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Aspects of hairy black holes in spontaneously-broken Einstein-Yang-Mills systems: Stability analysis and Entropy considerations

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Abstract

We analyze (3+1)-dimensional black-hole space-times in spontaneously broken Yang-Mills gauge theories that have been recently presented as candidates for an evasion of the scalar-no-hair theorem. Although we show that in principle the conditions for the no-hair theorem do not apply to this case, however we prove that the 'spirit' of the theorem is not violated, in the sense that there exist instabilities, in both the sphaleron and gravitational sectors. The instability analysis of the sphaleron sector, which was expected to be unstable for topological reasons, is performed by means of a variational method. As shown, there exist modes in this sector that are unstable against linear perturbations. Instabilities exist also in the gravitational sector. A method for counting the gravitational unstable modes, which utilizes a catastrophe-theoretic approach is presented. The rôle of the catastrophe functional is played by the mass functional of the black hole. The Higgs vacuum expectation value (v.e.v.) is used as a control parameter, having a critical value beyond which instabilities are turned on. The (stable) Schwarzschild solution is then understood from this point of view. The catastrophe-theory appproach facilitates enormously a universal stability study of non-Abelian black holes, which goes beyond linearized perturbations. Some elementary entropy considerations are also presented that support the catastrophe theory analysis, in the sense that 'high-entropy' branches of solutions are shown to be relatively more stable than 'low-entropy' ones. As a partial result of this entropy analysis, it is also shown that there exist *logarithmic* divergencies in the entropy of matter (scalar) fields near the horizon, which are up and above the linear divergencies, and, unlike them, they cannot be absorbed in a renormalization of the gravitational coupling constant of the theory. The associated part of the entropy violates the classical Bekenstein-Hawking formula which is a proportionality relation between black-hole entropy and horizon area. Such logarithmic divergencies, which are associated with the presence of non-abelian gauge and Higgs fields, persist in the 'extreme case', where linear divergencies disappear.

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1 Introduction

The surprising discovery of the Bartnik-McKinnon (BM) non-trivial particle-like structure [1] in the Einstein-Yang-Mills system opened many possibilities for the existence of non-trivial solutions to Einstein-non-Abelian-gauge systems. Indeed, soon after its discovery, many other self-gravitating structures with non-Abelian gauge fields have been discovered [3]. These include black holes with non-trivial hair, thereby leading to the possibility of evading the no-hair conjecture [2]. The physical reason for the existence of these classical solutions is the 'balance' between the non-Abelian gauge-field repulsion and the gravitational attraction. Such a balance allows for dressing black hole solutions by non-trivial configurations (outside the horizon) of fields that are not associated with a Gauss-law, thereby leading to an 'apparent' evasion of the no-hair conjecture.

Among such black-hole solutions, a physically interesting case is that of a spontaneously broken Yang-Mills theory in a non-trivial black-hole space-time (EYMH) [4]. This system has been recently examined from a stability point of view, and found to possess an instability [5], thereby making the physical importance of the solution rather marginal, but also indicating another dimension of the no-hair conjecture, not considered in the original analysis, that of stability.

In this article, we shall give more details of this stability considerations by extending the analysis to incorporate counting of the unstable modes, and going beyond the linear-stability case by employing catastrophe theory [6] in order to analyse instabilities in the gravitational sector of the solution. Catastrophe theory is a powerful mathematical tool to study or explain a variety of change of states in nature, and in particular a *discontinuous* change of states that occurs eventually despite a gradual (smooth) change of certain parameters of the system. In the case at hand, the catastrophe functional, which exhibits a discontinuous change in its behaviour, will be the mass of the black hole space time, whilst the control parameter, whose smooth change turns on the catastrophe at a given critical value, will be the vacuum expectation value (v.e.v.) of the Higgs field. The advantage of using the v.e.v. of the Higgs field as the control parameter, rather than the horizon radius as was done in [6], is that it will allow us to relate the stability of the EYMH black holes to that of the Schwarzschild solution, which is well known to be stable. The type of catastrophe encountered will be that of a *fold* catastrophe. The catastrophe-theoretic approach allows for a universal stability study of non-abelian black hole solutions that goes beyond linearised perturbations; the particular use of the Higgs v.e.v. as a control parameter in the case of the EYMH systems allows an *exact* counting of the unstable modes.

As part of our analysis, we shall make an attempt to associate the above catastrophe-theoretic considerations with some 'thermodynamic/information-theoretic' aspects of black hole physics, and in particular with the entropy of the black hole. By computing explicitly the entropy of quantum fluctuations of (scalar) matter fields near the horizon we shall show that 'high-entropy' branches of the solution possess less unstable modes (in the gravitational sector) than the 'low-entropy' ones. As a partial, but not less important, result of this part of our analysis, we shall also show that the entropy of the black hole possess linear *and* logarithmic divergencies. The linear divergencies do not violate the Bekenstein-Hawking formula relating entropy to the classical horizon area. The only difference is the divergent proportionality factors in front, which, however, can be absorbed in a conjectured renormalization of Newton's constant in the model [7, 8]. This is not the case with the logarithmic divergencies though. The latter persist even in 'extreme black-hole' cases, where the linear divergencies disappear. They clearly violate the Bekenstein-Hawking formula. In our case they owe their presence to the non-Abelian gauge and Higgs fields. The presence of logarithmic divergencies in black hole physics has been noted in ref. [8], but only in examples involving truncation from (3+1)-dimensional space-times to (1 + 1) dimensions, and in that reference their presence had been attributed to this bad truncation of the four-dimensional black hole spectrum. Later on, however, such logarithmic divergencies have been confirmed to exist in string-inspired dilatonic black holes in (3+1) dimensions [9]. Their presence in our EYMH system, and in general in non-Abelian black holes as we shall show, indicates that such logarithmic divergencies are *generic* in black hole space-times with non-conventional hair, and probably indicates information loss, even in extreme cases, associated with the presence of space time boundaries. This probably implies that the entropy of the black hole is not only associated with classical geometric factors, but is a much more complicated phenomenon related to information carried by the various (internal) black hole states. The latter phenomenon could be associated with, and may be offer ways out of, the usual difficulties of reconciling quantum mechanics with canonical quantum gravity.

The structure of the article is as follows. In section 2 we shall discuss the no hair conjecture for black holes space-times with non-trivial scalar field configurations by following a modern approach due to Bekenstein [10]. We shall show that the proof of the no-hair theorem fails for the case of the EYMH system, in accordance with the explicit solution found in ref. [4]. In section 3 we shall present a stability analysis of the system based on linear perturbations. We shall demonstrate the existence of instabilities in the sphaleron sector, following a variational approach which is an extension of the approach of Volkov and Gal'tsov [11] to study particle-like solutions. We shall also present arguments for counting the unstable modes in the sphaleron sector of the theory. In section 4 we shall present a method for counting the unstable modes in the gravitational sector by going beyond the linearised-perturbation analysis using catastrophe theory, with the mass functional of the black hole as the catastrophe functional and the Higgs v.e.v. as the control parameter. In section 5, in connection with the latter approach, we shall estimate the entropy of the various branches of the solution using a WKB approximation. We shall show that the high-entropy branch of solutions is relatively more stable (in the sense of possesssing fewer unstable modes) than the low-entropy branch. As we have already mentioned, we shall also discuss the existence of logarithmic divergencies in the entropy, associated with the presence of non-trivial hair in the black hole, in certain extreme cases, and we shall argue about an explicit violation of the classical Bekenstein-Hawking entropy formula, indicating a different (information oriented) rôle of the black hole entropy. Conclusions and outlook will be presented in section 6. Some technical aspects of our approach will be discussed in two appendices.

2 Bypassing Bekenstein's no-hair theorem in EYMH systems

Recently, Bekenstein presented a modern elegant proof of the no-hair theorem for black holes, which covers a variety of cases with scalar fields [10]. The theorem is formulated in such a way so as to rule out a multicomponent scalar field dressing an asymptotically flat, static, spherically-symmetric black hole. The basic assumption of the theorem is that the scalar field is minimally coupled to gravity and bears a non-negative energy density as seen by any observer, and the proof relies on very general principles, such as energy-momentum conservation and the Einstein equations. From the positivity assumption and the conservation equations for the energy momentum tensor T_{MN} of the theory, $\nabla^M T_{MN} = 0$, one obtains for a spherically-symmetric space-time background the condition that near the horizon the radial component of the energy-momentum tensor and its first derivative are negative

$$T^r_r < 0, \qquad (T^r_r)' < 0 \tag{1}$$

with the prime denoting differentiation with respect to r. This implies that in such systems there must be regions in space, outside the horizon where both quantities in (1) change sign. This contradicts the results following from Einstein's equations though [10], and this *contradiction* constitutes the proof of the no-hair theorem, since the only allowed non-trivial configurations are Schwarzschild black holes. We note, in passing, that there are known exceptions to the original version of the no-hair theorem [2], such as conformal scalar fields coupled to gravity, which come from the fact that in such theories the scalar fields diverge at the horizon of the black hole [12].

The interest for our case is that the theorem rules out the existence of non-trivial hair due to a Higgs field with a double (or multiple) well potential, as is the case for spontaneous symmetry breaking. Given that stability issues are not involved in the proof, it is of interest to reconcile the results of the theorem with the situation in our case of EYMH systems, where at least we know that an explicit solution with non-trivial hair exists [4], albeit unstable [5]. As we shall show below, the formal reason for bypassing the modern version of the no-hair theorem [10] lies in the violation of the key relation among the components of the stress tensor, $T^t_t = T^\theta_\theta$, shown to hold in the case of ref. [10]. The physical reason for the 'elusion' of the above no-hair conjecture lies in the fact that the presence of the repulsive non-Abelian gauge interactions balance the gravitational attraction, by producing terms that make the result (1) inapplicable in the present case. Below we shall demonstrate this in a mathematically rigorous way.

To this end, consider the EYMH theory with Lagrangian

$$\mathcal{L}_{EYMH} = -\frac{1}{4\pi}\left\{\frac{1}{4}|F_{MN}|^2 + \frac{1}{8}\phi^2|A_M|^2 + \frac{1}{2}|\partial_M\phi|^2 + V(\phi)\right\} \tag{2}$$

where A_M denotes the Yang-Mills field, F_{MN} its field strength, ϕ is the Higgs field and $V(\phi)$ its potential. All the indices are contracted with the help of the background gravitational tensor g_{MN}. In the spirit of Bekenstein's modern version of the no-hair theorem, we now examine the energy-momentum tensor of the model (2). It can be written in the form

$$8\pi T_{MN} = -\mathcal{E}g_{MN} + \frac{1}{4\pi}\left\{F_{MP}F_N{}^P + \frac{\phi^2}{4}A_MA_N + \partial_M\phi\partial_N\phi\right\} \tag{3}$$

with $\mathcal{E} \equiv -\mathcal{L}_{EYMH}$.

Consider, now, an observer moving with a four-velocity u^M . The observer sees a local energy density

$$\rho = \mathcal{E} + \frac{1}{4\pi}\left\{u^MF_{MP}F_N{}^Pu^N + \frac{\phi^2}{4}(u^MA_M)^2 + (u^M\partial_M\phi)^2\right\}, \qquad u^Mu_M = -1. \tag{4}$$

To simplify the situation let us consider a space-time with a time-like killing vector, and suppose that the observer moves along this killing vector. Then $u^M\partial_M\phi = 0$ and by an appropriate gauge choice $u^MA_M = 0 = u^MF_{MN}$. This gauge choice is compatible with the spherically-symmetric ansatz for A_M of ref. [4]. Under these assumptions,

$$\rho = \mathcal{E} \tag{5}$$

and the requirement that the local energy density as seen by any observer is non-negative implies

$$\mathcal{E} \geq 0. \tag{6}$$

We are now in position to proceed with the announced proof of the bypassing of the no-hair theorem of ref. [10] for the EYMH black hole of ref. [4]. To this end we consider a spherically-symmetric ansatz for the space-time metric g_{MN}, with an invariant line element of the form

$$ds^2 = -e^\Gamma dt^2 + e^\Lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad \Gamma = \Gamma(r),\ \Lambda = \Lambda(r). \tag{7}$$

To make the connection with the black hole case we further assume that the space-time is asymptotically flat.

From the conservation of the energy-momentum tensor, following from the invariance of the effective action under general co-ordinate transformations, one has for the r-component of the conservation equation

$$[(-g)^{\frac{1}{2}}T^r_r]' - \frac{1}{2}(-g)^{\frac{1}{2}}\left(\frac{\partial}{\partial r}g_{MN}\right)T^{MN} = 0 \tag{8}$$

with the prime denoting differentiation with respect to r. The spherical symmetry of the space time implies that $T^\theta_\theta = T^\varphi_\varphi$. Hence, (8) can be written as

$$(e^{\frac{\Gamma+\Lambda}{2}}r^2T^r_r)' - \frac{1}{2}e^{\frac{\Gamma+\Lambda}{2}}r^2\left[\Gamma' T^t_t + \Lambda' T^r_r + \frac{4}{r}T^\theta_\theta\right] = 0. \tag{9}$$

Observing that the terms containing Λ cancel, and integrating over the radial coordinate r from the horizon r_h to a generic distance r, one obtains

$$T^r_r(r) = \frac{e^{-\frac{\Gamma}{2}}}{2r^2}\int_{r_h}^{r} dr e^{\frac{\Gamma}{2}}r^2\left[\Gamma' T^t_t + \frac{4}{r}T^\theta_\theta\right] \tag{10}$$

Note that the assumption that scalar invariants, such as $T_{MN}T^{MN}$ are finite on the horizon (in order that the latter is regular), implies that the boundary terms on the horizon vanish in (10).

It is then straightforward to obtain

$$(T^r_r)' = \frac{1}{2}\left[\Gamma' T^t_t + \frac{4}{r}T^\theta_\theta\right] - \frac{e^{-\frac{\Gamma}{2}}}{r^2}(e^{\frac{\Gamma}{2}}r^2)' T^r_r. \tag{11}$$

Next, we consider Yang-Mills fields of the form [4]

$$A = (1+\omega(r))[-\hat{\tau}_\phi d\theta + \hat{\tau}_\theta \sin\theta d\varphi] \tag{12}$$

where τ_i, $i = r, \theta, \varphi$ are the generators of the $SU(2)$ group in spherical-polar coordinates. Ansatz (12) yields

$$\begin{aligned} T^t_t &= -\mathcal{E} \\ T^r_r &= -\mathcal{E} + \mathcal{F} \\ T^\theta_\theta &= -\mathcal{E} + \mathcal{J} \end{aligned} \tag{13}$$

with (see Appendix A for details of the relevant quantities),

$$\begin{aligned} \mathcal{F} &\equiv \frac{e^{-\Lambda}}{4\pi}\left[\frac{2\omega'^2}{r^2} + \phi'^2\right] \\ \mathcal{J} &\equiv \frac{1}{4\pi}\left[\frac{\omega'}{r^2}e^{-\Lambda} + \frac{(1-\omega^2)^2}{r^4} + \frac{\phi^2}{4r^2}(1+\omega^2)\right]. \end{aligned} \tag{14}$$

Substituting (14) in (13) yields

$$T^r_r(r) = \frac{e^{-\frac{\Gamma}{2}}}{r^2}\int_{r_h}^{r}\left\{-(e^{\frac{\Gamma}{2}}r^2)'\mathcal{E} + \frac{2}{r}\mathcal{J}\right\}dr \tag{15}$$

$$(T^r_r)'(r) = -\frac{e^{-\frac{\Gamma}{2}}}{r^2}(e^{\frac{\Gamma}{2}}r^2)'\mathcal{F} + \frac{2}{r}\mathcal{J} \tag{16}$$

where $\mathcal{E}$ is expressed as

$$\mathcal{E} = \frac{1}{4\pi}\left[\frac{(\omega')^2}{r^2}e^{-\Lambda} + \frac{(1-\omega^2)^2}{2r^4} + \frac{\phi^2(1+\omega)^2}{4r^2} + \frac{1}{2}(\phi')^2e^{-\Lambda} + \frac{\lambda}{4}(\phi^2 - v^2)^2\right]. \tag{17}$$

We now turn to the Einstein equations for the first time, following the analysis of ref. [10]. Our aim is to examine whether there is a contradiction with the requirement of the non-negative energy density. These equations read for our system

$$\begin{aligned} e^{-\Lambda}(r^{-2} - r^{-1}\Lambda') - r^{-2} &= 8\pi T^t_t = -8\pi\mathcal{E} \\ e^{-\Lambda}(r^{-1}\Gamma' + r^{-2}) - r^{-2} &= 8\pi T^r_r. \end{aligned} \tag{18}$$

Integrating out the first of these yields

$$e^{-\Lambda} = 1 - \frac{8\pi}{r}\int_{r_h}^{r}\mathcal{E}r^2dr - \frac{2\mathcal{M}_{\prime}}{r} \tag{19}$$

where $\mathcal{M}_{\prime}$ is a constant of integration.

The requirement for asymptotic flatness of space-time implies the following asymptotic behaviour for the energy-density functional $\mathcal{E} \sim O(r^{-3})$ as $r \to \infty$, so that $\Lambda \sim O(r^{-1})$. In order that $e^{\Lambda} \to \infty$ at the horizon, $r \to r_h$, $\mathcal{M}_{\prime}$ is fixed by

$$\mathcal{M}_{\prime} = \frac{r_h}{2}. \tag{20}$$

The second of the equations (18) can be rewritten in the form

$$e^{-\frac{\Gamma}{2}}r^{-2}(r^2e^{\frac{\Gamma}{2}})' = \left[4\pi r T^r_r + \frac{1}{2r}\right]e^{\Lambda} + \frac{3}{2r}. \tag{21}$$

Consider, first, the behaviour of T^r_r as $r \to \infty$. Asymptotically, $e^{\frac{\Gamma}{2}} \to 1$, and so the leading behaviour of $(T^r_r)'$ is

$$(T^r_r)' = \frac{2}{r}[\mathcal{J} - \mathcal{F}]. \tag{22}$$

We, now, note that the fields ω and ϕ have masses $\frac{v}{2}$ and $\mu = \sqrt{\lambda}v$ respectively. From the field equations and the requirement of finite energy density their behaviour at infinity must then be

$$\begin{aligned} \omega(r) &\sim -1 + ce^{-\frac{v}{2}r} \\ \phi(r) &\sim v + ae^{-\sqrt{2}\mu r} \end{aligned} \tag{23}$$

for some constants c and a. Hence, the leading asymptotic behaviour of $\mathcal{J}$ and $\mathcal{F}$ is

$$\begin{aligned}\mathcal{J} &\sim \frac{1}{4\pi}\left[\frac{c^2v^2}{4r^2}e^{-vr}+\frac{2c^2}{r^4}e^{-vr}+\frac{v^2c^2}{4r^2}e^{-vr}\right]\\ \mathcal{F} &\sim \frac{1}{4\pi}\left[\frac{c^2v^2}{2r^2}e^{-vr}+2a^2\mu^2e^{-\sqrt{2}\mu r}\right]\end{aligned} \tag{24}$$

since $e^{-\Lambda}\to 1$ asymptotically.

The leading behaviour of $(T^r_r)'$, therefore, is

$$(T^r_r)'\sim\frac{1}{4\pi}\left[\frac{2c^2}{r^4}e^{-vr}-2a^2\mu^2e^{-2\sqrt{2}\mu r}\right]. \tag{25}$$

There are two possible cases: (i) $2\sqrt{2}\mu > v$ (corresponding to $\lambda > 1/8$); in this case $(T^r_r)' > 0$ asymptotically, (ii) $2\sqrt{2}\mu \le v$ (corresponding to $\lambda \le 1/8$) ; then, $(T^r_r)' < 0$ asymptotically.

Since $\mathcal{J}$ vanishes exponentially at infinity, and $\mathcal{E}\sim O[r^{-3}]$ as $r\to\infty$, the integral defining $T^r_r(r)$ converges as $r\to\infty$ and $|T^r_r|$ decreases as r^{-2}.

Thus, in case (i) above, T^r_r is negative and increasing as $r\to\infty$, and in case (ii) T^r_r is positive and decreasing.

Now turn to the behaviour of T^r_r at the horizon. When $r\simeq r_h$, $\mathcal{E}$ and $\mathcal{J}$ are both finite, and Γ' diverges as $1/(r-r_h)$. Thus the main contribution to T^r_r as $r\simeq r_h$ is

$$T^r_r(t)\simeq\frac{e^{-\Gamma/2}}{r^2}\int_{r_h}^{r}(-e^{\Gamma/2}r^2)\frac{\Gamma'}{2}\mathcal{E}dr \tag{26}$$

which is finite.

At the horizon, $e^{\Gamma}=0$; outside the horizon, $e^{\Gamma}>0$. Hence $\Gamma'>0$ sufficiently close to the horizon, and, since $\mathcal{E}\ge 0$, $T^r_r<0$ for r sufficiently close to the horizon.

Since $\mathcal{F}\sim O[r-r_h]$ at $r\simeq r_h$, $(T^r_r)'$ is finite at the horizon and the leading contribution is

$$(T^r_r)'(r_h)\simeq -\frac{\Gamma'}{2}\mathcal{F}+\frac{2}{r}\mathcal{J}. \tag{27}$$

From ref. [4] we record the relation

$$re^{-\Lambda}\frac{\Gamma'}{2}=e^{-\Lambda}\left[\omega'^2+\frac{1}{2}r^2(\phi')^2\right]-\frac{1}{2}\frac{(1-\omega^2)^2}{r^2}-\frac{1}{4}\phi^2(1+\omega^2)^2+\frac{m}{r}-\frac{\lambda}{4}(\phi^2-v^2)^2r^2 \tag{28}$$

where $e^{-\Lambda} = 1 - \frac{2m(r)}{r}$. Hence,

$$(T_r^r)' = -\frac{e^{-\Lambda}}{4\pi r}\left[\frac{2(\omega')^2}{r^2} + (\phi')^2\right]\left\{(\omega')^2 + \frac{1}{2}r^2(\phi')^2 - \frac{1}{2}e^{\Lambda}\frac{(1-\omega^2)}{r^2} \right. \tag{29}$$

$$\left. -\frac{\phi^2}{4}e^{\Lambda}(1+\omega)^2 + \frac{m}{r}e^{\Lambda}\frac{\lambda}{4}(\phi^2 - v^2)r^2e^{\Lambda}\right\}$$

$$+\frac{1}{2\pi r}\left[\frac{(\omega')^2}{r^2}e^{\Lambda} + \frac{(1-\omega^2)^2}{r^4} + \frac{\phi^2}{4r^2}(1+\omega)^2\right]. \tag{30}$$

For $r \simeq r_h$, this expression simplifies to

$$\begin{aligned}(T_r^r)'(r_h) &\simeq \mathcal{J}(r_h)\left[\frac{2}{r_h} + \frac{4\pi}{r_h}\tilde{\mathcal{F}}(r_h)\right] \\ &\quad -\tilde{\mathcal{F}}(r_h)\left[\frac{1}{2} + \frac{1}{2}\frac{(1-\omega^2)^2}{r_h^2} - \frac{\lambda}{4}(\phi^2 - v^2)^2 r_h^2\right] \\ &= \tilde{\mathcal{F}}(r_h)\left[\frac{(1-\omega^2)^2}{2r_h^3} + \frac{\phi^2}{4r_h}(1+\omega^2)^2 + \frac{\lambda}{4}r_h(\phi^2 - v^2)^2 - \frac{1}{2r_h}\right] \\ &\quad +\frac{2}{r_h}\mathcal{J}\end{aligned} \tag{31}$$

where $\tilde{\mathcal{F}} = e^{\Lambda}\mathcal{F}(r) = \frac{1}{4\pi}[\frac{2(\omega')^2}{r^2} + (\phi')^2]$.

Consider for simplicity the case $r_h = 1$. Then, from the field equations [4] (see also Appendix A)

$$\omega_h' = \frac{1}{\mathcal{D}}\left[\frac{1}{4}\phi_h^2(1+\omega_h) - \omega_h(1-\omega_h^2)\right] = \frac{\mathcal{A}}{\mathcal{D}} \tag{32}$$

$$\phi_h' = \frac{1}{\mathcal{D}}\left[\frac{1}{2}\phi_h(1+\omega_h)^2 + \lambda\phi_h(\phi_h^2 - v^2)\right] = \frac{\mathcal{B}}{\mathcal{D}} \tag{33}$$

where

$$\mathcal{D} = 1 - (1-\omega_h^2)^2 - \frac{1}{2}\phi_h^2(1+\omega_h)^2 - \frac{1}{2}\lambda(\phi_h^2 - v^2)^2. \tag{34}$$

Then the expression (31) becomes

$$(T_r^r)'(r_h) = \frac{1}{4\pi\mathcal{D}}\left[8\pi\mathcal{D}\mathcal{J} - \mathcal{A}^2 - \frac{1}{2}\mathcal{B}^2\right] = \frac{\mathcal{C}}{4\pi\mathcal{D}}. \tag{35}$$

From the field equations [4] (see Appendix A),

$$\mathcal{D} = 1 - 2m_h' \tag{36}$$

which is always positive because the black holes are non-extremal. (See Appendix A for further discussion of this point.) Thus the sign of $(T_r^r)'(r_h)$ is the same as that of $\mathcal{C}$. Simplifying, we have

$$\mathcal{C} = c_1 + c_2 + c_3 + c_4 + c_5 \tag{37}$$

where

$$
\begin{aligned}
c_1 &= (1-\omega_h^2)^2\omega_h^2(3-2\omega_h^2) && (38)\\
c_2 &= \frac{1}{8}\phi_h^2(1+\omega_h)(1-\omega_h^2)(12\omega_h^3+12\omega_h^2-7\omega_h-9) && (39)\\
c_3 &= \frac{1}{16}\phi_h^4(1+\omega_h)^2(4\omega_h^2+8\omega_h+5) && (40)\\
c_4 &= -\lambda(\phi_h^2-v^2)^2\left[(1-\omega_h^2)^2+\frac{1}{2}\lambda\phi_h^2\right] && (41)\\
c_5 &= \frac{1}{4}\lambda\phi_h^2(v^2-\phi_h^2)(1+\omega_h)^2(2-v^2+\phi_h^2). && (42)
\end{aligned}
$$

The first term is always positive since $|\omega_h| \leq 1$. The cubic in c_2 possesses a local maximum at $\omega_h = -0.886$ where it has the value -1.724 and a local minimum at $\omega_h = 0.219$ where it equals -9.831. The cubic has a single root at $\omega_h = 0.8215$, and thus is positive for $\omega_h > 0.8215$ and negative for $\omega_h < 0.8215$. The quadratic in c_3 is always positive and possesses no real roots. It has a minimum value of 1 when $\omega_h = -1$. The term c_4 is always negative and c_5 is always positive since $|\phi_h| \leq v$.

In order to assess whether or not $\mathcal{C}$ as a whole is positive or negative, we shall consider each branch of black hole solutions in turn.

Firstly, consider the $k = 1$ branch of solutions. As v increases from 0 up to $v_{max} = 0.352$, ω_h increases monotonically from 0.632 to 0.869. The derivative of the first term is

$$
\frac{dc_1}{d\omega_h} = 2\omega_h(1-\omega_h^2)(3-13\omega_h+8\omega_h^2) \tag{43}
$$

where the quartic has roots given by $\omega_h^2 = 1.347$ or 0.278. This derivative is negative for $\omega_h \in (0.632, 0.869)$ and hence the first term decreases as v increases, and is bounded below by its value at the bifurcation point $v = v_{max}$, namely

$$
c_1 \geq 0.0674. \tag{44}
$$

The cubic in c_2 increases as ω_h increases from 0.632 to 0.869, and is bounded below by its value when $\omega_h = 0.632$

$$
12\omega_h^3 + 12\omega_h^2 - 7\omega_h - 9 \geq -5.602. \tag{45}
$$

Along this branch of solutions, it is also true that

$$
\begin{aligned}
(1-\omega_h^2) &\leq 1-(0.632)^2 = 0.601\\
1+\omega_h &\leq 2\\
\phi_h &\leq 0.19v \leq 0.0669. && (46)
\end{aligned}
$$

Altogether this gives

$$
c_2 \geq -5.602 \times \frac{1}{8} \times (0.0669)^2 \times 2 \times 0.601 = -3.767 \times 10^{-3}. \tag{47}
$$

The quadratic in c_3 increases as ω_h increases along the branch and so is bounded above by its value when $\omega_h = 0.869$;

$$4\omega_h^2 + 8\omega_h + 5 \leq 14.973. \tag{48}$$

Thus,

$$c_3 \geq -\frac{1}{16} \times (0.0669)^4 \times 2^2 \times 14.973 = -7.498 \times 10^{-5}. \tag{49}$$

For the fourth term, since

$$(\phi_h^2 - v^2)^2 \leq v^4 \tag{50}$$

we have

$$c_4 \geq -0.15 \times (0.352)^4 \times \left((0.601)^2 + \frac{1}{2} \times 0.15 \times (0.0669)^2\right) = -8.326 \times 10^{-4} \tag{51}$$

and finally,

$$c_5 \geq 0. \tag{52}$$

Thus, adding these expressions up, one obtains

$$\mathcal{C} \geq 0.0627 \geq 0 \tag{53}$$

so that $(T_r^r)'(r_h) > 0$ along the whole of the $k = 1$ branch of black hole solutions.

For the quasi-$k = 0$ branch, as v decreases from v_{max} down to 0, ω_h increases monotonically from 0.869, subject to the inequality

$$0.869 < \omega_h < 1 - 0.1v^2. \tag{54}$$

Hence, along this branch,

$$(1 - \omega_h^2)^2 = (1 - \omega_h)^2(1 + \omega_h)^2 \geq (0.1v^2)^2 \times (1.869)^2 = 0.0349v^4. \tag{55}$$

Thus the first term is bounded below as follows:

$$c_1 \geq 0.0349v^4 \times (0.869)^2 \times 1 = 0.0264v^4. \tag{56}$$

Since $\omega_h > 0.869$, the cubic in c_2 is positive all along this branch, so that c_2 and c_5 are positive. The quadratic in c_3 is bounded above by its value when $\omega_h = 2$, i.e.

$$4\omega_h^2 + 8\omega_h + 5 \leq 37. \tag{57}$$

All along the quasi-$k = 0$ branch,

$$\phi_h \leq 0.5v^2. \tag{58}$$

This gives

$$c_3 \geq -\frac{1}{16} \times (0.5v^2)^4 \times 2^2 \times 37 = -0.578v^8. \tag{59}$$

Since $v \leq 0.352$,

$$c_3 \geq -0.578 \times (0.352)^4 \times v^4 = -8.874 \times 10^{-3} v^4. \tag{60}$$

Finally, for c_4 we have

$$\begin{aligned} c_4 &\geq -0.15 v^4 \left[\frac{1}{2} \times 0.15 \times (0.5)^2 v^4 + (1 - (0.869)^2)^2 \right] \\ &= -8.992 \times 10^{-3} v^4 - 2.813 \times 10^{-3} v^8 \\ &\geq -8.992 \times 10^{-3} v^4 - 2.813 \times 10^{-3} \times (0.352)^4 v^4 \\ &= -9.035 \times 10^{-3} v^4. \end{aligned} \tag{61}$$

In total this gives

$$\mathcal{C} \geq 0.0264 v^4 - 8.874 \times 10^{-3} v^4 - 9.035 \times 10^{-3} v^4 = 8.491 \times 10^{-3} v^4 \geq 0. \tag{62}$$

In conclusion, $(T^r_r)'(r_h)$ is positive for all the black hole solutions having one node in ω, regardless of the value of the Higgs mass v.

Let us now check on possible contradictions with Einstein's equations.

Consider first the case $\lambda > 1/8$. Then, as $r \to \infty$, $T^r_r < 0$ and $(T^r_r)' > 0$. As $r \to r_h$, $T^r_r < 0$ and $(T^r_r)' > 0$. Hence there is no contradiction with Einstein's equations in this case.

Consider now the case $\lambda \leq 1/8$. In this case, as $r \to \infty$, $T^r_r > 0$ and $(T^r_r)' < 0$, whilst as $r \to r_h$, $T^r_r < 0$ and $(T^r_r)' > 0$. Hence, there is an interval $[r_a, r_b]$ in which $(T^r_r)'$ is positive and there exists a 'critical' distance $r_c \in (r_a, r_b)$ at which T^r_r changes sign.

However, unlike the case when the gauge fields are absent [10], here there is *no contradiction* with the result following from Einstein equations, because $(T^r_r)' > 0$ in some open interval close to the horizon, as we have seen above.

In conclusion the method of ref. [10] cannot be used to prove a 'no-scalar-hair' theorem for the EYMH system, as expected from the existence of the explicit solution of ref. [4]. The *key* difference is the presence of the positive term $\frac{2}{r}\mathcal{J}$ in the expression (16) for $(T^r_r)'$. This term is dependent on the Yang-Mills field and vanishes if this field is absent. Thus, there is a sort of 'balancing' between the gravitational attraction and the non-Abelian gauge field repulsion, which is responsible for the existence of the classical non-trivial black-hole solution of ref. [4]. However, as we shall discuss below, this solution is not stable against (linear) perturbations of the various field configurations [5]. Thus, although the 'letter' of the 'no-scalar-hair' theorem of ref. [10], based on non-negative scalar-field-energy density, is violated, its 'spirit' is maintained in the sense that there exist instabilities which that the solution cannot be formed as a result of collapse of stable matter.

3 Instability analysis of sphaleron sector of the EYMH black hole

The black hole solutions of ref. [4] in the EYMH system resemble the sphaleron solutions in $SU(2)$ gauge theory and one would expect them to be unstable for topological reasons. Below we shall confirm this expectation by proving [5] the existence of unstable modes in the sphaleron sector of the EYMH black hole system (for notation and definitions see Appendix A).

Recently, an instability proof of sphaleron solutions for arbitrary gauge groups in the EYM system has been given [13, 14]. The method consists of studying linearised radial perturbations around an equilibrium solution, whose detailed knowledge is not necessary to establish stability. The stability is examined by mapping the system of algebraic equations for the perturbations into a coupled system of differential equations of Schrödinger type [13, 14]. As in the particle case of ref. [1], the instability of the solution is established once a bound state in the respective Schrödinger equations is found. The latter shows up as an imaginary frequency mode in the spectrum, leading to an exponentially growing mode. There is an elegant physical interpretation behind this analysis, which is similar to the Cooper pair instability of super-conductivity. The gravitational attraction balances the non-Abelian gauge field repulsion in the classical solution [1], but the existence of bound states implies imaginary parts in the quantum ground state which lead to instabilities of the solution, in much the same way as the classical ground state in super-conductivity is not the absolute minimum of the free energy.

However, this method cannot be applied directly to the black hole case, due to divergences occuring in some of the expressions involved. This is a result of the singular behaviour of the metric function at the physical space-time boundaries (horizon) of the black hole.

3.1 Linearized perturbations and instabilities

It is the purpose of this section to generalise the method of ref. [13] to incorporate the black hole solution of the EYMH system of ref. [4]. By constructing appropriate trial linear radial perturbations, following ref. [14, 11], we show the existence of bound states in the spectrum of the coupled Schrödinger equations, and thus the instability of the black hole. Detailed knowledge of the black hole solutions is not actually required, apart from the fact that the existence of an horizon leads to modifications of the trial perturbations as compared to those of ref. [13, 14], in order to avoid divergences in the respective expressions [11].

We start by sketching the basic steps [13, 11] that will lead to a study of the stability of a classical solution $\phi_s(x,t)$ with finite energy in a (generic) classical field theory. One considers small perturbations $\delta\phi(x,t)$ around $\phi_s(x,t)$, and specifies [13] the time-dependence as

$$\delta\phi(x,t) = \exp(-i\Omega t)\Psi(x). \tag{63}$$

The linearised system (with respect to such perturbations), obtained from the equations of motion, can be cast into a Schrödinger eigenvalue problem

$$\mathcal{H}\Psi = \Omega^2 A \Psi \tag{64}$$

where the operators $\mathcal{H}$, A are assumed independent of the 'frequency' Ω. As we shall show later on, this is indeed the case of our black hole solution of the EYMH system. In that case it will also be shown that $\mathcal{H}$ is a self-adjoint operator with respect to a properly defined inner (scalar) product in the space of functions $\{\Psi\}$ [13], and the A matrix is positive definite, $<\Psi|A|\Psi>> 0$. A criterion for instability is the existence of an imaginary frequency mode in (64)

$$\Omega^2 < 0. \tag{65}$$

This is usually difficult to solve analytically in realistic models, and usually numerical calculations are required [15]. A less informative method which admits analytic treatment has been proposed recently in ref. [13, 11], and we shall follow this for the purposes of the present work. The method consists of a variational approach which makes use of the following functional defined through (64):

$$\Omega^2(\Psi) = \frac{<\Psi|\mathcal{H}|\Psi>}{<\Psi|A|\Psi>} \tag{66}$$

with Ψ a *trial* function. The lowest eigenvalue is known to provide a *lower* bound for this functional. Thus, the criterion of instability, which is equivalent to (65), in this approach reads

$$\begin{aligned} \Omega^2(\Psi) &< 0 \\ <\Psi|A|\Psi> &< \infty. \end{aligned} \tag{67}$$

The first of the above conditions implies that the operator $\mathcal{H}$ is not positive definite, and therefore negative eigenvalues do exist. The second condition, on the *finiteness* of the expectation value of the operator A, is required to ensure that Ψ lies in the Hilbert space containing the domain of $\mathcal{H}$. In certain cases, especially in the black hole case, there are divergences due to singular behaviour of modes at, say, the horizons, which could spoil these conditions (67). The advantage of the above variational method lies in the fact that it is an easier task to choose appropriate trial functions Ψ that satisfy (67) than solving the original eigenvalue problem (64). In what follows we shall apply this second method to the black hole solution of ref. [4].

For completeness, we first review basic formulas of the spherically symmetric black hole solutions of the EYMH system [4]. The space-time metric takes the form [4]

$$ds^2 = -N(t,r)S^2(t,r)dt^2 + N^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{68}$$

and we assume the following ansatz for the non-Abelian gauge potential [4, 13]

$$A = a_0\tau_r dt + a_1\tau_r dr + (\omega+1)[-\tau_\varphi d\theta + \tau_\theta \sin\theta d\varphi] + \tilde{\omega}[\tau_\theta d\theta + \tau_\varphi \sin\theta d\varphi] \tag{69}$$

where $\omega, \tilde{\omega}$ and $a_i, i = 0, 1$ are functions of t, r. The τ_i are appropriately normalised spherical generators of the SU(2) group in the notation of ref. [13].

The Higgs doublet assumes the form

$$\tilde{\Phi} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_2 + i\psi_1 \\ \phi - i\psi_3 \end{pmatrix} \qquad ; \qquad \boldsymbol{\psi} = \psi\hat{\boldsymbol{r}} \tag{70}$$

with the Higgs potential

$$V(\tilde{\Phi}) = \frac{\lambda}{4}(\tilde{\Phi}^\dagger\tilde{\Phi} - v^2)^2 \tag{71}$$

where v denotes the v.e.v. of $\tilde{\Phi}$ in the non-trivial vacuum.

The quantities ω, ϕ satisfy the static field equations

$$\begin{aligned} N\omega'' + \frac{(NS)'}{S}\omega' &= \frac{1}{r^2}(\omega^2-1)\omega + \frac{\phi^2}{4}(\omega+1) \\ N\phi'' + \frac{(NS)'}{S}\phi' + \frac{2N}{r}\phi' &= \frac{1}{2r^2}\phi(\omega+1)^2 + \lambda\phi(\phi^2 - v^2) \end{aligned} \tag{72}$$

where the prime denotes differentiation with respect to r. For later use, we also mention that a dot will denote differentiation with respect to t.

If we choose a gauge in which $\delta a_0 = 0$, the linearised perturbation equations decouple into two sectors [13] . The first consists of the gravitational modes δN, δS, $\delta\omega$ and $\delta\phi$ and the second of the matter perturbations δa_1, $\delta\tilde{\omega}$ and $\delta\psi$. For our analysis in this section it will be sufficient to concentrate on the matter perturbations, setting the gravitational perturbations δN and δS to zero, because an instability will show up in this sector of the theory. An instability study in the gravitational sector will be discussed in the following section 4. The equations for the linearised matter perturbations take the form [13]

$$\mathcal{H}\Psi + A\ddot{\Psi} = 0 \tag{73}$$

with,

$$\Psi = \begin{pmatrix} \delta a_1 \\ \delta\tilde{\omega} \\ \delta\psi \end{pmatrix} \tag{74}$$

and,

$$A = \begin{pmatrix} Nr^2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & r^2 \end{pmatrix} \tag{75}$$

and the components of $\mathcal{H}$ are

$$\begin{aligned}
\mathcal{H}_{a_1a_1} &= 2(NS)^2\left(\omega^2 + \frac{r^2}{8}\phi^2\right) \\
\mathcal{H}_{\tilde{\omega}\tilde{\omega}} &= 2p_*^2 + 2NS^2\left(\frac{\omega^2-1}{r^2} + \frac{\phi^2}{4}\right) \\
\mathcal{H}_{\psi\psi} &= 2p_*\frac{r^2}{2}p_* + 2NS^2\left(\frac{(-\omega+1)^2}{4} + \frac{r^2}{2}\lambda(\phi^2 - v^2)\right) \\
\mathcal{H}_{a_1\tilde{\omega}} &= -2iNS[(p_*\omega) - \omega p_*] \\
\mathcal{H}_{\tilde{\omega}a_1} &= -2i[p_*NS\omega + NS(p_*\omega)] \\
\mathcal{H}_{a_1\psi} &= \frac{ir^2}{2}NS[(p_*\phi) - \phi p_*] \\
\mathcal{H}_{\psi a_1} &= ip_*\frac{r^2}{2}NS\phi + i\frac{r^2}{2}NS(p_*\phi) \\
\mathcal{H}_{\tilde{\omega}\psi} &= \mathcal{H}_{\psi\tilde{\omega}} = -\phi NS^2
\end{aligned} \tag{76}$$

where the operator p_* is

$$p_* \equiv -iNS\frac{d}{dr}. \tag{77}$$

Upon specifying the time-dependence (63)

$$\Psi(r,t) = \Psi(r)e^{i\sigma t} \qquad ; \qquad \Psi(r) = \begin{pmatrix} \delta a_1(r) \\ \delta\tilde{\omega}(r) \\ \delta\psi(r) \end{pmatrix} \tag{78}$$

one arrives easily to an eigenvalue problem of the form (64), which can then be extended to the variational approach (67).

To this end, we choose as trial perturbations the following expressions (c.f. [13])

$$\begin{aligned}
\delta a_1 &= -\omega' Z \\
\delta\tilde{\omega} &= (\omega^2 - 1)Z \\
\delta\psi &= -\frac{1}{2}\phi(\omega+1)Z
\end{aligned}$$

where Z is a function of r to be determined.

One may define the inner product

$$< \Psi|X >\equiv \int_{r_h}^{\infty} \overline{\Psi} X \frac{1}{NS} dr \tag{79}$$

where r_h is the position of the horizon of the black hole. The operator $\mathcal{H}$ is then symmetric with respect to this scalar product. Following ref. [13], consider the expectation value

$$< \Psi|A|\Psi >= \int_{r_h}^{\infty} dr \frac{1}{NS} Z^2 \left[Nr^2(\omega')^2 + 2(\omega^2-1)^2 + \frac{r^2}{4}\phi^2(\omega+1)^2 \right] \tag{80}$$

which is clearly positive definite for real Z. Its finiteness will be examined later, and depends on the choice of the function Z.

Next, we proceed to the evaluation of the expectation value of the Hamiltonian $\mathcal{H}$ (77); after a tedious calculation one obtains

$$\begin{aligned} < \Psi|\mathcal{H}|\Psi > &= \int_{r_h}^{\infty} drSZ^2\{-2N(\omega')^2 + 2P^2N(\omega^2-1)^2 \\ &+\frac{1}{4}P^2Nr^2\phi^2(\omega+1)^2 - \frac{2}{r^2}(\omega^2-1)^2 - \frac{1}{2}\phi^2(\omega+1)^2\} \\ &+ \text{ boundary terms} \end{aligned} \tag{81}$$

where $P \equiv \frac{1}{Z}\frac{dZ}{dr}$. The boundary terms will be shown to vanish so we omit them in the expression (82). The final result is

$$\begin{aligned} < \Psi|\mathcal{H}|\Psi > &= \int_{r_h}^{\infty} drS \left\{ -2N(\omega')^2 - \frac{2}{r^2}(\omega^2-1)^2 - \frac{1}{2}\phi^2(\omega+1)^2 \right\} \\ &+ \int_{r_h}^{\infty} dr \left\{ \frac{2}{r^2}(\omega^2-1)^2 + \phi^2(\omega+1)^2 + 2N(\omega')^2 \right\} S(1-Z^2) \\ &+ \int_{r_h}^{\infty} drSN \left(\frac{dZ}{dr} \right)^2 \left[2(\omega^2-1)^2 + \frac{1}{4}r^2\phi^2(\omega+1)^2 \right]. \end{aligned} \tag{82}$$

The first of these terms is manifestly negative. To examine the remaining two, we introduce the 'tortoise' co-ordinate r^* defined by [11]

$$\frac{dr^*}{dr} = \frac{1}{NS} \tag{83}$$

and define a sequence of functions $Z_k(r^*)$ by [11]

$$Z_k(r^*) = Z\left(\frac{r^*}{k}\right) \qquad ; \qquad k = 1, 2, \ldots \tag{84}$$

where

$$\begin{aligned} Z(r^*) &= Z(-r^*), \\ Z(r^*) &= 1 \quad \text{for } r^* \in [0, a] \\ -D \leq \frac{dZ}{dr^*} &< 0, \quad \text{for } r^* \in [a, a+1] \\ Z(r^*) &= 0 \quad \text{for } r^* > a+1 \end{aligned} \tag{85}$$

where a, D are arbitrary positive constants. Then, for each value of k the vacuum expectation values of $\mathcal{H}$ and A are finite, $< \Psi|\mathcal{H}|\Psi >< \infty$, and $< \Psi|A|\Psi >< \infty$, with $Z = Z_k$, and all boundary terms vanish. This justifies *a posteriori* their being dropped in eq. (82). The integrands in the second and third terms of eq. (82) are uniformly convergent and tend to zero as $k \to \infty$. Hence, choosing k sufficiently large the dominant contribution in (82) comes from the first term which is negative.

This confirms the existence of bound states in the Schrödinger equation (73), (64), and thereby the instability (67) of the associated black hole solution of ref. [4] in the coupled EYMH system. The above analysis reveals the existence of at least one negative *odd-parity* eigenmode in the spectrum of the EYMH black hole.

3.2 Counting sphaleron-like unstable modes in the EYMH system

The exact number of such negative modes is an interesting question and we next proceed to investigate it. Recently, a method for determining the number of the sphaleron-like unstable modes has been applied by Volkov et al. [16] to the gravitating sphaleron case. We have been able to extend it to the present EYMH black hole. The method consists of mapping the system of linearized perturbations to a system of coupled Scrödinger-like equations. Counting of unstable modes is then equivalent to counting bound states of the quantum-mechanical analogue system. It is important to notice that due to the fact that the EYMH black hole solution is not known analytically, but only numerically, it will be necessary to make certain physically plausible assumptions concerning certain analyticity requirements [17] for the solutions of the analogue system. This is equivalent to requiring that the conditions for the validity of perturbation theory in ordinary quantum mechanics be applied to this problem. Details are described below.

Working in the gauge $\delta a_0 = 0$, and denoting the derivative with respect to the tortoise coordinate (83) by a prime, we can write the linearised perturbations in the sphaleron sector of the EYMH system as [13]:

$$\begin{aligned} 2N^2S^2\left(\omega^2 + \frac{r^2}{8}\phi^2\right)\delta a_1 + 2NS(\omega\delta\tilde{\omega}' - \omega'\delta\tilde{\omega}) & \\ + \frac{1}{2}r^2NS(\phi'\delta\psi - \phi\delta\psi') &= Nr^2\sigma^2\delta a_1 \\ 2(NS\omega\delta a_1)' + 2NS\omega'\delta a_1 + \delta\tilde{\omega}'' + NS^2\phi\delta\psi & \\ -\frac{2}{r^2}S^2\left((\omega^2-1) + \frac{\phi^2}{4}\right)\delta\tilde{\omega} &= -2\sigma^2\delta\tilde{\omega} \end{aligned}$$

$$\frac{1}{2}(NSr^2\phi\delta a_1)' + \frac{1}{2}r^2NS\phi'\delta a_1 - NS^2\phi\delta\tilde{\omega} - (r^2\delta\psi')'$$
$$+2NS^2\left(\frac{(1-\omega)^2}{4} + \frac{1}{2}r^2\lambda(\phi^2 - v^2)\right)\delta\psi \quad = \quad r^2\sigma^2\delta\psi \tag{86}$$

together with the Gauss constraint

$$\sigma^2\left\{\left(\frac{r^2}{S}\delta a_1\right)' + 2\omega\tilde{\omega} - r^2\frac{\phi}{2}\delta\psi\right\} = 0. \tag{87}$$

Define $\delta\xi = r\delta\psi$, $\delta\alpha = \frac{r^2}{2S}\delta a_1$. Then

$$\sigma^2\delta\alpha = f(r^*) \tag{88}$$

where

$$f(r^*) = NS\omega^2\delta a_1 + \frac{N}{8}Sr^2\phi^2\delta a_1 - \omega'\delta\tilde{\omega} + \omega\delta\tilde{\omega} + \frac{N}{4}S\phi\delta\xi + \frac{r}{4}(\phi'\delta\xi - \phi\delta\xi') \tag{89}$$

and

$$f'(r^*) = \sigma^2\left(-\omega\delta\tilde{\omega} + \frac{r}{4}\phi\delta\xi\right) \tag{90}$$

and the Gauss constraint becomes

$$\sigma\left(\delta\alpha' + \omega\delta\tilde{\omega} - r\frac{\phi}{4}\delta\xi\right) = 0. \tag{91}$$

Next, we define a 'strong' Gauss constraint by

$$\delta\alpha' = -\omega\delta\tilde{\omega} + \frac{r}{4}\phi\delta\xi \tag{92}$$

even when $\sigma = 0$.

Using (89) and (92) we may write

$$\begin{aligned}\delta\tilde{\omega} &= \frac{r^2}{P}\phi^2\left(\frac{\delta\alpha'}{r^2\phi^2}\right)' - \frac{Q}{P}\delta\alpha \\ \delta\xi &= \frac{4\omega^2}{Pr\phi}\left(\frac{\delta\alpha'}{\omega^2}\right)' - \frac{4Q\omega}{Pr\phi}\delta\alpha\end{aligned} \tag{93}$$

where

$$\begin{aligned}P(r^*) &= -2\omega' + 2\omega\phi' + 2\omega\frac{NS}{r} \\ Q(r^*) &= 2\frac{NS^2}{r^2}\omega^2 + \frac{1}{4}NS^2\phi^2 - \sigma^2.\end{aligned} \tag{94}$$

If we substitute these expressions into the equation for $\delta\tilde{\omega}$ we obtain the following equation for $\delta\alpha$:

$$\begin{aligned}
-\delta\alpha^{(\mathrm{iv})} + \left(\frac{2P'}{P} + HP\right)\delta\alpha''' \\
+\left\{\frac{P''}{P} - \frac{2P'^2}{P^2} + 2H'P + Q - \sigma^2 + NS^2 J\right\}\delta\alpha'' + \\
\left\{H''P + 2P\left(\frac{Q}{P}\right)' + \sigma^2 HP + \right. \\
\left. NS^2\left(-\frac{2\omega P}{r^2} - \frac{H}{r^2}(\omega^2-1)P - \frac{H}{4}\phi^2 P + \frac{4}{r^2}\omega'\right)\right\}\delta\alpha' + \\
\left\{P\left(\frac{Q}{P}\right)'' - 2\omega P\left(\frac{NS^2}{r^2}\right)' - \frac{4PNS^2}{r^2}\omega' + \sigma^2 Q - NS^2 QJ\right\}\delta\alpha &= 0 \qquad (95)
\end{aligned}$$

where

$$\begin{aligned}
H &\equiv \frac{2NS}{rP} + \frac{2\phi'}{\phi P} \\
J &\equiv -\frac{2\omega}{r^2} + \frac{\omega^2-1}{r^2} + \frac{\phi^2}{4}. \qquad (96)
\end{aligned}$$

Alternatively, we can eliminate $\delta\xi$ to obtain the following pair of coupled Schrödinger equations [14, 16]:

$$\begin{aligned}
\sigma^2\delta\alpha &= -\delta\alpha'' + \left(\frac{2\phi'}{\phi} + \frac{2NS}{r}\right)\delta\alpha' + \left(\frac{2NS^2}{r^2}\omega^2 + \frac{N}{4}S^2\phi^2\right)\delta\alpha + P\delta\tilde{\omega} \\
\sigma^2\delta\tilde{\omega} &= -\delta\tilde{\omega}'' - \frac{2N}{r^2}S^2(1+\omega)\delta\alpha' - \left\{\frac{4\omega' NS^2}{r^2} + 2\omega(\frac{NS^2}{r^2})'\right\}\delta\alpha \\
&\quad + \frac{NS^2}{r^2}\left\{(\omega-1)^2 + \frac{r^2\phi^2}{4}\right\}\delta\tilde{\omega}. \qquad (97)
\end{aligned}$$

To proceed, it appears necessary to make the following assumptions:

- We assume that the equilibrium solutions are continuous functions of the Higgs mass v.
- We also assume that given a Schrödinger-like equation $-\Psi'' + V\Psi = E\Psi$, where the potential V depends continuously on some parameter v, then the bound state energies also depend continuously on v. This can be proven rigorously if we make the physically plausible assumption of analyticity of the operators involved in the above system. The proof then relies on the powerful Kato-Rellich theorem of analytic operators [17]. This analyticity requirement is the case if perturbation theory is used to solve the Schrödinger system with potential $V(v+\delta v)$ in terms of the spectrum of $V(v)$, implying that the changes in the bound state energies δE due to the infinitesimal shift in the parameter

v are also infinitesimal. It is also the case where *variational* methods are applicable. This assumption implies that the eigenvalues of the discrete spectrum of the above equations (97), i.e. the bound-state energies for $\sigma^2 < 0$, are also continuous as v varies continuously.

The above assumptions had to be made because in the case of the EYMH black hole system, the solution is not known analytically but only numerically, and therefore the issue of analyticity of the various operators involved with respect to the Higgs field v.e.v., v, cannot be rigorously established.

We now notice that the continuous spectrum of the equations (97) is given by $\sigma^2 > 0$. Hence, the number of negative modes will change by one whenever a mode is either absorbed into the continuum or emerges from it.

For $\sigma^2 = 0$ the equations possess pure "gauge mode" solutions of the form

$$\delta\alpha = \frac{r^2\Omega'}{2NS^2}, \qquad \delta\tilde{\omega} = -\omega\Omega, \qquad \delta\xi = \frac{r\phi}{2}\Omega \tag{98}$$

where

$$\left(\frac{r^2\Omega'}{2NS^2}\right)' = \left(\omega^2 + \frac{r^2}{8}\phi^2\right)\Omega. \tag{99}$$

Thus, near the event horizon, $\Omega \sim O[(r-r_h)^k]$, where $k = 0$ or 1, and at infinity, $\Omega \sim e^{\pm\frac{v}{2}}$ upon choosing $S(\infty) = 1$. Hence, there is a single non-degenerate, non-normalisable eigenmode with $\sigma^2 = 0$.

For the fourth-order equation with $\sigma^2 = 0$, $\delta\alpha \sim O[(r-r_h)^k]$ near the horizon, where $k = 0$ (twice, corresponding to the pure "gauge mode" solutions) or $k = \frac{-1\pm\sqrt{5}}{2}$.

In the latter case, $\delta\tilde{\omega}, \delta\xi \sim O[(r-r_h)^{k-1}]$, and so will not remain bounded near the horizon. Hence there is a single non-degenerate zero mode for non-zero v.

From the above it follows that the number of negative modes of the system (97) cannot change at any non-zero value of v, including the bifurcation point v_{max}, by continuity. The negative eigenvalues of this system are non-degenerate and hence cannot themselves bifurcate at some value of v.

The only possible place where the number of negative modes may change is at $v = 0$. Let $\phi = v\tilde{\phi}, \delta\xi = v\delta\tilde{\xi}$. Then the system of equations (97) becomes

$$-\delta\alpha'' + \left(\frac{2\tilde{\phi}'}{\tilde{\phi}} + \frac{2NS}{r}\right)\delta\alpha' + \left(\frac{2NS^2\tilde{\phi}^2}{4}\right)\delta\alpha$$

$$
\begin{aligned}
&+\left(-2\omega' + 2\omega\frac{\tilde{\phi}'}{\tilde{\phi}} + 2\omega\frac{NS}{r}\right)\delta\tilde{\omega} &=& \ \sigma^2\delta\alpha \\
-\delta\tilde{\omega}'' - 2\frac{NS^2}{r^2}(1+\omega)\delta\alpha' + \frac{NS^2}{r^2}&\left[(\omega-1)^2 + v^2\frac{r^2\tilde{\phi}^2}{4}\right]\delta\tilde{\omega} && \\
&-\left[4\omega'\frac{NS^2}{r^2} + 2\omega\left(\frac{NS^2}{r^2}\right)'\right]\delta\alpha &=& \ \sigma^2\delta\tilde{\omega} \qquad (100)
\end{aligned}
$$

together with the Gauss constraint $\delta\alpha' = -\omega\delta\tilde{\omega} + \frac{1}{4}v^2 r\tilde{\phi}\delta\tilde{\xi}$.

Consider the $k = n$ branch of the EYMH solutions. In this branch, $\frac{\tilde{\phi}'}{\tilde{\phi}}$ has a well-defined limit as $v \longrightarrow 0$, and the system (97) is continuous at $v = 0$. In this case, the Gauss constraint reduces to $\delta\alpha' = -\omega\delta\tilde{\omega}$, and substituting in (100) yields the equation

$$
-\delta\alpha'' + \frac{2}{\omega}\omega'\delta\alpha' + \frac{2NS^2}{r^2}\omega^2\delta\alpha = \sigma^2\delta\alpha. \qquad (101)
$$

This is the equation studied in ref. [16] where it was shown that there are exactly n negative eigenvalues. Furthermore, there is a single non-degenerate zero mode given by

$$
\delta\alpha = \frac{r^2\Omega'}{2NS^2} \qquad (102)
$$

where

$$
\left(\frac{r^2\Omega'}{2NS^2}\right)' = \omega^2\Omega. \qquad (103)
$$

As before, near the horizon $\Omega \sim O[(r - r_h)]$ or $\Omega \sim O[1]$ whilst at infinity $\Omega \sim r$ or $r^{\frac{1}{2}}$, giving a single eigenmode with zero eigenvalue. Thus, the number of negative eigenvalues does not change at $v = 0$ for this branch of solutions.

For the quasi $k = n - 1$ branch of solutions $\frac{\tilde{\phi}'}{\tilde{\phi}}$ does not have a well-defined limit as $v \longrightarrow 0$, and the system (97) is not continuous at $v = 0$. Hence, by continuity, we can conclude that both the $k = n$ and the quasi $k = n - 1$ black holes have *exactly* n unstable modes in the *sphaleron* sector.

4 Instabilities in the Gravitational Sector - Catastrophe theory approach

It is the aim of this section to prove the existence and count the exact number of unstable modes in the gravitational sector of the solutions. In the first part we shall study the conditions for the existence of unstable modes in a linearized framework, and we shall study the possibility of a change in the stability of the system as one varies the Higgs v.e.v. v continuously from 0 up to the bifurcation point (c.f.

figure 2). In the second part, which will deal with the change of the stability of the system at the bifurcation point, we shall go beyond linearized perturbations by applying catastrophe theory. It should be stressed that although catastrophe theory was first employed by the authors of [6], our approach in this section is somewhat different, and has certain advantages, not least of which is that we are able to exploit the known stability of the Schwarzschild black hole to draw conclusions about the non-trivial EYMH black holes.

4.1 Linearized perturbations

The linearised perturbation equations for the gravitational sector are:

$$
\begin{aligned}
-\delta\omega'' + U_{\omega\omega}\delta\omega + U_{\omega\phi}\delta\phi &= \sigma^2\delta\omega \\
-\delta\phi'' - \frac{2NS}{r}\delta\phi' + U_{\phi\omega}\delta\omega + U_{\phi\phi}\delta\phi &= \sigma^2\delta\phi
\end{aligned}
\tag{104}
$$

where the prime denotes differentiation with respect to the tortoise coordinate (83) as before, and

$$
\begin{aligned}
U_{\omega\omega} &= \frac{NS^2}{r^2}\left[3\omega^2 - 1 + \frac{1}{4}r^2\phi^2 - 4r^2\omega'^2\left(\frac{N}{r} + \frac{(NS)'}{S}\right) + \frac{\omega(\omega^2-1)}{r}\omega' + 2r\omega'\phi^2\right] \\
U_{\omega\phi} &= \frac{NS^2}{r^2}\left[\frac{1}{2}(1+\omega)\phi r^2 - 2\phi'\omega' r^3\left(\frac{N}{r} + \frac{(NS)'}{S}\right) + 2r\phi'\omega(\omega^2-1)\right. \\
&\quad \left. +\phi\omega'(1+\omega)^2 r + \frac{1}{2}\phi'\phi^2(1+\omega) + 2\lambda r^3\phi\omega'(\phi^2 - v^2)\right] \\
U_{\phi\omega} &= \frac{2}{r^2}U_{\omega\phi} \\
U_{\phi\phi} &= \frac{NS^2}{r^2}\left[\frac{1}{2}(1+\omega)^2 + \lambda r^2(3\phi^2 - \tilde{v}^2) - 2r^3(\phi')^2\left(\frac{N}{r} + \frac{(NS)'}{S}\right)\right. \\
&\quad \left. +2\phi'\phi(1+\omega)^2 r + 4\lambda\phi\phi' r^3(\phi^2 - \tilde{v}^2)\right].
\end{aligned}
\tag{105}
$$

The continuity argument described previously when applied here, implies that the number of negative eigenvalues of the system (104,105) can change only when there is a zero mode.

Suppose that for some $v \neq 0$ there is such a zero mode of (104). Given the background solution (ω, ϕ) of the field equations, then $(\omega + \delta\omega, \phi + \delta\phi)$, together with the corresponding metric functions, will also be a solution of the field equations.

As $r \to \infty$, then

$$
\begin{pmatrix} \delta\omega \\ \delta\phi \end{pmatrix} \sim O[e^{-vr^*}] \qquad r \to -\infty \tag{106}
$$

and

$$
\begin{pmatrix} \delta\omega \\ \delta\phi \end{pmatrix} \sim \text{const} \qquad r \to -\infty. \tag{107}
$$

Also, since in this sector the change in the mass function $m(r)$ is given by (see Appendix A)

$$\delta m(r) = 2N\frac{d\omega}{dr}\delta\omega(r) + r^2 N\frac{d\phi}{dr}\delta\phi(r) \tag{108}$$

we have that for this zero mode $\delta m \to 0$ as $r^* \to \pm\infty$. Hence, for *fixed* parameters r_h, λ, v, g there are *two solutions* of the original field equations $\begin{pmatrix} \omega \\ \phi \end{pmatrix}$, and $\begin{pmatrix} \omega+\delta\omega \\ \phi+\delta\phi \end{pmatrix}$, satisfying the required boundary conditions. For our purposes we shall *assume* that the solution of ref. [4] is *unique*. Compatibility with the above analysis, then, requires the latter to be valid only at the *bifurcation point* (c.f. figure 2). Hence, there is a single zero mode at $v = v_{max}$. For any other v the zero mode is absent.

Let $\delta\phi \equiv v\delta\tilde{\phi}$, $\phi \equiv v\tilde{\phi}$. The equations (105) become

$$\begin{aligned} -\delta\omega'' + U_{\omega\omega}\delta\omega + v^2\tilde{U}_{\omega\phi}\delta\tilde{\phi} &= \sigma^2\delta\omega \\ -\delta\tilde{\phi} - \frac{2NS}{r}\delta\tilde{\phi}' + \tilde{U}_{\phi\omega}\delta\omega + U_{\phi\phi}\delta\tilde{\phi} &= \sigma^2\delta\tilde{\phi} \end{aligned} \tag{109}$$

where $U_{\omega\phi} \equiv v\tilde{U}_{\omega\phi}, \tilde{U}_{\phi\omega} \equiv v\tilde{U}_{\phi\omega}$.

These equations have a well-defined limit as $v \to 0$, and are continuous at $v = 0$. At $v = 0$ the equations reduce to

$$\delta\omega'' + U_{\omega\omega}\delta\omega = \sigma^2\delta\omega \tag{110}$$

where

$$U_{\omega\omega} = \frac{NS^2}{r^2}\left[3\omega^2 - 1 - 4r(\omega')^2\left(\frac{N}{r} + \frac{(NS)'}{S}\right) + \frac{\omega(\omega^2-1)}{r}\delta\omega'\right]. \tag{111}$$

If this equation possesses a zero mode, then $\delta\omega \to$ const, as $r \to \infty$ for this mode. As $r \to \infty$, $N \to 1 - \frac{m}{r} + O[e^{-r}]$, $\omega \to -1 + O[e^{-r}]$, $S \to 1 + O[e^{r}]$, and the equation takes the form

$$\frac{d^2}{dr^2}(\delta\omega) = -\frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1}\frac{d}{dr}(\delta\omega) + \frac{2}{r^2}\left(1 - \frac{m}{r}\right)^{-1}\delta\omega. \tag{112}$$

Suppose that $\delta\omega = \sum_{n=0}^{\infty} a_n r^{-\rho-n}$, $a_0 \neq 0$ as $r \to \infty$. Then $\rho = 2$ or -1. Let $\delta\omega = r^2 f(r)$. Then, $f(r)$ satisfies the equation

$$f'' + f'\left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{4}{r} - \frac{6m}{r^2}\right) = 0. \tag{113}$$

The solution to this equation as $r \to \infty$ assumes the form

$$f = \frac{A}{8\mathcal{M}^3}\log\left(\frac{r - 2\mathcal{M}}{r}\right) + \frac{A}{4\mathcal{M}^2 r^2} + \frac{A}{4\mathcal{M}r^2} + B \qquad r \to \infty \tag{114}$$

with A, B arbitrary constants and $\mathcal{M} = \lim_{r\to\infty} m(r)$, so that

$$\delta\omega \to Br^2 + \frac{Ar^2}{8\mathcal{M}^3}\log\left(\frac{r-2\mathcal{M}}{r}\right) + \frac{Ar}{4\mathcal{M}^2} + \frac{A}{4\mathcal{M}}. \tag{115}$$

Hence $\delta\omega$ can remain bounded as $r \to \infty$ only if $A = B = 0$, i.e. only the trivial solution exists. Thus, there are no non-trivial zero modes for $v = 0$.

We, therefore, conclude that along each branch of solutions the number of negative modes remains constant from $v = 0$ to $v = v_{max}$.

4.2 Bifurcation points and catastrophe theory

In order to determine what happens at $v = v_{max}$ we appeal to catastrophe theory [6]. Our aim is to study the possibility of a change of the stability of the system at v_{max}. To this end, we have to determine a certain function (*catastrophe functional*) in the black hole solution which changes *discontinuously* despite the smooth change of certain (control) parameters of the system. As we shall show below, in the case at hand the rôle of the catastrophe functional is played by the mass function of the black hole, whilst the control parameter is the Higgs v.e.v. v. At the bifurcation point v_{max} we shall find a *fold* catastrophe which affects the relevant stability of the branches of the solution. In addition, the catastrophe-theoretic approach allows for an *exact* counting of the unstable modes in the various branches. For notations and mathematical definitions on catastrophe theory we refer the interested reader to Appendix B.

We should note at this point that although catastrophe theory seems powerful enough to yield a universal stability study of all kinds of non-Abelian black holes [6], however one should express some caution in drawing conclusions about absolute stability. Indeed, catastrophe theory gives information about instabilities of certain modes of the system. If catastrophe theory gives a stable branch of solution, this does not mean that the system is completely stable, given that there may be other instabilities in sectors where catastrophe theory does not apply. In our EYMH system this is precisely the case with the sphaleron sector. However safe conclusions can be reached, within the framework of catastrophe theory, regarding *relative* stability of branches of solutions, and it is in this sense that we shall use it here in order to count the number of unstable modes of the various branches of the EYMH system. Having expressed these cautionary remarks we are now ready to proceed with our catastrophe-theoretic analysis.

The mass functional $\mathcal{M}$ (c.f. appendix A) can be re-written as a functional of the matter fields only as follows:

First note that $\mathcal{M} = m(\infty)$. Let $\mu(r) \equiv m(r) - m(r_h) = m(r) - \frac{r_h}{2}$. Then, using a prime to denote d/dr

$$\begin{aligned} \mu'(r) = m'(r) &= \frac{1}{2}\left[\left(1 - \frac{2m}{r}\right)(2(\omega')^2 + r^2(\phi')^2)\right] \\ &+ \frac{r^2}{2}\left[\frac{(1-\omega^2)^2}{r^4} + \frac{\phi^2}{2r^2}(1+\omega)^2 + \frac{\lambda}{2}(\phi^2 - v^2)^2\right] \\ &= \frac{1}{2}\left[\left(1 - \frac{r_h}{r}\right)(2(\omega')^2 + r^2(\phi')^2)\right] \\ &+ \frac{r^2}{2}\left[\frac{(1-\omega^2)^2}{r^4} + \frac{\phi^2(1+\omega)^2}{2r^2} + \frac{\lambda}{2}(\phi^2 - \tilde{v}^2)^2\right] \\ &- \frac{\mu}{r}(2(\omega')^2 + r^2(\phi')^2). \end{aligned} \tag{116}$$

The last term on the right-hand-side can be written in terms of the metric function δ (cf appendix A)

$$-\frac{\mu}{r}(2(\omega')^2 + r^2(\phi')^2) = \mu\delta'. \tag{117}$$

Solving for μ gives

$$\mu(r) = e^{\delta(r)} \int_{r_h}^{r} \mathcal{K}[\omega, \phi] e^{-\delta(r')} dr' \tag{118}$$

where

$$\begin{aligned} \mathcal{K}[\omega, \phi] &\equiv \frac{1}{2}\left[\left(1 - \frac{r_h}{r}\right)(2(\omega')^2 + r^2(\phi')^2)\right] \\ &+ \frac{r^2}{2}\left\{\frac{(1-\omega^2)^2}{r^4} + \frac{\phi^2(1+\omega)^2}{2r^2} + \frac{\lambda}{2}(\phi^2 - v^2)^2\right\}. \end{aligned} \tag{119}$$

Hence, setting $\delta(\infty) = 0$ we obtain

$$\mathcal{M} = \frac{r_h}{2} + \int_{r_h}^{\infty} \mathcal{K}[\omega, \phi] e^{-\delta(r)} dr. \tag{120}$$

Varying this functional with respect to the matter fields yields the correct equations of motion [18]. Thus, the equilibrium solutions of the field equations will be stationary points of the functional $\mathcal{M}$.

If we plot the solution curve in $(v, \delta_0, \mathcal{M})$ space, then the resulting curve is smooth (c.f. figure 1). For the *Catastrophe Theory* (c.f. Appendix B) we consider δ_0 as a variable, and v as a *control* parameter. The Whitney surface could be defined in our case as follows. For each v, consider a smoothly varying set of functions $\omega_\delta, \phi_\delta$ indexed by the value of δ_0 they give:

$$\delta_0 = \int_{r_h}^{\infty} \frac{1}{r}(2(\omega_\delta')^2 + r^2(\phi_\delta')^2) dr \tag{121}$$

such that $\omega_\delta, \phi_\delta$ are the appropriate solutions to the field equations when δ_0 lies on the solution curve. Then, the solution curve represents the curve of extremal points of this Whitney surface, $\delta\mathcal{M} = 0$.

The projections of this curve onto the (v, δ_0) and $(\delta_0, \mathcal{M})$ planes are also smooth curves. The catastrophe map χ projects the solution curve onto the $(v, \mathcal{M})$ plane

$$\chi \ : \ (v, \delta_0, \mathcal{M}) \rightarrow (v, \mathcal{M}). \tag{122}$$

This yields the curve shown in figure 2. This map is regular except at the point $(v = v_{max}, \mathcal{M} = \mathcal{M}_{max})$, where it is singular. This point is the *bifurcation set* B.

Since the Whitney surface describes a one-parameter (v) family of functions of a single variable (δ), and the bifurcation point is a single point, we have a fold catastrophe, as found in ref. [6] from a different point of view. A more detailed comparison, of the results of that reference with ours will be made at the end of the section.

Catastrophe theory tells us that the stability of the system will change at the point B on the solution curve, and, furthermore, that the branch of solutions (including the point B) having the higher mass (for the same value of v) will be *unstable*, relative to the other branch. Hence, from our previous continuity considerations, the $k = n$ branch of solutions will have *exactly one more* negative mode than the quasi- $k = n - 1$ branch.

The catastrophe theory analysis applies to the gravitational sector rather than the sphaleronic sector, since gravitational perturbations correspond to *small* changes in the functions ω and ϕ, whilst keeping the functional form of $\mathcal{M}$ *fixed*. On the contrary, sphaleronic perturbations keep the functions ω and ϕ fixed, affecting the functional form of $\mathcal{M}$. As we discussed in the previous section, the number of unstable sphaleron modes is the *same* for the $k = n$ and quasi-$k = n - 1$ branches.

All that remains is to determine the number of negative modes of the quasi-$k = 0$ branch of solutions. From the above considerations, this will be equal to the number of negative modes of the $v = 0$ limiting case of this branch of solutions, which is nothing other than the Schwarzschild black hole. The gravitational perturbation equation is in this case, where a prime now denotes d/dr^*,

$$-\delta\omega'' + U_{\omega\omega}\delta\omega = \sigma^2\delta\omega \tag{123}$$

where

$$U_{\omega\omega} = \frac{NS^2}{r^2}\left[3\omega^2 - 1 - 4r(\omega')^2\left(\frac{N}{r} + \frac{(NS)'}{S}\right) + \frac{\omega(\omega^2-1)}{r}\delta\omega'\right]. \tag{124}$$

For the Schwarzschild solution $N = 1 - \frac{r_h}{r}, S = 1, \omega = 1$. Equation (123), then, reduces to

$$-\delta\omega'' + \frac{2}{r^2}\left(1 - \frac{r_h}{r}\right)\delta\omega = \sigma^2\delta\omega \tag{125}$$

which has the form of a standard one-dimensional Schrödinger equation with potential

$$V(r^*) = \frac{2}{r^2}\left(1-\frac{r_h}{r}\right) \qquad \frac{d}{dr^*} \equiv \left(1-\frac{r_h}{r}\right)\frac{d}{dr}. \tag{126}$$

As $r^* \to \pm\infty$, $V(r^*) \to 0$. On the other hand, for finite r^*, $-\infty < r^* < \infty$, the potential is positive definite, $V(r^*) > 0$.

Then, by a standard theorem of quantum mechanics [20], the Schrödinger equation (125,126) has no bound states. Thus, the Schwarzschild solution (and, hence, the quasi-$k = 0$ branch of solutions) has no negative gravitational modes, a known result in agreement with the no-hair conjecture.

Working inductively through the various branches of solutions (the $v = 0$ limit of the $k = n$ branch is the same as that for the quasi-$k = n - 1$ branch after replacing ω by $-\omega$) we find that the $k = n$ branch possesses exactly n unstable gravitational modes, and the quasi-$k = n - 1$ branch exactly $n - 1$ negative modes. This result has been conjectured but not proven in ref. [21].

Before closing the section we would like to compare our results with those of ref. [6]. In ref. [6] the authors also used catastrophe theory to draw conclusions about the stability of the EYMH black holes, but their approach was somewhat different. There the writers fixed the parameter $\lambda = 0.125$ and also fixed the Higgs mass v. They then varied the horizon radius r_h and for each solution calculated the value of the mass functional $\mathcal{M}$, the field strength at the horizon B_h given by

$$B_h = |F^2|^{\frac{1}{2}} \Big|_{\text{horizon}} \tag{127}$$

and the Hawking-Bekenstein entropy $S = \pi r_h$, to give a smooth solution curve in $(\mathcal{M}, B_h, S)$ space. The projection of this curve on to the $(\mathcal{M}, S)$ plane has the same qualitative features as our figure 2. Here we have also fixed $\lambda = 0.15$, and in addition we have fixed $r_h = 1$ and varied the Higgs mass v from 0 up to the bifurcation point. Torii et al concluded that the $k = 1$ branch of solutions was more unstable than the quasi-$k = 0$ branch of solutions. The advantage of our approach is that, by interpolating between the various coloured black hole solutions [19], beginning with the Schwarzschild solution, we will be able to calculate the exact number of unstable modes of each branch of solutions and not just give qualitative information concerning their relative stability.

5 Entropy considerations

It remains now to associate the above catastrophic considerations with some elementary 'thermodynamic' properties of the black hole solutions, and in particular their entropy. Our aim in this section is to give elementary estimates of the entropy of the various branches, assuming thermodynamic equilibrium of the black hole with a surrounding heat bath. Such estimates will allow an association of the stability issues with the amount of entropy carried by the various branches of the solution. In particular we shall argue that the 'high-entropy' branch has relatively fewer unstable modes than the 'low-energy' ones and, thus, is relatively more stable. We shall employ approximate WKB semi-classical methods for the evaluation of the entropy. We shall also study the conditions under which such estimates are valid.

The calculation of the entropy of the black hole will be made on the basis of calculating the entropy of quantum matter fields in the black hole space time. This will constitute only a partial contribution to the total black hole entropy. A complete calculation requires a proper quantization of the gravitational field, which at present is not possible, given the non-renormalizability of (local) quantum gravity. Ignoring such back reaction effects of the matter fields to the (quantum) geometry of space time results in ultra-violet divergences in the calculated entropy of the matter fields [7, 8]. Such divergences can be absorbed in a renormalization of the gravitational (Newton's) constant. This is so because the entropy is proportional to the area of the black hole horizon, with the divergent contributions appearing as multiplicative factors.

In what follows we shall estimate the entropy of a scalar field propagating in the EYMH black hole background. Anticipating a path integral formalism for quantum gravity, we shall compute only the entropy which is due to quantum fluctuations of the scalar field in the black hole background. The part that contains the classical solutions to the equations of motion contributes to the 'classical' entropy, associated with the classical geometry of space time. This part is known to be proportional to 1/4 of the horizon area [7, 8]. The quantum-scalar-field entropy part will also turn out to be proportional to the horizon area, but the proportionality coefficient is linearly divergent as the ultraviolet cut-off is removed, exactly as it happens in the corresponding computation for the Schwarzschild black hole [7]. Absorbing the divergence into a conjectured renormalization of the gravitational constant will enable us to estimate the entropy of the various branches of the EYMH black hole solution, and relate this to the above-mentioned catastrophe-theoretic arguments.

As we shall show, this will be possible only in the *non-extremal* case, which is the case of the numerical solutions studied in ref. [4] and in the present work. Among the solutions, however, there exist some *extremal* cases, for which the Hawking temperature - which is defined by assuming thermal equilibrium of the black hole system with a surrounding heat bath - vanishes. In such a case, the linearly divergent entropy of the scalar field vanishes. However, there are non trivial *logarithmically* divergent contributions to the black hole entropy which cannot be absorbed in a renormalization of the gravitational constant. Moreover, the classical Bekenstein-Hawking entropy formula seems to be violated by such contributions to the black-hole entropy. The situation is similar to the case of a scalar field in an extreme

(3+1)-dimensional dilatonic black hole background [9], and seems to be generic to black holes with non-conventional hair. We shall briefly comment on this issue at the end of the section.

We shall be brief in our discussion and concentrate only in basic new results, relevant for our discussion above. For details in the formalism we refer the interested reader to the existing literature [7, 8]. To start with, we note that the metric for the EYMH black holes is given by:

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right) e^{-2\delta(r)} dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (128)$$

Consider a scalar field of mass μ propagating in this spacetime [7], satisfying the Klein-Gordon equation:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) - \mu^2\Phi = 0. \quad (129)$$

Since the metric is spherically symmetric, consider solutions of the wave equation of the form

$$\Phi(t, r, \theta, \varphi) = e^{-iEt} f_{El}(r) Y_{lm_l}(\theta, \varphi) \quad (130)$$

where $Y_{lm_l}(\theta, \varphi)$ is a spherical harmonic and E is the energy of the wave. The wave equation separates to give the following radial equation for $f_{El}(r)$

$$\begin{aligned} \left(1 - \frac{2m(r)}{r}\right)^{-1} E^2 f_{El}(r) & \\ + \frac{e^{\delta(r)}}{r^2}\frac{d}{dr}\left[e^{-\delta(r)} r^2 \left(1 - \frac{2m(r)}{r}\right)\frac{df_{El}(r)}{dr}\right] & \\ - \left[\frac{l(l+1)}{r^2} + \mu^2\right] f_{El}(r) &= 0. \end{aligned} \quad (131)$$

The "brick wall" boundary condition is assumed [7], namely, the wave function is cut off just outside the horizon,

$$\Phi = 0 \text{ at } r = r_h + \epsilon \quad (132)$$

where r_h is the black hole horizon radius, and ϵ is a small, positive, fixed distance which will play the rôle of an ultraviolet cut-off. We also impose an infra-red cut-off at a very large distance L from the horizon:

$$\Phi = 0 \text{ at } r = L, \text{ where } L \gg r_h. \quad (133)$$

Hence f satisfies

$$f_{El}(r) = 0 \text{ when } r = r_h + \epsilon \text{ or } r = L. \quad (134)$$

In anticipation of being able to use a WKB approximation, define functions $K(r)$ and $h(r)$ by

$$K^2(r,l,E) = \left(1-\frac{2m(r)}{r}\right)^{-1}\left[E^2\left(1-\frac{2m(r)}{r}\right)^{-1}-\frac{l(l+1)}{r^2}-\mu^2\right] \quad (135)$$

$$h(r) = e^{-\delta(r)}r^2\left(1-\frac{2m(r)}{r}\right). \quad (136)$$

Then the equation for $f_{El}(r)$ becomes

$$\frac{1}{h(r)}\frac{d}{dr}\left[h(r)\frac{d}{dr}f_{El}(r)\right]+K^2(r,l,E)f_{El}(r)=0. \quad (137)$$

Now define a function $u(r)$ by

$$f_{El}(r)=\frac{u(r)}{\sqrt{h(r)}}. \quad (138)$$

Then $u(r)$ satisfies

$$\frac{d^2u}{dr^2}+\left[K^2+\frac{1}{4h^2}\left(\frac{dh}{dr}\right)^2-\frac{1}{2h}\frac{d^2h}{dr^2}\right]u=0. \quad (139)$$

The WKB approximation for u will be valid if

$$\left|\frac{1}{4h^2}\left(\frac{dh}{dr}\right)^2-\frac{1}{2h}\frac{d^2h}{dr^2}\right| \ll \left|K^2\right| \quad (140)$$

and

$$\left|\frac{dK}{dr}\right| \ll \left|K^2\right|. \quad (141)$$

The first inequality is required so that u can be taken to satisfy the equation

$$\frac{d^2u}{dr^2}+K^2u=0 \quad (142)$$

where K is now the radial wave number, and the second inequality is required so that the approximation to the wave function

$$u(r)\sim\frac{1}{\sqrt{K(r)}}\exp\left[\pm i\int K(r)dr\right] \quad (143)$$

is valid. Assuming, for the present, that the WKB approximation is valid, define the radial wave-number K as above whenever the right-hand-side of (135) is non-negative. Define $K^2=0$ otherwise. Then the number of radial modes n_K is given by

$$\pi n_K=\int_{r_h+\epsilon}^{L} drK(r,l,E) \quad (144)$$

where the fact that n_K must be an integer restricts the possible values of E. For fixed energy E, the total number N of solutions with energy less than or equal to E is

$$\begin{aligned}\pi N &= \int (2l+1)\pi n_K dl \\ &= \int_{r_h+\epsilon}^{L} \left(1-\frac{2m(r)}{r}\right)^{-1} dr \int (2l+1) dl \\ &\times \left[E^2 - \left(\frac{l(l+1)}{r^2}+\mu^2\right)\left(1-\frac{2m(r)}{r}\right)\right]^{\frac{1}{2}} \end{aligned} \tag{145}$$

where the integration is performed over all values of l such that the argument of the square root is positive.

The Hawking temperature of the black hole is given by

$$T^{-1} = \beta = \frac{4\pi r_h e^{\delta_h}}{1-2m_h'} \tag{146}$$

where δ_h is fixed by the requirement that $\delta(\infty)=0$, and $' = d/dr$. Assume further that $\beta^{-1} \ll 1$. It should be noted that $T=0$ in the extreme case $m_h' = 0.5$. Further comments on the entropy in this situation will be made at the end of the section. However, as discussed in Appendix A, this situation does not arise for the black holes we are concerned with. The free energy F of the system is given by

$$\begin{aligned} e^{-\beta F} &= \sum e^{-\beta E} \\ &= \prod_{n_K,l,m_l} \frac{1}{1-\exp(-\beta E)}. \end{aligned} \tag{147}$$

Hence

$$\begin{aligned} \beta F &= \sum_{n_K,l,m_l} \log(1-e^{-\beta E}) \\ &\simeq \int dl(2l+1) \int dn_K \log(1-e^{-\beta E}) \end{aligned} \tag{148}$$

for large β, integrating over appropriate l, E. Integrating by parts,

$$\begin{aligned} F &= -\frac{1}{\beta}\int dl(2l+1)\int d(\beta E)\frac{n_K}{\exp(\beta E)-1} \\ &= -\frac{1}{\pi}\int dl(2l+1)\int dE \frac{1}{\exp(\beta E)-1}\int_{r_h+\epsilon}^{L} dr \\ &\times \left(1-\frac{2m(r)}{r}\right)^{-1}\left[E^2-\left(1-\frac{2m(r)}{r}\right)\left(\frac{l(l+1)}{r^2}+\mu^2\right)\right]^{\frac{1}{2}} \end{aligned} \tag{149}$$

where we have substituted for n_K from (144). The l integration can be performed explicitly,

$$\begin{aligned}
&\int dl(2l+1)\left[E^2\left(1-\frac{2m(r)}{r}\right)\left(\frac{l(l+1)}{r^2}+\mu^2\right)\right]^{\frac{1}{2}} \\
&= \frac{2}{3}r^2\left(1-\frac{2m(r)}{r}\right)^{-1}\left(E^2-\left(1-\frac{2m(r)}{r}\right)\mu^2\right)^{\frac{3}{2}} \qquad (150)
\end{aligned}$$

to give

$$F=-\frac{2}{3\pi}\int dE\frac{1}{\exp(\beta E)-1}\int_{r_h+\epsilon}^{L} dr r^2\left(1-\frac{2m(r)}{r}\right)^{-2}\left[E^2-\left(1-\frac{2m(r)}{r}\right)\mu^2\right]^{\frac{3}{2}}. \qquad (151)$$

Introduce a dimensionless radial co-ordinate x by

$$x=\frac{r}{r_h}. \qquad (152)$$

Then

$$F=-\frac{2r_h^3}{3\pi}\int dE\frac{1}{\exp(\beta E)-1}\int_{1+\hat{\epsilon}}^{\hat{L}} dx x^2\left(1-\frac{2\hat{m}(x)}{x}\right)^{-2}\left[E^2-\left(1-\frac{2\hat{m}(x)}{x}\right)\mu^2\right]^{\frac{3}{2}} \qquad (153)$$

where $\hat{\epsilon}=\frac{\epsilon}{r_h}$, $\hat{L}=\frac{L}{r_h}$, and $\hat{m}(x)=\frac{m(xr_h)}{r_h}$.

The contribution to F for large values of x is

$$F_0=-\frac{2}{9\pi}L^3\int_{\mu}^{\infty} dE\frac{(E^2-\mu^2)^{\frac{3}{2}}}{\exp(\beta E)-1} \qquad (154)$$

which is the expression for the free energy in flat space. The contribution for x near 1 diverges as $\epsilon\to 0$. For x near 1, the leading order term in the integrand in (153) is

$$E^3(x-1)^{-2}(1-2\hat{m}'_h)^2 \qquad (155)$$

where $\hat{m}'_h=\hat{m}'(1)=m'(r_h)$. This gives the leading order divergence in F

$$\begin{aligned}
F_{div} &= -\frac{2r_h^3}{3\pi}\frac{(1-2\hat{m}'_h)^{-2}}{\hat{\epsilon}}\int dE\frac{E^3}{\exp(\beta E)-1} \\
&= -\frac{2\pi^3}{45\hat{\epsilon}}\frac{r_h^3(1-\hat{m}'_h)^{-2}}{\beta^4} \\
&= -\frac{2\pi^3}{45\epsilon}\frac{r_h^4(1-\hat{m}'_h)^{-2}}{\beta^4}. \qquad (156)
\end{aligned}$$

The total energy U and entropy S are given by

$$U=\frac{\partial}{\partial\beta}(\beta F)=\frac{2\pi^3}{15\epsilon}\frac{r_h^4(1-2\hat{m}'_h)^{-2}}{\beta^4} \qquad (157)$$

$$S = \beta^2 \frac{\partial F}{\partial \beta} = \frac{8\pi^3}{45\epsilon} \frac{r_h^4 (1 - 2\hat{m}_h')^{-2}}{\beta^3}. \tag{158}$$

Substituting for β from (146), obtain

$$S = \frac{r_h}{360\epsilon}(1 - 2m_h')e^{-3\delta_h}. \tag{159}$$

Before discussing the implications of this formula, it is necessary to ascertain when the approximations used are valid. Firstly, $\beta \gg 1$ if $r_h \gg 1$ or $1 - 2m_h' \ll 1$. In the first case, non-trivial (viz. non-Schwarzschild) solutions exist only for very small values of v, the Higgs mass [6] and these solutions will be very close to the Schwarzschild solution having mass $M = r_h/2$. In the second case, the black hole is very nearly extremal. For $1 = 2m_h'$ exactly, the above analysis does not apply. However, $1 - 2m_h' \ll 1$ for large n [19] and for any value of v for which a non-trivial solution exists. We shall discuss the physical implications of this (nearly) extremal case at the end of the section.

Secondly, consider the validity of the WKB approximation. The principal contribution to the free energy F comes from the region where K is large; in particular, above we have concentrated on x close to 1. It is expected that the WKB approximation will be valid when K is large. For K large, it may be approximated by

$$K = E\left(1 - \frac{2m(r)}{r}\right)^{-1}. \tag{160}$$

For r near r_h, then

$$K = E(1 - 2m_h')^{-1}(r - r_h)^{-1} + 0(1) \tag{161}$$

whence

$$\frac{dK}{dr} = -E(1 - 2m_h')^{-1}(r - r_h)^{-2}. \tag{162}$$

Hence

$$\left|\frac{dK}{dr}\right| \ll |K^2| \text{ if } \left|\frac{E}{1 - 2m_h'}\right| \gg 1. \tag{163}$$

Similarly, for r near r_h we may approximate h by

$$h = e^{-\delta} r^2 (1 - 2m_h')(r - r_h) + 0(r - r_h)^2. \tag{164}$$

Then

$$\frac{1}{4h^2}\left(\frac{dh}{dr}\right)^2 - \frac{1}{2h}\left(\frac{d^2h}{dr^2}\right) = \frac{1}{4(r - r_h)^2} + 0(r - r_h)^{-1}, \tag{165}$$

so that

$$\left|\frac{1}{4h^2}\left(\frac{dh}{dr}\right)^2 - \frac{1}{2h}\frac{d^2h}{dr^2}\right| \ll |K^2| \text{ if } \left|\frac{4E}{1 - 2m_h'}\right| \gg 1. \tag{166}$$

Thus, for black hole solutions with large n, the WKB approximation is valid except for small values of E. Now return to the expression for the entropy (158),

$$S \equiv S_{linear} = \frac{r_h}{360\epsilon}(1 - 2m'_h)e^{-3\delta_h}. \tag{167}$$

We notice first that the entropy is positive, due to the fact that for the solutions $m'_h < 1/2$ to avoid naked singularities. Having said that, we now fix n and consider the two branches of black hole solutions, the $k = n$ and quasi-$k = n - 1$ solutions. The linear divergence r_h/ϵ is a common multiplicative factor in all branches, and thus can be absorbed in a renormalization of the gravitational constant [8]. This can be done as follows: Re-write $r_h/\epsilon = 4\pi r_h^2 \frac{1}{\epsilon 4\pi r_h}$, where $A = 4\pi r_h^2$ is the horizon area, G_0 is the bare graviational coupling constant (which, by convention, had been set to one in the previous formulae), and $r_h\epsilon = 2m_h\epsilon$ may be considered as the invariant distance (cut-off) of the brick wall from the horizon. The classical Bekenstein-Hawking entropy formula is then still valid, but with the renormalized gravitational constant G_R replacing the bare (classical) one G_0

$$S_{classical} + S_{linear} = (\frac{1}{4G_0} + O[\frac{1}{\epsilon}])A = \frac{1}{4G_R}A \tag{168}$$

Such a renormalization may be thought of as expressing quantum matter back reaction effects to the space-time geometry. Doing this in our case, we observe from (167) that for each v, the $k = n$ solution has larger m'_h and δ_h than the quasi-$k = n - 1$ solution. Hence the $k = n$ solution has a lower entropy than the quasi-$k = n - 1$ solution, in agreement with Torii et al [6].

Before closing the section we would like to make some important comments concerning the extreme case $m'(r_h) = 1/2$, for which the Hawking temperature (146) *vanishes*. In this case the linearly-divergent part of the entropy (159) also vanishes, but this is not the case for the next-to-leading order *logarithmically* divergent part[1].

The logarithmic divergent part of the free energy can be found from (153) by requiring the following expansion

$$\hat{m}(x) = \hat{m}_h + \hat{m}'_h(x-1) + \frac{1}{2}\hat{m}''_h(x-1)^2 + \dots \qquad \hat{m}_h = \frac{1}{2}. \tag{169}$$

Using the trick $x = 1 + (x - 1)$ we can write down the identities:

$$\begin{aligned} x^{-1} &= 1 - (x-1) + (x-1)^2 + \dots \\ x^{-2} &= 1 + 2(x-1) + (x-1)^2. \end{aligned} \tag{170}$$

[1]It should be noted that the logarithmic divergent parts exist also in the non-extreme case, but there they are suppressed by the dominant linearly divergent terms. It can be easily checked that for the solutions of ref. [4], their presence does not affect the entropy considerations above, based on the linearly divergent term.

Hence

$$
\begin{aligned}
1-\frac{2\hat{m}(x)}{x} &= (1-2\hat{m}_h')(x-1)+(2\hat{m}_h-1-\hat{m}_h'')(x-2)^2+\dots \\
\left(1-\frac{2\hat{m}(x)}{x}\right)^{-2} &= (1-2\hat{m}_h')^{-2}(x-2)^{-2} \\
&\times\left\{1+\frac{2(x-1)(2\hat{m}_h'-1-\hat{m}_h'')}{2\hat{m}_h'-1}+\dots\right\}. \qquad (171)
\end{aligned}
$$

Substituting in (153) we obtain for the next-to-leading order divergence of the free energy

$$
\begin{aligned}
F_{nlo} &= \frac{2r_h^3}{3\pi}\frac{2(2-4\hat{m}_h'+\hat{m}_h'')}{(1-\hat{m}_h')^3}\int dE\frac{E^3}{e^{\beta E}-1}\int_{1+\hat{\epsilon}}dx\frac{1}{x-1} \\
&+\frac{2r_h^3}{3\pi}\frac{3}{2}\mu^2\frac{1}{1-2\hat{m}_h'}\int dE\frac{E}{e^{\beta E}-1}\int_{1+\hat{\epsilon}}\frac{dx}{x-1}. \qquad (172)
\end{aligned}
$$

Using the formulae

$$
\begin{aligned}
\int_0^\infty dE\frac{E^3}{e^{\beta E}-1} &= \frac{\pi^4}{15\beta^4} \\
\int_0^\infty dE\frac{E}{e^{\beta E}-1} &= \frac{\pi^2}{6\beta^2} \qquad (173)
\end{aligned}
$$

the expression (172) reduces to

$$
F_{nlo}=\frac{4}{45}r_h^3\frac{\pi^4}{\beta^4}\frac{2-4\hat{m}_h'+\hat{m}_h''}{(1-2\hat{m}_h')^3}\log\hat{\epsilon}-\frac{1}{6}r_h^3\frac{\pi}{\beta^2}\mu^2\frac{1}{1-2\hat{m}_h'}\log\hat{\epsilon}. \qquad (174)
$$

From (158) the corresponding next-to-leading contribution to the entropy in the extremal case $\hat{m}_h' = 1/2$ (where the linear divergence vanishes) is given by the following expression :

$$
S_{nlo}=\left[\frac{1}{3}r_h^2\mu^2e^{-\delta_h}-\frac{1}{180}e^{-3\delta_h}\hat{m}_h''\right]\log\left(\frac{\epsilon}{r_h}\right) \qquad (175)
$$

Thus, we observe that in the extremal case the entropy diverges logarithmically with the ultraviolet cut-off, in a similar spirit to the case of the dilatonic black hole background [9]. In our case, however, the horizon area does not vanish, because there is no dilaton field exponentially coupled to the graviton. Thus, one could hope that the divergent contribution (175) could be absorbed in the renormalization of the gravitational constant, so that a formal Bekenstein-Hawking expression for the entropy is still valid. However, as we see from (175), for generic scalar fields this cannot be the case, due to terms that spoil the proportionality of S_{nlo} to the black hole horizon area $A = 4\pi r_h^2$. Indeed, let us analyze the various contributions in (175).

The first term can be absorbed into a renormalization of the gravitational constant, and respects the classical formula (168). This is not the case with the second term however. From equation (202) of Appendix A, we observe that there are contributions that depend on the (boundary) horizon values of the fields ϕ_h and ω_h which are not proportional to the horizon area A,

$$m_h'' = \frac{1}{r_h}\left(1 - \frac{\phi_h^2}{2\omega_h}(1+\omega_h)^2\right) \tag{176}$$

Thus, the associated contribution to the black hole entropy seems not to be related to geometric aspects of the black hole background.

One is tempted to interpret such contributions as being associated with information loss across the horizon. This is supported by the fact that the logarithmic divergencies disappear for black holes whose horizon is vanishing in the sense of $r_h \to \epsilon$. For consistency with the interpretation as loss of information, the *positivity* requirement of the relevant contribution to the black hole entropy has to be imposed. Returning to formula (175) we observe that the above requirement implies $m_h'' > 0$. In the present case we do not know whether extremal solutions exist. *A priori* there is no reason why such solutions should not exist in the EYMH system. If such a solution exists, the above-mentioned positivity requirement will impose restrictions on the boundary (horizon) values of the hair fields of the black hole background. From the case at hand, it seems that the ambiguities in sign are associated with the presence of the non-abelian gauge field component ω. Indeed, from (176), it is immediately seen that the contributions of the scalar Higgs field ϕ alone to m_h'' are manifestly positive. The terms that could lead to negative logarithmic contributions to the entropy are associated with the field ω and vanish for $\omega = -1$.

This phenomenon is somewhat similar to what is happening in the case of a (spin one) gauge field in the presence of an ordinary black hole background. If one integrates quantum fluctuations of a spin one field in a gravity background, there is an induced coefficient in front of the Einstein curvature term in the effective action whose sign is negative for space-time dimensions less than 8 [22]. Notice that such sign ambiguities do not occur for scalar fields in conventional black hole backgrounds. In our case, there are gauge fields present in the black hole background associated with non-conventional hair. The sign ambiguities found above in the logarithmically-divergent contributions to the entropy (175) occur already when one considers quantum fluctuations of scalar fields. This is associated with negative signatures of terms that involve the gauge field hair background in the effective action.

Some comments are now in order concerning the the so-called *entanglement* entropy [23] of fields in background space times with event horizons or other space-time boundaries. The entanglement entropy is obtained from the density matrix of the field upon tracing over degrees of freedom that cross the event horizon or lie in the interior of the black hole, and therefore is closely associated with loss of information. The entanglement entropy is always positive. This immediately implies a difference from the ordinary black hole entropy, computed above, for the case of spin one fields [22]. On the other hand, for scalar fields in ordinary black hole backgrounds both entropies are identical, since in that case sign ambiguities in the entropy do not arise. On the other hand, our computation for the extreme EYMH case, provided the latter exists, has shown that, in general, one should expect a difference between the two entropies even in the case of scalar fields propagating in such (extreme) non-Abelian black hole backgrounds.

There exists, of course, the interesting possibility that the entanglement entropy of scalar fields in this extreme black hole background can be identified with the logarithmic entropy terms (175), in which case the latter must be positive definite. This, as we discussed above, would imply restrictions on the boundary (horizon) values of the gauge hair for the extreme black hole to exist. The restrictions seem to be relatively mild though. As an example of the kind of the situation one encounters in such cases, consider the case where extreme EYMH black hole solutions exist. From (176), we observe that positivity of m''_h implies restrictions on the size of ω_h, $2\omega_h > \phi_h^2(1+\omega_h)^2 > 0$, which is a mild restriction.

However, all these are mere speculations at this stage. One has to await for a complete analytic solution of the EYMH black hole problem before reaches any conclusions regarding entropy production and information loss in extreme cases. Therefore, we leave any further considerations on such issues for future work.

6 Conclusions and Outlook

In this work, we have analyzed in detail black holes in (3+1)-dimensional Einstein-Yang-Mills-Higgs systems. We have argued that the conditions for the no-hair theorem are violated, which allows for the existence of Higgs and non-Abelian hair. This analytic work supports the numerical evidence for the existence of hair found in [4]. This is due to a balance between the gauge field repulsion and the gravitational attraction. However we have shown that the above black holes are unstable, and therefore cannot be formed by gravitational collapse of stable matter. Although the instability of the black hole sphaleron sector was expected for topological reasons, however our analysis in this work, which includes an *exact* counting of the unstable modes in this sector, acquires value in the sense that we have managed to describe rigorously the sphaleron black holes from a mathematical point of view. In the gravitational sector we have used catastrophe theory to classify and count the unstable modes. Our method of using as a catastrophe functional the black hole mass and as a control parameter the Higgs field v.e.v. proved advantageous over existing methods of similar origin [6] in that we managed to understand the connection with the Schwarzschild black holes from a stability/catastrophe-theoretic point of view. The above analysis, although applied to a specific class of systems, however is quite general and the various steps can be applied to other self-gravitating structures in order to reach conclusions related to the existence of non-trivial hair and their stability. For instance, we can tackle the problem of moduli hair of black holes in string-inspired dilaton-coupled higher derivative gravity [24]. The presence of Gauss-Bonnet combinations in such systems shares many similarities with the case of the non-Abelian black holes, and it would be interesting to study in detail the possibility of having non-trivial hair (to all orders in the Regge slope α') and its stability, following the methods advocated in the present work.

In addition to the question of the stability of non-conventional hairy solutions, the above analysis has revealed another important aspect concerning the information theoretic content of these (3+1)-dimensional hairy black holes, namely the existence of logarithmic divergent contributions to the entropy of matter (quantum (scalar) fields) near the horizon. Such contributions owe their existence to the non-trivial hair of the black hole, and they modify the Bekenstein-Hawking entropy formula, by yielding contributions that do not depend on the horizon area. Our findings can be compared to a similar situation characterizing extreme (string-inspired) black holes [9]. There, the deviation from the Bekenstein-Hawking entropy was seen to occur by the fact that in the extreme case, due to the presence of the dilaton, the effective horizon area vanishes, whilst the entropy did not vanish. In our case, despite the non vanishing entropy in the extreme case, the logarithmically-divergent entropy contributions violate explicitly the classical entropy-area formula by yielding contributions that are independent of the horizon area. This kind of entropy is clearly associated with loss of information across the horizon but it is not described in terms of classical geometric characteristics of the black hole. If true in a full quantum theory of gravity, this phenomenon might explain the information paradox. The question of associating this entropy with the entanglement entropy of fields in the EYMH background is left open in the present work. We hope to come back to this issue in the near future.

Whether a full quantum theory of gravity could make sense of such divergencies or not remains to be seen. There are conjectures/indications that string theory, which is believed to be a mathematically consistent, finite theory of quantum gravity, yields finite extensive quantities at the horizon [8], if string states, which in a generalized sense are gauged states, are properly taken into account [25]. However, our understanding of these issues, which are associated with the incompatibility - at present at least - of canonical quantum gravity with quantum mechanics, is so incomplete that any claim or attempt to relate the above issues to realistic computations involving quantum black hole physics would be inappropriate. We think, however, that it is interesting to point out yet another contradiction of quantum mechanics and general relativity associated with the proper quantization of extended objects possessing space-time boundaries.

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Appendix A

Notation and conventions

Throughout this paper we use the sign conventions of Misner, Thorne and Wheeler [26] for the metric and curvature tensors. In particular, the signature of the metric is $(-+++)$. For the EYMH system, we write the most general spherically symmetric metric in the form

$$ds^2 = -NS^2 dt^2 + N^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \tag{177}$$

where N and S are functions of t and r only and can be written in terms of the mass function m and the function δ as

$$N(t,r) = 1 - \frac{2m(t,r)}{r}, \qquad S(t,r) = e^{-\delta(t,r)}. \tag{178}$$

This latter form of the metric is particularly useful for black hole space-times. Following ref. [4], we take the most general spherically symmetric SU(2) gauge potential in the form

$$A = a_0 \tau_r dt + a_1 \tau_r dr + (1+\omega)[\tau_\theta \sin\theta d\varphi - \tau_\varphi d\theta] + \tilde{\omega}[\tau_\theta d\theta + \tau_\varphi \sin\theta d\varphi] \tag{179}$$

where a_0, a_1, ω and $\tilde{\omega}$ are functions of t and r alone and the τ_i are given by

$$\begin{aligned} \tau_r &= \tau_1 \sin\theta\cos\varphi + \tau_2 \sin\theta \sin\varphi + \tau_3 \cos\theta & (180)\\ \tau_\theta &= \tau_1 \cos\theta\cos\varphi + \tau_2 \cos\theta\sin\varphi - \tau_3 \sin\theta & (181)\\ \tau_\varphi &= -\tau_1 \sin\varphi + \tau_2 \cos\varphi & (182) \end{aligned}$$

with τ_i, $i = 1,2,3$ the usual Pauli spin matrices. The complex Higgs doublet assumes the form

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_2 + i\psi_1 \\ \phi - i\psi_3 \end{pmatrix} \tag{183}$$

where a suitable spherically symmetric ansatz is

$$\boldsymbol{\psi} = \psi(t,r)\hat{\boldsymbol{r}}, \qquad \phi = \phi(t,r). \tag{184}$$

Then the EYMH Lagrangian is [4]

$$\begin{aligned} \mathcal{L}_{EYMH} = & -\frac{1}{4\pi}\left[\frac{1}{4}|F|^2 + \frac{1}{8}(\phi^2 + |\psi|^2)|A|^2 + \frac{1}{2}g^{MN}\left[\partial_M\phi\partial_N\phi + (\partial_M\psi)\cdot(\partial_N\psi)\right]\right.\\ & \left. +V(\phi^2+|\psi|^2) + \frac{1}{2}g^{MN}A_M\cdot[\psi\times\partial_N\psi + \psi\partial_N\phi - \phi\partial_N\psi]\right] \end{aligned} \tag{185}$$

and the Higgs potential is

$$V(\phi^2) = \frac{\lambda}{4}(\phi^2 - v^2)^2. \tag{186}$$

For the equilibrium static solutions, a_0, a_1, $\tilde{\omega}$ and ψ all vanish and the remaining functions depend on r only. The metric functions $m(r)$ and $\delta(r)$ are required by the Einstein equations to satisfy the following, where $' = d/dr$:

$$\begin{aligned} m'(r) &= \frac{1}{2}\left[\left(1-\frac{2m(r)}{r}\right)(2\omega'^2+r^2\phi'^2)\right] \\ &+\frac{r^2}{2}\left[\frac{(1-\omega^2)^2}{r^4}+\frac{\phi^2}{2r^2}(1+\omega)^2+\frac{\lambda}{2}(\phi^2-v^2)^2\right] \qquad (187) \\ \delta'(r) &= -\frac{1}{r}(2\omega'^2+r^2\phi'^2) \qquad (188) \end{aligned}$$

subject to the boundary conditions $m(r_h) = \frac{r_h}{2}$ in order for a regular event horizon at $r = r_h$, and, in order for the spacetime to be asymptotically flat, $\delta(\infty) = 0$. For an asymptotically flat spacetime, it is also the case that $m(r) \to M$ as $r \to \infty$, where M is a constant equal to the ADM mass of the black hole. Integrating (187) from r_h to ∞ we obtain:

$$\begin{aligned} \mathcal{M}-\frac{r_h}{2} &= m(\infty)-m(r_h) = \int_{r_h}^{\infty} m'(r)dr \\ &= \int_{r_h}^{\infty}\frac{1}{2}\left[\left(1-\frac{2m}{r}\right)(2\omega'^2+r^2\phi'^2)\right] \\ &+\frac{r^2}{2}\left[\frac{(1-\omega^2)^2}{r^4}+\frac{\phi^2}{2r^2}(1+\omega)^2+\frac{\lambda}{2}(\phi^2-v^2)^2\right] \qquad (189) \end{aligned}$$

This equation defines the mass functional $\mathcal{M}$ as an integral of the fields over the spacetime.

Finally we define the 'tortoise' co-ordinate r^* by

$$\frac{dr^*}{dr} = \frac{1}{NS}. \qquad (190)$$

Numerical solution of equilibrium equations

The static field equations for the metric functions and matter fields are:

$$\begin{aligned} m'(r) &= \frac{1}{2}\left[\left(1-\frac{2m}{r}\right)(2\omega'^2+r^2\phi'^2)\right] \\ &+\frac{r^2}{2}\left[\frac{(1-\omega^2)^2}{r^4}+\frac{\phi^2}{2r^2}(1+\omega)^2+\frac{\lambda}{2}(\phi^2-v^2)^2\right] \qquad (191) \\ \delta'(r) &= -\frac{1}{r}(2\omega'^2+r^2\phi'^2) \qquad (192) \\ N\omega'' &= -\frac{(NS)'}{S}\omega'+\frac{1}{r^2}(\omega^2-1)\omega+\frac{\phi^2}{4}(1+\omega) \qquad (193) \\ N\phi'' &= -\frac{(NS)'}{S}\phi'-\frac{2N}{r}\phi'+\frac{\phi}{2r^2}(1+\omega)^2+\lambda\phi(\phi^2-v^2) \qquad (194) \end{aligned}$$

For finite energy solutions, we require that $\omega(\infty) = -1$, $\phi(\infty) = v$ and $\delta(\infty) = 0$ in order for spacetime to be asymptotically flat. These equations trivially possess the Reissner-Nordström solution given by

$$m \equiv \frac{r_h}{2}, \qquad \omega \equiv -1, \qquad \phi \equiv v, \qquad \delta \equiv 0. \tag{195}$$

Non-trivial solutions do not occur in closed form, so a numerical method of solution is necessary as in [4]. We set the horizon radius $r_h = 1$ and $\lambda = 0.15$ (cf. $\lambda = 0.125$ in ref. [4]).

From the above equations, if the function $\delta(r)$ satisfies (192) then $\delta(r)+$ constant will also be a valid solution. To make the numerical solution easier, we set $\delta(r_h) = 0$ (so that $\delta(\infty) = 0$ will not be satisfied) when integrating outwards from r_h. An appropriate constant can then be added to $\delta(r)$, after the field equations have been solved, to ensure that the boundary condition at infinity holds.

With this transformation, there are two unknowns at the event horizon, $\omega(r_h)$ and $\phi(r_h)$, since the field equations yield

$$\begin{aligned} \omega'(1) &= \frac{\frac{1}{4}\phi_h^2(1+\omega_h) - \omega_h(1-\omega_h^2)}{1-(1-\omega_h^2)^2 - \frac{1}{2}\phi_h^2(1+\omega_h)^2 - \frac{\lambda}{2}(\phi_h^2 - v^2)^2} \\ \phi'(1) &= \frac{\frac{1}{2}\phi_h(1+\omega_h)^2 + \lambda\phi_h(\phi_h^2 - v^2)}{1-(1-\omega_h^2)^2 - \frac{1}{2}\phi_h^2(1+\omega_h)^2 - \frac{\lambda}{2}(\phi_h^2 - v^2)^2} \end{aligned} \tag{196}$$

where

$$\omega_h = \omega(r_h) = \omega(1) \qquad \phi_h = \phi(r_h) = \phi(1). \tag{197}$$

Solving the field equations (191)–(194) is therefore a two-parameter shooting problem. The procedure is to take initial 'guesses' for the unknowns ω_h and ϕ_h and then integrate the differential equations out from r_h using a standard ordinary differential equation solver, attempting to satisfy the boundary conditions for large r. The initial starting values for ω_h and ϕ_h are then adjusted until these boundary conditions are satisfied (see [27] for further details of the algorithm used).

For each fixed value of the Higgs mass v, there are many solutions which can be indexed by the number of nodes k of the potential function $\omega(r)$. Here we concentrate on the case $k = 1$. Then, for each v, there are two solutions which can be ascribed to one of two families of solutions: the $k = 1$ branch or the quasi-$k = 0$ branch, depending on the behaviour of the families as $v \to 0$. The quasi-$k = 0$ branch of solutions approaches the Schwarzschild solution $\omega \equiv 1$, $\phi \equiv 0$ as $v \to 0$, whereas the $k = 1$ branch of solutions approaches the first coloured black hole of [19] as $v \to 0$, with $\phi \equiv 0$. As v increases, the two branches of solutions join up at $v = v_{max} = 0.352$. This phenomenon does not occur for $\lambda = 0.125$, as found by Greene, Mathur and O'Neill [4]. However, they conjectured that the two branches of solutions would converge of some value of λ. We stress here that our approach is somewhat different from that of ref. [6], where the field equations were solved for fixed Higgs mass v and varying r_h, whereas we have fixed r_h and varied v.

For each value of the Higgs mass v, we calculated the quantities

$$\begin{aligned}\mathcal{M} &= \frac{r_h}{2} + \int_{r_h}^{\infty} \left\{ \frac{1}{2}\left[\left(1 - \frac{2m}{r}\right)(2\omega'^2 + r^2\phi'^2) \right] \right. \\ &\quad \left. + r^2 \left[\frac{(1-\omega^2)^2}{r^4} + \frac{\phi^2}{2r^2}(1+\omega)^2 + \frac{\lambda}{2}(\phi^2 - v^2)^2 \right] \right\} dr \qquad (198) \\ \delta_0 &= \int_{r_h}^{\infty} \frac{1}{r}(2\omega'^2 + r^2\phi'^2) dr \qquad (199)\end{aligned}$$

for each of the two solutions. The resulting solution curve plotted in $(v, \delta_0, \mathcal{M})$ space is shown in figure 1. The projection of this curve on to the $(v, \mathcal{M})$ plane are shown in figure 2.

One issue that is important, especially when we come to consider the thermodynamics and entropy of the black holes, is whether or not they are extremal. An extremal black hole occurs when N has a double zero at the event horizon, and is caused physically by an inner horizon moving outwards until it coincides with the outermost event horizon. Mathematically, the condition for extremality is that

$$m'(1) = \frac{1}{2}. \qquad (200)$$

From the field equations (194), we have

$$m'(1) = \frac{1}{2}\left[(1-\omega_h^2)^2 + \frac{1}{2}\phi_h(1+\omega_h)^2 + \frac{\lambda}{2}(\phi_h^2 - v^2)^2 \right] \qquad (201)$$

$$m''(r_h) = \frac{1}{r_h}\left(1 - \frac{\phi_h^2}{2\omega_h}(1+\omega_h)^2 \right) \qquad (202)$$

where in the last relation we have kept an explicit r_h dependence for calculational convenience. There is thus no *a priori* reason why this quantity should not be equal to one half for some equilibrium solution. For the solutions on the $k = 1$ and quasi-$k = 0$ branches, we can however place the following bounds on $m'(1)$. The first term is decreasing for ω_h positive and increasing, and hence is bouded above by its value for the smallest value of ω_h along these branches, which is $\omega_h = 0.632$, whence

$$(1-\omega_h^2)^2 \leq 0.360. \qquad (203)$$

Along both these branches, $\phi_h \leq 0.19v$ which gives the following bound on the second term,

$$\frac{1}{2}\phi_h^2(1+\omega_h)^2 \leq 2 \times (0.19v)^2 \leq 2 \times 0.19^2 \times 0.352^2 = 8.95 \times 10^{-3}. \qquad (204)$$

Finally, for the last term we have

$$\frac{\lambda}{2}(\phi_h^2 - v^2)^2 \leq \frac{\lambda}{2}v^4 \leq 0.15 \times 0.5 \times 0.352^4 = 1.15 \times 10^{-3}. \tag{205}$$

Adding together all the contributions, we find that

$$m'(1) \leq 0.5 \times (0.360 + 8.95 \times 10^{-3} + 1.15 \times 10^{-3}) = 0.185 \leq 0.5 \tag{206}$$

and hence all the equilibrium black holes considered here are non-extremal.

Linear perturbation equations

Consider small, time-dependent perturbations about the equilibrium solutions discussed above, within the initial ansatz for the metric and matter field functions. We use a δ to denote one of these small perturbation quantities, all other quantities are assumed to be static equilibrium functions. Following ref. [13], we set $\delta a_0 = 0$ so that the field configurations remain purely magnetic. With this choice, the perturbation equations decouple into two independent coupled systems. The first concerns δa_1, $\delta\tilde{\omega}$ and $\delta\psi$ only. The equations take the form, with a prime denoting d/dr^* where r^* is the tortoise co-ordinate:

$$\begin{aligned}
-Nr^2\delta\ddot{a}_1 &= 2N^2S^2\left(\omega^2 + \frac{r^2}{8}\phi^2\right)\delta a_1 + 2NS(\omega\delta\tilde{\omega}' - \omega'\delta\tilde{\omega}) \\
&\quad + \frac{1}{2}r^2NS(\phi'\delta\psi - \phi\delta\psi') && (207) \\
2\delta\ddot{\tilde{\omega}} &= 2(NS\omega\delta a_1)' + 2NS\omega'\delta a_1 + \delta\tilde{\omega}'' + NS^2\phi\delta\psi \\
&\quad - \frac{2}{r^2}S^2\left(\omega^2 - 1 + \frac{\phi^2}{4}\right)\delta\tilde{\omega} && (208) \\
-r^2\delta\ddot{\psi} &= \frac{1}{2}(NSr^2\phi\delta a_1)' + \frac{1}{2}r^2NS\phi'\delta a_1 - NS^2\phi\delta\tilde{\omega} - (r^2\delta\psi')' \\
&\quad + 2NS^2\left(\frac{(1-\omega)^2}{4} + \frac{1}{2}r^2\lambda(\phi^2 - v^2)\right)\delta\psi && (209) \\
0 &= \partial_t\left\{\left(\frac{r^2}{S}\delta a_1\right)' + 2\omega\tilde{\omega} - \frac{r^2}{2}\phi\delta\psi\right\}. && (210)
\end{aligned}$$

This final equation is known as the *Gauss constraint* equation, since it represents an additional constraint on the field perturbations rather than an equation of motion. This system of coupled equations is referred to as the *sphaleronic sector* because it does not involve any perturbations of the metric functions.

The remaining perturbation equations form the *gravitational sector* and concern the perturbations of the metric functions and also $\delta\omega$ and $\delta\phi$:

$$-\delta\ddot{\omega} = -\delta\omega'' + U_{\omega\omega}\delta\omega + U_{\omega\phi}\delta\phi \tag{211}$$

$$-\delta\ddot{\phi} = -\delta\phi'' + U_{\phi\omega}\delta\omega + U_{\phi\phi}\delta\phi \tag{212}$$

where the U's are complicated functions of N, S, ω and ϕ and are given explicitly in section 4, equation 105. The equations governing the behaviour of δm and δS are derived from the linearised Einstein equations and are:

$$\frac{d}{dr}(S\delta m) = \frac{d}{dr}\left(2NS\frac{d\omega}{dr}\delta\omega + r^2 NS\frac{d\phi}{dr}\delta\phi\right) \tag{213}$$

$$\delta\dot{m} = 2N\frac{d\omega}{dr}\delta\dot{\omega} + r^2 N\frac{d\phi}{dr}\delta\dot{\phi} \tag{214}$$

$$\delta\left(\frac{1}{S}\frac{dS}{dr}\right) = \frac{4}{r}\frac{d\omega}{dr}\frac{d\delta\omega}{dr} + 2r\frac{d\phi}{dr}\frac{d\delta\phi}{dr}. \tag{215}$$

From (213) δm has the form

$$\delta m = 2N\frac{d\omega}{dr}\delta\omega + Nr^2\frac{d\phi}{dr}\delta\phi + \frac{f(t)}{S} \tag{216}$$

where $f(t)$ is an arbitrary function of t. Compare this with the following, which results from integrating (214):

$$\delta m = 2N\frac{d\omega}{dr}\delta\omega + Nr^2\frac{d\phi}{dr}\delta\phi + g(r) \tag{217}$$

where $g(r)$ is an arbitrary function of r. Comparing (216) with (217), we see that $f(t) \equiv 0 \equiv g(r)$ and

$$\delta m = 2N\frac{d\omega}{dr}\delta\omega + Nr^2\frac{d\phi}{dr}\delta\phi. \tag{218}$$

We consider periodic perturbations of the form

$$\delta\omega(r,t) = \delta\omega(r)e^{i\sigma t} \tag{219}$$

and similarly for the other pertubation quantities. When substituted into the perturbation equations for each of the two sectors, the equations studied in detail in sections 3 and 4 are derived.

Appendix B

Definitions and results of catastrophe theory

Consider a family of functions

$$f : X \times C \to I\!R \qquad f(x,c) = f_c(x) \tag{220}$$

Here X and C are both manifolds known as the state space and control space respectively. In other words, we have a family of functions of the variable x, the members of the family being indexed by c. From now on we take both X and C to be intervals of the real line. Then f maps out a surface $z = f(x,c)$ in $I\!R^3$ which is known as the *Whitney surface* [28].

The catastrophe manifold is defined as the subset of $X \times C$ at which

$$\frac{d}{dx} f_c(x) = 0, \tag{221}$$

namely it is the set of all critical points of the family of functions. In section 4, critical points of the functional $\mathcal{M}$ correspond to solutions of the field equations, and hence the catastrophe manifold corresponds to the projection of the solution curve onto the $(x,c) = (\delta_0, v)$ plane.

The catastrophe map χ is the restriction to the catastrophe manifold of the natural projection

$$\pi : X \times C \to C, \qquad \pi(x,c) = c. \tag{222}$$

This can easily be extended to a projection of the solution curve on to the (c,z) plane:

$$\chi(x, c, z = f(x,c)) = (c, z = f(x,c)). \tag{223}$$

The singularity set is the set of singular points of χ in the catastrophe manifold, and the image of the singularity set in C is called the *bifurcation set* B. Here both manifolds X and C are of dimension 1, and hence χ will be singular whenever its derivative vanishes.

The first result we require is that the singularity set is the set of points (x,c) at which $f_c(x)$ has a degenerate critical point, in other words, both

$$\frac{d}{dx} f_c(x) = 0 \text{ and } \frac{d^2}{dx^2} f_c(x) = 0. \tag{224}$$

This implies that the set B is the place where the number and nature of the critical points of the family of functions $f_c(x)$ change (see [28] for more details of these results).

In our case, where both X and C are one-dimensional, the only possibility is that the bifurcation set B either is empty (in which case there is no catastrophe) or B contains a single point (when a *fold catastrophe* occurs). We observe in section 4 that the latter situation arises.

The catastrophe manifold is a curve $\mathcal{C}$ in the (x, c) plane, with the point B lying on it. On one side of the point B, points lying on $\mathcal{C}$ correspond to maxima of the functions $f_c(x)$ whilst on the other side of B they represent minima. In section 4 the value of f corresponds to energy, so that minima of f will represent (relatively) stable objects, whilst maxima of f will represent (relatively) unstable configurations.

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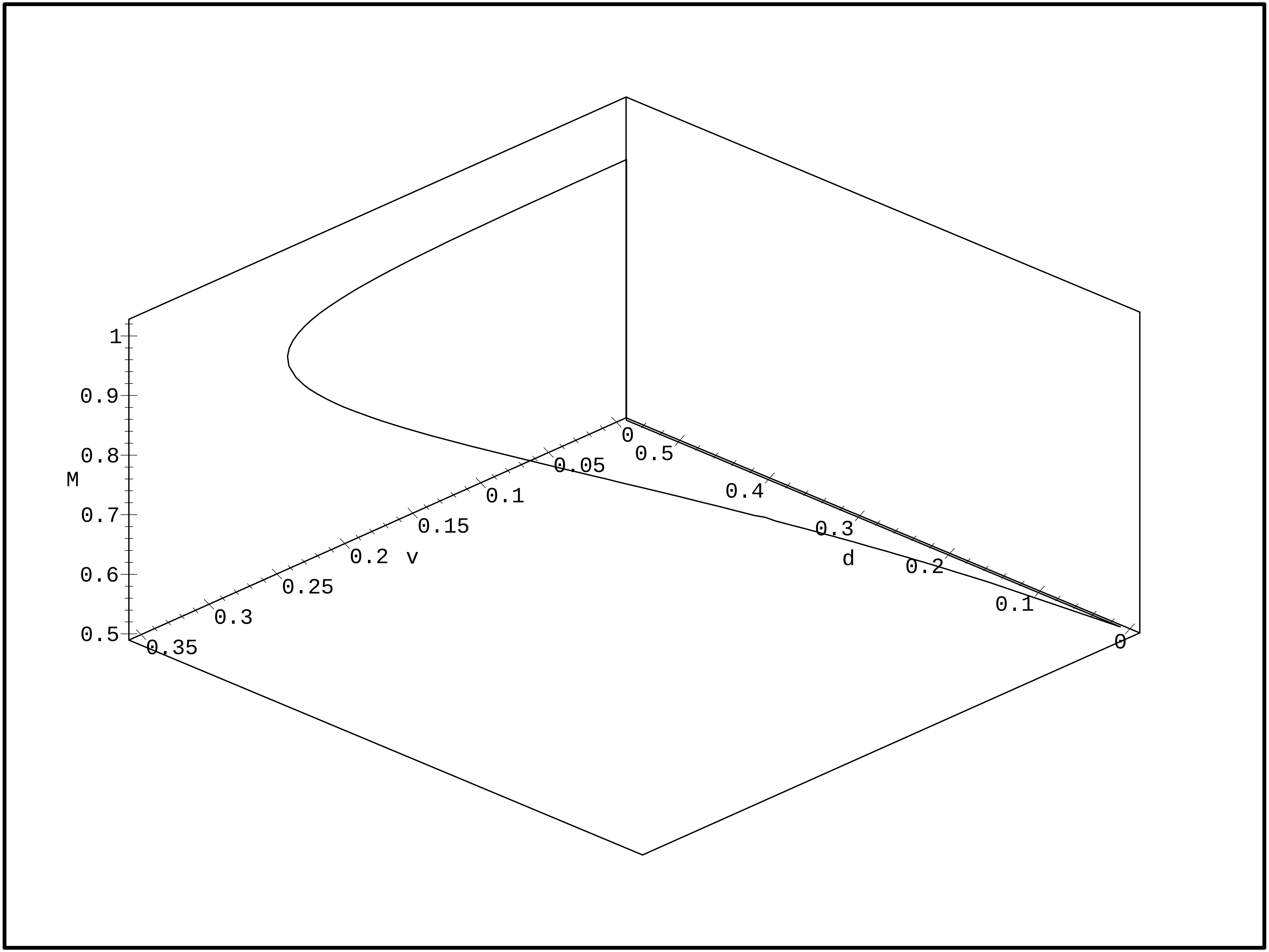
Figure Captions

Figure 1 Solution curve for black holes in EYMH theory with one node of the gauge field component ω, in $(v, \delta_0, \mathcal{M})$ parameter space: v is the Higgs v.e.v., δ_0 is the black hole parameter defined in Appendix A and $\mathcal{M}$ is the mass functional of the black hole. Notice that the solution curve is smooth, but this is not true for its projection onto the $(v, \mathcal{M})$ plane, see Figure 2 below.

Figure 2 Projection of the solution curve of Figure 1 onto the $(v, \mathcal{M})$ plane. The cusp at $v = v_{max}$ indicates the rôle of $\mathcal{M}$ as a fold catastrophe functional, with v the appropriate control parameter. The upper branch of solutions (quasi-$k = 0$), corresponding to higher entropy, is more stable relative to the lower branch ($k = 1$).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0.6
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v
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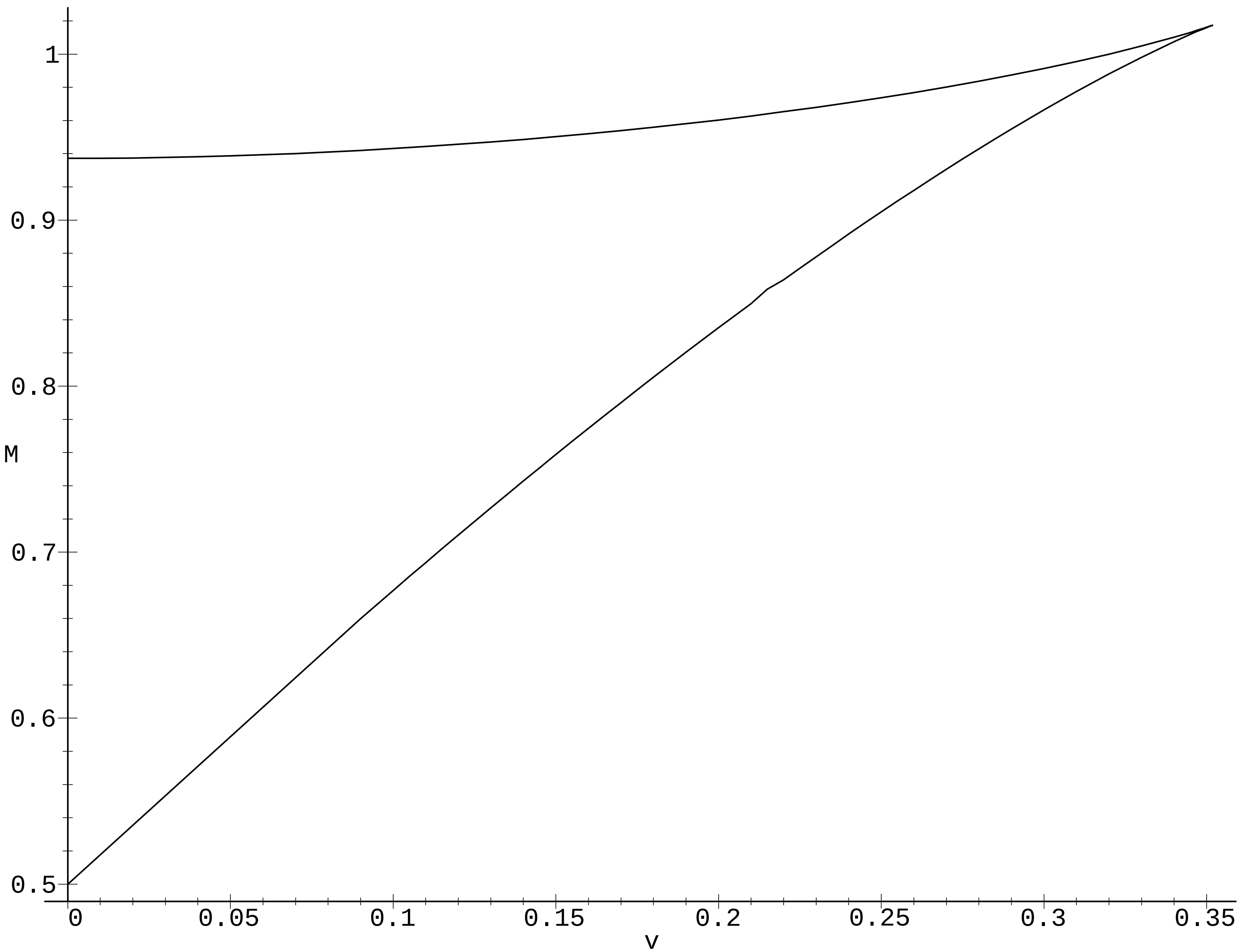
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