$ for the closed $\mathcal{O}$-subalgebra of R_τ^{red} generated by the set

$$S := \{\operatorname{tr} \rho_\tau(\operatorname{Frob}_{\mathfrak{q}}) \mid \mathfrak{q} \notin \Sigma\}.$$

Lemma 2.3. *The image of $R^{\mathrm{ps}} \to R_\tau^{\mathrm{red}}$ is R_τ^{tr} and hence R_τ^{tr} is an object in the category* $\operatorname{LCN}(E)$..

Proof. This is clear (cf. [CV03] Theorem 3.11) since R^{ps} is topologically generated by $T(\operatorname{Frob}_{\mathfrak{p}})$ (and R_τ^{tr} is closed). $\square$

2.3. **Selmer groups.** For a crystalline p-adic G_Σ-module M (finitely generated or cofinitely generated over $\mathcal{O}$ - for precise definitions cf. [BK13], section 5) we define the Selmer group $H_\Sigma^1(F, M)$ to be the subgroup of $H_{\mathrm{cont}}^1(F_\Sigma, M)$ consisting of cohomology classes which are crystalline at all primes $\mathfrak{p}$ of F dividing p. Note that we place no restrictions at the primes in Σ that do not lie over p. For more details cf. [loc.cit.].

We are now going to state our assumptions. The role of the first one is to rigidify the problem of deforming the representations τ_j appearing on the diagonal of the

residual representations. The role of the second is to rule out characteristic zero upper triangular deformations.

Assumption 2.4. Assume that $R_{\tau_j} = \mathcal{O}$ and denote by $\tilde{\tau}_j$ the unique lifts of τ_j to $\mathrm{GL}_{n_j}(\mathcal{O})$.

Assumption 2.5 ("Bloch-Kato conjecture")**.** One has the following bound:

$$\#H^1_\Sigma(F, \mathrm{Hom}_\mathcal{O}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_\mathcal{O} E/\mathcal{O}) \leq \#\mathcal{O}/L,$$

for some non-zero $L \in \mathcal{O}$.

Remark 2.6. In applications the constant L will be the special L-value at zero of the Galois representation $\mathrm{Hom}_\mathcal{O}(\tilde{\tau}_2, \tilde{\tau}_1)$ divided by an appropriate period.

For the remainder of this section we will work under the above two assumptions.

2.4. **Ideal of reducibility.** Let A be a Noetherian Henselian local (commutative) ring with maximal ideal $\mathfrak{m}_A$ and residue field $\mathbf{F}$ and let R be an A-algebra. We recall from [BC09] Proposition 1.5.1 the definition of the ideal of reducibility of a (residually multiplicity free) pseudo-representation $T : R \to A$ of dimension n, for which we assume that

$$T = \mathrm{tr}\, \tau_1 + \mathrm{tr}\, \tau_2 \quad \mathrm{mod}\; \mathfrak{m}_A$$

Definition 2.7 ([BC09] Proposition 1.5.1 and Definition 1.5.2)**.** There exists a smallest ideal I of A such that $T \bmod I$ is the sum of two pseudo-characters T_1, T_2 with $T_i = \mathrm{tr}\, \tau_i \bmod \mathfrak{m}_A$. We call this smallest ideal the *ideal of reducibility* of T and denote it by I_T.

Definition 2.8. We will write $I^{\mathrm{ps}} \subset R^{\mathrm{ps}}$ for the ideal of reducibility of the universal pseudo-deformation $T^{\mathrm{ps}} : R^{\mathrm{ps}}[G_\Sigma] \to R^{\mathrm{ps}}$, $I_\tau \subset R_\tau$ for the ideal of reducibility of $\mathrm{tr}\, \rho_\tau : R_\tau[G_\Sigma] \to R_\tau$, $I_\tau^{\mathrm{red}} \subset R_\tau^{\mathrm{red}}$ for the ideal of reducibility of $\mathrm{tr}\, \rho_\tau^{\mathrm{red}} : R_\tau^{\mathrm{red}}[G_\Sigma] \to R_\tau^{\mathrm{red}}$ and I_τ^{tr} for the ideal of reducibility of $\mathrm{tr}\, \rho_\tau^{\mathrm{red}} : R_\tau^{\mathrm{tr}}[G_\Sigma] \to R_\tau^{\mathrm{tr}}$.

Lemma 2.9. *Let I_0 be the smallest closed ideal of R_τ^{tr} containing the set*

$$\{\mathrm{tr}\, \rho_\tau^{\mathrm{red}}(\mathrm{Frob}_v) - \mathrm{tr}\, \tilde{\tau}_1(\mathrm{Frob}_v) - \mathrm{tr}\, \tilde{\tau}_2(\mathrm{Frob}_v) \mid v \notin \Sigma\}.$$

Then I_0 equals the ideal of reducibility $I_\tau^{\mathrm{tr}} \subset R_\tau^{\mathrm{tr}}$.

Proof. By the Chebotarev density theorem we get $\mathrm{tr}\, \rho_\tau^{\mathrm{red}} = \mathrm{tr}\, \tilde{\tau}_1 + \mathrm{tr}\, \tilde{\tau}_2 \pmod{I_0}$, hence $I_0 \supset I_\tau^{\mathrm{tr}}$. Conversely, we know from the definition of the ideal of reducibility that $\mathrm{tr}\, \rho_\tau^{\mathrm{red}} \pmod{I_\tau^{\mathrm{tr}}}$ is given by the sum of two pseudo-characters reducing to $\mathrm{tr}\, \tau_i$. By Assumption 2.4 and Theorems 7.6 and 7.7 of [BK13] (see also [Böc11] Theorem 2.4.1) these two pseudo-characters must equal $\mathrm{tr}\, \tilde{\tau}_i \pmod{I_\tau^{\mathrm{tr}}}$. This shows that $I_\tau^{\mathrm{tr}} \supset I_0$. $\square$

Corollary 2.10. *The quotient $R_\tau^{\mathrm{tr}}/I_\tau^{\mathrm{tr}}$ is cyclic.* $\square$

Remark 2.11. Combined with Lemma 7.11 of [BK13] this shows that for any pseudo-deformation $T : A[G_\Sigma] \to A$ of $\mathrm{tr}\, \tau_1 + \mathrm{tr}\, \tau_2$ with ideal of reducibility I_T for which there is a surjection $R_\tau^{\mathrm{tr}} \to A$, the quotient A/I_T is cyclic.

Proposition 2.12. *The module R_τ/I_τ is a torsion $\mathcal{O}$-module.*

Proof. Fix $\sigma \in \mathbf{Z}_+$ and set $S := R_\tau/I_\tau$. Suppose that S is not torsion. Let $\phi : S \twoheadrightarrow R := S/\varpi^\sigma S$ be the canonical surjection (of $\mathcal{O}$-algebras). Let $A := \phi(\mathcal{O})$. We first claim that $A = \mathcal{O}/\varpi^\sigma \mathcal{O}$. Clearly $\varpi^\sigma = 0$ in $S/\varpi^\sigma S$, so we just need to prove that $\varpi^{\sigma-1} \notin \varpi^\sigma S$. Suppose on the contrary that $\varpi^{\sigma-1} \in \varpi^\sigma S$. Then there exists $s \in S$ such that

$$(2.2) \qquad \varpi^{\sigma-1}(1 - \varpi s) = 0 \quad \text{in } S.$$

Since the residue field of S is $\mathcal{O}/\varpi = \mathbf{F}$, we see that ϖ is not a unit in S, and hence $1 - \varpi s$ is a unit in S. Thus (2.2) implies that $\varpi^{\sigma-1} = 0$ in S, which leads to a contradiction and hence we have proved that $A = \mathcal{O}/\varpi^\sigma \mathcal{O}$.

We now use the following lemma.

Lemma 2.13. *There exists an $\mathcal{O}$-submodule $B \subset R$ such that*

$$R = A \oplus B$$

as $\mathcal{O}$-modules.

Proof. This follows from the following result.

Lemma 2.14 (Lemma 6.8(ii), p.222 in [Hun80])**.** *Let A' be a module over a PID R' such that $p^n A' = 0$ and $p^{n-1} A' \neq 0$ for some prime $p \in R'$ and a positive integer n. Let a be an element of A' of order p^n. Then there is a submodule C' of A' such that $A' = R'a \oplus C'$.*

Apply Lemma 2.14 for $R' = \mathcal{O}$, $A' = R$, $p = \varpi$, $n = \sigma$, $a = \psi(1)$. Then $R'a = A$. $\qquad\square$

We now finish the proof of Proposition 2.12. Let e be an $\mathcal{O}$-module generator of A. Write $\rho_I : G \to \mathrm{GL}_n(R)$ for the deformation corresponding to the canonical map $R_\tau \twoheadrightarrow R$. Then we can write

$$\rho_I = \begin{bmatrix} \tilde{\tau}_1 & \alpha e + \beta \\ & \tilde{\tau}_2 \end{bmatrix},$$

where $\alpha : G \to M_{n_1 \times n_2}(\mathcal{O})$ and $\beta : G \to M_{n_1 \times n_2}(B)$ are maps (here we identify $\tilde{\tau}_j$ with its composition with $\mathcal{O} \to R$). Define

$$\rho_I^+ : G \to \mathrm{GL}_n(A) \quad g \mapsto \begin{bmatrix} \tilde{\tau}_1(g) & \alpha(g)e \\ & \tilde{\tau}_2(g) \end{bmatrix}.$$

We must check that ρ_I^+ is a homomorphism. This follows easily from the fact that ρ_I is a homomorphism and the fact that A is a direct summand of R. Moreover, note that the image of α is not contained in $M_{n_1 \times n_2}(\varpi \mathcal{O})$ because ρ_I reduces to τ which is not semi-simple.

Note that ρ_I^+ is an upper-triangular deformation into $\mathrm{GL}_n(\mathcal{O}/\varpi^\sigma)$. Moreover, since ρ_I^+ reduces to τ, it gives rise to an element in $H^1_\Sigma(F, \mathrm{Hom}_\mathcal{O}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes E/\mathcal{O})$ which generates an $\mathcal{O}$-submodule isomorphic to $\mathcal{O}/\varpi^\sigma$. Since σ was arbitrary we conclude that $H^1_\Sigma(F, \mathrm{Hom}_\mathcal{O}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes E/\mathcal{O})$ must be infinite which contradicts Assumption 2.5. This concludes the proof of Proposition 2.12. $\qquad\square$

Remark 2.15. If $\dim_\mathbf{F} H^1_\Sigma(F, \mathrm{Hom}(\tau_2, \tau_1)) = 1$ then R_τ/I_τ and $R_\tau^{\mathrm{red}}/I_\tau^{\mathrm{red}}$ are cyclic $\mathcal{O}$-modules by Corollary 7.12 in [BK13] which combined with Proposition 2.12 implies finiteness of R_τ/I_τ and $R_\tau^{\mathrm{red}}/I_\tau^{\mathrm{red}}$. On the other hand given that $\dim_\mathbf{F} H^1_\Sigma(F, \mathrm{Hom}(\tau_2, \tau_1)) > 1$ it is easy to construct an upper-triangular (not necessarily crystalline) lift of τ to $\mathbf{F}[[X]]$ which would suggest that in general R_τ/I_τ,

and even $R_\tau^{\mathrm{red}}/I_\tau^{\mathrm{red}}$ (since $\mathbf{F}[[X]]$ is reduced), may have positive Krull dimension. Indeed, to see this, let f be a cohomology class corresponding to τ and let g be a cohomology class linearly independent from f. Then the representation

$$\rho = \begin{bmatrix} \tau_1 & \tau_2(f + gX) \\ 0 & \tau_2 \end{bmatrix}$$

is a non-trivial lift of τ to $\mathrm{GL}_n(\mathbf{F}[[X]])$. In particular there is no guarantee that R_τ^{red} is a finitely generated $\mathcal{O}$-module. Since our method of proving modularity relies on that property we will restrict in the following section to the 'characteristic zero' part of R_τ^{red} of which we will demand that it is finite over $\mathcal{O}$.

2.5. **The ring R_τ^0.** Set $\mathcal{P}(\tau) := \{\mathfrak{p} \in \mathrm{Spec}(R_\tau) \mid R_\tau/\mathfrak{p} = \mathcal{O}\}$. For the rest of this article we assume the following:

Assumption 2.16. Assume that $\mathcal{P}(\tau)$ is finite.

We then define R_τ^0 to be the image of R_τ in $\prod_{\mathfrak{p} \in \mathcal{P}(\tau)} \mathcal{O}$. It is clear that R_τ^0 is a finitely generated $\mathcal{O}$-module and an object in $\mathrm{LCN}(E)$. Note that the canonical surjection $R_\tau \twoheadrightarrow R_\tau^0$ factors through R_τ^{red}. Write ρ_τ^0 for the composition of ρ_τ with the map $\varphi_\tau : R_\tau \twoheadrightarrow R_\tau^0$. Write I_τ^0 for the ideal of reducibility of $\mathrm{tr}\,\rho_\tau^0$. By [BK13], Lemma 7.11, we have $\varphi_\tau(I_\tau) \subset I_\tau^0$ (in fact equality holds since the opposite inclusion is obvious) and thus φ_τ induces a surjection $R_\tau/I_\tau \twoheadrightarrow R_\tau^0/I_\tau^0$.

Lemma 2.17. *The quotient R_τ^0/I_τ^0 is finite.*

Proof. This follows immediately from Proposition 2.12 and the surjectivity of $R_\tau/I_\tau \to R_\tau^0/I_\tau^0$. $\qquad\square$

Define $R_\tau^{\mathrm{tr},0} \subset R_\tau^0$ to be the closed $\mathcal{O}$-subalgebra generated by the set

$$S := \{\mathrm{tr}\,\rho_\tau^0(\mathrm{Frob}_\mathfrak{q}) \mid \mathfrak{q} \notin \Sigma\}.$$

Lemma 2.18. *The image of R_τ^{tr} under $\varphi_\tau : R_\tau \twoheadrightarrow R_\tau^0$ is $R_\tau^{\mathrm{tr},0}$. Thus $R_\tau^{\mathrm{tr},0}$ is an object in the category* $\mathrm{LCN}(E)$.

Proof. It is clear that $R_\tau^{\mathrm{tr},0} \subset \varphi_\tau(R_\tau^{\mathrm{tr}})$. On the other hand $S \subset \varphi_\tau(R_\tau^{\mathrm{tr}})$, so the equality holds because S is dense in $R_\tau^{\mathrm{tr},0}$. $\qquad\square$

We will write $I_\tau^{\mathrm{tr},0} \subset R_\tau^{\mathrm{tr},0}$ for the ideal of reducibility of $\mathrm{tr}\,\rho_\tau^0$. By Lemma 2.18 and Lemma 7.11 in [BK13] we get that $\varphi_\tau(I_\tau^{\mathrm{tr}}) \subset I_\tau^{\mathrm{tr},0}$ (in fact equality holds) and thus φ_τ induces a surjection $R_\tau^{\mathrm{tr}}/I_\tau^{\mathrm{tr}} \twoheadrightarrow R_\tau^{\mathrm{tr},0}/I_\tau^{\mathrm{tr},0}$. By Remark 2.11 the quotient $R_\tau^{\mathrm{tr},0}/I_\tau^{\mathrm{tr},0}$ is a cyclic $\mathcal{O}$-module.

2.6. **Generic irreducibility of ρ_τ^0.**

Lemma 2.19. *For any τ as in (2.1) $\rho_\tau^0 \otimes_{R_\tau^0} \mathcal{F}$ is irreducible. Here $\mathcal{F}$ is any of the fields $\mathcal{F}_s$ in $\mathcal{F}_\tau^0 = \prod_s \mathcal{F}_s$, where $\mathcal{F}_\tau^0$ is the total ring of fractions of R_τ^0.*

Proof. First note that since R_τ^0 is a finitely generated $\mathcal{O}$-module and since E is assumed to be sufficiently large we may assume that all of the fields $\mathcal{F}_s$ are equal to E. If any of the representations $\rho_\tau^0 \otimes_{R_\tau^0} \mathcal{F}$ is reducible write $\rho = \bigoplus_{j=1}^s \rho_j$ for its semi-simplification with each ρ_j irreducible, $j = 1, 2, \ldots, s$. Then by compactness of G_Σ for each $1 \leq j \leq s$ there exists a G_Σ-stable $\mathcal{O}$-lattice inside the representation space of ρ_j. This implies that $\mathrm{tr}\,\rho_j(\sigma) \in \mathcal{O}$ for all $\sigma \in G_\Sigma$ and all $1 \leq j \leq s$. Hence $\mathrm{tr}\,\rho$ splits over $\mathcal{O}$ into the sum of traces of ρ_j. Since ρ_τ^0 is a deformation of τ we

easily conclude that $\rho = \rho_1 \oplus \rho_2$ with ρ_j (with respect to some lattice) being a deformation of τ_j ($j = 1, 2$). Using the fact that ρ_τ^0 is a deformation of τ we now deduce that there is an $\mathcal{O}$-lattice inside the space of $\rho_\tau^0 \otimes_{R_\tau^0} \mathcal{F}$ with respect to which $\rho_\tau^0 \otimes_{R_\tau^0} \mathcal{F}$ is block-upper-triangular (with correct dimensions) and non-semi-simple. When we reduce it modulo ϖ^m, the upper-right shoulder will give rise to an element of order ϖ^m in $H_\Sigma^1(F, \mathrm{Hom}_\mathcal{O}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_\mathcal{O} E/\mathcal{O})$. Since m is arbitrary this contradicts Assumption 2.5. $\square$

3. The rings $\mathbf{T}_\tau$

Let us now define the rings $\mathbf{T}_\tau$ that will correspond to R_τ^0 on the Hecke side.

Proposition 3.1. *If $\rho : G_\Sigma \to \mathrm{GL}_n(E)$ is irreducible and satisfies*

$$\tag{3.1} \overline{\rho}^{\mathrm{ss}} \cong \tau_1 \oplus \tau_2$$

then there exists a lattice inside E^n so that with respect to that lattice the mod ϖ reduction $\overline{\rho}$ of ρ has the form

$$\overline{\rho} = \begin{bmatrix} \tau_1 & * \\ 0 & \tau_2 \end{bmatrix}$$

and is non-semi-simple.

Proof. This is a special case of [Urb01], Theorem 1.1, where the ring $\mathcal{B}$ in [loc.cit.] is a discrete valuation ring $= \mathcal{O}$. $\square$

For each representation τ as in (2.1) let Φ_τ be the set of (inequivalent) characteristic zero deformations of τ, i.e. crystalline at $\mathfrak{p} \mid p$ Galois representations $\rho : G_\Sigma \to \mathrm{GL}_n(\mathcal{O})$ whose reduction equals τ. Also, let $\Phi_{\tau,E}$ be the set of (inequivalent) crystalline at $\mathfrak{p} \mid p$ Galois representations $\rho : G_\Sigma \to \mathrm{GL}_n(E)$ such that there exists a G_Σ-stable lattice L in the space of ρ so that the mod ϖ-reduction of ρ_L equals τ.

The following is a higher-dimensional analogue of Lemma 2.13(ii) from [SW99]:

Proposition 3.2. *One has $\Phi_{\tau,E} \cap \Phi_{\tau',E} = \emptyset$ if $\tau \not\cong \tau'$.*

Proof. Let $\rho : G_\Sigma \to \mathrm{GL}_n(E)$ be a representation such that $\overline{\rho}^{\mathrm{ss}} = \tau_1 \oplus \tau_2$ and let T equal its trace. Suppose there exist two lattices L_i in the representation space of ρ such that the reductions of the corresponding representations ρ_{L_i} are given by τ and τ' with $\tau \not\cong \tau'$ as in (2.1). We now consider the classes c_{L_i} of the cocycles corresponding to $\overline{\rho}_{L_i}$ in $\mathrm{Ext}^1_{\mathbf{F}[G_\Sigma]/\ker T}(\tau_2, \tau_1)$. Using Assumption 2.5 above and Corollary 7.8 in [BK13] we conclude that the quotient $\mathcal{O}/I_T$ is finite. Thus arguing as in the proof of Proposition 1.7.4 in [BC09] but using Proposition 3.1 in [BK13] instead of generic irreducibility of T to conclude that $\ker T = \ker \rho$ (see [BC09], Proof of Proposition 1.7.2, on how this equality - which follows from Proposition 1.6.4 in [loc.cit.] in the generically irreducible case - is used) we obtain that the existence of ρ_{L_i} with trace T and non-split reduction as in (2.1) implies that $\mathrm{Ext}^1_X(\tau_2, \tau_1)$ is 1-dimensional, where $X := (\mathcal{O}[G_\Sigma]/\ker T)/\varpi(\mathcal{O}[G_\Sigma]/\ker T)$.

First note that $X = \mathcal{O}[G_\Sigma]/(\varpi \mathcal{O}[G_\Sigma] + \ker T)$. Secondly one clearly has that $\ker(\mathcal{O}[G_\Sigma] \twoheadrightarrow \mathbf{F}[G_\Sigma]) = \varpi \mathcal{O}[G_\Sigma]$. These two facts imply that the map $\mathcal{O}[G_\Sigma] \to X$ factors through $\mathcal{O}[G_\Sigma] \twoheadrightarrow \mathbf{F}[G_\Sigma]$ and that the kernel of the resulting surjection $\mathbf{F}[G_\Sigma] \twoheadrightarrow X$ equals $(\ker T)\mathbf{F}[G_\Sigma]$. Thus we have $X = \mathbf{F}[G_\Sigma]/(\ker T)\mathbf{F}[G_\Sigma]$, so by the above we conclude that $\mathrm{Ext}^1_{\mathbf{F}[G_\Sigma]/\ker T}(\tau_2, \tau_1)$ is one-dimensional. This means the corresponding representations of $\mathbf{F}[G_\Sigma]/\ker T$ are isomorphic. Since $\ker T =$

$\ker \rho$ (as noted above) the reductions both factor through this quotient of $\mathbf{F}[G_\Sigma]$, and so they are isomorphic as representation of $\mathbf{F}[G_\Sigma]$, in contradiction to our assumption. $\qquad\square$

The following notation will remain in force throughout the paper.

Notation 3.3. Write $\mathfrak{T}$ for the set of isomorphism classes of residual representations of the form (2.1). Set $\Phi = \bigcup_{\tau \in \mathfrak{T}} \Phi_\tau$.

Remark 3.4. Assumption 2.16 that $\mathcal{P}(\tau)$ is finite is equivalent to assuming that the set Φ_τ is a finite set.

We now fix subsets $\Pi_\tau \subset \Phi_\tau$ and $\Pi \subset \Phi$ of deformations. In our later application these will be taken to correspond to all the modular deformations corresponding to cuspforms of a particular weight and level which are congruent to a fixed Eisenstein series. In particular Π_τ may be empty.

Whenever $\Pi_\tau \neq \emptyset$ we obtain an $\mathcal{O}$-algebra map $R_\tau \to \prod_{\rho \in \Pi_\tau} \mathcal{O}$. This induces a map

$$(3.2) \qquad\qquad R_\tau^{\mathrm{tr}} \to \prod_{\rho \in \Pi_\tau} \mathcal{O}.$$

Definition 3.5. We (suggestively) write $\mathbf{T}_\tau$ for the image of the map (3.2) - note that this also depends on the choice of the set Π_τ - and denote the resulting surjective $\mathcal{O}$-algebra map $R_\tau^{\mathrm{tr}} \twoheadrightarrow \mathbf{T}_\tau$ by ϕ_τ. Also we will write $\mathbf{T}$ for the image of $\phi : R^{\mathrm{ps}} \to \prod_{\rho \in \Pi} \mathcal{O}$, where ϕ is induced from the traces of the deformations ρ_π. Finally we will write $J_\tau \subset \mathbf{T}_\tau$ for the ideal of reducibility of the pseudo-representation $\mathbf{T}_\tau \otimes_{R_\tau^{\mathrm{tr}}, \phi_\tau} \mathrm{tr}\, \rho_\tau : \mathbf{T}_\tau[G_\Sigma] \to \mathbf{T}_\tau$ and $J \subset \mathbf{T}$ for the ideal of reducibility of the pseudo-representation $T^{\mathrm{ps}} \otimes_{R^{\mathrm{ps}}, \phi} \mathbf{T} : \mathbf{T}[G_\Sigma] \to \mathbf{T}$.

Lemma 3.6. *The maps $R_\tau^{\mathrm{tr}} \twoheadrightarrow \mathbf{T}_\tau$ and $R^{\mathrm{ps}} \twoheadrightarrow \mathbf{T}$ factor through $R_\tau^{\mathrm{tr},0}$ and the image of R^{ps} inside $R^{\mathrm{ps}} \otimes_{\mathcal{O}} E$ respectively.*

Proof. Clearly the kernel of $R_\tau \to \prod_{\rho \in \Pi_\tau} \mathcal{O}$ contains $\bigcap_{\mathfrak{p} \in \mathcal{P}(\tau)} \mathfrak{p}$. Thus the map $R_\tau \to \prod_{\rho \in \Pi_\tau} \mathcal{O}$ factors through R_τ^0. Then the claim follows since $\varphi_\tau(R_\tau^{\mathrm{tr}}) = R_\tau^{\mathrm{tr},0}$ by Lemma 2.18. $\qquad\square$

Lemma 3.7. *The quotient $\mathbf{T}_\tau / J_\tau$ is cyclic and one has $J_\tau = \phi_\tau(I_\tau^{\mathrm{tr}})$.*

Proof. The first part is a consequence of Lemma 2.9 and was already mentioned in Remark 2.11.

By Lemma 7.11 in [BK13] we know that $J_\tau \supset \phi_\tau(I_\tau^{\mathrm{tr}})$. For the opposite inclusion we argue as follows. We need to show that $\phi_\tau \circ \mathrm{tr}\, \rho_\tau \equiv \Psi_1' + \Psi_2' \mod \phi_\tau(I_\tau^{\mathrm{tr}})$ for Ψ_1', Ψ_2' pseudo-representations.

Put $B = \mathbf{T}$, $A = R_\tau^{\mathrm{tr}}$ and write φ for $\phi_\tau : R_\tau^{\mathrm{tr}} \to \mathbf{T}$ and T_B for $\mathbf{T}_\tau \otimes_{R_\tau^{\mathrm{tr}}, \phi_\tau} \mathrm{tr}\, \rho_\tau$. Let $x \in B[G_\Sigma]$. Since φ is surjective there exists $y \in A[G_\Sigma]$ such that $\varphi(y) = x$. Then by definition of T_B we have $T_B(x) = \varphi \circ T(y) = \varphi(\Psi_1(y) + \Psi_2(y) + i)$ for some pseudo-representations Ψ_1, Ψ_2 and $i \in I_\tau^{\mathrm{tr}}$. Now set $\Psi_j'(x) := \varphi \circ \Psi_j(y)$ for $j = 1, 2$. $\qquad\square$

Corollary 3.8. *One has $J_\tau = \phi_\tau(I_\tau^{\mathrm{tr},0})$.*

Proof. By Lemma 3.6 the map ϕ_τ factors through $R_\tau^{\mathrm{tr},0}$. By abuse of notation we will denote the induced map also by ϕ_τ as in the statement of the Corollary. Then since $\varphi_\tau(I_\tau^{\mathrm{tr}}) = I_\tau^{\mathrm{tr},0}$ (with φ_τ as in section 2.5) we get the corollary. $\qquad\square$

Lemma 3.9. *The quotient* $\mathbf{T}/J$ *is cyclic.*

Proof. For this we prove as in Lemma 2.9 that J is equal to the smallest closed ideal of T generated by the set $\{(\phi \circ T^{\mathrm{ps}})(\mathrm{Frob}_v) - \mathrm{tr}\,\tilde{\tau}_1(\mathrm{Frob}_v) - \mathrm{tr}\,\tilde{\tau}_2(\mathrm{Frob}_v) \mid v \notin \Sigma\}$. We note that Assumption 2.4 can again be applied as Definition 3.5 tells us that ϕ is induced by the traces of the crystalline deformations ρ_π. $\qquad\square$

4. The lattice $\mathcal{L}$ and modular extensions

We will make a frequent use of the following result that is due to Urban [Urb01]. Let $\mathcal{B}$ be a Henselian and reduced local commutative algebra that is a finitely generated $\mathcal{O}$-module. Since $\mathcal{O}$ is assumed to be sufficiently large and $\mathcal{B}$ is reduced we have

$$\mathcal{B} \subset \hat{\mathcal{B}} = \prod_{i=1}^{s} \mathcal{O} \subset \prod_{i=1}^{s} E = \mathcal{F}_{\mathcal{B}},$$

where $\hat{\mathcal{B}}$ stands for the normalization of $\mathcal{B}$ and $\mathcal{F}_{\mathcal{B}}$ for its total ring of fractions. Write $\mathfrak{m}_{\mathcal{B}}$ for the maximal ideal of $\mathcal{B}$. For any finitely generated free $\mathcal{F}_{\mathcal{B}}$-module M, any $\mathcal{B}$-submodule $N \subset M$ which is finitely generated as a $\mathcal{B}$-module and has the property that $N \otimes_{\mathcal{B}} \mathcal{F}_{\mathcal{B}} = M$ will be called a $\mathcal{B}$-*lattice*.

Theorem 4.1 ([Urb01] Theorem 1.1). *Let $\mathcal{R}$ be a $\mathcal{B}$-algebra, and let ρ be an absolutely irreducible representation of $\mathcal{R}$ on $\mathcal{F}_{\mathcal{B}}^n$ (i.e., ρ composed with each of the projections $\mathcal{F}_{\mathcal{B}} \twoheadrightarrow E$ is absolutely irreducible) such that there exist two representations ρ_i for $i = 1, 2$ in $M_{n_i}(\mathcal{B})$ and I a proper ideal of $\mathcal{B}$ such that*

 (i) *the coefficients of the characteristic polynomial of ρ belong to $\mathcal{B}$;*
 (ii) *the characteristic polynomials of ρ and $\rho_1 \oplus \rho_2$ are congruent modulo I;*
 (iii) *$\overline{\rho}_1 := \rho_1 \bmod \mathfrak{m}_{\mathcal{B}}$ and $\overline{\rho}_2 := \rho_2 \bmod \mathfrak{m}_{\mathcal{B}}$ are absolutely irreducible;*
 (iv) *$\overline{\rho}_1 \not\cong \overline{\rho}_2$.*

Then there exist an $\mathcal{R}$-stable $\mathcal{B}$-lattice $\mathcal{L}$ in $\mathcal{F}_{\mathcal{B}}^n$ and a $\mathcal{B}$-lattice $\mathcal{T}$ of $\mathcal{F}_{\mathcal{B}}$ such that we have the following exact sequence of $\mathcal{R}$-modules:

$$0 \to \rho_1 \otimes_{\mathcal{B}} \mathcal{T}/I\mathcal{T} \to \mathcal{L} \otimes_{\mathcal{B}} \mathcal{B}/I \to \rho_2 \otimes_{\mathcal{B}} \mathcal{B}/I \to 0$$

which splits as a sequence of $\mathcal{B}$-modules. Moreover, $\mathcal{L}$ has no quotient isomorphic to $\overline{\rho}_1$.

Since we will not only use Theorem 4.1 itself but also the construction of the lattice $\mathcal{L}$ let us briefly summarize how $\mathcal{L}$ is built (for details cf. [loc.cit.], p. 490-491). Let ρ_i be the composition of the representation ρ with the projection $\mathcal{B} \twoheadrightarrow \mathcal{O}$ onto the ith component of $\hat{\mathcal{B}}$. Urban shows that we can always conjugate ρ_i (over E) so that the mod ϖ-reduction of (the conjugate of ρ_i which we will from now on denote by) ρ_i has the form

$$(4.1) \qquad\qquad \begin{bmatrix} \overline{\rho}_1 & * \\ 0 & \overline{\rho}_2 \end{bmatrix}.$$

Set $\rho_{\hat{\mathcal{B}}} := (\rho_i)_i$. It is also shown in [loc.cit.] that the matrices $\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix}$ are in the image of $\rho_{\hat{\mathcal{B}}}$. One then defines the lattice $\mathcal{L}$ to be the $\mathcal{B}$-submodule of $\hat{\mathcal{B}}^n$ generated by $\rho(r)^t \begin{bmatrix} 0, 0, \ldots, 0, 1 \end{bmatrix}$, where r runs over $\mathcal{R}$ and set $\mathcal{L}^1 := \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \mathcal{L}$ and $\mathcal{L}^2 := \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} \mathcal{L}$.

Let $\mathfrak{T}$ be as in Notation 3.3. Let $\mathbf{T}$, $\Pi \subset \Phi$ and $\Pi_\tau \subset \Phi_\tau$ be as in section 3. Write $\tilde{\mathbf{T}} = \prod_{\pi \in \Pi} \mathcal{O}$ for the normalization of $\mathbf{T}$. Let ρ in Theorem 4.1 be $\rho_\Pi = \prod_{\tau \in \mathfrak{T}} \prod_{\rho_\pi \in \Pi_\tau} \rho_\pi = \prod_{\pi \in \Pi} \rho_\pi$ and $\rho_i = \tilde{\tau}_i$, $i = 1, 2$, where $\tilde{\tau}_i : G_\Sigma \to \mathrm{GL}_{n_i}(\mathcal{O})$ is a fixed crystalline deformation of τ_i which we from now on assume exists. (If one works under Assumption 2.4, then the $\tilde{\tau}_i$'s are unique, but we do not need this uniqueness for the arguments of this section.) Note that the reduction of ρ_π already has the form (4.1), so we can take $\rho_{\hat{\mathcal{B}}} = \rho_\Pi$ and define lattices $\mathcal{L}$, $\mathcal{L}^1$ and $\mathcal{L}^2$ as above (with $\mathcal{B} = \mathbf{T}$, $\mathcal{R} = \mathbf{T}[G_\Sigma]$). The G_Σ-action on $\mathcal{L}$ is then via restriction of ρ_Π to $\mathcal{L}$. Write $\mathfrak{m} = \mathfrak{m}_{\mathbf{T}}$ for the maximal ideal of the local ring $\mathbf{T}$ and let J as in section 3 be its reducibility ideal.

By Theorem 4.1 (and Lemmas 1.1 and 1.5(ii) in [Urb01]) there exists a $\mathbf{T}$-lattice $\mathcal{T}$ and a short exact sequence of $\mathbf{T}[G_\Sigma]$-modules (which splits as a sequence of $\mathbf{T}$-modules):

$$(4.2) \qquad 0 \to \mathcal{L}^1 \otimes_{\mathbf{T}} \mathbf{T}/J \to \mathcal{L} \otimes_{\mathbf{T}} \mathbf{T}/J \to \mathcal{L}^2 \otimes_{\mathbf{T}} \mathbf{T}/J \to 0$$

with

$$\mathcal{L}^1 \otimes_{\mathbf{T}} \mathbf{T}/J \cong \tilde{\tau}_1 \otimes_{\mathcal{O}} \mathcal{T}/J\mathcal{T} \quad \text{and} \quad \mathcal{L}^2 \otimes_{\mathbf{T}} \mathbf{T}/J \cong \tilde{\tau}_2 \otimes_{\mathcal{O}} \mathbf{T}/J$$

as $\mathbf{T}$-modules where the $\mathbf{T}$-action on $\tilde{\tau}_1 \otimes_{\mathcal{O}} \mathcal{T}/J\mathcal{T}$ is via the second factor.

Note that we have the following identification

$$(4.3) \qquad \mathrm{Hom}_{\mathcal{O}}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_{\mathcal{O}} \mathcal{T}/J\mathcal{T} \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{T}/J}(\mathcal{L}^2 \otimes_{\mathbf{T}} \mathbf{T}/J, \mathcal{L}^1 \otimes_{\mathbf{T}} \mathbf{T}/J).$$

Let $s : \mathcal{L}^2 \otimes_{\mathbf{T}} \mathbf{T}/J \to \mathcal{L} \otimes_{\mathbf{T}} \mathbf{T}/J$ be a section of $\mathbf{T}/J$-modules of (4.2). Using (4.3) as in [Klo09], p.159-160, we define a cohomology class $c \in H^1(F_\Sigma, \mathrm{Hom}_{\mathcal{O}}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_{\mathcal{O}} \mathcal{T}/J\mathcal{T})$ by

$$g \mapsto (\lambda_2 \otimes t \mapsto s(\lambda_2 \otimes t) - g \cdot s(g^{-1} \cdot \lambda_2 \otimes t)).$$

We also define a map

$$\iota_J : \mathrm{Hom}_{\mathcal{O}}(\mathcal{T}/J\mathcal{T}, E/\mathcal{O}) \to H^1(F_\Sigma, \mathrm{Hom}_{\mathcal{O}}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_{\mathcal{O}} E/\mathcal{O}), \quad f \mapsto (1 \otimes f)(c).$$

Let us just briefly remark that ι_J is independent of the choice of the section s. From now on we will make the following assumption on the quotient $\mathbf{T}/J$.

Assumption 4.2. One has

$$\#\mathbf{T}/J \geq \#\mathcal{O}/L$$

with L as in Assumption 2.5.

Remark 4.3. In Section 7 we will describe a particular setup for $n = 2$ and F an imaginary quadratic field under which Assumptions 2.5 and 4.2 are satisfied. However, we expect that these conditions hold also for other CM fields (for $n = 2$), and have therefore presented the results of this and the following sections under these two general assumptions.

Lemma 4.4. *If Assumptions 2.5 and 4.2 hold, then the map*

$$\iota_J : \mathrm{Hom}_{\mathcal{O}}(\mathcal{T}/J\mathcal{T}, E/\mathcal{O}) \to H^1(F_\Sigma, \mathrm{Hom}_{\mathcal{O}}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_{\mathcal{O}} E/\mathcal{O})$$

is injective and its image equals $H^1_\Sigma(F, \mathrm{Hom}_{\mathcal{O}}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_{\mathcal{O}} E/\mathcal{O}))$.

Proof. For the injectivity of ι_J and for the fact that its image lands in the Selmer group one follows the strategy in [Ber05], p.119-120 which was later spelled out in a higher dimensional case in [Klo09], Lemmas 9.25 and 9.26. Let us outline the argument here. Let $f \in \ker \iota_J$ and set $K_f := (\mathcal{T}/J\mathcal{T})/\ker f$, $I_f := (E/\mathcal{O})/\mathrm{Im} f$ and $\tilde{T} := \mathrm{Hom}_{\mathcal{O}}(\tilde{\tau}_2, \tilde{\tau}_1)$. Tensoring the exact sequence $0 \to K_f \xrightarrow{f} E/\mathcal{O} \to I_f \to 0$ with $\otimes_{\mathcal{O}} \tilde{T}$ we get the exactness of the bottom row of the following commutative diagram:

(4.4)
$$H^1(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} \mathcal{T}/J\mathcal{T})$$

$$\downarrow{\phi} \qquad \searrow{H^1(1 \otimes f)}$$

$$H^0(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} I_f) \longrightarrow H^1(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} K_f) \xrightarrow{H^1(1 \otimes f)} H^1(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} E/\mathcal{O}).$$

Clearly $H^1(1 \otimes f) \circ \phi(c) = 0$ and since the first term in the bottom row vanishes (as a consequence of absolute irreducibility of $\tilde{\tau}_i$ and the fact that $\tilde{\tau}_1 \not\cong \tilde{\tau}_2$) we get $\phi(c) = 0$. Assuming $f \neq 0$, one constructs an $\mathcal{O}$-module $A \subset \mathcal{T}/J\mathcal{T}$ containing $\ker f$ such that $(\mathcal{T}/J\mathcal{T})/A = \mathcal{O}/\varpi$. It is easy to show that $\phi(c) = 0$ implies the splitting of the following exact sequence of $\mathbf{T}[G_\Sigma]$-modules

$$0 \to \tau_1 \to (\mathcal{L}/J\mathcal{L})/(\varpi\mathcal{L} + \tilde{\tau}_1 \otimes_{\mathcal{O}} A) \to \tau_2 \to 0$$

contradicting the fact that $\mathcal{L}$ has no quotient isomorphic to τ_1 (cf. Theorem 4.1). This proves injectivity of ι_J.

On the other hand the fact that $\mathrm{Im}(\iota_J)$ is contained in the Selmer group can be deduced from the fact that each of the representations ρ_π making up $\rho = \rho_\Pi = \prod_{\pi \in \Pi} \rho_\pi$ is crystalline because it implies that the cohomology class c is also crystalline (see [Klo09], proof of Lemma 9.25 for more details).

Using [Klo09], Lemma 9.21 (which is just a slightly expanded version of Theorem 4.1) we get $\mathrm{Fitt}_{\mathbf{T}} \mathcal{T} = 0$.

By Lemma 3.9 we know that $\mathbf{T}/J = \mathcal{O}/\varpi^n$ for some n. Recall property 4 from the Appendix in [MW84]: For an R-module M and an ideal $I \subset R$ we have

(4.5)
$$\mathrm{Fitt}_{R/I}(M/IM) = \mathrm{Fitt}_R(M) + I \subset R/I.$$

Since $\mathrm{Fitt}_{\mathbf{T}}(\mathcal{T}) = 0$ this implies that $\mathrm{Fitt}_{\mathcal{O}/\varpi^n}(\mathcal{T}/J\mathcal{T}) = \mathrm{Fitt}_{\mathbf{T}/J}(\mathcal{T}/J\mathcal{T}) = (0)$ in $\mathbf{T}/J = \mathcal{O}/\varpi^n$.

Note that ϖ^n annihilates $\mathcal{T}/J\mathcal{T}$, so using (4.5) again we get $\mathrm{Fitt}_{\mathcal{O}/\varpi^n}(\mathcal{T}/J\mathcal{T}) = \mathrm{Fitt}_{\mathcal{O}/\varpi^n}((\mathcal{T}/J\mathcal{T})/\varpi^n) = \mathrm{Fitt}_{\mathcal{O}}(\mathcal{T}/J\mathcal{T}) + \varpi^n\mathcal{O}$.

Together this shows that $\mathrm{Fitt}_{\mathcal{O}}(\mathcal{T}/J\mathcal{T})$ maps to the 0-ideal in $\mathcal{O}/\varpi^n$, i.e.

$$\mathrm{Fitt}_{\mathcal{O}}(\mathcal{T}/J\mathcal{T}) \subset \varpi^n\mathcal{O} = \mathrm{Fitt}_{\mathcal{O}}(\mathbf{T}/J).$$

By property 11 in the Appendix of [MW84] we know that $\varpi^{\mathrm{length}_{\mathcal{O}}(\mathcal{T}/J\mathcal{T})}\mathcal{O} \subset \mathrm{Fitt}_{\mathcal{O}}(\mathcal{T}/J\mathcal{T})$. This, combined with Assumption 4.2 and Assumption 2.5, implies that ι_J must in fact surject onto the Selmer group. $\qquad\square$

Since (4.2) splits as a sequence of $\mathbf{T}/J$-modules we can tensor it with $\otimes_{\mathbf{T}/J}\mathbf{F}$ and obtain an exact sequence of $\mathbf{F}[G_\Sigma]$-modules

$$0 \to \mathcal{L}^1 \otimes_{\mathbf{T}} \mathbf{F} \to \mathcal{L} \otimes_{\mathbf{T}} \mathbf{F} \to \mathcal{L}^2 \otimes_{\mathbf{T}} \mathbf{F} \to 0$$

with

$$\mathcal{L}^1 \otimes_{\mathbf{T}} \mathbf{F} \cong \mathcal{T} \otimes_{\mathbf{T}} \tau_1 \quad \text{and} \quad \mathcal{L}^2 \otimes_{\mathbf{T}} \mathbf{F} \cong \tau_2.$$

Arguing as above (with $\mathfrak{m}_{\mathbf{T}}$ instead of J) we again obtain an injective map

$$\iota : \operatorname{Hom}_{\mathcal{O}}(\mathcal{T}/J\mathcal{T}, \mathbf{F}) \to H^1(F_\Sigma, \operatorname{Hom}_{\mathbf{F}}(\tau_2, \tau_1)).$$

Lemma 4.5. *Suppose that Assumptions 2.5 and 4.2 hold. The map $\iota : \operatorname{Hom}_{\mathcal{O}}(\mathcal{T}/J\mathcal{T}, \mathbf{F}) \to H^1(F_\Sigma, \operatorname{Hom}_{\mathbf{F}}(\tau_2, \tau_1))$ is injective and its image equals $H^1_\Sigma(F, \operatorname{Hom}_{\mathbf{F}}(\tau_2, \tau_1))$.*

Proof. We have the following commutative diagram

$$
\begin{array}{ccc}
\operatorname{Hom}_{\mathcal{O}}(\mathcal{T}/J\mathcal{T}, E/\mathcal{O}) & \xrightarrow{\iota_J} & H^1(F_\Sigma, \operatorname{Hom}_{\mathcal{O}}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_{\mathcal{O}} E/\mathcal{O}) \\
\uparrow & & \uparrow \\
\operatorname{Hom}_{\mathcal{O}}(\mathcal{T}/J\mathcal{T}, \mathbf{F}) & \xrightarrow{\iota} & H^1(F_\Sigma, \operatorname{Hom}_{\mathbf{F}}(\tau_2, \tau_1))
\end{array}
$$

Denote the right vertical arrow by f. Lemma 4.4 implies that the image of $f \circ \iota$ is contained in the ϖ-torsion of $H^1_\Sigma(F, \operatorname{Hom}_{\mathcal{O}}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_{\mathcal{O}} E/\mathcal{O})$. Moreover, by Proposition 5.8 in [BK13], we know that the ϖ-torsion in $H^1_\Sigma(F, \operatorname{Hom}_{\mathcal{O}}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_{\mathcal{O}} E/\mathcal{O})$ coincides with $f(H^1_\Sigma(F, \operatorname{Hom}_{\mathbf{F}}(\tau_2, \tau_1)))$. Since f is injective, this implies that the image of ι is contained in the Selmer group. Hence it remains to show that ι_J is an isomorphism on ϖ-torsion. But this is clear since ι_J is an isomorphism by Lemma 4.4. $\square$

Lemma 4.6. *Suppose that Assumptions 2.5 and 4.2 hold. Write $\overline{\rho}_\Pi$ for $\prod_{\rho_\pi \in \Pi} \overline{\rho}_\pi$. The $\mathbf{F}[G_\Sigma]$-module $\mathcal{L} \otimes_{\mathbf{T}} \mathbf{F}$ coincides with the $\mathbf{F}$-subspace of $\prod_{\rho_\pi \in \Pi} \mathbf{F}^n$ generated by $\overline{\rho}_\Pi(r)e_n$, where r runs over $\mathbf{F}[G_\Sigma]$, e_n is a column matrix in $\mathbf{F}^n$ whose last entry is 1 and all the other ones are zero.*

Proof. By definition of $\mathcal{L}$, every element of $\mathcal{L} \otimes_{\mathbf{T}} \mathbf{F}$ can be written as $\sum_i t_i \rho_\Pi(g_i)e_n \otimes a_i$ with $t_i \in \mathbf{T}$, $a_i \in \mathbf{F}$ and $g_i \in G_\Sigma$. Writing $\overline{t}_i$ for the image of t_i under the canonical map $\mathbf{T} \twoheadrightarrow \mathbf{F}$ we can re-write the above sum as $\sum_i \rho_\Pi(g_i)e_n \otimes a_i\overline{t}_i$ and $a_i\overline{t}_i \in \mathbf{F}$. It suffices to show now that for every $g \in G_\Sigma$ we get $\rho_\Pi(g)e_n \otimes 1 = \overline{\rho}_\Pi(g)e_n \otimes 1$. Write

$$\rho_\Pi(g) = \begin{bmatrix} a_{11}(g) + a'_{11}(g) & a_{12}(g) + a'_{12}(g) \\ a'_{21}(g) & a_{22}(g) + a'_{22}(g) \end{bmatrix},$$

where a_{11}, a'_{11} are $(n-1) \times (n-1)$-matrices, a_{22}, a'_{22} are scalars and the other matrices have sizes determined by these two and the entries of $a'_{ij}(g)$ lie in $\varpi\mathcal{O} \oplus \varpi\mathcal{O} \oplus \cdots \oplus \varpi\mathcal{O}$ Thus,

$$\rho_\Pi(g)e_n \otimes 1 = \begin{bmatrix} a_{12}(g) + a'_{12}(g) \\ a_{22}(g) + a'_{22}(g) \end{bmatrix} \otimes 1 = \begin{bmatrix} a_{12}(g) \\ a_{22}(g) \end{bmatrix} \otimes 1 + \begin{bmatrix} a'_{12}(g)/\varpi \\ a'_{22}(g)/\varpi \end{bmatrix} \otimes \varpi,$$

and the latter tensor is zero. This proves the lemma. $\square$

Let us now turn to the 2-dimensional situation, where every τ is (up to a twist) of the form

$$\tau = \begin{bmatrix} 1 & * \\ 0 & \chi \end{bmatrix}$$

for a Galois character χ. Note that $\mathrm{Hom}_{\mathcal{O}}(\mathcal{T}/J\mathcal{T}, \mathbf{F}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{T} \otimes_{\mathbf{T}} \mathbf{T}/J, \mathbf{F}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{T} \otimes_{\mathbf{T}} \mathbf{T}/\mathfrak{m}, \mathbf{F}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{T} \otimes_{\mathbf{T}} \mathbf{F}, \mathbf{F})$.

Proposition 4.7. *Suppose that Assumptions 2.5 and 4.2 hold. The image of $\iota :$ $\mathrm{Hom}(\mathcal{T} \otimes_{\mathbf{T}} \mathbf{F}, \mathbf{F}) \hookrightarrow H^1_{\Sigma}(F, \chi^{-1})$ is spanned by extensions τ such that $\Pi_{\tau} \neq \emptyset$.*

Proof. Let Π be as above and $\overline{\Pi}$ be a subset Π consisting of representatives of distinct isomorphism classes of residual representations (i.e., one element from every non-empty Π_{τ}). By Lemma 4.6 the lattice $\mathcal{L} \otimes_{\mathbf{T}} \mathbf{F}$ is generated by vectors

$$x = \begin{bmatrix} (1, 1, \ldots, 1) & \chi(g)\alpha(g) \\ (0, 0, \ldots, 0) & (\chi(g), \chi(g), \ldots, \chi(g)) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let us explain the notation: There are r elements in $\overline{\Pi}$ (which we will denote by $\tau^1, \tau^2, \ldots, \tau^r$), $\sigma := \sum_{i=1}^{r} s_i$ elements in Π. Moreover, α is a σ-tuple of functions such that $\alpha(g)$ equals

$$(\alpha_{1,1} f_1(g), \ldots, \alpha_{1,s_1} f_1(g), \alpha_{2,1} f_2(g), \ldots, \alpha_{2,s_2} f_2(g), \ldots, \alpha_{r,1} f_r(g), \ldots \alpha_{r,s_r} f_r(g)) \in \mathbf{F}^{\sigma},$$

where the $\alpha_{i,j}$ are elements of $\mathbf{F}^{\times}$. We get that x equals

$$\chi(g) \begin{bmatrix} (\alpha_{1,1} f_1(g), \ldots, \alpha_{1,s_1} f_1(g), \alpha_{2,1} f_2(g), \ldots, \alpha_{2,s_2} f_2(g), \ldots, \alpha_{r,1} f_r(g), \ldots \alpha_{r,s_r} f_r(g)) \\ (1, 1, \ldots, 1) \end{bmatrix}.$$

Let α^j be the jth entry of α. Then we conclude that $\mathcal{L} \otimes_{\mathbf{T}} \mathbf{F} \cong \mathbf{T} \otimes_{\mathbf{T}} V = V$, where V is the subspace of $(\mathbf{F} \oplus \mathbf{F})^{\#\Pi}$ spanned over $\mathbf{F}$ by the set vectors of the form

$$\left(\begin{bmatrix} \chi(g)\alpha^1(g) \\ \chi(g) \end{bmatrix}, \begin{bmatrix} \chi(g)\alpha^2(g) \\ \chi(g) \end{bmatrix}, \ldots, \begin{bmatrix} \chi(g)\alpha^{\sigma}(g) \\ \chi(g) \end{bmatrix} \right).$$

For $j \in \{1, 2, \ldots, \sigma\}$ define integers $n(j) \in \{1, 2, \ldots, r\}$ and $m(j) \in \{1, 2, \ldots, s_{n(j)}\}$ by the equality

$$\alpha^j(g) = \alpha_{n(j),m(j)} f_{n(j)}(g).$$

The G_{Σ}-action on V is via $\overline{\rho}_{\Pi}$, hence $h \in G_{\Sigma}$ acts on $\begin{bmatrix} \chi(g)\alpha^j(g) \\ \chi(g) \end{bmatrix}$ via the $n(j)$th residual representation in $\overline{\Pi}$, i.e., by multiplication by $\tau^{n(j)}(h)$. In particular all the vectors in V have the form

$$(4.6) \qquad v = \left(\begin{bmatrix} a_1 \\ a \end{bmatrix}, \begin{bmatrix} a_2 \\ a \end{bmatrix}, \ldots, \begin{bmatrix} a_{\sigma} \\ a \end{bmatrix} \right).$$

By definition we have

$$(4.7) \qquad \mathcal{L}^2 \otimes_{\mathbf{T}} \mathbf{F} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathcal{L} \otimes_{\mathbf{T}} \mathbf{F} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} V \cong \mathbf{F}(\chi)$$

as $\mathbf{F}[G_{\Sigma}]$-modules, where we write $\mathbf{F}(\chi)$ for the one-dimensional $\mathbf{F}$-vector space on which G_{Σ} acts via χ. The surjective $\mathbf{F}[G_{\Sigma}]$-module map $V \twoheadrightarrow \mathbf{F}(\chi)$ is given by sending a vector v as in (4.6) to a. Write V' for the kernel of this map. Identifying V' with $\mathcal{L}^1 \otimes_{\mathbf{T}} \mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{L} \otimes_{\mathbf{T}} \mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} V$ provides us with a splitting (only as $\mathbf{F}$-vector spaces) of the short exact sequence of $\mathbf{F}[G_{\Sigma}]$-modules

$$0 \to V' \to V \to \mathbf{F}(\chi) \to 0.$$

Since the G_Σ-action on $\mathcal{L}^1 \otimes_{\mathbf{T}} \mathbf{F}$ is trivial, we have $V' = \mathcal{L}^1 \otimes_{\mathbf{T}} \mathbf{F} = \mathcal{T} \otimes_{\mathbf{T}} \mathbf{F}$. Clearly, we may assume that the vectors in V' all have the form

$$(4.8) \qquad\qquad v_0 = \left(\begin{bmatrix} a_1 \\ 0 \end{bmatrix}, \begin{bmatrix} a_2 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} a_\sigma \\ 0 \end{bmatrix} \right).$$

Let $\phi_j \in \mathrm{Hom}_{\mathbf{F}}(V', \mathbf{F}) = \mathrm{Hom}(\mathcal{T} \otimes_{\mathbf{T}} \mathbf{F}, \mathbf{F})$ be the homomorphism sending v_0 as in (4.8) to a_j.

Then the map ι sends ϕ_j to the cocycle $\alpha_{n(j),m(j)} f_{n(j)}$, i.e, to the residual representation $\begin{bmatrix} 1 & \chi \alpha_{n(j),m(j)} f_{n(j)} \\ 0 & \chi \end{bmatrix}$, which is isomorphic to the residual representation of the $n(j)$th element of $\overline{\Pi}$. So, this proves that the image of ι is spanned by modular extensions. $\qquad\square$

Corollary 4.8. *If Assumptions 2.5 and 4.2 are satisfied then the space $H^1_\Sigma(F, \chi^{-1})$ has a basis consisting of extensions τ such that $\Pi_\tau \neq \emptyset$.*

Proof. This follows from Lemma 4.5 and Proposition 4.7. $\qquad\square$

Remark 4.9. Corollary 4.8 does not imply that $\Pi_\tau \neq \emptyset$ for all isomorphism classes $\tau \in \mathfrak{T}$. In fact, if we replace $\mathbf{F}$ by its finite extension $\mathbf{F}'$ of degree m, then the order of $\mathfrak{T}$ increases (since it is given by $\# H^1_\Sigma(F, \mathrm{Hom}(\tau_2, \tau_1))/(q-1)$, where q is the order of the residue field), while the number of modular forms, i.e., $\sum_{\tau \in \mathfrak{T}} \# \Pi_\tau$ remains the same.

5. BOUNDING THE SIZE OF $\prod_i \mathbf{T}_i / \phi(I_i)$

In this section we keep in force Assumptions 2.5 and 4.2. Moreover, we work in the two-dimensional setup and we set $\tau_1 = 1$ and $\tau_2 = \chi$ (which can always be achieved by twisting by a Galois character). Let $\mathcal{B} := \{e_1, \ldots, e_s\}$ be a basis of $H^1_\Sigma(F, \mathrm{Hom}(\tau_2, \tau_1)) = H^1_\Sigma(F, \chi^{-1})$ consisting of 'modular' extensions, i.e., extensions τ such that $\Pi_\tau \neq \emptyset$ (cf. Corollary 4.8) and write τ^i for the corresponding residual representations. Let us write $\mathbf{T}_i$ for $\mathbf{T}_{\tau^i}$. Similarly let us write J_i and Π_i for J_{τ^i} and Π_{τ^i} respectively. Write $p_i : \mathbf{T} \twoheadrightarrow \mathbf{T}_i$ for the canonical projection. Consider the map $\mathbf{T} \to \prod_{i=1}^s \mathbf{T}_i$. Let $J \subset \mathbf{T}$ be as in section 3. Set $J_i = p_i(J)$. Note that J_i is an ideal because p_i is surjective.

Let us begin with an observation that there is no reason to expect that the canonical map

$$\mathbf{T}/J \to \prod_{i=1}^s \mathbf{T}_i / J_i$$

should in general be injective or surjective. In fact Lemma 3.9 shows that $\mathbf{T}/J$ is cyclic over $\mathcal{O}$. However, as we shall see below the orders of both sides are equal provided that the basis $\mathcal{B}$ is unique up to scaling and that all of the ideals J_i are principal.

Proposition 5.1. $\# \prod_{i=1}^s \mathbf{T}_i / J_i \leq \# \mathbf{T}/J$.

Proof. In this proof we follow mostly the notation of [Klo09], section 9. Let Π_i be as before. As in section 4 we use Theorem 4.1 to get a lattice $\mathcal{L}_i \subset \prod_{\pi \in \Pi_i} \rho_\pi$ and a finitely generated $\mathbf{T}_i$-module $\mathcal{T}_i$ such that the following sequence of $\mathbf{T}_i / J_i[G_\Sigma]$-modules is exact:

$$(5.1) \qquad 0 \to (\mathcal{T}_i / J_i \mathcal{T}_i) \otimes_{\mathcal{O}} \tilde{\tau}_1 \to \mathcal{L}_i / J_i \to (\mathbf{T}_i / J_i) \otimes_{\mathcal{O}} \tilde{\tau}_2 \to 0.$$

Similarly to the situation in section 4, the sequence splits as a sequence of $\mathbf{T}_i$-modules, hence after tensoring with $\mathbf{F}$ we obtain a short exact sequence:

$$(5.2) \qquad 0 \to (\mathcal{T}_i/J_i\mathcal{T}_i) \otimes_{\mathbf{T}_i} \tau_1 \to \mathcal{L}_i/J_i \otimes_{\mathbf{T}_i} \mathbf{F} \to (\mathbf{T}_i/J_i) \otimes_{\mathbf{T}_i} \tau_2 \to 0.$$

Fix i. As in the proof of Lemma 4.5 the sequence (5.2) gives rise to an injection

$$(5.3) \qquad \iota_i : \operatorname{Hom}(\mathcal{T}_i/J_i\mathcal{T}_i, \mathbf{F}) \hookrightarrow H^1_\Sigma(F, \operatorname{Hom}_{\mathcal{O}}(\tau_2, \tau_1)).$$

Arguing exactly as in the proof of Proposition 4.7 we see that the image of ι_i is one-dimensional and is spanned by the cohomology classes corresponding to the isomorphism class of τ^i, i.e., for every $i = 1, 2, \ldots, s$ one has $\operatorname{Im}(\iota_i) \subset \langle e_i \rangle$. This implies that $\operatorname{Hom}(\mathcal{T}_i/J_i\mathcal{T}_i, \mathbf{F})$ is one-dimensional and hence $\mathcal{T}_i/J_i\mathcal{T}_i$ is a cyclic $\mathcal{O}$-module, and hence a cyclic $\mathbf{T}_i/J_i (= \mathcal{O}/\varpi^{d_i})$-module (cf. Lemma 3.7). On the other hand, again using Lemma 9.21 of [Klo09], we get that $\operatorname{Fitt}_{\mathbf{T}_i} \mathcal{T}_i = 0$ and this implies (as in the proof of Lemma 4.4) that

$$\operatorname{val}_p(\#\mathbf{T}_i/J_i) \leq \operatorname{val}_p(\#\mathcal{T}_i/J_i\mathcal{T}_i).$$

This combined with the fact that $\mathcal{T}_i/J_i\mathcal{T}_i$ is a cyclic $\mathbf{T}_i/J_i$-module implies that $\mathbf{T}_i/J_i \cong \mathcal{T}_i/J_i\mathcal{T}_i$. In particular this implies that the lattice $\mathcal{L}_i/J_i \cong (\mathbf{T}_i/J_i)^2$ as $\mathbf{T}_i$-modules.

Let $\tilde{\rho}_i : G \to \operatorname{GL}_2(\mathbf{T}_i/J_i)$ be the representation given by the short exact sequence $0 \to (\mathbf{T}_i/J_i) \otimes \tilde{\tau}_1 \to \mathcal{L}_i/J_i \to (\mathbf{T}_i/J_i) \otimes \tilde{\tau}_2 \to 0$ (coming from the sequence (5.1) and the fact that $\mathcal{T}_i/J_i\mathcal{T}_i \cong \mathbf{T}_i/J_i$). One has $\mathbf{T}_i/J_i = \mathcal{O}/\varpi^{d_i}$ and since $\tilde{\rho}_i$ reduces to τ^i we must have $d_i \leq r_i$, where $\mathcal{O}c_i \cong \mathcal{O}/\varpi^{r_i}\mathcal{O}$. So, in particular we get that $\sum_i d_i \leq \sum r_i$. Combining Assumptions 2.5 with 4.2 we obtain the claim of Proposition 5.1. $\qquad\qquad\qquad\square$

Our goal is now to prove the opposite inequality, which under some additional assumption will follow from a more general commutative algebra result which was proved by the authors and Kenneth Kramer in [BKK14] and which we will now present.

Let $s \in \mathbf{Z}_+$ and let $\{n_1, n_2, \ldots, n_s\}$ be a set of s positive integers. Set $n = \sum_{i=1}^s n_i$. Let $A_i = \mathcal{O}^{n_i}$ with $i \in \{1, 2, \ldots, s\}$. Set $A = \prod_{i=1}^s A_i = \mathcal{O}^n$. Let $\varphi_i : A \twoheadrightarrow A_i$ be the canonical projection. Let $T \subset A$ be a (local complete) $\mathcal{O}$-subalgebra which is of full rank as an $\mathcal{O}$-submodule and let $J \subset T$ be an ideal of finite index. Set $T_i = \varphi_i(T)$ and $J_i = \varphi_i(J)$. Note that each T_i is also a (local complete) $\mathcal{O}$-subalgebra and the projections $\varphi_i|_T$ are local homomorphisms. Then J_i is also an ideal of finite index in T_i.

Theorem 5.2 ([BKK14], Theorem 2.1). *If $\#\mathbf{F}^\times \geq s - 1$ and each J_i is principal, then $\# \prod_{i=1}^s T_i/J_i \geq \#T/J$.*

Let V be a vector space and write $\mathbf{P}^1(V)$ for the set of all lines in V passing through the origin. There is a canonical map $V \setminus \{0\} \to \mathbf{P}^1(V)$ sending a vector v to the line spanned by v.

Let $\mathcal{S}$ be the set of all modular bases of $H^1_\Sigma(F, \operatorname{Hom}(\tau_2, \tau_1))$, i.e., the set of bases $\mathcal{B}' = \{e'_1, e'_2, \ldots, e'_s\}$ having the property that $\Pi_{\tau'_i} \neq \emptyset$, where τ'_i is the residual representation corresponding to the extension represented by e'_i. The set $\mathcal{S}$ is non-empty as $\mathcal{B} \in \mathcal{S}$.

Definition 5.3. We will say that $H^1_\Sigma(F, \operatorname{Hom}(\tau_2, \tau_1))$ has a *projectively unique modular basis* if the images of all the elements of $\mathcal{S}$ in $\mathbf{P}^1(H^1_\Sigma(F, \operatorname{Hom}(\tau_2, \tau_1)))$ agree.

In the case when $H^1_\Sigma(F, \mathrm{Hom}(\tau_2, \tau_1))$ has this property we will refer to any element of $\mathcal{S}$ as the projectively unique modular basis.

Note that it is possible to find $i_0 \in \{1, 2, \ldots, s\}$ such that the set $\mathcal{B}' := \mathcal{B} \cup \{e_\tau\} \setminus \{e_{i_0}\}$ is still a basis of $H^1_\Sigma(F, \mathrm{Hom}(\tau_2, \tau_1))$ (and one still has that $\Pi_{\tau'} \neq \emptyset$ for all $\tau' \in \mathcal{B}'$). Hence we can assume without loss of generality that $\mathcal{B} = \{e_1, e_2, \ldots, e_s\}$ with $\tau^1 = \tau$. In fact, if $H^1_\Sigma(F, \mathrm{Hom}(\tau_2, \tau_1))$ has a projectively unique modular basis, it follows that if $\mathcal{B}'$ is another modular basis, then the isomorphism classes of the residual representations corresponding to the elements of $\mathcal{B}'$ are the same as the isomorphism classes of the residual representations corresponding to the elements of $\mathcal{B}$.

Proposition 5.4. *Suppose $H^1_\Sigma(F, \mathrm{Hom}(\tau_2, \tau_1))$ has a projectively unique modular basis. If for each i, the ideal J_i of $\mathbf{T}_i$ is principal and $\mathbf{T}_i/J_i$ is finite, then $\# \prod_{i=1}^s \mathbf{T}_i/J_i \geq \#\mathbf{T}/J$.*

Proof. First note that our assumption that E be sufficiently large allows us to assume that $\#\mathbf{F}^\times$ satisfies the inequality in Theorem 5.2. Since $\mathbf{T}$ is a free $\mathcal{O}$-module of finite rank we set n to be that rank and define n_i to be the rank of $\mathbf{T}_i$. The assumption that $\mathcal{B}$ be projectively unique guarantees that every $\mathcal{O}$-algebra homomorphism has a corresponding residual Galois representation isomorphic to τ^i for some i. Hence $n = \sum_{i=1}^s n_i$. Finally, note that $\mathbf{T}/J$ is finite. Indeed, first note that if we consider $\mathbf{T}$ as a (full rank) $\mathcal{O}$-subalgebra of $\prod_{i=1}^s \mathbf{T}_i$ then for a sufficiently large exponent N, we have $p^N e_i \in \mathbf{T}$, where $e_i \in \prod_{i=1}^s \mathbf{T}_i$ is the idempotent corresponding to $\mathbf{T}_i$. On the other hand because $\mathbf{T}_i/J_i$ is finite for each i, there exists a positive integer M such that $p^M \in J_i$ for each i. Let $x_i \in J$ be a preimage of $p^M \in J_i$. Then $p^{N+M} = \sum_{i=1}^s x_i p^N e_i \in J$, hence T/J is torsion and thus finite. The Proposition now follows from Theorem 5.2 by taking $T = \mathbf{T}$. $\qquad\square$

Remark 5.5. Theorem 5.2 also has consequences for congruences between modular forms. Suppose that $\mathbf{T} = \mathbf{T}_\Sigma$ is the cuspidal Hecke algebra (acting on the space of automorphic forms over imaginary quadratic fields of weight 2 right invariant under a certain compact subgroup K_f) localized at a maximal ideal corresponding to an Eisenstein series, say $\mathcal{E}$. Let $J = J_\Sigma$ be the Eisenstein ideal corresponding to $\mathcal{E}$ (see section 7.3 for the details). Let $\mathcal{N}$ be the set of $\mathcal{O}$-algebra homomorphisms $\mathbf{T} \to \mathcal{O}$, i.e. to cuspidal Hecke eigencharacters congruent to the eigencharacter λ_0 of $\mathcal{E}$ mod ϖ. For $\lambda \in \mathcal{N}$ write m_λ for the largest positive integer such that $\lambda(T) \equiv \lambda_0(T)$ mod ϖ^{m_λ} for all $T \in \mathbf{T}$. Let e be the ramification index of E over $\mathbf{Q}_p$. As a consequence of Theorem 5.2 we get the following inequality (cf. Proposition 4.3 in [BKK14]

$$\frac{1}{e} \sum_{\lambda \in \mathcal{N}} m_\lambda \geq \mathrm{val}_\varpi(\#\mathbf{T}/J).$$

For many more applications of Theorem 5.2 see [loc.cit.].

Corollary 5.6. *Suppose that Assumptions 2.5 and 4.2 hold. If the modular basis $\mathcal{B}$ is projectively unique and if for each i, the ideal J_i of $\mathbf{T}_i$ is principal and $\mathbf{T}_i/J_i$ is finite, then*

$$\# \prod_{i=1}^s \mathbf{T}_i/J_i = \#\mathbf{T}/J \geq \#\mathcal{O}/L.$$

6. Urban's method applied to $R^{\mathrm{tr},0}$

In this section we again set $n = 2$, $\tau_1 = 1$ and $\tau_2 = \chi$ and we fix a residual representation

$$\tau : G_\Sigma \to \mathrm{GL}_2(\mathbf{F}), \quad \tau = \begin{bmatrix} 1 & \chi f \\ 0 & \chi \end{bmatrix}.$$

Let Φ_τ, Π_τ be as in section 3. From now on we will assume that Π_τ is non-empty. Also recall that we make Assumption 2.16, which by Remark 3.4 is equivalent to assuming Φ_τ is finite (in particular R_τ^0 is defined and finitely generated as an $\mathcal{O}$-module). The surjection $\phi : R_\tau^{\mathrm{tr},0} \twoheadrightarrow \mathbf{T}_\tau$ (cf. Definition 3.5 and Lemma 3.6) descends to a surjection

(6.1) $$R_\tau^{\mathrm{tr},0}/I_\tau^{\mathrm{tr},0} \twoheadrightarrow \mathbf{T}_\tau/J_\tau$$

(since by Lemma 3.8 $\phi_\tau(I_\tau^{\mathrm{tr},0}) = J_\tau$). The main goal of this section is to prove that under certain assumptions the map in (6.1) is an isomorphism (Theorem 6.3). Before we state the theorem let us demonstrate several properties of $R_\tau^{\mathrm{tr},0}$. In particular we will show that $R_\tau^{\mathrm{tr},0} \cong R_\tau^0$ (Theorem 6.2). In this section we also assume that Assumptions 2.4, 2.5 and 4.2 are satisfied.

Lemma 6.1. *The $\mathcal{O}$-rank of R_τ^0 equals the $\mathcal{O}$-rank of $R_\tau^{\mathrm{tr},0}$. In particular the normalizations (and the total rings of fractions) of R_τ^0 and $R_\tau^{\mathrm{tr},0}$ coincide.*

Proof. Write ρ for ρ_τ^0, i.e., $\rho : G_\Sigma \to \mathrm{GL}_2(R_\tau^0)$. We claim that we can conjugate ρ so that for every $g \in G_\Sigma$ we have $\rho(g) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a, c, d \in R_\tau^{\mathrm{tr},0}$. Indeed, since the characters on the diagonal of τ are distinct mod ϖ, we can find $\sigma \in G_\Sigma$ on which they differ, so that the eigenvalues of $\tau(\sigma)$ lift by Hensel's lemma to distinct eigenvalues of $\rho(\sigma)$ in R_τ^0, and we can conjugate (over R_τ^0) to have $\rho(\sigma)$ be diagonal with these lifted eigenvalues. For a general element $g \in G_\Sigma$, we then compare $\mathrm{tr}\,\rho(g)$ with $\mathrm{tr}\,\rho(\sigma g)$ and use that the eigenvalues are distinct mod ϖ to see that the two diagonal entries of $\rho(g)$ lie in $R_\tau^{\mathrm{tr},0}$. Similarly we show (cf. the proof of Lemma 3.27 in [DDT97]) that the lower-left entry also lies in $R_\tau^{\mathrm{tr},0}$.

Note that since R_τ^0 is a finitely generated $\mathcal{O}$-module (and $\mathcal{O}$ is assumed to be sufficiently large) we get an embedding

$$R_\tau^0 \hookrightarrow \hat{R}_\tau^0 \cong \prod_{i=1}^k \mathcal{O},$$

where $\hat{R}_\tau^0$ is the normalization of R_τ^0. For $i = 1, \ldots, k$, write ρ_i for the composition of ρ with the projection onto the ith component of $\hat{R}_\tau^0$. Suppose that the $\mathcal{O}$-rank of $R_\tau^{\mathrm{tr},0}$ is strictly smaller than the $\mathcal{O}$-rank of R_τ^0. Then there exist two minimal primes (after possibly renumbering the minimal primes we will call them $\mathfrak{p}_1$, $\mathfrak{p}_2$) of R_τ^0 which contract to the same minimal prime $\mathfrak{p}$ of $R_\tau^{\mathrm{tr},0}$. Hence we get the following

commutative diagram:

$$\begin{array}{ccc} R_\tau^{\mathrm{tr},0} & \hookrightarrow & R_\tau^0 \\ & & \\ \downarrow & & \downarrow \searrow \\ & R_\tau^0/\mathfrak{p}_1 & \quad R_\tau^0/\mathfrak{p}_2 \\ \downarrow & \nearrow {\sim} \quad {\sim} \nearrow & \\ R_\tau^{\mathrm{tr},0}/\mathfrak{p} & & \end{array}$$

This implies that the corresponding two deformations (to $\mathcal{O}$) $\rho_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $\rho_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ (since their a-, c- and d-entries (as functions) factor through $R_\tau^{\mathrm{tr},0}/\mathfrak{p}$) must satisfy $a_1 = a_2 =: a$, $c_1 = c_2 =: c$ and $d_1 = d_2 =: d$. In particular their traces are equal. Using Lemma 2.19 we see that both $\rho_1 \otimes_{\mathcal{O}} E$ and $\rho_2 \otimes_{\mathcal{O}} E$ are irreducible and thus by the Brauer-Nesbitt Theorem we conclude that $\rho_1 \otimes_{\mathcal{O}} E \cong \rho_2 \otimes_{\mathcal{O}} E$. Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GL}_2(E)$ be such that

$$(6.2) \qquad\qquad M\rho_1 = \rho_2 M.$$

Then an easy matrix calculation shows that

$$(6.3) \qquad\qquad Aa + Bc = aA + b_2 C$$

and

$$(6.4) \qquad\qquad Cb_1 + Dd = cB + dD$$

from which we get that $Cb_1 = Cb_2$. Suppose for the moment that there exists $g \in G_\Sigma$ such that $b_1(g) \neq b_2(g)$. Since $\mathcal{O}$ is a domain we conclude that $C = 0$. Since the representations ρ_1 and ρ_2 are irreducible over E, the function c cannot be identically zero. Using (6.4) we conclude that $B = 0$. Finally, computing the lower-left entries on both sides of (6.2) we get $Dc = cA$, so again using the fact c is not identically zero and that $\mathcal{O}$ is a domain we get that $A = D$. Thus, M is a non-zero scalar matrix. Hence we get a contradiction to our assumption on the existence of g and we conclude that $b_1 = b_2$. In particular ρ_1 and ρ_2 are identical deformations of τ which correspond to distinct minimal primes of R_τ^0. Hence ρ_1 and ρ_2 give rise to two different homomorphisms from R_τ^{red} to $\mathcal{O}$. This contradicts the bijectivity of the correspondence

$$\mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(R_\tau^{\mathrm{red}}, \mathcal{O}) \leftrightarrow \{\text{deformations of } \tau \text{ into } \mathcal{O}\}.$$

$\square$

Theorem 6.2. *There exists a deformation $\rho_\tau^{\mathrm{tr},0} : G_\Sigma \to \mathrm{GL}_2(R_\tau^{\mathrm{tr},0})$ of τ. The resulting canonical map $R_\tau^{\mathrm{red}} \to R_\tau^{\mathrm{tr},0}$ factors through R_τ^0 and induces an isomorphism $R_\tau^0 \cong R_\tau^{\mathrm{tr},0}$.*

Proof. We will (once again) apply Theorem 4.1 (due to Urban). In the notation of section 4 we will write $\mathcal{F}_\mathcal{B} = \mathcal{F}$ to be total ring of fractions of $\mathcal{B} = R_\tau^{\mathrm{tr},0} \subset \mathcal{F}$. Note that by Lemma 6.1, $\mathcal{F}$ is also the total ring of fractions of R_τ^0. Moreover, we take $\mathcal{R} = R_\tau^{\mathrm{tr},0}[G_\Sigma]$, $\rho = \rho_\tau \otimes_\mathcal{B} \mathcal{F} : G_\Sigma \to \mathrm{GL}_2(\mathcal{F})$ which induces a morphism $\rho : R_\tau^{\mathrm{tr},0}[G_\Sigma] \to M_2(\mathcal{F})$ of $R_\tau^{\mathrm{tr},0}$-algebras. As before, the representations denoted in

Theorem 4.1 by ρ_1 and ρ_2 are our unique lifts $\tilde{\tau}_1, \tilde{\tau}_2 : G_\Sigma \to \mathrm{GL}_2(\mathcal{O}) \hookrightarrow \mathrm{GL}_2(R_\tau^{\mathrm{tr},0})$ and we set $I = I_\tau^{\mathrm{tr},0}$. Note that conditions (i) and (ii) of Theorem 4.1 are satisfied respectively by the definition of $R_\tau^{\mathrm{tr},0}$ and of $I_\tau^{\mathrm{tr},0}$ and conditions (iii) and (iv) are satisfied by our assumptions on τ_1 and τ_2. Finally, the condition of irreducibility of ρ is satisfied by Lemma 2.19.

Hence we conclude from Theorem 4.1 that there exists an $R_\tau^{\mathrm{tr},0}[G_\Sigma]$-stable lattice $\mathcal{L} \subset \mathcal{F}^2$ and a finitely generated $R_\tau^{\mathrm{tr},0}$-module $\mathcal{T}_\tau \subset \mathcal{F}$ such that we have an exact sequence of $R_\tau^{\mathrm{tr},0}[G_\Sigma]$-modules:

$$(6.5) \qquad 0 \to \tilde{\tau}_1 \otimes_\mathcal{O} \mathcal{T}_\tau / I_\tau^{\mathrm{tr},0} \mathcal{T}_\tau \to \mathcal{L} \otimes_{R_\tau^{\mathrm{tr},0}} R_\tau^{\mathrm{tr},0}/I_\tau^{\mathrm{tr},0} \to \tilde{\tau}_2 \otimes_\mathcal{O} R_\tau^{\mathrm{tr},0}/I_\tau^{\mathrm{tr},0} \to 0.$$

It follows from [Urb01] Lemmas 1.1 and 1.5 that $\mathcal{L} = \mathcal{T}_\tau \oplus R_\tau^{\mathrm{tr},0}$ as $R_\tau^{\mathrm{tr},0}$-modules. We will now show that $\mathcal{T}_\tau \cong R_\tau^{\mathrm{tr},0}$. Indeed, the lattice $\mathcal{L}$ is defined as in section 4, but since we only work with a fixed residual representation τ, the representation ρ in Theorem 4.1 equals the $\prod_{\rho_\pi \in \Phi_\tau} \rho_\pi$. Using Lemma 4.6 for this representation (i.e., when we replace ρ_Π with ρ as above), we conclude that the G_Σ-module $\mathcal{L} \otimes_{R_\tau^{\mathrm{tr},0}} \mathbf{F}$ is the subspace of $\prod_{\rho_\pi \in \Phi_\tau} \mathbf{F}^2$ generated by $\overline{\rho}(r)e_2$ (with notation as in that lemma). This subspace is clearly isomorphic to τ as a G_Σ-module. So, the middle term in (6.5) after tensoring with $\mathbf{F}$ is two-dimensional, hence we must have $\mathcal{T}_\tau / I_\tau^{\mathrm{tr},0} \mathcal{T}_\tau \otimes_{R_\tau^{\mathrm{tr},0}} \mathbf{F} = \mathcal{T}_\tau / \mathfrak{m} \mathcal{T}_\tau = \mathbf{F}$, where $\mathfrak{m}$ is the maximal ideal of $R_\tau^{\mathrm{tr},0}$. Thus, by Nakayama's Lemma, we see that $\mathcal{T}_\tau$ is generated over $R_\tau^{\mathrm{tr},0}$ by one element, say x. Consider the surjective map $\phi : R_\tau^{\mathrm{tr},0} \to \mathcal{T}_\tau$ given by $r \mapsto rx$. Let $a \in \ker \phi$. Then a annihilates $\mathcal{T}_\tau$. However, by definition of $\mathcal{T}_\tau$ and the fact that $R_\tau^{\mathrm{tr},0}$ embeds into its ring of fractions $\mathcal{F}$ we can consider x and a as elements of $\mathcal{F} = \prod E$. If $a = (a_1, a_2, \ldots, a_s)$ and $x = (x_1, x_2, \ldots, x_s)$ and $a_j \neq 0$, then $x_j = 0$. However, this implies that $\mathcal{T} \otimes \mathcal{F} \neq \mathcal{F}$, which contradicts the fact that $\mathcal{T}_\tau$ is a lattice in $\mathcal{F}$ (cf. Theorem 4.1). Hence ϕ is injective and we indeed have $\mathcal{T}_\tau = R_\tau^{\mathrm{tr},0}$. Thus $\mathcal{L} = (R_\tau^{\mathrm{tr},0})^2$ as an $R_\tau^{\mathrm{tr},0}$-module and ρ gives rise to a representation $\rho_\tau^{\mathrm{tr},0} : G_\Sigma \to \mathrm{GL}_2(R_\tau^{\mathrm{tr},0})$.

By the above it is clear that $\rho_\tau^{\mathrm{tr},0}$ reduces to τ. Let us make sure that the resulting representation is crystalline. Indeed, the lattice $\mathcal{L}$ lives inside the finite direct product of the representations ρ_π for $\pi \in \Phi_\tau$ and each of the ρ_π is crystalline. Hence as a submodule of a finite direct sum of crystalline representations $\rho_\tau^{\mathrm{tr},0}$ is crystalline. This proves the first assertion.

By universality of R_τ^{red} we obtain an $\mathcal{O}$-algebra map $\phi : R_\tau^{\mathrm{red}} \to R_\tau^{\mathrm{tr},0}$. Since $R_\tau^{\mathrm{tr},0}$ is a subring of the direct product of finitely many copies of $\mathcal{O}$, the map ϕ clearly factors through R_τ^0. Let us denote the induced map $R_\tau^0 \to R_\tau^{\mathrm{tr},0}$ also by ϕ. We claim that ϕ is surjective. Indeed, by its definition $R^{\mathrm{tr},0}$ is generated by traces of ρ_τ^0. So, it is enough to show that the traces of ρ_τ^0 and $\rho_\tau^{\mathrm{tr},0} = \phi \circ \rho_\tau^0$ coincide. This follows from the construction of the lattice $\mathcal{L}$ which is a G_Σ-stable lattice inside the representation $\rho_\tau^0 \otimes \mathcal{F}$.

In particular, both representations ρ_τ^0 and $\rho_\tau^{\mathrm{tr},0}$ are isomorphic after tensoring with $\mathcal{F}$, hence they must have equal traces. Since both R_τ^0 and $R_\tau^{\mathrm{tr},0}$ are finitely generated $\mathcal{O}$-modules with the same rings of fractions (Lemma 6.1), the kernel of ϕ must be a torsion $\mathcal{O}$-module. This implies that the kernel must be zero (as R_τ^0 embeds into $\prod E$). This proves the second assertion. $\qquad \square$

Suppose that τ is such that $\Pi_\tau \neq \emptyset$. Recall that by Corollary 4.8 there exists a basis $\mathcal{B} = \{e_1, e_2, \ldots, e_s\}$ of $H_\Sigma^1(F, \mathrm{Hom}(\tau_2, \tau_1))$ such that $\Pi_{\tau^i} \neq \emptyset$ for all $i = 1, 2, \ldots, s$, where τ^i denotes the representation corresponding to the extension e_i. It is possible to find $i_0 \in \{1, 2, \ldots, s\}$ such that the set $\mathcal{B}' := \mathcal{B} \cup \{e_\tau\} \setminus \{e_{i_0}\}$ is

still a basis (and one still has that $\Pi_{\tau'} \neq \emptyset$ for all $\tau' \in \mathcal{B}'$). Hence we can assume without loss of generality that $\mathcal{B} = \{e_1, e_2, \ldots, e_s\}$ with $\tau^1 = \tau$.

Theorem 6.3. *Suppose Assumptions 2.4, 2.5, 2.16 and 4.2 are satisfied. Suppose moreover that the modular basis $\mathcal{B}$ is projectively unique and that for each $i = 1, 2, \ldots, s$ the corresponding ideal J_{τ^i} is principal. Then the map*

$$R_\tau^{\mathrm{tr},0}/I_\tau^{\mathrm{tr},0} \twoheadrightarrow \mathbf{T}_\tau/J_\tau$$

is an isomorphism.

Proof. In the proof of Theorem 6.2 we showed the existence of an $R_\tau^{\mathrm{tr},0}$-algebra morphism $\rho : R_\tau^{\mathrm{tr},0}[G_\Sigma] \to M_2(\mathcal{F})$ and an $R_\tau^{\mathrm{tr},0}[G_\Sigma]$-stable lattice $\mathcal{L} \subset \mathcal{F}^2$ together with a finitely generated $R_\tau^{\mathrm{tr},0}$-submodule $\mathcal{T}_\tau \subset \mathcal{F}$ such that we have an exact sequence of $R_\tau^{\mathrm{tr},0}[G_\Sigma]$-modules:

$$(6.6) \qquad 0 \to \tilde{\tau}_1 \otimes_{\mathcal{O}} \mathcal{T}_\tau/I_\tau^{\mathrm{tr},0}\mathcal{T}_\tau \to \mathcal{L} \otimes_{R_\tau^{\mathrm{tr},0}} R_\tau^{\mathrm{tr},0}/I_\tau^{\mathrm{tr},0} \to \tilde{\tau}_2 \otimes_{\mathcal{O}} R_\tau^{\mathrm{tr},0}/I_\tau^{\mathrm{tr},0} \to 0$$

which splits as a sequence of $R_\tau^{\mathrm{tr},0}/I_\tau^{\mathrm{tr},0}$-modules. As in section 4 we get a map:

$$\iota : \mathrm{Hom}_{\mathcal{O}}(\mathcal{T}_\tau/I_\tau^{\mathrm{tr},0}\mathcal{T}_\tau, E/\mathcal{O}) \to H^1(F_\Sigma, \mathrm{Hom}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_{\mathcal{O}} E/\mathcal{O}).$$

The fact that ι is injective and that its image is contained in the Selmer group is proved in the same way as Lemma 4.4. Tensoring (6.6) with $\mathbf{F}$ and arguing as in the proof of Proposition 5.1 (this time using that $R_\tau^{\mathrm{tr},0}/I_\tau^{\mathrm{tr},0}$ is cyclic by Remark 2.11) we see that $\mathcal{T}_\tau/I_\tau^{\mathrm{tr},0}\mathcal{T}_\tau$ is cyclic and that

$$(6.7) \qquad \mathrm{val}_p(\#\mathcal{T}_\tau/I_\tau^{\mathrm{tr},0}\mathcal{T}_\tau) \geq \mathrm{val}_p(\#R_\tau^{\mathrm{tr},0}/I_\tau^{\mathrm{tr},0}).$$

Using the above arguments for the rings $R_i^{\mathrm{tr},0}$ corresponding to the residual representation arising from e_i, and putting them together we obtain
$$(6.8)$$
$$\mathrm{val}_p(\#\prod_{i=1}^s R_i^{\mathrm{tr},0}/I_i^{\mathrm{tr},0}) \leq \mathrm{val}_p(\#\prod_{i=1}^s \mathcal{T}_i/I_i^{\mathrm{tr},0}\mathcal{T}_i) \leq \mathrm{val}_p(\#H_\Sigma^1(F, \mathrm{Hom}(\tilde{\tau}_2, \tilde{\tau}_1)\otimes E/\mathcal{O}))$$

(the first inequality comes from (6.7)). The assumption in Corollary 5.6 that $\mathbf{T}_i/J_i$ are finite is satisfied since this is true for $R_i^{\mathrm{tr},0}/I_i^{\mathrm{tr},0} = R_\tau^0/I_\tau^0$ by Theorem 6.2 and Lemma 2.17. Combining Assumptions 2.5, 4.2 with Corollary 5.6 we obtain that

$$(6.9) \qquad \mathrm{val}_p(\#H_\Sigma^1(F, \mathrm{Hom}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes E/\mathcal{O})) \leq \mathrm{val}_p(\#\prod_{i=1}^s \mathbf{T}_i/J_i).$$

Combining (6.8) with (6.9) we conclude that the maps

$$R_i^{\mathrm{tr},0}/I_i^{\mathrm{tr},0} \twoheadrightarrow \mathbf{T}_i/J_i$$

must be isomorphisms for every $i = 1, 2, \ldots, s$. This completes the proof of Theorem 6.3.   □

7. Imaginary quadratic case

For the remainder of the article we specialize to the case $n = 2$, F imaginary quadratic, $\tau_1 = 1$ and $\tau_2 = \chi : G_\Sigma \to \mathbf{F}^\times$ an anticyclotomic character (i.e., we assume that $\chi(cgc) = \chi(g^{-1})$ for all $g \in G_\Sigma$ and $c \in G_{\mathbf{Q}}$ a complex conjugation). Recall again our assumption 2.16 that $\mathcal{P}(\tau)$ is finite (so that R_τ^0 is defined and finitely generated as an $\mathcal{O}$-module).

7.1. **Principality of the ideal of reducibility.** Let $f \in H^1_\Sigma(F, \chi^{-1})$ be non-zero. Set $\tau = \begin{bmatrix} 1 & \chi f \\ & \chi \end{bmatrix}$ with universal deformation ring R_τ.

Proposition 7.1. *Suppose Assumption 2.4 is satisfied and write Ψ for $\tilde{\tau}_2$. Then the ideal of reducibility I^0_τ of R^0_τ is principal.*

We begin with the following lemma.

Lemma 7.2. *There exists $\rho^{0,\mathrm{opp}}_\tau : G_\Sigma \to \mathrm{GL}_2(R^0_\tau)$ such that $\overline{\rho}^{0,\mathrm{opp}}_\tau = \begin{bmatrix} \chi & * \\ & 1 \end{bmatrix}$ and is non-split.*

Proof. Let c be the complex conjugation. Define ρ'_τ by

$$\rho'_\tau(g) = \rho^0_\tau(cgc).$$

Then

$$\overline{\rho}'_\tau = \begin{bmatrix} 1 & \chi(cgc)f(cgc) \\ & \chi(cgc) \end{bmatrix} = \begin{bmatrix} 1 & \chi^{-1}(g)f(cgc) \\ & \chi^{-1}(g) \end{bmatrix}.$$

If this is split, then $f' \in H^1_\Sigma(F, \chi)$ defined by $f'(g) = f(cgc)$ is the zero cohomology class. However, $a \mapsto a^c$ gives an isomorphism of $H^1_\Sigma(F, \chi^{-1})$ onto $H^1_\Sigma(F, \chi)$ (cf. Lemma 7.1.5 in [Ber05]), so $f' \neq 0$ since $f \neq 0$. Now set $\rho^{0,\mathrm{opp}}_\tau = \rho'_\tau \otimes \Psi$. $\square$

Proof of Proposition 7.1. We first note that the ideal of reducibility of $\mathrm{tr}\rho^{0,\mathrm{opp}}_\tau$ equals I^0_τ. By Proposition 3.1 in [BK13] we know that $\ker \rho^0_\tau = \ker \mathrm{tr}\, \rho^0_\tau$ and $\ker \rho^{0,\mathrm{opp}}_\tau = \ker \mathrm{tr}\, \rho^{0,\mathrm{opp}}_\tau$ since R^0_τ/I^0_τ is finite by Lemma 2.17. Noting that $\rho^0_\tau(R^0_\tau[G_\Sigma]) = \rho^{0,\mathrm{opp}}_\tau(R^0_\tau[G_\Sigma])$ we therefore have an isomorphism

$$R^0_\tau[G_\Sigma]/\ker \mathrm{tr}\rho^0_\tau \cong R^0_\tau[G_\Sigma]/\ker \mathrm{tr}\rho^{0,\mathrm{opp}}_\tau.$$

This means that applying [BC09] Proposition 1.7.4 with ρ^0_τ and $\rho^{0,\mathrm{opp}}_\tau$ establishes condition (1) in [BC09] Proposition 1.7.5, so we can conclude that I^0_τ is principal. (Note that the generic irreducibility assumption in the propositions in [BC09] can be replaced by $\ker\rho = \ker T$). $\square$

Corollary 7.3. *Suppose Assumptions 2.4, 2.5, 2.16 and 4.2 are satisfied. Then the ideal J_τ of $\mathbf{T}_\tau$ is principal.*

Proof. Let $\phi_\tau : R^{\mathrm{tr},0}_\tau \twoheadrightarrow \mathbf{T}_\tau$ be the canonical surjection. By Corollary 3.8 we know that $\phi(I^{\mathrm{tr},0}_\tau) = J_\tau$. The claim follows from combining Theorem 6.2 with Proposition 7.1. $\square$

Remark 7.4. For other fields F (e.g. CM fields), principality of the ideal of reducibility would hold for conjugate self-dual representations (see Theorem 2.11 of [BK13]).

7.2. **Selmer groups.** In this subsection we discuss Assumptions 2.4 and 2.5 for certain characters χ, for which we will prove our main results.

Example 7.5. Fix a prime $\mathfrak{p}$ lying over p and denote by $i_\mathfrak{p}$ the fixed embedding $\overline{F} \hookrightarrow \overline{F}_\mathfrak{p}$ and $i_\infty : \overline{F} \hookrightarrow \mathbf{C}$. Let $\tau_2 = \chi$ be a p-adic Galois character of the following form: Let ϕ_1, ϕ_2 be two Hecke characters of infinity types z and z^{-1} respectively, and set $\phi = \phi_1/\phi_2$. Assume that Σ contains the set S_ϕ of primes dividing $\mathfrak{M}_1\mathfrak{M}_2\mathfrak{M}^c_1\mathfrak{M}^c_2 d_F p$, where $\mathfrak{M}_i$ denotes the conductor of ϕ_i. Let $\phi_\mathfrak{p} : G_\Sigma \to \mathcal{O}^*$ denote the $\mathfrak{p}$-adic Galois character corresponding to ϕ defined by $\phi_\mathfrak{p}(\mathrm{Frob}_\mathfrak{q}) =$

$i_{\mathfrak{p}}(i_\infty^{-1}(\phi(\varpi_{\mathfrak{q}})))$ for $\mathfrak{q} \notin \Sigma$. Set $\Psi := \phi_{\mathfrak{p}}\epsilon$, where ϵ is the p-adic cyclotomic character, and $\chi = \overline{\Psi}$. Assume also that if $\mathfrak{q} \in \Sigma$, then $\#k_{\mathfrak{q}} \not\equiv 1 \pmod{p}$. Under this assumption Assumption 2.4 will be satisfied by Corollary 9.7 in [BK13].

Let $L^{\mathrm{int}}(0, \phi)$ be the special L-value attached to ϕ as in [BK09]. Write W for $\mathrm{Hom}_{\mathcal{O}}(\Psi, 1) \otimes E/\mathcal{O}$. In this case we adapt Assumption 2.5 as follows:

Conjecture 7.6. $\#H_f^1(F, W) \leq \#\mathcal{O}/\varpi^m$, where $m = \mathrm{val}_\varpi(L^{\mathrm{int}}(0, \phi))$.

Note that this conjecture implies Assumption 2.5 for $\Sigma = \Sigma_p = \{\mathfrak{p} \mid p\}$. However, our conclusions hold for all sets $\Sigma \supset \Sigma_p$ for which $H_\Sigma^1 = H_f^1$ (see Lemma 5.6 of [BK13]).

Remark 7.7. Conjecture 7.6 can in many cases be deduced from the Main conjecture proven by Rubin [Rub91]. If $\phi^{-1} = \psi^2$ for ψ a Hecke character associated to a CM elliptic curve, then one can argue as follows. By Proposition 4.4.3 in [Dee99] and using that $H_f^1(F, W) \cong H_f^1(F, W^c)$, we have $\#H_f^1(F, W) = \#H_f^1(F, E/\mathcal{O}(\phi_{\mathfrak{p}}^{-1}))$. Thus we can use Corollary 4.3.4 in [Dee99] which together with the functional equation satisfied by $L(0, \phi)$ implies the desired inequality.

7.3. **Link of rings $\mathbf{T}_\tau$ to an actual Hecke algebra.** In this section we recall from [Ber09] an Eisenstein ideal bound for a Hecke algebra $\mathbf{T}(\Sigma)$ acting on cuspidal automorphic forms. We also recall results about Galois representations associated to such forms and use this to relate $\mathbf{T}(\Sigma)$ to the ring $\mathbf{T}$ defined in Section 3.

Continuing with the notation of Example 7.5 we assume $\phi = \phi_1/\phi_2$ is unramified. For an ideal $\mathfrak{N}$ in $\mathcal{O}_F$ and a finite place $\mathfrak{q}$ of F put $\mathfrak{N}_{\mathfrak{q}} = \mathfrak{N}\mathcal{O}_{F,\mathfrak{q}}$. We define

$$U^1(\mathfrak{N}_{\mathfrak{q}}) = \{k \in \mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{q}}) \mid \det(k) \equiv 1 \mod \mathfrak{N}_{\mathfrak{q}}\}.$$

Now put

$$(7.1) \qquad K_f := \prod_{\mathfrak{q}\mid\mathfrak{M}_1} U^1(\mathfrak{M}_{1,\mathfrak{q}}) \subset \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2(\mathbf{A}_f).$$

Denote by $S_2(K_f)$ the space of cuspidal automorphic forms of $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2(\mathbf{A})$ of weight 2, right-invariant under K_f (for more details see Section 3.1 of [Urb95]). Put $\gamma = \phi_1\phi_2$ and write $S_2(K_f, \gamma)$ for the forms with central character γ.

From now on, let Σ be a finite set of places of F containing

$$S_\phi := \{\mathfrak{q} \mid \mathfrak{M}_1\mathfrak{M}_1^c\} \cup \{\mathfrak{q} \mid pd_F\}.$$

We denote by $\mathbf{T}(\Sigma)$ the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(S_2(K_f, \gamma))$ generated by the Hecke operators $T_{\mathfrak{q}}$ for all places $\mathfrak{q} \notin \Sigma$.

Let $J(\Sigma) \subset \mathbf{T}(\Sigma)$ be the ideal generated by

$$\{T_{\mathfrak{q}} - \phi_1(\varpi_{\mathfrak{q}}) \cdot \#k_{\mathfrak{q}} - \phi_2(\varpi_{\mathfrak{q}}) \mid \mathfrak{q} \notin \Sigma\}.$$

Definition 7.8. Denote by $\mathfrak{m}(\Sigma)$ a maximal ideal of $\mathbf{T}(\Sigma)$ containing the image of $J(\Sigma)$. We set $\mathbf{T}_\Sigma := \mathbf{T}(\Sigma)_{\mathfrak{m}(\Sigma)}$. Moreover, set $J_\Sigma := J(\Sigma)\mathbf{T}_\Sigma$. We refer to J_Σ as the *Eisenstein ideal* of $\mathbf{T}_\Sigma$.

Theorem 7.9 ([Ber09] Theorem 14)**.** *Let $p > 3$ and assume $\ell \not\equiv \pm 1 \mod p$ for $\ell \mid d_F$. Let ϕ be an unramified Hecke character of infinity type $\phi^{(\infty)}(z) = z^2$. There exist Hecke characters ϕ_1, ϕ_2 with $\phi_1/\phi_2 = \phi$ such that their conductors are divisible only by ramified primes or inert primes not congruent to $\pm 1 \mod p$ and such that*

$$\#(\mathbf{T}_\Sigma/J_\Sigma) \geq \#(\mathcal{O}/(L^{\mathrm{int}}(0, \phi))).$$

The space $S_2(K_f, \psi)$ is isomorphic as a $G(\mathbf{A}_f)$-module to $\bigoplus \pi_f^{K_f}$ for automorphic representations π of a certain infinity type (see Theorem 7.10 below) with central character ψ. Here π_f denotes the restriction of π to $\mathrm{GL}_2(\mathbf{A}_f)$ and $\pi_f^{K_f}$ stands for the K_f-invariants.

For $g \in G(\mathbf{A}_f)$ we have the usual Hecke action of $[K_f g K_f]$ on $S_2(K_f)$ and $S_2(K_f, \psi)$. For primes $\mathfrak{q}$ such that the vth component of K_f is $\mathrm{GL}_2(\mathcal{O}_{F,v})$ we define $T_\mathfrak{q} = [K_f \begin{bmatrix} \varpi_\mathfrak{q} & \\ & 1 \end{bmatrix} K_f]$.

Combining the work of Taylor, Harris, and Soudry with results of Friedberg-Hoffstein and Laumon/Weissauer, one can show the following (see [BH07] for general case of cuspforms of weight k and forthcoming work for general CM fields by C. P. Mok (with a similar assumption on the central character) and Harris-Taylor-Thorne-Lan (without such an assumption)):

Theorem 7.10 ([BH07] Theorem 1.1). *Given a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_F)$ with π_∞ isomorphic to the principal series representation corresponding to*

$$\begin{bmatrix} t_1 & * \\ & t_2 \end{bmatrix} \mapsto \left(\frac{t_1}{|t_1|} \right) \left(\frac{|t_2|}{t_2} \right)$$

and cyclotomic central character ψ (i.e., $\psi^c = \psi$), let Σ_π denote the set consisting of the places of F lying above p, the primes where π or π^c is ramified, and the primes ramified in $F/\mathbf{Q}$.

Then there exists a finite extension E of $F_\mathfrak{p}$ and a Galois representation

$$\rho_\pi : G_{\Sigma_\pi} \to \mathrm{GL}_2(E)$$

such that if $\mathfrak{q} \notin \Sigma_\pi$, then ρ_π is unramified at $\mathfrak{q}$ and the characteristic polynomial of $\rho_\pi(\mathrm{Frob}_\mathfrak{q})$ is $x^2 - a_\mathfrak{q}(\pi)x + \psi(\varpi_\mathfrak{q})(\#k_\mathfrak{q})$, where $a_\mathfrak{q}(\pi)$ is the Hecke eigenvalue corresponding to $T_\mathfrak{q}$. Moreover, ρ_π is absolutely irreducible.

Regarding the crystallinity of the representations ρ_π we make the following conjecture (see Section 2.1 for the definition of a short crystalline Galois representation, and note that we assume $p > 3$):

Conjecture 7.11. *If π is unramified at $\mathfrak{q} \mid p$ then $\rho_\pi|_{G_\mathfrak{q}}$ is crystalline and short.*

This has now been proven in many cases by A. Jorza [Jor10]. Note that for the choice of characters ϕ_i as in Theorem 7.9 the cuspforms occurring in $S_2(K_f, \psi)$ are unramified at $\mathfrak{q} \mid p$.

Definition 7.12. Let χ be the mod ϖ reduction of $\phi_\mathfrak{p}\epsilon$. We now define the subsets Π as the set of (strict equivalence classes of Galois) deformations of residual representations of the form (2.1), one for each ρ_π associated to an automorphic representation π occurring in $S_2(K_f, \gamma)_{\mathfrak{m}(\Sigma)}$ and define Π_τ to be the subset with residual representation isomorphic to (a twist by $\phi_{2,\mathfrak{p}}$ of) $\tau = \begin{bmatrix} 1 & * \\ 0 & \chi \end{bmatrix}$. (Note that $\Pi_\tau \cap \Pi_{\tau'} = \emptyset$ for $\tau \not\cong \tau'$ by Proposition 3.2).

For every τ one has the natural surjective map $\mathbf{T} \twoheadrightarrow \mathbf{T}_\tau$ resulting from the surjections $R^{\mathrm{ps}} \twoheadrightarrow R_\tau^{\mathrm{tr}}$.

Lemma 7.13. *If Π is the set of modular deformations defined above then the ring $\mathbf{T} \subset \prod_{\rho_\pi \in \Pi} \mathcal{O}$ defined in the previous section can be identified with the Hecke*

algebra $\mathbf{T}_\Sigma$. Furthermore, $\mathbf{T}_\tau$ agrees with the quotient of $\mathbf{T}_\Sigma$ acting on the subspace of automorphic forms spanned by eigenforms π such that $\rho_\pi \in \Pi_\tau$.

Proof. We will just prove the first part (concerning $\mathbf{T}$ and $\mathbf{T}_\Sigma$ - the proof for $\mathbf{T}_\tau$ being analogous). We define the following $\mathcal{O}$-algebra map:

$$f : \mathbf{T}_\Sigma \to \mathbf{T} \subset \prod_{\rho_\pi \in \Pi} \mathcal{O} : \ T_{\mathfrak{q}} \mapsto (a_{\mathfrak{q}}(\pi))_{\rho_\pi \in \Pi},$$

where we use that $a_{\mathfrak{q}}(\pi) = \operatorname{tr} \rho_\pi(\operatorname{Frob}_{\mathfrak{q}})$ and therefore $(a_{\mathfrak{q}}(\pi))_{\rho_\pi \in \Pi} = T(\operatorname{Frob}_{\mathfrak{q}})$.

We check that this map f is injective: By definition, $\mathbf{T}_\Sigma \hookrightarrow \oplus_{\rho_\pi \in \Pi} \operatorname{End}_{\mathcal{O}}(V_\pi^{K_f})$, where we denote by V_π the representation space of π. Since T_v acts on π by $a_v(\pi)$, the image in each summand is given by the $\mathcal{O}$-algebra generated by the $a_v(\pi)$'s. Hence injectivity follows.

For surjectivity first note that $f(\mathbf{T}_\Sigma) \supset S := \{T(\operatorname{Frob}_{\mathfrak{q}}) \mid \mathfrak{q} \notin \Sigma\}$. Since f is injective let us identify $\mathbf{T}_\Sigma$ with $f(\mathbf{T}_\Sigma)$. Clearly $\mathbf{T}_\Sigma$ is local, complete with respect to its maximal ideal $\mathfrak{m}_\Sigma := \mathfrak{m}(\Sigma)\mathbf{T}_\Sigma$ and $\mathbf{T}$ has the same properties derived from the properties of R^{ps}. Moreover, looking at the residue fields we see that $\mathfrak{m}_{\mathbf{T}} \cap \mathbf{T}_\Sigma = \mathfrak{m}_\Sigma$, so the $\mathfrak{m}_\Sigma$-adic topology on $\mathbf{T}_\Sigma$ is the subspace topology induced from the $\mathfrak{m}_{\mathbf{T}}$-adic topology on $\mathbf{T}$. Then using Theorem 8.1 in [Mat89] again we see that the closure of $\mathbf{T}_\Sigma$ in $\mathbf{T}$ equals the completion of $\mathbf{T}_\Sigma$. But since $\mathbf{T}_\Sigma$ is already complete and $\mathbf{T}$ is topologically generated by S (i.e., the closure of S in $\mathbf{T}$ equals $\mathbf{T}$), we conclude that $\mathbf{T}_\Sigma = \mathbf{T}$, hence we are done. $\square$

8. Main result

In this section we will state our main theorems (Theorems 8.1, 8.2 and 8.5) for the two-dimensional Galois representations over imaginary quadratic fields considered in section 7. In this case many of the assumptions introduced throughout the paper can be proven to hold. However, we would like to stress that the conclusions are still valid if instead one assumes Assumptions 2.4, 2.5 and 4.2. To make this section self-contained we will repeat all the assumptions in the case of an imaginary quadratic field which were made in section 7.

Let F be an imaginary quadratic field, $p > 3$, $p \nmid \#\operatorname{Cl}_F d_F$, and assume $\ell \not\equiv \pm 1 \mod p$ for $\ell \mid d_F$. Let ϕ be an unramified Hecke character of infinity type $\phi^{(\infty)}(z) = z^2$ and write χ for the mod ϖ-reduction of $\phi_{\mathfrak{p}}\epsilon$. Furthermore, assume that Σ contains the set of places S_ϕ (containing the primes dividing the conductors of the two auxiliary characters ϕ_i from Theorem 7.9).

Let $\tau : G_\Sigma \to \operatorname{GL}_2(\mathbf{F})$ be a non-semi-simple representation of the form

$$\tau = \begin{bmatrix} 1 & * \\ 0 & \chi \end{bmatrix}.$$

Theorem 8.1. *Suppose Conjecture 7.6 is satisfied (see Remark 7.7). There exists an $\mathbf{F}$-basis $\mathcal{B}$ of $H^1_\Sigma(F, \chi^{-1})$ which is modular, i.e., such that if $b \in \mathcal{B}$ and $f : G_\Sigma \to \mathbf{F}(\chi^{-1})$ is a cocycle representing b, then the residual representation*

$$\rho_f : G_\Sigma \to \operatorname{GL}_2(\mathbf{F}), \quad \rho_f(g) = \begin{bmatrix} 1 & f(g)\chi(g) \\ 0 & \chi(g) \end{bmatrix}$$

is (up to a twist) the reduction (mod ϖ) of a representation $\rho_\pi : G_\Sigma \to \operatorname{GL}_2(\mathcal{O})$ attached to an automorphic representation π of $\operatorname{GL}_2(\mathbf{A}_F)$.

Proof. This follows from Corollary 4.8. Note that Assumption 4.2 is satisfied by Theorem 7.9 and Lemma 7.13 (which implies that $J \subseteq J_\Sigma$ since $T \mod J_\Sigma$ is the sum of two characters; in fact, $J = J_\Sigma$ by the proof of Lemma 3.9) and Conjecture 7.6 replaces Assumption 2.5 in Corollary 4.8. $\qquad\square$

From now on assume that Conjecture 7.6 is satisfied. Then by Theorem 8.1 the Selmer group $H^1_\Sigma(F, \chi^{-1})$ has a modular basis $\mathcal{B}$. From now on assume also that $\mathcal{B}$ is indeed projectively unique. Then, as discussed in section 5, the set of modular residual representations (i.e., those residual representations $\tau' = \begin{bmatrix} 1 & * \\ 0 & \chi \end{bmatrix}$ for which $\Pi_{\tau'} \neq \emptyset$) is in one-to-one correspondence with the elements of $\mathcal{B}$ given by sending a modular extension to the corresponding residual representation. In particular, the extension corresponding to τ must be (up to scaling) among the elements of $\mathcal{B}$. Hence, by rescaling one of the elements of $\mathcal{B}$ and reordering we may assume that $\mathcal{B} = \{e_1 = e_\tau, e_2, \ldots, e_s\}$.

We assume further now that if $\mathfrak{q} \in \Sigma$, then $\#k_\mathfrak{q} \not\equiv 1 \pmod{p}$. Using the principality of the ideal of reducibility (see Proposition 7.1) we can prove a modularity result for residually reducible representations:

Theorem 8.2. *Assume that the Galois representations ρ_π for π occurring in $S_2(K_f, \Psi)_{\mathfrak{m}(\Sigma)}$ (for notation please see Section 7.3) are crystalline at $v \mid p$. Also suppose that $\#H^1_f(F, E/\mathcal{O}((\phi_\mathfrak{p}\epsilon))^{-1}) \leq \#\mathcal{O}/\varpi^m$, where $m = \mathrm{val}_\varpi(L^{\mathrm{int}}(0, \phi))$.*

Let $\rho : G_\Sigma \to \mathrm{GL}_2(E)$ be a continuous, irreducible representation which is crystalline at $\mathfrak{p} \mid p$ and write $\tau : G_\Sigma \to \mathrm{GL}_2(\mathbf{F})$ for its $\mod \varpi$ reduction with respect to some lattice in E^2. Suppose that $\tau^{\mathrm{ss}} \cong 1 \oplus \chi$. Assume that the sets $\Phi_{\tau'}$ for $\tau' \in \mathcal{B}$ are finite. Then ρ is modular, i.e., there exists an automorphic representation π' of $\mathrm{GL}_2(\mathbf{A}_F)$ such that $\rho \cong \rho_{\pi'}$.

Remark 8.3. As discussed in Remark 7.7 and the paragraph following Conjecture 7.11 the first two assumptions should be satisfied in the majority of cases by work of Jorza and Rubin.

Remark 8.4. We note that when $\mathrm{rk}_\mathcal{O}\mathbf{T} = \dim_\mathbf{F} H^1_\Sigma(F, \mathrm{Hom}(\tau_2, \tau_1))$ it is easy to see that $H^1_\Sigma(F, \mathrm{Hom}(\tau_2, \tau_1))$ has a projectively unique modular basis. It may be possible to check numerically in specific examples if $\mathrm{rk}_\mathcal{O}\mathbf{T} = \dim_\mathbf{F} H^1_\Sigma(F, \mathrm{Hom}(\tau_2, \tau_1))$ holds, but we have unfortunately not been able to carry this out.

Proof. This is a summary of the arguments carried out so far. As in the proof of Theorem 8.1 we note that Assumption 4.2 is satisfied. Also Assumption 2.4 is satisfied by Corollary 9.7 in [BK13] (see also discussion in Example 7.5). Let $R^{\mathrm{tr},0}_i$, $\mathbf{T}_i$ be as before the $\mathcal{O}$-subalgebra of $R^0_{\tau_i}$ (defined as the image of $R^{\mathrm{red}}_{\tau_i}$ inside $\prod_{\mathfrak{p} \in \mathcal{P}_{\tau_i}} \mathcal{O}$) generated by traces and the Hecke algebra (respectively) corresponding to e_i. We denote the corresponding ideals of reducibility and the Eisenstein ideal by $I^{\mathrm{tr},0}_i$ and J_i respectively. We get for every $i = 1, 2, \ldots, s$ a commutative diagram with surjective arrows:

(8.1)
$$
\begin{array}{ccc}
R^{\mathrm{tr},0}_i & \xrightarrow{\phi_i} & \mathbf{T}_i \\
\downarrow & & \downarrow \\
R^{\mathrm{tr},0}_i/I^{\mathrm{tr},0}_i & \longrightarrow & \mathbf{T}_i/J_i.
\end{array}
$$

By Theorem 6.3 and Corollary 7.3 the bottom map is an isomorphism for every $i = 1, 2, \ldots, s$.

By Theorem 6.2 we obtain a canonical map $R_\tau^{\mathrm{red}} \to R_\tau^{\mathrm{tr},0}$ which factors through an isomorphism $R_\tau^0 \cong R_\tau^{\mathrm{tr},0}$. We know by Proposition 7.1 that I_τ^0 is principal. Using this and diagram (8.1) for $i = \tau$ (and our conclusion that the bottom map in (8.1) is an isomorphism) we can apply Theorem 4.1 from [BK13] to conclude that ϕ_τ is an isomorphism. Then any $\mathcal{O}$-algebra map $R_\tau \twoheadrightarrow \mathcal{O}$ factors through $R_\tau^0 \cong \mathbf{T}_\tau$, so any deformation to $\mathcal{O}$ is modular. $\qquad\square$

Let Ψ be as in example 7.5. The property of $H_\Sigma^1(F, \chi^{-1})$ of having a projectively unique modular basis can be replaced by the assumption that $H_\Sigma^1(F, \Psi^{-1} \otimes E/\mathcal{O})$ is an $\mathbf{F}$-vector space.

Theorem 8.5. *Let the notation and assumptions be the same as in Theorem 8.2 except that we do not demand that $H_\Sigma^1(F, \chi^{-1})$ has a projectively unique modular basis, but instead assume that $\varpi H_\Sigma^1(F, \Psi^{-1} \otimes E/\mathcal{O}) = 0$. Then, as before, ρ is modular.*

Proof. We need to reprove Theorem 6.3. Our assumption that ϖ annihilates the Selmer group along with injectivity of ι in the proof of Theorem 6.3 implies that ϖ annihilates $\mathcal{T}_\tau / I_\tau^{\mathrm{tr},0} \mathcal{T}_\tau$. We showed in the proof of Theorem 6.2 that $\mathcal{T}_\tau \cong R_\tau^{\mathrm{tr},0}$. Hence we conclude that ϖ annihilates $R_\tau^{\mathrm{tr},0} / I_\tau^{\mathrm{tr},0}$. As discussed towards the end of section 2.5, the module $R_\tau^{\mathrm{tr},0} / I_\tau^{\mathrm{tr},0}$ is a cyclic $\mathcal{O}$-module, hence we must have $R_\tau^{\mathrm{tr},0} / I_\tau^{\mathrm{tr},0} \cong \mathbf{F}$. Since $\mathbf{T}_\tau / J_\tau$ is a non-zero $\mathcal{O}$-module, this implies that the map

$$R_\tau^{\mathrm{tr},0} / I_\tau^{\mathrm{tr},0} \twoheadrightarrow \mathbf{T}_\tau / J_\tau$$

must be injective. $\qquad\square$

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