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Homogeneous spaces of Dirac groupoids

M. Jotz

Abstract

A Poisson structure on a homogeneous space of a Poisson groupoid is *homogeneous* if the action of the Lie groupoid on the homogeneous space is compatible with the Poisson structures. According to a result of Liu, Weinstein and Xu, Poisson homogeneous spaces of a Poisson groupoid are in correspondence with suitable Dirac structures in the Courant algebroid defined by the Lie bialgebroid of the Poisson groupoid. We show that this correspondence result fits into a more natural context: the one of *Dirac groupoids*, which are objects generalizing Poisson groupoids and multiplicative closed 2-forms on groupoids.

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Contents

1 Introduction

Poisson groups were introduced by Drinfel′d [7, 6] as classical counterpart of quantum groups. A Poisson group (G, π) is a Lie group G with a *multiplicative* Poisson bivector π, i.e. such that the multiplication $G \times G \to G$ is a Poisson map. The dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G inherits then a Lie algebra structure from the multiplicative Poisson structure on G. The two Lie algebras are compatible; they form together a Lie bialgebra. Drinfel′d showed in[7] that this defines a bijection between simply connected Poisson Lie groups and Lie bialgebras. The product $\mathfrak{g} \times \mathfrak{g}^*$ carries then the induced structure of a quadratic Lie algebra, i.e. of a Lie algebra with a nondegenerate symmetric pairing. Drinfel′d established then in [8] a bijection between a special class of maximal isotropic subalgebras of $\mathfrak{g} \times \mathfrak{g}^*$ and Poisson homogeneous spaces of the Poisson group. A homogeneous space of a Poisson group is a Poisson structure on a homogeneous space of the Lie group, such that the transitive action of the group on the homogeneous space is a Poisson map.

Liu, Weinstein and Xu studied the counterpart of this last result in the situation of Poisson groupoids. They defined in [18] the notion of Courant algebroid, generalizing the structure found by Courant [5] on the Pontryagin bundle $TM \oplus T^*M$ of a smooth manifold M. They also showed that a *Lie bialgebroid structure* on a pair (A, A^*) of vector bundles induces a Courant algebroid structure on the direct sum $A \oplus A^*$. Lie bialgebroids (with first component an integrable Lie algebroid) where shown by Mackenzie and Xu [23, 24] to be in one-one correspondence with source simply connected Poisson groupoids. Liu, Weinstein and Xu proved in [19] that Poisson homogeneous spaces of a Poisson groupoid correspond to a special class of Dirac structures in the corresponding Courant algebroid.

This paper shows that *any* Dirac structure in this Courant algebroid corresponds in fact to a *Dirac homogeneous space* of the Poisson groupoid. We have shown in [10] the counterpart of this result in the special case of Poisson groups, and more generally of *Dirac groups* and their homogeneous spaces.

Let us now be more explicit. Let $G \rightrightarrows M$ be a Lie groupoid endowed with a Poisson bivector field $\pi_G \in \Gamma(\bigwedge^2 TG)$. The bivector field π_G is *multiplicative* if the vector bundle map $\pi_G^\sharp : T^*G \to TG$ is a Lie groupoid morphism over some map $A^* \to TM$ [23], where A^* is the dual of the Lie algebroid A of $G \rightrightarrows M$, and $TG \rightrightarrows TM$ and $T^*G \rightrightarrows A^*$ are endowed with the tangent and cotangent Lie groupoid structures (see [4], [27], [22]). Equivalently, the graph of $\pi_G^\sharp$, $\operatorname{Graph}(\pi_G^\sharp) \subseteq TG \oplus T^*G$, is a subgroupoid of the *Pontryagin groupoid* $(TG \oplus T^*G) \rightrightarrows (TM \oplus A^*)$. The pair $(G \rightrightarrows M, \pi_G)$ is then a *Poisson groupoid.* Poisson groupoids were introduced by Weinstein [30] as a common generalization of Poisson groups and symplectic groupoids (see also [16] and references therein).

A Poisson groupoid $(G \rightrightarrows M, \pi_G)$ induces a Lie algebroid structure on A^* [23] and a Courant algebroid structure on the direct sum $A \oplus A^*$ [18]. The pair (A, A^*) is the *Lie bialgebroid* associated to $(G \rightrightarrows M, \pi_G)$. If $G \rightrightarrows M$ is a target-simply connected Lie groupoid, and if (A, A^*) has a Lie bialgebroid structure, then there exists a unique multiplicative Poisson structure π_G on G such that $(G\rightrightarrows M, \pi_G)$ is a Poisson groupoid with Lie bialgebroid (A, A^*) [24].

In the same spirit, a closed 2-form ω_G on a Lie groupoid $G\rightrightarrows M$ is multiplicative if the map $\omega_G^\flat : TG \to T^*G$ associated to ω_G is a Lie groupoid morphism over a map $\lambda : TM \to A^*$. It is shown in [3] and [1] that multiplicative closed 2-forms on a Lie groupoid $G\rightrightarrows M$ are in one-one correspondence with IM-2-forms: vector bundle maps $\sigma : A \to T^*M$ satisfying some compatibility conditions with the Lie algebroid structure on A.

Dirac structures unify Poisson brackets and closed 2-forms in the sense that the graphs of the vector bundle morphisms $\pi^\sharp : T^*M \to TM$ and $\omega^\flat : TM \to T^*M$ associated to a Poisson bivector π on M and a closed 2-form $\omega \in \Omega^2(M)$ define Dirac structures on the manifold M. Hence, it is natural to ask how to recover the two results above on classification of multiplicative Poisson bivectors and closed 2-forms on a Lie groupoid in terms of data on its algebroid, which are by nature very different, as special cases of a more general result about the infinitesimal description of *Dirac groupoids*. These objects have been defined in [26] (see also [17]); a Dirac groupoid is a groupoid endowed with a Dirac structure that is a subgroupoid of the Pontryagin groupoid $(TG \oplus T^*G) \rightrightarrows (TM \oplus A^*)$. This paper is the first part of a series of papers devoted to the study of the infinitesimal description of Dirac groupoids [11, 15].

We show in this paper that, given a Dirac groupoid $(G\rightrightarrows M, \mathsf{D})$, there is an induced Lie algebroid structure on the units $U = \mathsf{D} \cap (TM \oplus A^*)$ of the multiplicative Dirac structure (Theorem 3.15). This was predicted by [26] and, since U is the graph of the anchor map of A^* in the Poisson case, generalizes the construction of a Lie algebroid structure on A^* from a multiplicative Poisson structure on $G\rightrightarrows M$. We show then that the Courant algebroid structure on $TG \oplus T^*G$ defines naturally a Courant algebroid B over M (Theorem 3.23). In the Poisson case, we recover exactly the Courant algebroid $A \oplus A^* \to M$ defined by the Lie bialgebroid (A, A^*). This new approach therefore shows that the Courant algebroid $A \oplus A^*$ defined by the Lie bialgebroid (A, A^*) of a Poisson groupoid as a suitable restriction and reduction of the ambient Courant algebroid structure on $TG \oplus T^*G$.

We then focus on Dirac homogeneous spaces of Dirac groupoids. A *Poisson homogeneous space* (X, π_X) of a Poisson groupoid $(G\rightrightarrows M, \pi_G)$ is a homogeneous space X of $G\rightrightarrows M$ endowed with a Poisson structure π_X that is compatible with the action of $G\rightrightarrows M$ on $J : X \to M$ (see [19] for more details).

The paper [8] proves that the Poisson homogeneous spaces of a Poisson group are classified by a special class of Lagrangian subalgebras of the double Lie algebra $\mathfrak{g} \times \mathfrak{g}^*$ defined by the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ of the Poisson group. This result is extended to Poisson homogeneous spaces of Poisson groupoids in [18], and to *Dirac homogeneous spaces* of *Dirac groups* in [10]. Poisson homogeneous spaces of a Poisson groupoid are in one-one correspondence with a special class of Dirac structures in the Courant algebroid $A \oplus A^*$.

Here, we establish a one-one correspondence between $(G\rightrightarrows M, \mathsf{D})$-homogeneous Dirac manifolds and maximal isotropic subspaces of B satisfying a natural condition, both in the closed and general cases. Dirac homogeneous spaces of a Dirac groupoid are related in this manner to Dirac structures in B (Theorem 4.16). In the case of almost Dirac structures, we classify the homogeneous spaces in terms of an action of the bisections of $G\rightrightarrows M$ on the vector bundle B. This classification theorem for Dirac homogeneous spaces of Dirac groupoids unifies the theorems in [8], [18] and [10].

The problem is easier to tackle in the Lie group case than in the Lie groupoid setting. The Lie bialgebra of a Dirac group can be defined using the theory of Poisson groups: multiplicative Dirac structures on a Lie group are only a slight generalization of the graphs of multiplicative bivector fields [25, 10]. In the present paper, we first construct the object B that will play the role of the Lie bialgebroid in this more general setting. Because we find a suitable object for the classification of the homogeneous spaces, our classification theorem suggests that a lot of information about the Dirac groupoid is contained in the triple (A, U, B).

In [11], we completely describe *Dirac algebroids* via the associated Lie algebroid U and the Courant algebroid B. A Dirac algebroid is a Lie algebroid endowed with a compatible Dirac structure [26]. Since Dirac algebroids are the infinitesimal counterpart in the sense of [26] of Dirac groupoids, this leads to a method to recover a Dirac groupoid from the triple (A, U, B). This is explained in [15].

Outline of the paper. Some background about Lie groupoids, their Lie algebroids and Courant algebroids are recalled in §2.1 and some generalities about Dirac manifolds are recalled in §2.2.

The definition of a Dirac groupoid is given in §3.1. In §3.2, we study the set U of units of the Dirac structure, seen as a subgroupoid of $(TG\oplus T^*G)\rightrightarrows(TM\oplus A^*)$. We show that there is a Lie algebroid structures on this vector bundle over M. The existence of Lie algebroid structures on the cores K^{s} and K^{t} is then a consequence. We find a canonical Courant algebroid over M associated to a closed Dirac groupoid $(G\rightrightarrows M, \mathsf{D})$. In the case of a Poisson groupoid, this is the Courant algebroid defined by the Lie bialgebroid. In §3.4, we prove that there is an induced action of the bisections of $G\rightrightarrows M$ on the vector bundle defined in §3.3.

Dirac homogeneous spaces of Dirac groupoids are defined in Section 4. We prove then our main theorem (Theorem 4.16) on the correspondence between (closed) Dirac homogeneous spaces of a (closed) Dirac groupoid and Lagrangian subspaces (subalgebroids) of the Courant algebroid B.

Notations and conventions. Let M be a smooth manifold. We will denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the spaces of smooth sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $\mathsf{E}\to M$, the space of sections of E will be written $\Gamma(\mathsf{E})$. We will write $\mathrm{Dom}(\sigma)$ for the open subset of the smooth manifold M where the local section $\sigma\in\Gamma(\mathsf{E})$ is defined.

The *Pontryagin bundle* of M is the direct sum $TM \oplus T^*M \to M$. The zero section in TM will be considered as a trivial vector bundle over M and written 0_M, and the zero section in T^*M will be written 0^*_M. The pullback or restriction of a vector bundle $\mathsf{E} \to M$ to an embedded submanifold N of M will be written $\mathsf{E}|_N$. In the special case of the tangent and cotangent spaces of M, we will write T_NM and T^*_NM. The annihilator in T^*M of a smooth subbundle $F \subseteq TM$ will be written $F^\circ \subseteq T^*M$. Let finally $f : M \to N$ be a surjective submersion. Then the kernel of Tf is a smooth subbundle of TM. We write $T^fM = \ker(Tf)$.

A groupoid G with base M will be written $G \rightrightarrows M$. The set M will be considered most of the time as a subset of G, that is, the unity 1_p will be identified with p for all $p \in M$. The manifolds underlying Lie groupoids will always assumed to be Hausdorff.

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2 Review of necessary background

2.1 Lie groupoids and Lie algebroids

The general theory of Lie groupoids and their Lie algebroids can be found in [22]. We fix here some notation and conventions.

For $g \in G$, the *right translation* by g is written $R_g : \mathsf{s}^{-1}(\mathsf{t}(g)) \to \mathsf{s}^{-1}(\mathsf{s}(g))$, and the *left translation* $L_g : \mathsf{t}^{-1}(\mathsf{s}(g)) \to \mathsf{t}^{-1}(\mathsf{t}(g))$.

A *bisection* of $G \rightrightarrows M$ is a smooth map $\kappa : M \to G$ which is right-inverse to $\mathsf{t} : G \to M$ and is such that $\mathsf{s} \circ \kappa$ is a diffeomorphism. The set of bisections of G is denoted by $\mathcal{B}(G)$. If $\kappa : M \to G$ is a bisection of $G \rightrightarrows M$, then the *right translation by* κ is defined as follows:

$$R_\kappa : G \to G, \qquad g \mapsto R_{\kappa(\mathsf{s}(g))}(g) = g \cdot \kappa(\mathsf{s}(g)).$$

We will also use the *left translation by* κ,

$$L_\kappa : G \to G, \qquad g \mapsto L_{\kappa((s\circ\kappa)^{-1}(\mathsf{t}(g)))}(g).$$

The set $\mathcal{B}(G)$ of bisections of G has the structure of a group. For $\kappa, \lambda \in \mathcal{B}(G)$, the product $\lambda \star \kappa$ is given by $\lambda \star \kappa : M \to G$, $(\lambda \star \kappa)(p) = \lambda(p) \cdot \kappa((\mathsf{s} \circ \lambda)(p))$ for all $p \in M$.

We will also consider *local bisections* of G and local right translations in the following, without always saying it explicitly. We will write $\mathcal{B}_U(G)$ for the set of local bisections of $G \rightrightarrows M$ with the domain of definition $U \subseteq M$.

The Lie algebroid of a Lie groupoid In this paper, the Lie algebroid of the Lie groupoid $G \rightrightarrows M$ is $A := T^{\mathsf{t}}_M G$, equipped with the anchor map $T\mathsf{s}|_A$ and the Lie bracket defined by the left invariant vector fields. We write $(A, \rho, [\cdot\,,\cdot])$ for the *Lie algebroid of the Lie* *groupoid* G. Note that the vector field $a^l \in \mathfrak{X}(G)$, for $a \in \Gamma(A)$, satisfies $a^l \sim_{\mathsf{s}} \rho(a) \in \mathfrak{X}(M)$. We write $(\tilde{A}, \tilde{\rho}, [\cdot\,,\cdot]_{\tilde{A}})$ for the Lie algebroid of $G \rightrightarrows M$ defined by right-invariant vector fields.

We also recall the definition of the *exponential map for a Lie groupoid*:

Proposition 2.1 *[22, Proposition 3.6.1] Let $G \rightrightarrows M$ be a Lie groupoid, choose $a \in \Gamma(A)$ and set $W = \mathrm{Dom}(a)$. For all $p \in W$ there exists an open neighborhood V of p in W, a flow neighborhood for a, an $\varepsilon > 0$ and a unique smooth family of local bisections $\mathrm{Exp}(ta) \in \mathcal{B}_V(G)$, $|t| < \varepsilon$, such that:*

1. $\frac{d}{dt}\big|_{t=0} \mathrm{Exp}(ta) = a$,
2. $\mathrm{Exp}(0 \cdot a) = \mathrm{Id}_V$,
3. $\mathrm{Exp}((t+s)a) = \mathrm{Exp}(ta) \cdot \mathrm{Exp}(sa)$, *if* $|t|, |s|, |s+t| < \varepsilon$,
4. $\mathrm{Exp}(-ta) = (\mathrm{Exp}(ta)) \in V$,
5. $\{\mathsf{s} \circ \mathrm{Exp}(ta) : V \to V_t\}$ *is a local* 1*-parameter group of transformations for* $\rho(a) \in \mathfrak{X}(M)$.

Let $G \rightrightarrows M$ be a Lie groupoid and let C_p be the connectedness component of p in $\mathsf{t}^{-1}(p)$. Then the union

$$C(G) := \bigcup_{p \in M} C_p$$

is a wide Lie subgroupoid of $G \rightrightarrows M$ (see [22]), the *identity-component subgroupoid* of $G \rightrightarrows M$. The set of values $\mathrm{Exp}(ta)(p)$, for all $a \in \Gamma(A)$, $p \in M$ and $t \in \mathbb{R}$ where this makes sense, is the identity-component subgroupoid $C(G)$ of $G \rightrightarrows M$ (see [24],[22]). Hence, if $G \rightrightarrows M$ is t-connected, that is, if all the t-fibers of G are connected, then $G = C(G)$ is the set of values of $\mathrm{Exp}(ta)(p)$, $a \in \Gamma(A)$, $p \in M$ and $t \in \mathbb{R}$ where defined.

Note that, in the same manner, the flow of a right invariant vector field b^r is the left translation by a family of bisections $\{L_t\}$ of G satisfying $\mathsf{s} \circ L_t = \mathrm{Id}$ on their domains of definition and such that $\mathsf{t} \circ L_t$ are diffeomorphisms on their images. Hence, the flow of b^r commutes with the flow of a^l for any left invariant vector field a^l and we get the fact that $\left[b^r, a^l\right] = 0$ for all $b \in \Gamma(T^{\mathsf{s}}_M G)$ and $a \in \Gamma(A)$.

The tangent prolongation of a Lie groupoid Let $G \rightrightarrows M$ be a Lie groupoid. Applying the tangent functor to each of the maps defining G yields a Lie groupoid structure on TG with base TM, source $T\mathsf{s}$, target $T\mathsf{t}$ (these maps will be written s and t in the following) and multiplication $T\mathsf{m} : T(G \times_M G) \to TG$. The identity at $v_p \in T_pM$ is $1_{v_p} = T_p\epsilon v_p$. This defines the *tangent prolongation of* $G \rightrightarrows M$ or the *tangent groupoid associated to* $G \rightrightarrows M$. We write $v_g \star v_h = T\mathsf{m}(v_g, v_h)$ for compatible $v_g, v_h \in TG$.

The cotangent Lie groupoid defined by a Lie groupoid If $G \rightrightarrows M$ is a Lie groupoid, then there is an induced Lie groupoid structure on $T^*G \rightrightarrows A^* = (TM)^\circ$. The source map $\hat{\mathsf{s}} : T^*G \to A^*$ is given by

$$\hat{\mathsf{s}}(\alpha_g) \in A^*_{\mathsf{s}(g)}G \text{ for } \alpha_g \in T^*_g G, \qquad \hat{\mathsf{s}}(\alpha_g)(a_{\mathsf{s}(g)}) = \alpha_g(T_{\mathsf{s}(g)} L_g a_{\mathsf{s}(g)})$$

for all $a_{\mathsf{s}(g)} \in A_{\mathsf{s}(g)}G$, and the target map $\hat{\mathsf{t}} : T^*G \to A^*$ is given by

$$\hat{\mathsf{t}}(\alpha_g) \in A^*_{\mathsf{t}(g)}G, \qquad \hat{\mathsf{t}}(\alpha_g)(a_{\mathsf{t}(g)}) = \alpha_g\big(T_{\mathsf{t}(g)} R_g (a_{\mathsf{t}(g)} - T_{\mathsf{t}(g)} \mathsf{s} a_{\mathsf{t}(g)})\big)$$

for all $a_{\mathsf{t}(g)} \in A_{\mathsf{t}(g)}G$. If $\hat{\mathsf{s}}(\alpha_g) = \hat{\mathsf{t}}(\alpha_h)$, then the product $\alpha_g \star \alpha_h$ is defined by

$$(\alpha_g \star \alpha_h)(v_g \star v_h) = \alpha_g(v_g) + \alpha_h(v_h)$$

for all composable pairs $(v_g, v_h) \in T_{(g,h)}(G \times_M G)$.

This Lie groupoid structure was introduced in [4] (see also [27] and [22]).

The Pontryagin groupoid defined by a Lie groupoid If $G \rightrightarrows M$ is a Lie groupoid, there is hence an induced Lie groupoid structure on $\mathsf{P}_G = TG \oplus T^*G$ over $TM \oplus A^*$. We will write $\mathbb{T}\mathsf{t}$ for the target map $\mathsf{P}_G \to TM \oplus A^*$, and $\mathbb{T}\mathsf{s} : \mathsf{P}_G \to TM \oplus A^*$ for the source map. Here again, we write $p_g \star p_h$ for the product[1] of compatible $p_g, p_h \in \mathsf{P}_G$.

The proof of the following two lemmas is straightforward.

Lemma 2.2 *Let $G \rightrightarrows M$ be a Lie groupoid. Choose $g, h \in G$ and $\kappa \in \mathcal{B}(G)$. Choose $(v_h, \alpha_h) \in \mathsf{P}_G(h)$, $(v_g, \alpha_g) \in \mathsf{P}_G(g)$ such that $\mathbb{T}\mathsf{s}\,(v_g, \alpha_g) = \mathbb{T}\mathsf{t}\,(v_h, \alpha_h)$. Then*

$$T_{g\star h} R_\kappa (v_g \star v_h) = v_g \star (T_h R_\kappa v_h) \tag{1}$$

and

$$\alpha_g \star \Big(\big(T_{R_\kappa(h)} R_\kappa^{-1}\big)^* \alpha_h \Big) = \big(T_{R_\kappa(g\star h)} R_\kappa^{-1}\big)^* (\alpha_g \star \alpha_h). \tag{2}$$

Lemma 2.3 *Let $G \rightrightarrows M$ be a Lie groupoid. Choose $g \in G$ and set $p = \mathsf{t}(g)$. Then, for all $\alpha_p \in T^*_p M$, we have*

$$-(T_{g^{-1}} \mathsf{s})^* \alpha_p = ((T_g \mathsf{t})^* \alpha_p)^{-1}.$$

Remark 2.4 If $(v_p, (T_p\mathsf{s})^*\alpha_p)$ is such that $T_p\mathsf{t} v_p = 0_p$, then $\mathbb{T}\mathsf{t}(v_p, (T_p\mathsf{s})^*\alpha_p) = (0_p, 0_p)$. It is easy to check that if $g \in G$ is such that $\mathsf{s}(g) = p$, then $(0_g, 0_g) \star (v_p, (T_p\mathsf{s})^*\alpha_p) = (T_p L_g v_p, (T_g\mathsf{s})^*\alpha_p)$ for all $g \in \mathsf{s}^{-1}(p)$. $\triangle$

[1] Note that in order to simplify the notation, we also use the symbol $\star$ for the products in $TG \rightrightarrows TM$, in $T^*G \rightrightarrows A^*$, and for the product of bisections. It should nevertheless always be clear from the context to which multiplication this symbol refers.

Homogeneous spaces Let $G \rightrightarrows M$ be a Lie groupoid and X a set with a map $J : X \to M$. Consider the set $G \times_M X = \{(g, x) \in G \times X \mid \mathsf{s}(g) = J(x)\}$. A groupoid action of $G \rightrightarrows M$ on $J : X \to M$ is a map $\Phi : G \times_M X \to X$, $\Phi(g, x) = g \cdot x = gx$ such that $J(g \cdot x) = \mathsf{t}(g)$ for all $(g, x) \in G \times_M X$, $g \cdot (h \cdot x) = (g \cdot h) \cdot x$ for all $(h, x) \in G \times_M X$, and $g \in G$ such that $\mathsf{s}(g) = \mathsf{t}(h)$, and $J(x) \cdot x = x$ for all $x \in X$.

Let $G \rightrightarrows M$ be a Lie groupoid and $H \rightrightarrows M$ a wide subgroupoid of G. Define the equivalence relation

$$g \sim_H g' \iff \exists\, h \in H \text{ such that } g \cdot h = g'$$

on G and $G/H := \{gH \mid g \in G\}$, where $gH = \{g \cdot h \mid \mathsf{s}(g) = \mathsf{t}(h) \text{ and } h \in H\}$. The map t factors to a map $J : G/H \to M$, $J(gH) = \mathsf{t}(g)$ for all $gH \in G/H$. The multiplication $\mathsf{m} : G \times_M G \to G$ factors to a groupoid action Φ of $G \rightrightarrows M$ on $J : G/H \to M$, $\Phi(g, g'H) = (g \cdot g')H$ for all $(g, g'H) \in G \times_M (G/H)$.

The topological space G/H with H a wide subgroupoid of $G \rightrightarrows M$ is a *homogeneous space* of $G \rightrightarrows M$ [19].

Example 2.5 Let $G \rightrightarrows M$ be a groupoid. The two extreme examples of homogeneous spaces of G are the following.

1. In the case where the wide subgroupoid is M, the equivalence classes are $gM = \{g \cdot p \mid p \in M, p = \mathsf{s}(g)\} = \{g\}$ and the quotient is just $G/M = G$, where $G \rightrightarrows M$ acts on $\mathsf{t} : G \to M$ via the multiplication.

2. If the wide subgroupoid is G itself, then the equivalence classes are $gG = \{g \cdot h \mid h \in G, \mathsf{t}(h) = \mathsf{s}(g)\} = \mathsf{t}^{-1}(\mathsf{t}(g))$ and the quotient is $G/G = M$, with projection equal to the target map $\mathsf{t} : G \to G/G \simeq M$. $G \rightrightarrows M$ acts on $\mathrm{Id}_M : M \to M$ via $\Phi : G \times_M M \to M$, $(g, p) \mapsto \mathsf{t}(g \cdot p) = \mathsf{t}(g)$. $\Diamond$

Assume that H *is a* t*-connected wide Lie subgroupoid of* G and that G/H is a smooth manifold such that the projection $q : G \to G/H$ is a smooth surjective submersion.

Consider the vector bundle $AH = T^{\mathsf{t}}_M H \subseteq T_M H \subseteq T_M G$ over M and the subbundle $\mathcal{H} \subseteq TG$ defined as the left invariant image of AH, i.e. $\mathcal{H}(g) = T_{\mathsf{s}(g)} L_g A_{\mathsf{s}(g)} H$ for all $g \in G$. Then $\mathcal{H} = \ker Tq$ and G/H is the leaf space of the foliation on G defined by the *involutive subbundle* $\mathcal{H} \subseteq TG$.

Consider the set $\mathcal{B}(H)$ of (local) bisections $\kappa : U \subseteq M \to H$ of H such that $\mathsf{t} \circ \kappa = \mathrm{Id}_U$ and $\mathsf{s} \circ \kappa$ is a diffeomorphism. We have $gH = \{R_\kappa(g) \mid \kappa \in \mathcal{B}(H)\}$ and G/H is the quotient of G by the right action of $\mathcal{B}(H)$ on G. A function $f \in C^\infty(G)$ pushes forward to the quotient G/H if and only if it is invariant under R_κ for all bisections $\kappa \in \mathcal{B}(H)$.

Courant algebroids A Courant algebroid [18, 28] over a manifold M is a vector bundle $\mathsf{E} \to M$ equipped with a fibrewise nondegenerate symmetric bilinear form $\langle \cdot\,, \cdot \rangle$, a bilinear bracket $[\![\cdot\,, \cdot]\!]$ on the smooth sections $\Gamma(\mathsf{E})$, and a vector bundle map $\rho \colon \mathsf{E} \to TM$ over the identity called the anchor, which satisfy the following conditions

1. $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$,

2. $\rho(e_1)\langle e_2, e_3\rangle = \langle [\![e_1, e_2]\!], e_3\rangle + \langle e_2, [\![e_1, e_3]\!]\rangle$,

3. $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \mathcal{D}\langle e_1, e_2\rangle$

for all $e_1, e_2, e_3 \in \Gamma(\mathsf{E})$ and $\varphi \in C^\infty(M)$. Here, we use the notation $\mathcal{D} := \rho^* \circ \mathbf{d}\colon C^\infty(M) \to \Gamma(\mathsf{E})$, using $\langle \cdot\,, \cdot\rangle$ to identify E with E^*:

$$\langle \mathcal{D}\varphi, e\rangle = \rho(e)(\varphi)$$

for all $\varphi \in C^\infty(M)$ and $e \in \Gamma(E)$. The following conditions

4. $\rho([\![e_1, e_2]\!]) = [\rho(e_1), \rho(e_2)]$,

5. $[\![e_1, \varphi e_2]\!] = \varphi[\![e_1, e_2]\!] + (\rho(e_1)\varphi)e_2$

are then also satisfied. They are often part of the definition in the literature, but it was already observed in [29] that they follow from (1) – (3).

Example 2.6 Let (A, A^*) be a pair of dual vector bundles, both endowed with Lie algebroid structures $(A \to M, \rho, [\cdot\,, \cdot])$ and $(A^* \to M, \rho_\star, [\cdot\,, \cdot]_\star)$.

The direct sum $A \oplus A^*$ is naturally endowed with an anchor map $\mathbf{c} := \rho + \rho_\star : A \oplus A^* \to TM$, $\mathbf{c}(x, \xi) = \rho(x) + \rho_\star(\xi)$ and the symmetric bracket $\langle \cdot\,, \cdot\rangle$ given by

$$\langle (x_m, \xi_m), (y_m, \eta_m)\rangle = \xi_m(y_m) + \eta_m(x_m) \tag{3}$$

for all $m \in M$, $x_m, y_m \in T_mM$ and $\xi_m, \eta_m \in T^*_mM$. Define the bracket

$$[\![(X, \xi), (Y, \eta)]\!]_{A\oplus A^*} = ([X, Y] + \pounds_\xi Y - \mathbf{i}_\eta \mathbf{d}_{A^*} X, [\xi, \eta]_\star + \pounds_X \eta - \mathbf{i}_Y \mathbf{d}_A \xi)$$

for all $(X, \xi), (Y, \eta) \in \Gamma(A \oplus A^*)$. Then $(A \oplus A^*, \mathbf{c}, \langle \cdot\,, \cdot\rangle, [\![\cdot\,, \cdot]\!]_{A\oplus A^*})$ is a Courant algebroid if and only if (A, A^*) is a Lie bialgebroid[2].

Here, $\pounds_X : \Gamma(A^*) \to \Gamma(A^*)$ is the derivation that is dual to the derivation $[X, \cdot] : \Gamma(A) \to \Gamma(A)$, i.e. $\langle \pounds_X \eta, Y\rangle = \rho(X)\langle \eta, Y\rangle - \langle \eta, [X, Y]\rangle$. In the same manner, $\pounds_\xi : \Gamma(A) \to \Gamma(A)$ is dual to $[\xi, \cdot]_\star : \Gamma(A^*) \to \Gamma(A^*)$. The operator $\mathbf{d}_A : \Gamma(\Lambda^\bullet A^*) \to \Gamma(\Lambda^{\bullet+1} A^*)$ is the differential associated to $(A, \rho, [\cdot\,, \cdot])$. In the case of sections of A^*, it is hence simply

$$\mathbf{d}_A \xi(X, Y) = \rho(X)(\xi(Y)) - \rho(Y)(\xi(X)) - \xi([X, Y]),$$

$\xi \in \Gamma(A^*)$, $X, Y \in \Gamma(A)$.

Note that in the case of the Lie bialgebroid (TM, T^*M), where T^*M is endowed with the trivial Lie algebroid structure, we get the standard Courant algebroid structure on the Pontryagin bundle $TM \oplus T^*M$, with anchor pr_{TM} and the *Courant-Dorfman bracket* on its set of sections:

$$[\![(X, \alpha), (Y, \beta)]\!] = ([X, Y], \pounds_X \beta - \mathbf{i}_Y \mathbf{d}\alpha)\,,$$

$X, Y \in \mathfrak{X}(M)$, $\alpha, \beta \in \Omega^1(M)$. $\Diamond$

[2] Since we do not need the explicit definition of a Lie bialgebroid, we will take this to be the definition.

Remark 2.7 Let $G\rightrightarrows M$ be a Lie groupoid. Then

$$\ker \mathbb{T}\mathsf{t} = (\ker \mathbb{T}\mathsf{s})^\perp \text{ and } \ker \mathbb{T}\mathsf{s} = (\ker \mathbb{T}\mathsf{t})^\perp \tag{4}$$

relative to the pairing (3) on $TG\oplus T^*G$. Later results will show that the sets of sections of $\ker \mathbb{T}\mathsf{t}$ and $\ker \mathbb{T}\mathsf{s}$ are both closed under the Courant-Dorfman bracket on $\Gamma(TG\oplus T^*G)$.△

2.2 Dirac structures

As we have seen in Example 2.6, the *Pontryagin bundle* $\mathsf{P}_M := TM \oplus T^*M$ of a smooth manifold M is endowed with the non-degenerate symmetric fiberwise pairing of signature $(\dim M, \dim M)$ given by (3). An *almost Dirac structure* (see [5]) on M is a Lagrangian vector subbundle $\mathsf{D} \subset \mathsf{P}_M$. That is, D coincides with its orthogonal relative to (3) and so its fibers are necessarily $\dim M$-dimensional.

Let (M, D) be a Dirac manifold. For each $m \in M$, the Dirac structure D defines a subspace $\mathsf{G}_0(m) = \mathsf{D}(m) \cap T_mM \subset T_mM$ by

$$\mathsf{G}_0(m) := \{v_m \in T_mM \mid (v_m, 0) \in \mathsf{D}(m)\}.$$

The almost Dirac structure D is a *Dirac structure* if

$$[\![\Gamma(\mathsf{D}), \Gamma(\mathsf{D})]\!] \subset \Gamma(\mathsf{D}). \tag{5}$$

The restriction of the Courant bracket to the sections of a Dirac bundle is skew-symmetric and satisfies the Jacobi identity.

Note that in the following, we work in the general setting of almost Dirac structures. To simplify the notation, we will simply call almost Dirac structures "Dirac structures" and always state it explicitely if the integrability condition (5) **is assumed to be satisfied. We will say in this case that the Dirac structure is *closed*.**[3]

(Closed) Dirac structures can be defined in the same manner in an arbitrary Courant algebroid.

The class of Dirac structures given in the next example will be very important in the following.

Example 2.8 Let M be a smooth manifold endowed with a globally defined bivector field $\pi \in \Gamma\left(\bigwedge^2 TM\right)$. Then the subdistribution $\mathsf{D}_\pi \subseteq \mathsf{P}_M$ defined by

$$\mathsf{D}_\pi(m) = \left\{(\pi^\sharp(\alpha_m), \alpha_m) \mid \alpha_m \in T_m^*M\right\} \quad \text{for all } m \in M,$$

where $\pi^\sharp : T^*M \to TM$ is defined by $\pi^\sharp(\alpha) = \pi(\alpha, \cdot) \in \mathfrak{X}(M)$ for all $\alpha \in \Omega^1(M)$, is a Dirac structure on M. It is closed if and only if the bivector field satisfies $[\pi, \pi] = 0$, that is, if and only if (M, π) is a Poisson manifold. ◊

[3]We prefer the terminology "closed" to "integrable" because integrability of a Dirac structure can also signify that it is integrable as a Lie algebroid, i.e. it integrates to a presymplectic groupoid as in [2].

Dirac maps and Dirac reduction Let (M, D_M) and (N, D_N) be two Dirac manifolds and $F : M \to N$ a smooth map. Then F is a *forward Dirac map* if for all $n \in N$, $m \in F^{-1}(n)$ and $(v_n, \alpha_n) \in \mathsf{D}_N(n)$ there exists $(v_m, \alpha_m) \in \mathsf{D}_M(m)$ such that $T_mFv_m = v_n$ and $\alpha_m = (T_mF)^*\alpha_n$. The map F is a *backward Dirac map* if for all $m \in M$, $n = F(m)$ and $(v_m, \alpha_m) \in \mathsf{D}_M(m)$ there exists $(v_n, \alpha_n) \in \mathsf{D}_N(n)$ such that $T_mFv_m = v_n$ and $\alpha_m = (T_mF)^*\alpha_n$.

Assume that $G \rightrightarrows M$ is a Lie groupoid, and that G is endowed with a Dirac structure. Let H be a t-connected, wide Lie subgroupoid of G such that G/H has a smooth manifold structure and $q : G \to G/H$ is a smooth surjective submersion. Since G/H is the leaf space of $\mathcal{H}$, where $\mathcal{H}$ is the left invariant image of AH (see the previous section), we can apply the results in [32] (see also [13]) for Dirac reduction. Assume that the Dirac structure D on G is such that $\mathsf{D} \cap (TG \oplus \mathcal{H}^\circ)$ has constant rank on G and

$$[\![\Gamma(\mathsf{D}), \Gamma(\mathcal{H} \oplus \{0\})]\!] \subseteq \Gamma(\mathsf{D} + (\mathcal{H} \oplus \{0\})), \tag{6}$$

then D induces a Dirac structure $q(\mathsf{D})$ on the quotient G/H. The Dirac structure $q(\mathsf{D})$ on G/H is given by

$$\Gamma(q(\mathsf{D})) = \{(\bar{X}, \bar{\alpha}) \in \Gamma(\mathsf{P}_{G/H}) \mid \exists X \in \mathfrak{X}(G) \text{ such that } X \sim_q \bar{X} \text{ and } (X, q^*\bar{\alpha}) \in \Gamma(\mathsf{D})\}.$$

In other words, $q(\mathsf{D})$ is the forward Dirac image of D under $q : G \to G/H$. If the Dirac structure D is closed, then $q(\mathsf{D})$ is closed.

If $\mathcal{H} \oplus \{0\} \subseteq \mathsf{D}$, then $\mathsf{D} = q^*(q(\mathsf{D}))$, where for any Dirac structure $\bar{\mathsf{D}}$ on G/H, its *pullback* $q^*(\bar{\mathsf{D}})$ to G is the Dirac structure on G defined by

$$q^*(\bar{\mathsf{D}})(g) = \{(v_g, (T_gq)^*\alpha_{gH}) \in \mathsf{P}_G(g) \mid (T_gqv_g, \alpha_{gH}) \in \bar{\mathsf{D}}(gH)\}$$

for all $g \in G$. (The bundle $q^*(\mathsf{D})$ is the backward Dirac image of D under q.)

Note that if we can verify that

$$(R_\kappa^*X, R_\kappa^*\alpha) \in \Gamma(\mathsf{D}) \text{ for all } (X, \alpha) \in \Gamma(\mathsf{D}) \text{ and } \kappa \in \mathcal{B}(H), \tag{7}$$

then condition (6) is satisfied.

3 The geometry of Dirac groupoids.

3.1 Definition and examples

Definition 3.1 ([26]) *A Dirac groupoid is a Lie groupoid $G \rightrightarrows M$ endowed with a Dirac structure D such that $\mathsf{D} \subseteq TG \oplus T^*G$ is a Lie subgroupoid. The Dirac structure D is then said to be* multiplicative.

Note that in [26], Dirac manifolds are always closed by definition.

Example 3.2 Poisson groupoids were introduced in [30] and studied in [30], [31], [23] among other, see also [22]. It is shown in [23] that $(G\rightrightarrows M, \pi_G)$ is a Poisson groupoid if and only if the vector bundle map $\pi_G^\sharp : T^*G \to TG$ associated to π_G is a morphism of Lie groupoids over some map $a_* : A^* \to TM$ (the restriction of $\pi_G^\sharp$ to A^*). Using this, it is easy to see that $(G\rightrightarrows M, \pi_G)$ is a Poisson groupoid if and only if $(G\rightrightarrows M, \mathsf{D}_{\pi_G})$ is a closed Dirac groupoid. $\Diamond$

Example 3.3 Let $G\rightrightarrows M$ be a Lie groupoid. A 2-form ω_G on G is *multiplicative* if the partial multiplication map $\mathsf{m} : G \times_M G \to G$ satisfies $\mathsf{m}^*\omega_G = \mathrm{pr}_1^*\omega_G + \mathrm{pr}_2^*\omega_G$. The graph $\mathsf{D}_{\omega_G} = \mathrm{Graph}(\omega_G^\flat : TG \to T^*G) \subseteq \mathsf{P}_G$ is then multiplicative, and $(G\rightrightarrows M, \mathsf{D}_{\omega_G})$ is a Dirac groupoid, see [26], [1]. If the 2-form is closed, then the Dirac groupoid is closed.

Note that *presymplectic groupoids* have been studied in [3], [2]. These are Lie groupoids endowed with closed, multiplicative 2-forms satisfying some additional non degeneracy properties. $\Diamond$

Example 3.4 Let (G, D) be a Dirac group in the sense of [10]. We have seen there that it is a Dirac group in the sense of [26], that is, D is a subgroupoid of the Pontryagin groupoid $TG \oplus TG^* \rightrightarrows \{0\} \oplus \mathfrak{g}^*$. The set of units is here $\mathsf{D}(e) \cap (\{0\} \oplus \mathfrak{g}^*) = \{0\} \oplus \mathfrak{p}_1$ since we know that $\mathsf{D}(e)$ is equal to a direct sum $\mathfrak{g}_0 \oplus \mathfrak{g}_0^\circ \subseteq \mathfrak{g} \oplus \mathfrak{g}^*$, with $\mathfrak{g}_0$ an ideal in $\mathfrak{g}$ and $\mathfrak{g}_0^\circ \subseteq \mathfrak{g}^*$ its annihilator. $\Diamond$

Example 3.5 Consider a smooth Dirac manifold (M, D_M) and the pair groupoid $M \times M \rightrightarrows M$ associated to M. The tangent groupoid $T(M \times M) \rightrightarrows TM$ of $M \times M \rightrightarrows M$ is then $TM \times TM \rightrightarrows TM$, the pair groupoid associated to TM.

The dual $A^*(M \times M)$ is given by $A^*_{(m,m)}(M \times M) = \{(-\alpha_m, \alpha_m) \mid \alpha_m \in T^*_mM\}$ for all $m \in M$ and the structure of the cotangent groupoid $T^*(M \times M) \rightrightarrows A^*(M \times M)$ can be described as follows. If $(\alpha_m, \alpha_n) \in T^*_{(m,n)}(M \times M)$, it is easy to check that $\hat{\mathsf{t}}(\alpha_m, \alpha_n) = (\alpha_m, -\alpha_m)$ and $\hat{\mathsf{s}}(\alpha_m, \alpha_n) = (-\alpha_n, \alpha_n)$. The multiplication is then given by

$$(\alpha_m, \alpha_n) \star (-\alpha_n, \alpha_p) = (\alpha_m, \alpha_p).$$

It is easy to check that the Dirac structure $\mathsf{D}_M \ominus \mathsf{D}_M$, defined by

$$(\mathsf{D}_M \ominus \mathsf{D}_M)(m, n) = \left\{ ((v_m, -v_n), (\alpha_m, \alpha_n)) \in \mathsf{P}_{M\times M}(m, n) \,\middle|\, \begin{array}{c} (v_m, \alpha_m) \in \mathsf{D}_M(m) \\ \text{and } (v_n, \alpha_n) \in \mathsf{D}_M(n) \end{array} \right\}$$

for all $(m, n) \in M \times M$, is a multiplicative Dirac structure on $M \times M \rightrightarrows M$. This generalizes the fact that if (M, π_M) is a Poisson manifold, then $M \times M \rightrightarrows M$ endowed with $\pi_M \oplus (-\pi_M)$ is a Poisson groupoid.

We call the Dirac groupoid $(M \times M \rightrightarrows M, \mathsf{D}_M \ominus \mathsf{D}_M)$ the *pair Dirac groupoid* associated to (M, D_M). It is closed if and only if (M, D_M) is closed. $\Diamond$

Remark 3.6 In the Poisson case, it is known by results in [30] that any multiplicative Poisson structure on a pair groupoid is $\pi_M \oplus (-\pi_M)$ for some Poisson bivector π_M on

M. This is not true in general. For instance, let M be a smooth manifold with a smooth free action of a Lie group H with Lie algebra $\mathfrak{h}$. Then the diagonal action of H on $M \times M$ is by Lie groupoid morphisms, and its vertical space $\mathcal{V} \subseteq T(M \times M)$, $\mathcal{V}(m, n) = \{(\xi_M(m), \xi_M(n)) \mid \xi \in \mathfrak{h}\}$ for all $m, n \in M$, is multiplicative (see for instance [12]). The Dirac structure $\mathcal{V} \oplus \mathcal{V}^\circ$ is then multiplicative, but cannot be written as a pair Dirac structure on $M \times M$. $\triangle$

We now study some immediate properties of Dirac groupoids.

Definition 3.7 *1. Let $(G \rightrightarrows M, \mathsf{D})$ be a Dirac groupoid and U the set of units of D, i.e. the subdistribution $\mathsf{D} \cap (TM \oplus A^*)$ of $TM \oplus A^*$. We write $\rho_\star : U \to TM$ for the map $\mathrm{pr}_{TM}|_U$.*

2. We write $\ker \mathbb{T}\mathsf{s}$, respectively $\ker \mathbb{T}\mathsf{t}$ for the kernel $T^{\mathsf{s}}G \oplus (T^{\mathsf{t}}G)^\circ$ (respectively $T^{\mathsf{t}}G \oplus (T^{\mathsf{s}}G)^\circ$) of the source map $\mathbb{T}\mathsf{s} : \mathsf{P}_G \to TM \oplus A^*$ (respectively the target map $\mathbb{T}\mathsf{t} : \mathsf{P}_G \to TM \oplus A^*$). We denote by K^{s} the restriction to M of $\mathsf{D} \cap \ker \mathbb{T}\mathsf{s}$, i.e.

$$K^{\mathsf{s}} = \mathsf{D} \cap (T^{\mathsf{s}}_M G \oplus (T^{\mathsf{t}}_M G)^\circ) = (\mathsf{D} \cap \ker \mathbb{T}\mathsf{s})|_M.$$

In the same manner, we write $K^{\mathsf{t}} := \mathsf{D} \cap (T^{\mathsf{t}}_M G \oplus (T^{\mathsf{s}}_M G)^\circ) = (\mathsf{D} \cap \ker \mathbb{T}\mathsf{t})|_M$.

Theorem 3.8 *Let $(G \rightrightarrows M, \mathsf{D}_G)$ be a Dirac groupoid. Then the Dirac subspace $\mathsf{D}|_M$ splits as a direct sum*

$$\mathsf{D}|_M = U \oplus K^{\mathsf{t}}$$

and in the same manner

$$\mathsf{D}|_M = U \oplus K^{\mathsf{s}}.$$

The three intersections U, K^{s}, K^{t} of vector bundles over M are smooth and have constant rank on M.

Proof: Choose $p \in M$ and $(v_p, \alpha_p) \in \mathsf{D}(p)$. Then we have $\mathbb{T}\mathsf{t}(v_p, \alpha_p) \in \mathsf{D}(p)$ and hence also $(v_p, \alpha_p) - \mathbb{T}\mathsf{t}(v_p, \alpha_p) \in \mathsf{D}(p)$. We find that $v_p - T_p\mathsf{t}v_p \in T^{\mathsf{t}}_p G$ and $T_p\mathsf{t}(v_p) \in T_pM$, and in the same manner $\hat{\mathsf{t}}(\alpha_p) \in A^*_p G = (T_pM)^\circ$, by definition, and $\alpha_p - \hat{\mathsf{t}}(\alpha_p) \in (T^{\mathsf{s}}_p G)^\circ$.

Since

$$(v_p, \alpha_p) = \mathbb{T}\mathsf{t}(v_p, \alpha_p) + ((v_p, \alpha_p) - \mathbb{T}\mathsf{t}(v_p, \alpha_p)),$$

we have shown the first equality. The second formula can be shown in the same manner, using the map $\mathbb{T}\mathsf{s} : \mathsf{D}(p) \to \mathsf{D}(p) \cap (T_pM \times A^*_p G)$.

Next, we show that the intersection of D with $TM \oplus A^*$ is smooth. Choose $p \in M$ and $(v_p, \alpha_p) \in \mathsf{D}(p) \cap (T_pM \times A^*_p G)$. Since D is a smooth vector bundle on G, we find a section $(X, \alpha) \in \Gamma(\mathsf{D})$ defined on a neighborhood of p such that $(X, \alpha)(p) = (v_p, \alpha_p)$. The restriction $(X, \alpha)|_M$ is then a smooth section of $\mathsf{D}|_M$. We have $\mathbb{T}\mathsf{s}((X, \alpha)|_M) \in$

$\Gamma(\mathsf{D}\cap(TM\oplus A^*))$ and $\mathbb{T}\mathsf{s}(X,\alpha)(p)=(T_p\mathsf{s}v_p,\alpha_p|_{T_p^{\mathsf{t}}G})=(v_p,\alpha_p)$ since $v_p\in T_pM$ and $\alpha_p\in A_p^*G=(T_pM)^\circ$.

Thus, we have found a smooth section of $\mathsf{D}\cap(TM\oplus A^*)$ defined on a neighborhood of p in M and taking value (v_p,α_p) at p.

Since $(\mathsf{D}|_M)^\perp=\mathsf{D}|_M$ and $Tm\oplus A^*=(TM\oplus A^*)^\perp$ are smooth subbundles of $\mathsf{P}_G|_M$, we get from Proposition 4.4 in [14] that $\mathsf{D}\cap(TM\oplus A^*)$ has constant rank on M. By the splittings shown above and the fact that $\mathsf{D}|_M$ has constant rank on M, we find that the two other intersections have constant rank on M, and are thus smooth. $\square$

In the case of a Dirac group, the bundle $K^{\mathsf{s}}\to M$ is $\mathfrak{g}_0\to\{e\}$, as is shown in the next example. We will see later that K^{s} has a crucial role in the construction of the Courant algebroid associated to a Dirac groupoid $(G\rightrightarrows M,\mathsf{D})$. The fact that the left and right invariant images of this subspace are exactly the characteristic distribution of the Dirac structure is a very special and convenient feature in the group case, that makes the Dirac groups much easier to understand than arbitrary Dirac groupoids (see [10]).

Example 3.9 Let (G,D) be a Dirac group (Example 3.4) and set $\mathfrak{p}_1=\mathsf{P}_1(e)\subseteq\mathfrak{g}^*$ and $\mathfrak{g}_0=\mathsf{G}_0(e)\subseteq\mathfrak{g}$, where e is the neutral element of G We have $M=\{e\}$,

$$\mathsf{D}(e)\cap(T_eM\times(T_eM)^\circ)=\mathsf{D}(e)\cap(\{0\}\times\mathfrak{g}^*)=\{0\}\times\mathfrak{p}_1$$

and

$$\mathsf{D}(e)\cap(T_e^{\mathsf{s}}G\times(T_e^{\mathsf{t}}G)^\circ)=\mathsf{D}(e)\cap(\mathfrak{g}\times\{0\})=\mathfrak{g}_0\times\{0\}.$$

We recover hence the equality $\mathsf{D}(e)=\mathfrak{g}_0\times\mathfrak{p}_1$ [10].

In this particular case, D is a Poisson structure if and only if $\mathsf{D}(e)$ is equal to the set of units of $TG\oplus T^*G\rightrightarrows\{0\}\times\mathfrak{g}^*$, i.e. $\mathfrak{g}_0=\{0\}$ and $\mathfrak{p}_1=\mathfrak{g}^*$. This is not true in general.$\lozenge$

The following lemma is immediate.

Lemma 3.10 *Let $(G\rightrightarrows M,\mathsf{D})$ be a Dirac groupoid. For all $g\in G$, we have*

$$\mathsf{D}(g)\cap\ker\mathbb{T}\mathsf{t}=(0_g,0_g)\star K^{\mathsf{t}}_{\mathsf{s}(g)}$$

and

$$\mathsf{D}(g)\cap\ker\mathbb{T}\mathsf{s}=K^{\mathsf{s}}_{\mathsf{t}(g)}\star(0_g,0_g).$$

The intersections $\mathsf{D}\cap\ker\mathbb{T}\mathsf{t}$ and $\mathsf{D}\cap\ker\mathbb{T}\mathsf{s}$ have consequently constant rank on G.

Example 3.11 If $(G\rightrightarrows M,\pi_G)$ is a Poisson groupoid, then $\pi_G^\sharp(\mathbf{d}(\mathsf{s}^*f))\in\Gamma(T^{\mathsf{t}}G)$ for all $f\in C^\infty(M)$ (see [30]). The intersection $\mathsf{D}_{\pi_G}\cap\ker\mathbb{T}\mathsf{t}$ is hence spanned by the sections $(\pi_G^\sharp(\mathbf{d}(\mathsf{s}^*f)),\mathbf{d}(\mathsf{s}^*f))$, with $f\in C^\infty(M)$, and has constant rank. The intersection $\mathsf{D}_{\pi_G}\cap\ker\mathbb{T}\mathsf{s}$ is spanned by the sections $(\pi_G^\sharp(\mathbf{d}(\mathsf{t}^*f)),\mathbf{d}(\mathsf{t}^*f))$ with $f\in C^\infty(M)$. $\lozenge$

3.2 The units of a Dirac groupoid

In this section, we show how the set of units U of a closed multiplicative Dirac structure is endowed with a Lie algebroid structure. The Dirac structure is then an LA-groupoid. We then prove a crucial formula for the derivative of specials sections of the Dirac structure, the *star-sections*, along left-invariant vector fields.

Proposition 3.12 *Let u be a section of $\mathsf{D}\cap(TM\oplus A^*)=U$. Then there exists a smooth section d of D such that $d|_M=u$ and $\mathbb{T}\mathsf{s}\circ d=u\circ\mathsf{s}$.*

Following [21], we then say that the section d of D is a s-*star section* or simply *star section* and we write $d\sim_{\mathsf{s}} u$. Note that outside of M, d is unique up to sections of $\mathsf{D}\cap\ker\mathbb{T}\mathsf{s}$.

Proof: We have shown in Lemma 3.10 that $\mathsf{D}\cap\ker\mathbb{T}\mathsf{s}$ is a subbundle of D. Hence, we can consider the smooth vector bundle $\mathsf{D}/(\mathsf{D}\cap\ker\mathbb{T}\mathsf{s})$ over G. Since D is a Lie subgroupoid of $\mathsf{P}_G\rightrightarrows(TM\oplus A^*)$, we can consider the restriction to D of the source map, $\mathbb{T}\mathsf{s}:\mathsf{D}\to U$. Since $\mathsf{D}\cap\ker\mathbb{T}\mathsf{s}$ is the kernel of this map, we have an induced smooth vector bundle homomorphism $\overline{\mathbb{T}\mathsf{s}}:\mathsf{D}/(\mathsf{D}\cap\ker\mathbb{T}\mathsf{s})\to U$ over the source map $\mathsf{s}:G\to M$, that is bijective in every fiber. Hence, there exists a unique smooth section $[d]$ of $\mathsf{D}/(\mathsf{D}\cap\ker\mathbb{T}\mathsf{s})$ such that $\overline{\mathbb{T}\mathsf{s}}([d](g))=u(\mathsf{s}(g))$ for all $g\in G$. If $d\in\Gamma(\mathsf{D})$ is a representative of $[d]$ such that $d|_M=u$, then $\mathbb{T}\mathsf{s}\circ d=u\circ\mathsf{s}$. □

Lemma 3.13 *Choose star sections $d\sim_{\mathsf{s}} u$, $d'\sim_{\mathsf{s}} u'$ of D. Then, if $d=(X,\alpha)$ and $d'=(Y,\beta)$, the identity*

$$\langle(\pounds_{a^l}X,\pounds_{a^l}\alpha),(Y,\beta)\rangle=\mathsf{s}^*\Big(\langle(\pounds_{a^l}X,\pounds_{a^l}\alpha),(Y,\beta)\rangle|_M\Big) \tag{8}$$

holds for any section $a\in\Gamma(A)$.

Proof: Choose $g\in G$ and set $p=\mathsf{s}(g)$. For all $t\in(-\varepsilon,\varepsilon)$ for a small ε, we have

$$d(g\cdot\mathrm{Exp}(ta)(p))=d(g\cdot\mathrm{Exp}(ta)(p))\star\big(d(\mathrm{Exp}(ta)(p))\big)^{-1}\star d(\mathrm{Exp}(ta)(p)).$$

The product

$$d(g\cdot\mathrm{Exp}(ta)(p))\star\big(d(\mathrm{Exp}(ta)(p))\big)^{-1}$$

is an element of $\mathsf{D}(g)$ for all $t\in(-\varepsilon,\varepsilon)$ and will be written $\delta_t(g)$ to simplify the notation. Then

$$\begin{aligned}&\big(T_{R_{\mathrm{Exp}(ta)}(g)}R_{\mathrm{Exp}(-ta)}X(g\cdot\mathrm{Exp}(ta)(p)),\alpha(g\cdot\mathrm{Exp}(ta)(p))\circ T_gR_{\mathrm{Exp}(ta)}\big)\\&=\delta_t(g)\star\big(T_{\mathrm{Exp}(ta)}R_{\mathrm{Exp}(-ta)}X(\mathrm{Exp}(ta)(p)),\alpha(\mathrm{Exp}(ta)(p))\circ T_pR_{\mathrm{Exp}(ta)}\big)\end{aligned} \tag{9}$$

and so

$$
\begin{aligned}
& (\beta(\pounds_{a^l}X) + (\pounds_{a^l}\alpha)(Y))\,(g) \\
= & \left\langle d'(g), \left.\frac{d}{dt}\right|_{t=0} \left(\left(R^*_{\mathrm{Exp}(ta)}X\right)(g), \left(R^*_{\mathrm{Exp}(ta)}\alpha\right)(g)\right)\right\rangle \\
\overset{(9)}{=} & \left.\frac{d}{dt}\right|_{t=0} \left\langle d'(g)\star u'(p),\ \delta_t(g)\star\left(\left(R^*_{\mathrm{Exp}(ta)}X\right)(p), \left(R^*_{\mathrm{Exp}(ta)}\alpha\right)(p)\right)\right\rangle \\
= & \left.\frac{d}{dt}\right|_{t=0} \langle d'(g), \delta_t(g)\rangle + \left.\frac{d}{dt}\right|_{t=0} \left\langle u'(p), \left(\left(R^*_{\mathrm{Exp}(ta)}X\right)(p), \left(R^*_{\mathrm{Exp}(ta)}\alpha\right)(p)\right)\right\rangle \\
= & \left(\left.\frac{d}{dt}\right|_{t=0} 0\right) + \left\langle u', (\pounds_{a^l}X, \pounds_{a^l}\alpha)\right\rangle(p) = (\beta(\pounds_{a^l}X) + (\pounds_{a^l}\alpha)(Y))\,(\mathsf{s}(g)). \qquad \square
\end{aligned}
$$

Proposition 3.14 *Let $(G\rightrightarrows M, \mathsf{D})$ be a Dirac groupoid. Choose star sections $d \sim_{\mathsf{s}} u$, $d' \sim_{\mathsf{s}} u'$ of D. Then the Courant-Dorfman bracket $[\![d, d']\!]$ is again a star section.*

Proof: We write $d = (X, \alpha)$, $d' = (Y, \beta)$, $u = (\bar{X}, \xi)$ and $u' = (\bar{Y}, \eta)$. Since $X \sim_{\mathsf{s}} \bar{X}$ and $Y \sim_{\mathsf{s}} \bar{Y}$, we know that $[X, Y] \sim_{\mathsf{s}} [\bar{X}, \bar{Y}]$. Since $X|_M = \bar{X}$, $Y|_M = \bar{Y}$ the value of $[X, Y]$ on points in M is equal to the value of $[\bar{X}, \bar{Y}] \in \mathfrak{X}(M)$. We check that for all $p \in M$, we have $\hat{\mathsf{s}}\left((\pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha)(g)\right) = (\pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha)(p)$ for any $g \in \mathsf{s}^{-1}(p)$.

We have for any $a \in \Gamma(A)$:

$$\hat{\mathsf{s}}\left(\left(\pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha\right)(g)\right)(a(p)) = \left(\pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha\right)\left(a^l\right)(g).$$

Hence, we compute with (8)

$$
\begin{aligned}
\left(\pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha\right)(a^l) &= X(\beta(a^l)) + \beta(\pounds_{a^l}X) - Y(\alpha(a^l)) + a^l(\alpha(Y)) - \alpha(\pounds_{a^l}Y) \\
&= X(\mathsf{s}^*(\eta(a))) + \beta(\pounds_{a^l}X) - Y(\mathsf{s}^*(\xi(a))) + (\pounds_{a^l}\alpha)(Y) \\
&= \mathsf{s}^*\Big(\bar{X}(\eta(a)) + \bar{Y}(\pounds_{a^l}X) - \eta(\xi(a)) + (\pounds_{a^l}\alpha)(\bar{Y})\Big).
\end{aligned}
$$

Then we also have

$$\left(\pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha\right)(a(p)) = \left(\bar{X}(\eta(a)) + \eta(\pounds_{a^l}X) - \bar{Y}(\xi(a)) + (\pounds_{a^l}\alpha)(\bar{Y})\right)(p)$$

for $p \in M$. It is easy to check as well that $\pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha$ vanishes on TM. We get that $\left(\pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha\right)|_M$ is a section of $A^* = TM^\circ$ and

$$\hat{\mathsf{s}}\left(\left(\pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha\right)(g)\right) = \left(\pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha\right)(\mathsf{s}(g))$$

for all $g \in G$. $\square$

Theorem 3.15 *Let $(G\rightrightarrows M, \mathsf{D})$ be a Dirac groupoid. Then there is an induced skew-symmetric bracket*

$$[\cdot\,,\cdot]_\star \quad : \Gamma(U) \times \Gamma(U) \quad \to \quad \Gamma(TM \oplus A^*)$$

defined by $[u, u']_\star = [\![d, d']\!]|_M$ for any choice of star sections $d \sim_{\mathsf{s}} u$ and $d' \sim_{\mathsf{s}} u'$ of D. If $(G\rightrightarrows M, \mathsf{D})$ is closed, then $(U, [\cdot\,,\cdot]_\star, \rho_\star)$ is a Lie algebroid over M.

PROOF: We use the same notation as in the previous proof. By Proposition 3.14, we have

$$[\![(X,\alpha),(Y,\beta)]\!] \sim_{\mathsf{s}} \left([\bar{X},\bar{Y}],(\pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha)\,|_M\right).$$

Thus, we first have to show that the right-hand side of this equation does not depend on the choice of the sections d and d'. Choose a star section $\delta \sim_{\mathsf{s}} 0$ of D, i.e. $\delta = (Z,\gamma) \in \Gamma(\mathsf{D} \cap \ker \mathbb{T}\mathsf{s})$ with $\delta|_M = 0$. For any $a \in \Gamma(A)$, we find, as in the proof of Proposition 3.14, $(\pounds_Z\alpha - \mathbf{i}_X\mathbf{d}\gamma)\left(a^l\right) = -\langle \pounds_{a^l} d, \delta\rangle$ and hence, at any $p \in M$:

$$(\pounds_Z\alpha - \mathbf{i}_X\mathbf{d}\gamma)(a(p)) = -\langle \pounds_{a^l} d(p), (0_p, 0_p)\rangle = 0.$$

Thus, we find $(\pounds_Z\alpha - \mathbf{i}_X\mathbf{d}\gamma)(p) = 0_p$ since we know by the previous proposition that $(\pounds_Z\alpha - \mathbf{i}_X\mathbf{d}\gamma)(p) \in A_p^*G = T_pM^\circ$. We get hence $[\![\delta, d]\!]\,(p) = (0_p, 0_p)$ and the bracket on $\Gamma(U)$ is well-defined. Since the Courant-Dorfman bracket on sections of D is skew-symmetric, $[\cdot\,,\cdot]_\star$ is skew-symmetric.

If D is closed, then for all star sections $d, d' \in \Gamma(\mathsf{D})$, the bracket $[\![d, d']\!]$ is also a section of D and its restriction to M is a section of U. The Jacobi identity is satisfied by $[\cdot\,,\cdot]_\star$ because the Courant-Dorfman bracket on sections of D satisfies it. For any $u, u' \in \Gamma(U)$ and $f \in C^\infty(M)$, we have

$$\rho_\star\left[u, u'\right]_\star = \left[\bar{X}, \bar{Y}\right] = \left[\rho_\star(u), \rho_\star(u')\right]$$

and

$$\begin{aligned}
\left[u, f\cdot u'\right]_\star(p) &= [\![d, (\mathsf{s}^*f)d']\!]\,(p) = X(\mathsf{s}^*f)d'(p) + (\mathsf{s}^*f)\,[\![d, d']\!]\,(p)\\
&= \bar{X}(f)(p)\cdot u'(p) + f(p)\cdot\left[u, u'\right]_\star(p)\\
&= \rho_\star(u)(f)(p)\cdot u'(p) + f(p)\cdot\left[u, u'\right]_\star(p)
\end{aligned}$$

for all $p \in M$. □

Let $G\rightrightarrows M$ be a Lie groupoid, $TG\rightrightarrows TM$ its tangent prolongation and $(A \to M, \rho, [\cdot\,,\cdot]_\rho)$ a Lie algebroid over M. Let Ω be a smooth manifold. The quadruple $(\Omega; G, A; M)$ is an *LA-groupoid* [21] if Ω has both a Lie groupoid structure over A and a Lie algebroid structure over G such that the two structures on Ω commute in the sense that the maps defining the groupoid structure are all Lie algebroid morphisms. (The bracket on sections of U can be defined in the same manner with the target map, and the fact that the multiplication in $T^*G \oplus TG$ is a Lie algebroid morphism is shown in [26].) The double source map $(\tilde{q}, \tilde{\mathsf{s}}) : \Omega \to G \oplus A$ has furthermore to be a surjective submersion.

Recall from [5] that if D is closed, then $\mathsf{D} \to G$ has the structure of a Lie algebroid with the Courant-Dorfman bracket and the projection to TG as anchor. Thus, the previous theorem shows that the quadruple $(\mathsf{D}; G, U; M)$ is an LA-groupoid (see also [26]):

$$\begin{tikzcd}
\mathsf{D} \arrow[rr, shift left, "\mathbb{T}\mathsf{s}"] \arrow[rr, shift right, "\mathbb{T}\mathsf{t}"'] \arrow[dr, "\pi_{TG}"] \arrow[dd, "q"'] & & U \arrow[dr, "\rho_\star"] \arrow[dd, dashed] & \\
& TG \arrow[rr, shift left, "T\mathsf{s}"] \arrow[rr, shift right, "T\mathsf{t}"'] \arrow[dl] & & TM \arrow[dl] \\
G \arrow[rr, shift left, "\mathsf{s}"] \arrow[rr, shift right, "\mathsf{t}"'] & & M &
\end{tikzcd}$$

It is shown in [21] (see also [20]), that the bracket of two star sections is again a star section. Here, we have shown this fact in Proposition 3.14 and get as a consequence the fact that U has the structure of a Lie algebroid over M.

The next interesting object in [21] is the *core* K of Ω. It is defined as the pullback vector bundle across $\epsilon : M \hookrightarrow G$ of the kernel $\ker(\tilde{\mathsf{s}} : \Omega \to A)$. Hence, it is here exactly the vector bundle K^{s} over M. It comes equipped with the vector bundle morphisms $\delta_U : K^{\mathsf{s}} \to U$, $(v_p, \alpha_p) \mapsto \mathbb{T}\mathsf{t}(v_p, \alpha_p)$ and $\delta_{\widetilde{A}} : K^{\mathsf{s}} \to \widetilde{A}$, $(v_p, \alpha_p) \mapsto v_p$. We have then $\tilde{\rho} \circ \delta_{\widetilde{A}} = \rho_\star \circ \delta_U =: \mathsf{k}$. Furthermore, there is an induced bracket $[\cdot\,,\cdot]_{K^{\mathsf{s}}}$ on sections of K^{s} such that $(K^{\mathsf{s}}, [\cdot\,,\cdot]_{K^{\mathsf{s}}}, \mathsf{k})$ is a Lie algebroid over M. We prove directly this fact for our special situation in the following proposition.

Recall that if (v_p, α_p), $p \in M$, is an element of K^{s}_p, then α_p can be written $(T_p\mathsf{t})^*\beta_p$ with some $\beta_p \in T_p^*M$. Furthermore, if σ is a section of $K^{\mathsf{s}} \subseteq (T^{\mathsf{s}}G \oplus (T^{\mathsf{t}}G)^\circ)|_M$, then σ^r defined by $\sigma^r(g) = \sigma(\mathsf{t}(g)) \star (0_g, 0_g)$ for all $g \in G$ is a section of $\mathsf{D} \cap \ker \mathbb{T}\mathsf{s}$ by Lemma 3.10 and Remark 2.4.

Proposition 3.16 *Let $(G \rightrightarrows M, \mathsf{D})$ be a Dirac groupoid. Define $[\cdot\,,\cdot]_{K^{\mathsf{s}}} : \Gamma(K^{\mathsf{s}}) \times \Gamma(K^{\mathsf{s}}) \to \Gamma((\ker \mathbb{T}\mathsf{s})|_M)$ by*

$$([\sigma, \tau]_{K^{\mathsf{s}}})^r = [\![\sigma^r, \tau^r]\!]$$

for all sections $\sigma, \tau \in \Gamma(K^{\mathsf{s}})$, i.e. if $\sigma = (a, \mathsf{t}^*\theta|_M)$ and $\tau = (b, \mathsf{t}^*\omega|_M)$, then

$$[\sigma, \tau]_{K^{\mathsf{s}}} = \left([a, b]_{\widetilde{A}}, (\mathsf{t}^*(\pounds_{\tilde{\rho}(a)}\omega - \mathbf{i}_{\tilde{\rho}(b)}\mathbf{d}\theta))|_M\right).$$

If D is closed, this bracket has image in $\Gamma(K^{\mathsf{s}})$ and K^{s} has the structure of a Lie algebroid over M with the anchor map k defined by $\mathsf{k}(a_p, \theta_p) = \tilde{\rho}(a_p)$ for all $(a_p, \theta_p) \in K^{\mathsf{s}}_p$, $p \in M$.

Note that if D is closed, the space K^{t} has in a similar manner the structure of an algebroid over M.

Proof: Assume that D is closed. The bracket

$$[\![\sigma^r, \tau^r]\!] = [\![(a^r, \mathsf{t}^*\theta)\,, (b^r, \mathsf{t}^*\omega)]\!]$$

is then a section of D. The identity

$$[\![(a^r, \mathsf{t}^*\theta)\,, (b^r, \mathsf{t}^*\omega)]\!] = \left([a, b]_{\widetilde{A}}^{\,r}, \mathsf{t}^*(\pounds_{\tilde{\rho}(a)}\omega - \mathbf{i}_{\tilde{\rho}(b)}\mathbf{d}\theta)\right)$$

shows hence that $[\![\sigma^r, \tau^r]\!] \in \Gamma(\mathsf{D} \cap \ker \mathbb{T}\mathsf{s})$ is right invariant and consequently that $[\sigma, \tau]_{K^{\mathsf{s}}} \in \Gamma(K^{\mathsf{s}})$. $\square$

As in [21], we have thus four Lie algebroids over M:

$$\begin{tikzcd} K^{\mathsf{s}} \arrow[rr, "\delta_U"] \arrow[dd, "\delta_{\widetilde{A}}"'] \arrow[dr] & & U \arrow[dd, "\rho_\star"] \arrow[dl] \\ & M & \\ \widetilde{A} \arrow[rr, "\tilde{\rho}"'] \arrow[ur] & & TM \arrow[ul] \end{tikzcd}$$

Remark 3.17 The integrability of a mutliplicative Dirac structure is completely encoded in the square of Lie algebroids associated to it as above. Let $(G\rightrightarrows M, \mathsf{D})$ be a Dirac groupoid. Assume that $G\rightrightarrows M$ is t-connected. It is shown in [9] that D is closed if and only if:

1. the bracket $[\cdot\,,\cdot]_\star$ has image in $\Gamma(U)$ and satisfies the Jacobi identity and
2. the bracket $[\cdot\,,\cdot]_{K^{\mathsf{s}}}$ has image in $\Gamma(K^{\mathsf{s}})$. $\triangle$

Next, we compute the Lie algebroid $U \to M$ for our three "standard" examples.

Example 3.18 Let $(G\rightrightarrows M, \pi_G)$ be a Poisson groupoid and D_{π_G} the graph of the vector bundle homomorphism $\pi_G^\sharp : T^*G \to TG$ associated to π_G. The pair $(G\rightrightarrows M, \mathsf{D}_{\pi_G})$ is a closed Dirac groupoid. The set U of units of D_{π_G} equals here Graph $\left(\pi_G^\sharp\Big|_{A^*} : A^* \to TM\right)$ and is hence isomorphic to A^* as a vector bundle, via the maps $\Theta := \mathrm{pr}_{A^*} : U \to A^*$ and $\Theta^{-1} = \left(\pi_G^\sharp\Big|_{A^*}, \mathrm{Id}_{A^*}\right) : A^* \to U$ over Id_M.

The vector bundle A^* has the structure of a Lie algebroid over M with anchor map given by $A^* \to TM$, $\alpha_p \mapsto \pi_G^\sharp(\alpha_p) \in T_pM$ and with bracket the restriction to A^* of the bracket $[\cdot\,,\cdot]_{\pi_G}$ on $\Omega^1(G)$ defined by π_G: $[\alpha,\beta]_{\pi_G} = \pounds_{\pi_G^\sharp(\alpha)}\beta - \pounds_{\pi_G^\sharp(\beta)}\alpha - \mathbf{d}\pi_G(\alpha,\beta)$ for all $\alpha,\beta \in \Omega^1(G)$ [4]. Thus, A^* with this Lie algebroid structure and U are isomorphic as Lie algebroids via Θ and Θ^{-1}. $\Diamond$

Example 3.19 Let ω_G be a multiplicative closed 2-form on a Lie groupoid $G\rightrightarrows M$ and consider the associated multiplicative Dirac structure D_{ω_G} on G. The Lie algebroid $U \to M$ is here equal to

$$U = \mathrm{Graph}\left(\omega_G^\flat|_{TM} : TM \to A^*\right)$$

with anchor map $\rho_\star : U \to TM$ given by $\rho_\star\left(v_p, \omega_G^\flat(v_p)\right) = v_p$. The bracket of two sections $\left(\bar{X}, \omega_G^\flat(\bar{X})\right), \left(\bar{Y}, \omega_G^\flat(\bar{Y})\right) \in \Gamma(U)$ is simply given by

$$\left[\!\left[\left(\bar{X}, \omega_G^\flat(\bar{X})\right), \left(\bar{Y}, \omega_G^\flat(\bar{Y})\right)\right]\!\right] = \left([\bar{X},\bar{Y}], \omega_G^\flat\left([\bar{X},\bar{Y}]\right)\right).$$

The Lie algebroid U is obviously isomorphic to the tangent Lie algebroid $TM \to M$ of M. $\Diamond$

Example 3.20 Let (M, D_M) be a smooth Dirac manifold and $(M \times M \rightrightarrows M, \mathsf{D}_M \ominus \mathsf{D}_M)$ the associated pair Dirac groupoid as in Example 3.5. The set U is here defined by

$$U(m, m) = \{(v_m, v_m, \alpha_m, -\alpha_m) \mid (v_m, \alpha_m) \in \mathsf{D}_M(m)\}$$

for all $m \in M$. Hence, we have an isomorphism $U \to \mathsf{D}_M$ over the map $\mathrm{pr}_1 : \Delta_M \to M$. If (M, D_M) is closed, the Lie algebroid structure on U corresponds to the Lie algebroid structure on D_M (see [5]). $\Diamond$

We conclude this section with the following theorem, which statement is in our opinion the most important feature of Dirac groupoids. This result will be crucial for the constructions in the next sections and in [15].

Theorem 3.21 *Let $(G \rightrightarrows M, \mathsf{D})$ be a Dirac groupoid, $d \sim_{\mathsf{s}} u$ a star section of D and $a \in \Gamma(A)$. Then the derivative $\pounds_{a^l} d$ can be written as a sum*

$$\pounds_{a^l} d = \mathcal{L}_a d + (\sigma_{d,a})^l \tag{10}$$

with $\mathcal{L}_a d$ a star section of D and $\sigma_{d,a} \in \Gamma(\ker(\mathbb{T}\mathsf{t})|_M)$. We have $\mathcal{L}_a \xi \sim_{\mathsf{s}} \mathbb{T}\mathsf{t}\left(\pounds_{a^l} d|_M\right)$ in the sense that

$$\mathbb{T}\mathsf{s}\left(\mathcal{L}_a d(g)\right) = \mathbb{T}\mathsf{t}\left(\pounds_{a^l} d(\mathsf{s}(g))\right)$$

for all $g \in G$.

In addition, if $d \sim_s 0$, then $\pounds_{a^l} d \in \Gamma(\mathsf{D} \cap \ker \mathbb{T}\mathsf{s})$. In particular, its restriction to M is a section of K^{s}.

The following lemma will be useful for the proof of this theorem. The proof is easy and shall be omitted.

Lemma 3.22 *Let $G \rightrightarrows M$ be a Lie groupoid. Choose $\tau \in \Gamma((\ker \mathbb{T}\mathsf{t})|_M)$ and $\sigma \in \Gamma((\ker \mathbb{T}\mathsf{s})|_M)$. Then we have*

$$\left[\!\left[\tau^l, \sigma^r\right]\!\right] = 0. \tag{11}$$

In particular, we have for $a \in \Gamma(A)$:

$$\pounds_{a^l} \sigma^r = 0. \tag{12}$$

Proof (of Theorem 3.21): Note first that, in general, $\pounds_{a^l} d$ is a section of $\mathsf{D} + \ker \mathbb{T}\mathsf{t}$: for all $\sigma^r \in \Gamma(\mathsf{D} \cap \ker \mathbb{T}\mathsf{s})$, we have

$$\langle \pounds_{a^l} d, \sigma^r \rangle = \pounds_{a^l} \langle d, \sigma^r \rangle - \langle d, \pounds_{a^l} \sigma^r \rangle = 0$$

using $\mathsf{D} = \mathsf{D}^\perp$ and (12). This leads to $\pounds_{a^l} d \in \Gamma\left((\mathsf{D} \cap \ker \mathbb{T}\mathsf{s})^\perp\right) = \Gamma(\mathsf{D} + \ker \mathbb{T}\mathsf{t})$. We write here $d = (X, \alpha)$ and $u = (\bar{X}, \xi)$. Choose $g \in G$. Then

$$T_g\mathsf{s}(\pounds_{a^l} X)(g) = T_g\mathsf{s}\left[a^l, X\right](g) = \left[\rho(a), \bar{X}\right](\mathsf{s}(g))$$

and for any $b \in \Gamma(A)$

$$\begin{aligned}\hat{\mathsf{s}}(\pounds_{a^l}\alpha(g))(b(\mathsf{s}(g))) &= (\pounds_{a^l}\alpha)(b^l)(g) = (a^l(\mathsf{s}^*(\xi(b))) - \mathsf{s}^*(\xi([a,b]_A)))(g) \\ &= \big(\rho(a)(\xi(b)) - \xi([a,b]_A)\big)(\mathsf{s}(g)).\end{aligned}$$

This shows that $\mathbb{T}\mathsf{s}\left(\pounds_{a^l}d(g)\right)$ depends only on the values of a and u at $\mathsf{s}(g)$.

Set

$$\sigma^l_{d,a}(g) := (0_g, 0_g) \star \Big(\pounds_{a^l}d(\mathsf{s}(g)) - \mathbb{T}\mathsf{t}\left(\pounds_{a^l}d(\mathsf{s}(g))\right)\Big)$$

and

$$\mathcal{L}_a d(g) := \pounds_{a^l}d(g) - \sigma^l_{d,a}(g)$$

for all $g \in G$. Then $\sigma^l_{d,a}$ is a smooth left-invariant section of $\ker \mathbb{T}\mathsf{t}$ and $\mathcal{L}_a d$ is a star section since

$$\mathbb{T}\mathsf{s}(\mathcal{L}_a d(g)) = \mathcal{L}_a d(\mathsf{s}(g))$$

for all $g \in G$.

It remains hence to show that $\mathcal{L}_a d$ is a section of D. The equality

$$\langle \sigma^r, \mathcal{L}_a d\rangle = \langle \sigma^r, \pounds_{a^l}d\rangle - \left\langle \sigma^r, \sigma^l_{d,a}\right\rangle = 0$$

holds for all $\sigma^r \in \Gamma(\ker \mathbb{T}\mathsf{s} \cap \mathsf{D})$, and for all star sections $d' \sim_{\mathsf{s}} u'$ of D, we compute

$$\begin{aligned}&\left\langle d', \mathcal{L}_a d\right\rangle(g) = \left\langle d', \pounds_{a^l}d - \sigma^l_{d,a}\right\rangle(g) \\ &= \left\langle d', \pounds_{a^l}d\right\rangle(g) - \left\langle d'(g) \star u'(\mathsf{s}(g)), (0_g, 0_g) \star \left(\pounds_{a^l}d(\mathsf{s}(g)) - \mathbb{T}\mathsf{t}\left(\pounds_{a^l}d(\mathsf{s}(g))\right)\right)\right\rangle \\ &\overset{(8)}{=} \left\langle u'(\mathsf{s}(g)), \pounds_{a^l}d(\mathsf{s}(g))\right\rangle - \left\langle u'(\mathsf{s}(g)), \pounds_{a^l}d(\mathsf{s}(g)) - \mathbb{T}\mathsf{t}\left(\pounds_{a^l}d(\mathsf{s}(g))\right)\right\rangle \\ &= \left\langle u'(\mathsf{s}(g)), \mathbb{T}\mathsf{t}\left(\pounds_{a^l}d(\mathsf{s}(g))\right)\right\rangle = 0\end{aligned}$$

since $TM \oplus A^* = (TM \oplus A^*)^\perp$. Thus, we have shown that $\mathcal{L}_a d \in \Gamma(\mathsf{D}^\perp) = \Gamma(\mathsf{D})$.

For the proof of the second statement, assume that $d \sim_{\mathsf{s}} u = 0$. For all left invariant sections $\sigma = (b, \mathsf{s}^*\theta|_M)$ of $\ker \mathbb{T}\mathsf{t}|_M$, we have

$$\left\langle \pounds_{a^l}d, \sigma^l\right\rangle = \pounds_{a^l}\left\langle d, \sigma^l\right\rangle - \left\langle d, \pounds_{a^l}\sigma^l\right\rangle = \pounds_{a^l}\left(\mathsf{s}^*\langle u, \sigma\rangle\right) - \mathsf{s}^*\langle u, ([a,b]_A, \mathsf{s}^*(\pounds_{\rho(a)}\theta)|_M)\rangle = 0.$$

Choose any star section $d' \sim_{\mathsf{s}} u'$ of D. Then

$$\left\langle \pounds_{a^l}d, d'\right\rangle = \pounds_{a^l}\left\langle d, d'\right\rangle - \left\langle d, \pounds_{a^l}d'\right\rangle = 0$$

since $\pounds_{a^l}d' \in \Gamma\left((\mathsf{D} \cap \ker \mathbb{T}\mathsf{s})^\perp\right)$. We have also $\pounds_{a^l}d \in \Gamma\left((\mathsf{D} \cap \ker \mathbb{T}\mathsf{s})^\perp\right)$ and, because the star sections of D and the sections of $\mathsf{D} \cap \ker \mathbb{T}\mathsf{s}$ span D, this shows that $\pounds_{a^l}d \in \Gamma\left((\mathsf{D} + \ker \mathbb{T}\mathsf{t})^\perp\right) = \Gamma(\mathsf{D} \cap \ker \mathbb{T}\mathsf{s})$. □

3.3 The Courant algebroid associated to a closed Dirac groupoid

The dual space of U can be identified with $\mathsf{P}_G|_M/U^\perp$. Since

$$U^\perp = \mathsf{D}|_M + (TM \oplus A^*) = K^{\mathsf{t}} \oplus (TM \oplus A^*)$$

and

$$\mathsf{P}_G|_M = (TM \oplus A^*) + \ker \mathbb{T}\mathsf{t}|_M,$$

we have

$$U^* \simeq \frac{\ker \mathbb{T}\mathsf{t}|_M}{K^{\mathsf{t}}}.$$

Since $\mathsf{D}|_M \subseteq U \oplus \ker \mathbb{T}\mathsf{t}|_M$, we have $K^{\mathsf{s}} \subseteq U \oplus \ker \mathbb{T}\mathsf{t}|_M$ and the quotient

$$\mathsf{B} := \frac{U \oplus \ker \mathbb{T}\mathsf{t}|_M}{K^{\mathsf{s}}}$$

is a smooth vector bundle over M. Consider the map

$$\Psi : \ker \mathbb{T}\mathsf{t}|_M \oplus U \to \mathsf{B},$$

$$\Psi\left(\sigma + u\right) = \sigma + u + K^{\mathsf{s}} =: \sigma \oplus u$$

for all $\sigma \in \Gamma(\ker \mathbb{T}\mathsf{t}|_M)$ and $u \in \Gamma(U)$. If $\Psi(\sigma + u) = K^{\mathsf{s}}$, then we have $\sigma + u \in \Gamma(\mathsf{D}|_M)$ and hence $\sigma \in \Gamma(\mathsf{D}|_M)$ since $u \in \Gamma(\mathsf{D}|_M)$. This yields $\sigma \in \Gamma(K^{\mathsf{t}})$ and the map Ψ factors to a vector bundle morphism

$$\bar{\Psi} : U^* \oplus U \to \mathsf{B}$$

over the identity Id_M.

The map $\bar{\Psi}$ is surjective and a dimension count shows that it is an isomorphism.

Since $(\ker \mathbb{T}\mathsf{t}|_M \oplus U)^\perp = (\ker \mathbb{T}\mathsf{t}|_M + \mathsf{D}|_M)^\perp = K^{\mathsf{s}}$, the bracket $\langle \cdot\,, \cdot \rangle$ restricts to a non degenerate symmetric bracket on B:

$$\langle u \oplus \sigma, u' \oplus \tau' \rangle = \eta(a) + \xi(b) + \theta(\bar{Y} + \rho(b)) + \omega(\bar{X} + \rho(a))$$

where $u = (\bar{X}, \xi)$, $u' = (\bar{Y}, \eta)$, $\sigma = (a, \mathsf{s}^*\theta)$ and $\tau = (b, \mathsf{s}^*\omega)$.

Recall from Example 3.18 that if $(G \rightrightarrows M, \mathsf{D}_{\pi_G})$ is a Poisson groupoid, the bundle U is equal to $\mathrm{Graph}(\pi_G^\sharp|_{A^*}) \simeq A^*$, $\rho_\star(\xi) = \pi_G^\sharp(\xi)$ for all $\xi \in \Gamma(A^*)$ and the bracket on sections of U is the bracket induced by the Poisson structure. In the same manner, we have $U^* = \ker \mathbb{T}\mathsf{t}|_M/K = \ker \mathbb{T}\mathsf{t}|_M/\,\mathrm{Graph}\left(\pi_G^\sharp|_{(T_M^{\mathsf{s}}G)^\circ}\right)$ which is isomorphic as a vector bundle to A. The vector bundle B is thus the vector bundle underlying the Courant algebroid associated to $(G \rightrightarrows M, \pi)$. We will study this example in more detail in Example 3.24, where we will show that B carries a natural Courant algebroid structure that makes it isomorphic as a Courant algebroid to $A \oplus A^*$.

We show here that if the Dirac groupoid $(G \rightrightarrows M, \mathsf{D})$ is closed, the vector bundle $\mathsf{B} \to M$ always inherits the structure of a Courant algebroid from the ambient standard Courant algebroid structure of P_G.

Theorem 3.23 *Let $(G \rightrightarrows M, \mathsf{D})$ be a closed Dirac groupoid,*

$$\mathsf{B} = \frac{U \oplus \ker \mathbb{T}\mathsf{t}|_M}{K^{\mathsf{s}}} \to M$$

and set $\mathsf{b} : \mathsf{B} \to TM$, $\mathsf{b}((v_p, \alpha_p) + K^{\mathsf{s}}(p)) = T_p\mathsf{s} v_p$. Define

$$[\cdot\,,\cdot] : \Gamma(\mathsf{B}) \times \Gamma(\mathsf{B}) \to \Gamma(\mathsf{B})$$

by

$$\left[u \oplus \sigma, u' \oplus \tau\right] = \left[\!\!\left[d + \sigma^l, d' + \tau^l\right]\!\!\right]\Big|_M + K^{\mathsf{s}}$$

for all $\sigma, \tau \in \Gamma\left(\ker \mathbb{T}\mathsf{t}|_M\right)$, $u, u' \in \Gamma(U)$ and star sections $d \sim_{\mathsf{s}} u$, $d' \sim_{\mathsf{s}} u'$ of D. Then $(\mathsf{B}, \mathsf{b}, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle)$ is a Courant algebroid.

Proof: The map b is well-defined since $T_p\mathsf{s}v_p = 0_p$ for all $(v_p, \alpha_p) \in K^{\mathsf{s}}$. We show that the bracket on sections of B is well-defined, that is, that it has image in $\Gamma(\mathsf{B})$ and does not depend on the choice of the sections $u + \sigma$ and $u' + \tau$ representing $u \oplus \sigma$ and $u' \oplus \tau$, and neither on the choices of star sections d, d' over u and u'.

We have, writing $d = (X, \alpha)$, $d' = (Y, \beta)$, $u = (\bar{X}, \xi)$, $u' = (\bar{Y}, \eta)$, $\sigma^l = \left(a^l, \mathsf{s}^*\theta\right)$ and $\tau^l = \left(b^l, \mathsf{s}^*\omega\right)$:

$$\begin{aligned}
&[\![d + \sigma^l, d' + \tau^l]\!]\\
&= \left[\!\!\left[(X + a^l, \alpha + \mathsf{s}^*\theta), (Y + b^l, \beta + \mathsf{s}^*\omega)\right]\!\!\right]\\
&= [\![d, d']\!] + \pounds_{a^l} d' - \pounds_{b^l} d + \left([a, b]^l_{AG}, \mathsf{s}^*\left(\pounds_{\rho(a)+\bar{X}}\omega - \mathbf{i}_{\rho(b)+\bar{Y}}\mathbf{d}\theta + \mathbf{d}\langle\xi, b\rangle\right)\right)
\end{aligned} \tag{13}$$

By Theorems 3.15 and 3.21, the restriction of this to M is a section of $U \oplus \ker \mathbb{T}\mathsf{t}|_M$ and depends on the choice of the star sections d, d' only by sections of K^{s}.

If $u + \sigma \in \Gamma(K^{\mathsf{s}})$, then as above, we find that $\sigma \in \Gamma(K^{\mathsf{t}})$. The section $d + \sigma^l - (u + \sigma)^r$ is then a section of D that is a star section s-related to 0. Since by the considerations above, we know that

$$\left[\!\!\left[d + \sigma^l - (u + \sigma)^r, d' + \tau^l\right]\!\!\right]\Big|_M \in \Gamma(K^{\mathsf{s}})$$

and

$$\left[\!\!\left[(u + \sigma)^r, d' + \tau^l\right]\!\!\right]\Big|_M \in \Gamma(K^{\mathsf{s}})$$

for all star sections $d' \sim_{\mathsf{s}} u'$ of D and $\tau \in \Gamma((\ker(\mathbb{T}\mathsf{t}))|_M)$. This yields $[\![d + \sigma^l, d' + \tau^l]\!]\big|_M \in \Gamma(K^{\mathsf{s}})$. In the same manner, we get $[\![d' + \tau^l, d + \sigma^l]\!]\big|_M \in \Gamma(K^{\mathsf{s}})$ and we have shown that the bracket does not depend on the choice of the representatives for $u \oplus \sigma$ and $u' \oplus \tau$.

We next show that $(\mathsf{B}, \mathsf{b}, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle)$ is a Courant algebroid. The map

$$\mathcal{D} : C^\infty(M) \to \Gamma(\mathsf{B})$$

is simply given by

$$\mathcal{D}f = (0, \mathsf{s}^*\mathbf{d}f) + K^{\mathsf{s}}$$

since

$$\left\langle \mathcal{D}f, \overline{(v_m, \alpha_m)} \right\rangle = \mathsf{b}\left(\overline{(v_m, \alpha_m)}\right)(f) = T_m\mathsf{s}v_m(f)$$

for all $\overline{(v_m, \alpha_m)} \in \mathsf{B}(m)$. We check all the Courant algebroid axioms. Choose $u_1 \oplus \sigma_1, u_2 \oplus \sigma_2$ and $u_3 \oplus \sigma_3 \in \Gamma(\mathsf{B})$ and let f be an arbitrary element of $C^\infty(M)$. Choose also star sections $d_i \sim_{\mathsf{s}} u_i$ for $i = 1, 2, 3$. As before, we write $u_i = (\bar{X}_i, \xi_i)$ and $d_i = (X_i, \alpha_i)$. Note first that

$$\langle d_1 + \sigma_1^l, d_2 + \sigma_2^l \rangle = \mathsf{s}^* \langle u_1 \oplus \sigma_1, u_2 \oplus \sigma_2 \rangle \tag{14}$$

We now verify the three axioms for Courant algebroids (see page 8). By (13), the bracket

$$\left[\!\left[d_2 + \sigma_2^l, d_3 + \sigma_3^l \right]\!\right]$$

can be taken as the section extending

$$[u_2 \oplus \sigma_2, u_3 \oplus \sigma_3]$$

to compute its bracket with $u_1 \oplus \sigma_1$. Since P_G is a Courant algebroid, we have

$$\begin{aligned} &\left[\!\left[d_1 + \sigma_1^l, \left[\!\left[d_2 + \sigma_2^l, d_3 + \sigma_3^l \right]\!\right] \right]\!\right] \\ = &\left[\!\left[\left[\!\left[d_1 + \sigma_1^l, d_2 + \sigma_2^l \right]\!\right], d_3 + \sigma_3^l \right]\!\right] + \left[\!\left[d_2 + \sigma_2^l, \left[\!\left[d_1 + \sigma_1^l, d_3 + \sigma_3^l \right]\!\right] \right]\!\right] \end{aligned}$$

This restricts to

$$\begin{aligned} &[u_1 \oplus \sigma_1, [u_2 \oplus \sigma_2, u_3 \oplus \sigma_3]] \\ = &[[u_1 \oplus \sigma_1, u_2 \oplus \sigma_2], u_3 \oplus \sigma_3] + [u_2 \oplus \sigma_2, [u_1 \oplus \sigma_1, u_3 \oplus \sigma_3]] \end{aligned}$$

on M.

We have

$$\begin{aligned} &(X_1 + a_1^l)\langle d_2 + \sigma_2^l, d_3 + \sigma_3^l \rangle \\ = &\left\langle \left[\!\left[d_1 + \sigma_1^l, d_2 + \sigma_2^l \right]\!\right], d_3 + \sigma_3^l \right\rangle + \left\langle d_2 + \sigma_2^l, \left[\!\left[d_1 + \sigma_1^l, d_3 + \sigma_3^l \right]\!\right] \right\rangle \end{aligned}$$

By (13) and (14), the right-hand side restricts to

$$\langle [u_1 \oplus \sigma_1, u_2 \oplus \sigma_2], u_3 \oplus \sigma_3 \rangle + \langle u_2 \oplus \sigma_2, [u_1 \oplus \sigma_1, u_3 \oplus \sigma_3] \rangle$$

on M. Again by (14), the left-hand side equals

$$\mathsf{s}^*\left((\bar{X}_1+\rho(a_1))\langle u_2\oplus\sigma_2, u_3\oplus\sigma_3\rangle\right).$$

Since $\mathsf{b}(u_1\oplus\sigma_1)=\bar{X}_1+\rho(a_1)$, this proves

$$\begin{aligned}&\mathsf{b}(u_1\oplus\sigma_1)\langle u_2\oplus\sigma_2, u_3\oplus\sigma_3\rangle\\ =&\langle[u_1\oplus\sigma_1, u_2\oplus\sigma_2], u_3\oplus\sigma_3\rangle+\langle u_2\oplus\sigma_2,[u_1\oplus\sigma_1, u_3\oplus\sigma_3]\rangle\end{aligned}$$

Using (14), it is easy to see that the equality

$$\left[\!\left[d_1+\sigma_1^l, d_2+\sigma_2^l\right]\!\right]+\left[\!\left[d_2+\sigma_2^l, d_1+\sigma_1^l\right]\!\right]=\left(0,\mathbf{d}\langle d_1+\sigma_1^l, d_2+\sigma_2^l\rangle\right)$$

restricts to

$$[u_1\oplus\sigma_1, u_2\oplus\sigma_2]+[u_2\oplus\sigma_2, u_1\oplus\sigma_1]=\mathcal{D}\langle u_1\oplus\sigma_1, u_2\oplus\sigma_2\rangle \qquad \square$$

on M.

Example 3.24 In the special case of a Poisson groupoid $(G\rightrightarrows M,\mathsf{D}_{\pi_G})$, the obtained Courant algebroid is isomorphic to the Courant algebroid defined by the Lie bialgebroid associated to $(G\rightrightarrows M,\pi_G)$, see [18], [19]. This shows how the Courant algebroid structure on $A\oplus A^*$ induced by the Lie bialgebroid of the Poisson groupoid $(G\rightrightarrows M,\pi_G)$ can be related to the standard Courant algebroid structure on $\mathsf{P}_G=TG\oplus T^*G$.

The isomorphism $\Psi: A\oplus A^*\to\mathsf{B}$ is given by

$$\Psi(a,\xi)=(a+\rho_*(\xi),\xi)+K^{\mathsf{s}},$$

with inverse

$$\Psi^{-1}((v_p,\alpha_p)+K^{\mathsf{s}}(p))=(v_p-\pi_G^\sharp(\alpha_p),\hat{\mathsf{s}}(\alpha_p)).$$

The verification of the equalities $\Psi^{-1}\circ\Psi=\mathrm{Id}_{A\oplus A^*}$ and $\Psi\circ\Psi^{-1}=\mathrm{Id}_{\mathsf{B}}$ is easy. We check in [9] that

$$\Psi^{-1}\left([\Psi(a,\xi),\Psi(b,\eta)]_{\mathsf{B}}\right)=\ [(a,\xi),(b,\eta)]_{A\oplus A^*}$$

for $(a,\xi),(b,\eta)\in\Gamma(A\oplus A^*)$. Since the computations are long, but straightforward, we omit them here. $\Diamond$

Example 3.25 Consider a Lie groupoid $G\rightrightarrows M$ endowed with a closed multiplicative 2-form $\omega_G\in\Omega^2(G)$. The Courant algebroid B is given here by

$$\mathsf{B}=\left(\mathrm{Graph}(\omega_G^\flat|_{TM}:TM\to A^*)+\ker\mathbb{T}\mathsf{t}|_M\right)/\,\mathrm{Graph}\left(\omega_G^\flat|_{T_M^{\mathsf{s}}G}:T_M^{\mathsf{s}}G\to(T_M^{\mathsf{t}}G)^\circ\right).$$

We show in [9] that it is isomorphic as a Courant algebroid to the standard Courant algebroid $\mathsf{P}_M = TM \oplus T^*M$, via the maps

$$\Lambda : \mathsf{B} \to TM \oplus T^*M, \qquad \Lambda\left(\overline{(v_p, \alpha_p)}\right) = (T_p\mathsf{s}v_p, \beta_p),$$

where $(T_p\mathsf{s})^*\beta_p = \alpha_p - \omega_G^\flat(v_p)$, and

$$\Lambda^{-1} : TM \oplus T^*M \to \mathsf{B}, \qquad \Lambda^{-1}(v_p, \alpha_p) = \overline{(\epsilon(v_p), (T_p\mathsf{s})^*\alpha_p + \omega_G^\flat(\epsilon(v_p)))}.$$

Example 3.26 Consider the pair Dirac groupoid $(M \times M \rightrightarrows M, \mathsf{D}_M \ominus \mathsf{D}_M)$ associated to a closed Dirac manifold (M, D_M) (see Example 3.5). The vector bundle $\mathsf{B} \to \Delta_M$ is defined here by

$$\mathsf{B}(m, m) = \frac{U(m, m) + \{0\} \times T_mM \times \{0\} \times T_m^*M}{\{(v_m, 0_m, \alpha_m, 0_m) \mid (v_m, \alpha_m) \in \mathsf{D}_M(m)\}}$$

for all $m \in M$ (recall that U is given in Example 3.20). Hence, we get an isomorphism

$$\Pi : \mathsf{B} \to TM \oplus T^*M, \quad \overline{(v_m, w_m, \alpha_m, \beta_m)} \mapsto (w_m, \beta_m) \tag{15}$$

over $\mathrm{pr}_1 : \Delta_M \to M$, with inverse

$$\Pi^{-1} : TM \oplus T^*M \to \mathsf{B}, \quad (w_m, \beta_m) \mapsto \overline{(0_m, w_m, 0_m, \beta_m)}.$$

The Courant bracket on B is easily seen to correspond via this isomorphism to the standard Courant bracket on $TM \oplus T^*M$ (and hence, does not depend on D_M). $\Diamond$

3.4 Induced action of the group of bisections on B

We show here how the action of G on $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ found in [10] in the Lie group case can be generalized to the setting of Dirac groupoids. In this section, the Dirac groupoids that we consider are not necessarily closed. Hence, the vector bundle B exists, but does not necessarily have a Courant algebroid structure.

We begin with a lemma, which will also be useful in the following section about Dirac homogeneous spaces.

Lemma 3.27 *Let $G \rightrightarrows M$ be a Lie groupoid, $d \sim_{\mathsf{s}} u$ a star-section of $TG \oplus T^*G$ and $(v_p, \alpha_p) \in T_pG \times T_p^*G$ such that $\mathbb{T}\mathsf{t}(v_p, \alpha_p) = u(p)$. Then $(v_p, \alpha_p) = u(p) + \sigma(p)$ with some $\sigma \in \Gamma((\ker(\mathbb{T}\mathsf{t})|_M)$ and*

$$d(g) \star (v_p, \alpha_p) = d(g) + \sigma^l(g)$$

for any $g \in \mathsf{s}^{-1}(p)$.

PROOF: This proof is just a computation. We leave it to the reader. $\square$

Theorem 3.28 *Let $(G\rightrightarrows M, \mathsf{D})$ be a Dirac groupoid. Choose a bisection $\kappa \in \mathcal{B}(G)$ and consider*

$$r_\kappa : U \oplus \ker \mathbb{T}\mathsf{t}|_M \to \mathsf{B}$$

$$r_\kappa(v_p, \alpha_p) = \Big(T_{\kappa(p)^{-1}} R_\kappa(v_{\kappa(p)^{-1}} \star v_p), \big(T_{(\mathsf{s}\circ\kappa)(p)} R_\kappa^{-1}\big)^* \left(\alpha_{\kappa(p)^{-1}} \star \alpha_p\right)\Big) + K^{\mathsf{s}}((\mathsf{s}\circ\kappa)(p)),$$

where $(v_{\kappa(p)^{-1}}, \alpha_{\kappa(p)^{-1}}) \in \mathsf{D}(\kappa(p)^{-1})$ is such that

$$\mathbb{T}\mathsf{s}\left(v_{\kappa(p)^{-1}}, \alpha_{\kappa(p)^{-1}}\right) = \mathbb{T}\mathsf{t}(v_p, \alpha_p).$$

The map r_κ is well-defined and induces the right translation by κ,

$$\begin{array}{cccc} \rho_\kappa : & \mathsf{B} & \to & \mathsf{B} \\ & (v_p, \alpha_p) + K^{\mathsf{s}}(p) & \mapsto & r_\kappa(v_p, \alpha_p). \end{array}$$

The map $\rho : \mathcal{B}(G) \times \mathsf{B} \to \mathsf{B}$ is a right action.

Proof (of Theorem 3.28): First, we check that the map r_κ is well-defined, that is, that it has image in B and does not depend on the choices made.

Choose $p \in M$, $(v_p, \alpha_p) \in U(p) \times (\ker \mathbb{T}\mathsf{t})(p)$ and $\kappa \in \mathcal{B}(G)$. Set $\kappa(p) = g$. Since the map r_κ is linear in every fiber of $U \oplus (\ker \mathbb{T}\mathsf{t})|_M$, it suffices to show that the image of $(0_p, 0_p)$ is $K^{\mathsf{s}}(\mathsf{s}(g))$ for any choice of $(v_{g^{-1}}, \alpha_{g^{-1}}) \in \mathsf{D}(g^{-1})$ such that $\mathbb{T}\mathsf{s}(v_{g^{-1}}, \alpha_{g^{-1}}) = (0_p, 0_p)$ to prove that it is well-defined. Using (1) and (2), we get

$$\begin{aligned} T_{g^{-1}} R_\kappa(v_{g^{-1}} \star 0_p) &= v_{g^{-1}} \star (T_p R_\kappa 0_p) = v_{g^{-1}} \star 0_g \\ (\alpha_{g^{-1}} \star 0_p) \circ T_{\mathsf{s}(g)} R_{\kappa^{-1}} &= \alpha_{g^{-1}} \star (0_p \circ T_g R_{\kappa^{-1}}) = \alpha_{g^{-1}} \star 0_g. \end{aligned}$$

Thus, we have shown that

$$r_\kappa(0_p, 0_p) = (v_{g^{-1}}, \alpha_{g^{-1}}) \star (0_g, 0_g) \in \mathsf{D}(\mathsf{s}(g)) \cap \ker \mathbb{T}\mathsf{s} = K^{\mathsf{s}}(\mathsf{s}(g)).$$

Choose next $(v_p, \alpha_p) \in U \oplus (\ker \mathbb{T}\mathsf{t})|_M$ such that $(v_p, \alpha_p) \in K^{\mathsf{s}}(p)$, that is, such that $\overline{(v_p, \alpha_p)} = 0$ in B. Choose $(v_{g^{-1}}, \alpha_{g^{-1}}) \in \mathsf{D}(g^{-1})$ such that $\mathbb{T}\mathsf{s}(v_{g^{-1}}, \alpha_{g^{-1}}) = \mathbb{T}\mathsf{t}(v_p, \alpha_p)$. Then we have $T_{g^{-1}} R_\kappa(v_{g^{-1}} \star v_p) = T_{g^{-1}} R_\kappa(v_{g^{-1}} \star v_p \star 0_p) = v_{g^{-1}} \star v_p \star 0_g$, since $T_p \mathsf{s} v_p = 0$. We have also $\hat{\mathsf{s}}(\alpha_p) = 0$, and by (2):

$$(T_{\mathsf{s}(g)} R_\kappa^{-1})^*(\alpha_{g^{-1}} \star \alpha_p) = (T_{\mathsf{s}(g)} R_\kappa^{-1})^*(\alpha_{g^{-1}} \star \alpha_p \star 0_p) = \alpha_{g^{-1}} \star \alpha_p \star 0_g.$$

Thus, $r_\kappa(v_p, \alpha_p) = (v_{g^{-1}}, \alpha_{g^{-1}}) \star (v_p, \alpha_p) \star (0_g, 0_g) \in K^{\mathsf{s}}(\mathsf{s} \circ \kappa(p))$. The map $\rho_\kappa : \mathsf{B} \to \mathsf{B}$ is consequently well-defined.

The second claim of the theorem is easy to check. □

Example 3.29 Let (M, D) be a smooth Dirac manifold and consider the pair Dirac groupoid $(M \times M \rightrightarrows M, \mathsf{D} \ominus \mathsf{D})$ associated to it. The set of bisections of $M \times M \rightrightarrows M$ is equal to $\mathcal{B}(M \times M) = \{\mathrm{Id}_M\} \times \mathrm{Diff}(M)$. For $\kappa = (\mathrm{Id}_M, \phi_\kappa) \in \mathcal{B}(M \times M)$, $p := (m, m) \in$

Δ_M and $\overline{(v_m, w_m, \alpha_m, \beta_m)} \in \mathsf{B}$. Then it is easy to check that $\rho_\kappa\left(\overline{(v_m, w_m, \alpha_m, \beta_m)}\right)$ is given by

$$\rho_\kappa\left(\overline{(v_m, w_m, \alpha_m, \beta_m)}\right) = \overline{(0_n, T_m\phi_\kappa w_m, 0_n, (T_n\phi_\kappa^{-1})^*\beta_m)}.$$

Recall that B is isomorphic to P_M via (15). The action of $\mathcal{B}(M \times M)$ on B corresponds via this identification to the action of $\mathrm{Diff}(M)$ on P_M given by $\phi \cdot (v_m, \alpha_m) = (T_m\phi v_m, (T_{\phi(m)}\phi^{-1})^*\alpha_m)$ for all $\phi \in \mathrm{Diff}(M)$ and $(v_m, \alpha_m) \in \mathsf{P}_M(m)$. $\Diamond$

4 Classification of Dirac homogeneous spaces

We show here that Dirac structures in the Courant algebroid found in Section 3 correspond to Dirac homogeneous spaces of the Dirac groupoid.

We prove our main theorem (Theorem 4.16) about the correspondence between (closed) Dirac homogeneous spaces of a (closed) Dirac groupoid and Lagrangian subspaces (subalgebroids) of the Euclidean vector bundle (Courant algebroid) B. This result generalizes the result of [8] about the Poisson homogeneous spaces of Poisson Lie groups, of [19] about Poisson homogeneous spaces of Poisson groupoids and the result in [10] about the Dirac homogeneous spaces of Dirac groups. To be able to define the notion of a homogeneous Dirac structure on a homogeneous space of a Lie groupoid, we need the following proposition. The proof is straightforward and is left to the reader.

Proposition 4.1 *Let $G \rightrightarrows M$ be a Lie groupoid acting on a smooth manifold P with momentum map $\mathsf{J} : P \to M$. Then there is an induced action of $TG \rightrightarrows TM$ on $T\mathsf{J} : TP \to TM$.*

Assume that $P \simeq G/H$ is a smooth homogeneous space of G and let $q : G \to G/H$ be the projection and Φ the action of G on G/H. The map $\widehat{\mathsf{J}} : T^*(G/H) \to A^*$, $\widehat{\mathsf{J}}(\alpha_{gH}) = \hat{\mathsf{t}}\left((T_gq)^*\alpha_{gH}\right)$ for all $gH \in G/H$ is well-defined and $\widehat{\Phi} : T^*G \times_{A^*} T^*(G/H) \to T^*(G/H)$ given by

$$\left(\widehat{\Phi}(\alpha_{g'}, \alpha_{gH})\right)\left(T_{(g',gH)}\Phi(v_{g'}, v_{gH})\right) = \alpha_{g'}(v_{g'}) + \alpha_{gH}(v_{gH})$$

defines an action of $T^*G \rightrightarrows A^*$ on $\widehat{\mathsf{J}} : T^*(G/H) \to A^*$.

In the following, we write $\alpha_g \cdot \alpha_{g'H}$ for $\widehat{\Phi}(\alpha_g, \alpha_{g'H})$.

Corollary 4.2 *If G/H is a smooth homogeneous space of $G \rightrightarrows M$, there is an induced action $\mathbb{T}\Phi = (T\Phi, \widehat{\Phi})$ of*

$$(TG \oplus T^*G) \rightrightarrows (TM \oplus A^*)$$

on

$$\mathbb{T}\mathsf{J} := T\mathsf{J} \times \widehat{\mathsf{J}} : T(G/H) \oplus T^*(G/H) \to (TM \oplus A^*).$$

The following definition generalizes in a natural manner the notions of Poisson homogeneous space of a Poisson groupoid and Dirac homogeneous space of a Dirac group.

Definition 4.3 *Let* $(G\rightrightarrows M, \mathsf{D})$ *be a Dirac groupoid, and* G/H *a smooth homogeneous space of* $G\rightrightarrows M$ *endowed with a Dirac structure* $\mathsf{D}_{G/H}$*. The pair* $(G/H, \mathsf{D}_{G/H})$ *is a* Dirac homogeneous space *of the Dirac groupoid* $(G\rightrightarrows M, \mathsf{D})$ *if the induced action of* $(TG \oplus T^*G) \rightrightarrows (TM \oplus A^*)$ *on* $\mathbb{T}\mathsf{J} : (T(G/H) \times_{G/H} T^*(G/H)) \to (TM \oplus A^*)$ *restricts to an action of*

$$\mathsf{D} \rightrightarrows U \qquad on \qquad \mathbb{T}\mathsf{J}|_{\mathsf{D}_{G/H}} : \mathsf{D}_{G/H} \to U.$$

Example 4.4 Consider a Poisson homogeneous space $(G/H, \pi)$ of a Poisson groupoid $(G\rightrightarrows M, \pi_G)$, i.e. the graph $\operatorname{Graph}(\Phi) \subseteq G \times G/H \times \overline{G/H}$ is a coisotropic submanifold (see [19]).

Consider the Dirac groupoid $(G\rightrightarrows M, \mathsf{D}_{\pi_G})$ defined by $(G\rightrightarrows M, \pi_G)$ and the Dirac manifold $(G/H, \mathsf{D}_{G/H})$, defined by $\mathsf{D}_{G/H} = \operatorname{Graph}(\pi^\sharp : T^*(G/H) \to T(G/H))$. The verification of the fact that $(G/H, \mathsf{D}_{G/H})$ is a Dirac homogeneous space of the Dirac groupoid $(G\rightrightarrows M, \mathsf{D}_{\pi_G})$ is straightforward.

Conversely, if $\mathbb{T}\Phi$ restricts to an action of D_{π_G} on D_π, then the graph of the left action of G on G/H is coisotropic. $\Diamond$

Example 4.5 Let $(G\rightrightarrows M, \omega_G)$ be a presymplectic groupoid and H a wide subgroupoid of $G\rightrightarrows M$. Assume that G/H has a smooth manifold structure such that the projection $q : G \to G/H$ is a surjective submersion. Let ω be a closed 2-form on G/H such that the action $\Phi : G \times_M (G/H) \to G/H$ is a presymplectic groupoid action, i.e. $\Phi^*\omega = \operatorname{pr}_{G/H}^* \omega + \operatorname{pr}_G^* \omega_G$ [2]. Let D_ω be the graph of the vector bundle map $\omega^\flat : T(G/H) \to T^*(G/H)$ associated to ω. It is easy to check that the pair $(G/H, \mathsf{D}_\omega)$ is a closed Dirac homogeneous space of the closed Dirac groupoid $(G\rightrightarrows M, \mathsf{D}_{\omega_G})$, see Example 3.3. $\Diamond$

Example 4.6 Let $(G\rightrightarrows M, \mathsf{D})$ be a Dirac groupoid. Then $(\mathsf{t} : G \to M, \mathsf{D})$ is a Dirac homogeneous space of $(G\rightrightarrows M, \mathsf{D})$. $\Diamond$

4.1 The homogeneous Dirac structure on the classes of the units

Let $G\rightrightarrows M$ be a Lie groupoid and G/H a smooth homogeneous space of $G\rightrightarrows M$ endowed with a Dirac structure $\mathsf{D}_{G/H}$. Consider the Dirac bundle $\mathfrak{D} = q^*(\mathsf{D}_{G/H})|_M \subseteq \mathsf{P}_G|_M$ over the units M. More explicitly, we have

$$\mathfrak{D}(p) = \left\{ (v_p, \alpha_p) \in T_pG \times T_p^*G \;\middle|\; \begin{array}{c} \exists (v_{pH}, \alpha_{pH}) \in \mathsf{D}_{G/H}(pH) \quad \text{such that} \\ \alpha_p = (T_pq)^*\alpha_{pH} \text{ and } T_pq v_p = v_{pH} \end{array} \right\} \tag{16}$$

for all $p \in M$.

Proposition 4.7 *Let* $(G\rightrightarrows M, \mathsf{D})$ *be a Dirac groupoid and* $(G/H, \mathsf{D}_{G/H})$ *a Dirac homogeneous space of* $(G\rightrightarrows M, \mathsf{D})$*. Then* $\mathfrak{D} \subseteq \mathsf{P}_G|_M$ *defined as in* (16) *satisfies*

$$K^{\mathsf{s}} \subseteq \mathfrak{D} \subseteq U \oplus (\ker \mathbb{T}\mathsf{t})|_M. \tag{17}$$

Thus, the quotient $\bar{\mathfrak{D}} = \mathfrak{D}/K^{\mathsf{s}}$ *is a smooth subbundle of* B*. We have by definition* $AH \oplus \{0\} \subseteq \mathfrak{D}$*.*

PROOF: Choose $p \in M$ and $(v_p, \alpha_p) \in K^{\mathsf{s}}(p) = \mathsf{D}(p) \cap \ker \mathbb{T}\mathsf{s}$. Then $\mathbb{T}\mathsf{s}(v_p, \alpha_p) = (0_p, 0_p)$ and the product $(v_p, \alpha_p) \cdot (0_{pH}, 0_{pH})$ makes sense. Since $(0_{pH}, 0_{pH}) \in \mathsf{D}_{G/H}(pH)$, we have then $(T_p q v_p, \alpha_p \cdot 0_{pH}) = (v_p, \alpha_p) \cdot (0_{pH}, 0_{pH}) \in \mathsf{D}_{G/H}(pH)$. But $\alpha_p \cdot 0_{pH}$ is such that $(T_p q)^*(\alpha_p \cdot 0_{pH}) = \alpha_p \star ((T_p q)^* 0_{pH}) = \alpha_p$, and we get $(v_p, \alpha_p) \in \mathfrak{D}(p)$ by definition of $\mathfrak{D}$.

The inclusion $K^{\mathsf{s}} \subseteq \mathfrak{D}$ yields immediately $\mathfrak{D} = \mathfrak{D}^{\perp} \subseteq K^{\mathsf{s}\perp} = \mathsf{D}|_M + (\ker \mathbb{T}\mathsf{t})|_M = U \oplus (\ker \mathbb{T}\mathsf{t})|_M$. □

Theorem 4.8 *Let $(G \rightrightarrows M, \mathsf{D})$ be a Dirac groupoid and $\mathfrak{D}$ a Dirac subspace of $\mathsf{P}_G|_M$ satisfying (17). Then the set $\mathsf{L} = \mathsf{D} \cdot \mathfrak{D} \subseteq \mathsf{P}_G$ defined by*

$$\mathsf{L}(g) = \left\{ (v_g, \alpha_g) \star (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) \;\middle|\; \begin{array}{c} (v_g, \alpha_g) \in \mathsf{D}(g), \\ (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) \in \mathfrak{D}(\mathsf{s}(g)), \\ \mathbb{T}\mathsf{s}(v_g, \alpha_g) = \mathbb{T}\mathsf{t}(v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) \end{array} \right\}$$

is a Dirac structure on G and (G, L) is a Dirac homogeneous space of $(G \rightrightarrows M, \mathsf{D})$.

Note that $\mathsf{D} \cap \ker \mathbb{T}\mathsf{s} \subseteq \mathsf{L}$ by construction: for all $(v_g, \alpha_g) \in \mathsf{D}(g) \cap \ker \mathbb{T}\mathsf{s}$, we have $\mathbb{T}\mathsf{s}(v_g, \alpha_g) = (0_{\mathsf{s}(g)}, 0_{\mathsf{s}(g)}) \in \mathfrak{D}(\mathsf{s}(g))$ and hence $(v_g, \alpha_g) = (v_g, \alpha_g) \star (0_{\mathsf{s}(g)}, 0_{\mathsf{s}(g)}) \in \mathsf{L}(g)$.

PROOF: By Lemma 3.27, L is spanned by sections $d + \sigma^l$, such that $u + \sigma$ is a section of $\mathfrak{D}$ (with $d \sim_{\mathsf{s}} u$ a star section of U and $\sigma \in \Gamma((\ker \mathbb{T}\mathsf{t})|_M)$) and all the sections of $\mathsf{D} \cap \ker \mathbb{T}\mathsf{s}$. This shows that L is smooth.

Choose $(v_g, \alpha_g) \star (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)})$ and $(w_g, \beta_g) \star (w_{\mathsf{s}(g)}, \beta_{\mathsf{s}(g)}) \in \mathsf{L}(g)$, that is, with $(v_g, \alpha_g), (w_g, \beta_g) \in \mathsf{D}(g)$ and $(v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}), (w_{\mathsf{s}(g)}, \beta_{\mathsf{s}(g)}) \in \mathfrak{D}(\mathsf{s}(g))$. We have then

$$\begin{aligned} &\left\langle (v_g, \alpha_g) \star (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}), (w_g, \beta_g) \star (w_{\mathsf{s}(g)}, \beta_{\mathsf{s}(g)}) \right\rangle \\ &= \langle (v_g, \alpha_g), (w_g, \beta_g) \rangle + \langle (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}), (w_{\mathsf{s}(g)}, \beta_{\mathsf{s}(g)}) \rangle = 0. \end{aligned}$$

This shows $\mathsf{L} \subseteq \mathsf{L}^{\perp}$.

For the converse inclusion, choose $(w_g, \beta_g) \in \mathsf{L}(g)^{\perp}$. Then

$$(w_g, \beta_g) \in (\mathsf{D}(g) \cap \ker \mathbb{T}\mathsf{s})^{\perp} = (\mathsf{D} + \ker \mathbb{T}\mathsf{t})(g)$$

and consequently, we get the fact that $\mathbb{T}\mathsf{t}(w_g, \beta_g) \in \mathbb{T}\mathsf{t}(\mathsf{D}(g)) = U(\mathsf{t}(g))$. We write $\mathsf{t}(g) = p$ and $\mathbb{T}\mathsf{t}(w_g, \beta_g) = u(p)$ for some section $u \in \Gamma(U)$. Consider a section $d \in \Gamma(\mathsf{D})$ such that $d \sim_{\mathsf{s}} u$. Then we have for all $(v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) \in \mathfrak{D}(\mathsf{s}(g))$ and $(v_g, \alpha_g) \in \mathsf{D}(g)$ such that $\mathbb{T}\mathsf{t}(v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) = \mathbb{T}\mathsf{s}(v_g, \alpha_g)$:

$$\begin{aligned} \left\langle d(g^{-1}) \star (w_g, \beta_g), \left(v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}\right) \right\rangle &= \left\langle d(g^{-1}) \star (w_g, \beta_g), (v_g, \alpha_g)^{-1} \star (v_g, \alpha_g) \star (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) \right\rangle \\ &= \langle (w_g, \beta_g), (v_g, \alpha_g) \star (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) \rangle + \langle d(g^{-1}), (v_g, \alpha_g)^{-1} \rangle \\ &= 0, \end{aligned}$$

since $(v_g, \alpha_g) \star (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) \in \mathsf{L}(g)$ and $(v_g, \alpha_g)^{-1} \in \mathsf{D}(g^{-1})$. This proves that

$$d(g^{-1}) \star (w_g, \beta_g) \in \mathfrak{D}(\mathsf{s}(g))^{\perp} = \mathfrak{D}(\mathsf{s}(g)),$$

and hence, if we write $d(g^{-1})\star(w_g,\beta_g)=(w_{\mathsf{s}(g)},\beta_{\mathsf{s}(g)})\in\mathfrak{D}(\mathsf{s}(g))$,

$$(w_g,\beta_g)=\left(d(g^{-1})\right)^{-1}\star(w_{\mathsf{s}(g)},\beta_{\mathsf{s}(g)})\in\mathsf{L}(g).$$

The second claim is obvious since the restriction to L of the map $\mathbb{T}\mathsf{J}$ has image in $\mathbb{T}\mathsf{t}(\mathsf{D})=U$ and, by construction of L, the map $\mathsf{D}\times_U\mathsf{L}$, $((v_g,\alpha_g),(v_h,\alpha_h))\mapsto(v_g,\alpha_g)\star(v_h,\alpha_h)$ is a well-defined Lie groupoid action. □

Theorem 4.9 *In the situation of the preceding theorem, if $\mathfrak{D}$ is the restriction to M of the pullback $q^*(\mathsf{D}_{G/H})$ (as in (16)) for some Dirac homogeneous space $(G/H,\mathsf{D}_{G/H})$ of $(G\rightrightarrows M,\mathsf{D})$, then $\mathsf{L}=q^*(\mathsf{D}_{G/H})$.*

Proof: Choose $(v_g,\alpha_g)\in q^*\left(\mathsf{D}_{G/H}\right)(g)$. Then α_g is equal to $(T_gq)^*\alpha_{gH}$ for some $\alpha_{gH}\in T^*_{gH}(G/H)$ such that $(T_gqv_g,\alpha_{gH})\in\mathsf{D}_{G/H}(gH)$. Then $\mathbb{T}\mathsf{J}(T_gqv_g,\alpha_{gH})=\mathbb{T}\mathsf{t}(v_g,\alpha_g)\in U(\mathsf{t}(g))$ and there exists $(w_{g^{-1}},\beta_{g^{-1}})\in\mathsf{D}(g^{-1})$ such that

$$\mathbb{T}\mathsf{s}(w_{g^{-1}},\beta_{g^{-1}})=\mathbb{T}\mathsf{J}(T_gqv_g,\alpha_{gH}).$$

Set $p=\mathsf{s}(g)$ and consider $(u_{pH},\gamma_{pH}):=(w_{g^{-1}},\beta_{g^{-1}})\cdot(T_gqv_g,\alpha_{gH})\in\mathsf{D}_{G/H}(pH)$. Then we have

$$(T_pq)^*\gamma_{pH}=\beta_{g^{-1}}\star((T_gq)^*\alpha_{gH})=\beta_{g^{-1}}\star\alpha_g$$

by Proposition 4.1 about the action of $T^*G\rightrightarrows A^*$ on $\widehat{\mathsf{J}}:T^*(G/H)\to A^*$, and

$$u_{pH}=w_{g^{-1}}\cdot(T_gqv_g)=T_{(g^{-1},gH)}\Phi(w_{g^{-1}},T_gqv_g)=T_pq(w_{g^{-1}}\star v_g).$$

Thus, $(u_p,\gamma_p):=(w_{g^{-1}},\beta_{g^{-1}})\star(v_g,\alpha_g)$ is an element of $\mathfrak{D}(p)$, and we have $(v_g,\alpha_g)=(w_{g^{-1}},\beta_{g^{-1}})^{-1}\star(u_p,\gamma_p)$. Since D is multiplicative and $(w_{g^{-1}},\beta_{g^{-1}})\in\mathsf{D}(g^{-1})$, the pair $(w_{g^{-1}},\beta_{g^{-1}})^{-1}$ is an element of $\mathsf{D}(g)$ and we have shown that $(v_g,\alpha_g)\in\mathsf{L}(g)$.

Since $q^*(\mathsf{D}_{G/H})\subseteq\mathsf{L}$ is an inclusion of Dirac structures, we have then equality. □

Remark 4.10 Note that Theorem 4.9 shows that if $(G\rightrightarrows M,\mathsf{D})$ is a Dirac groupoid, a D-homogeneous Dirac structure on G/H is uniquely determined by its restriction to $q(M)\subseteq G/H$. △

Example 4.11 We have seen in Example 4.6 that if $(G\rightrightarrows M,\mathsf{D})$ is a Dirac groupoid, then $(\mathsf{t}:G\to M,\mathsf{D})$ is a Dirac homogeneous space of $(G\rightrightarrows M,\mathsf{D})$.

The space $\mathfrak{D}$ is here the direct sum $K^{\mathsf{s}}\oplus U$. The corresponding Dirac structure L is equal to D by construction. ♢

4.2 The classification

Recall that if $(G\rightrightarrows M,\mathsf{D})$ is a Dirac groupoid, then there is an induced action of the set of bisections $\mathcal{B}(G)$ of G on the vector bundle B associated to D (see Theorem 3.28). If H is a wide Lie subgroupoid of $G\rightrightarrows M$, this action restricts to an action of $\mathcal{B}(H)$ on B. We use this action to characterize D-homogeneous Dirac structures on G/H.

Theorem 4.12 *Let $(G \rightrightarrows M, \mathsf{D})$ be a Dirac groupoid, H a t-connected wide subgroupoid of G such that the homogeneous space G/H has a smooth manifold structure and $q: G \to G/H$ is a smooth surjective submersion. Let $\mathfrak{D}$ be a Dirac subspace of $\mathsf{P}_G|_M$ satisfying (17) and such that $AH \oplus \{0\} \subseteq \mathfrak{D}$. Then the following are equivalent:*

1. *$\mathfrak{D}$ is the pullback $q^*(\mathsf{D}_{G/H})|_M$ as in (16), where $\mathsf{D}_{G/H}$ is some D-homogeneous Dirac structure on G/H.*
2. *$\bar{\mathfrak{D}} = \mathfrak{D}/K^{\mathsf{s}} \subseteq \mathsf{B}$ is invariant under the induced action of $\mathcal{B}(H)$ on B.*
3. *The D-homogeneous Dirac structure $\mathsf{L} = \mathsf{D} \cdot \mathfrak{D} \subseteq \mathsf{P}_G$ as in Theorem 4.8 pushes-forward to a (D-homogeneous) Dirac structure on the quotient G/H.*

Note that, together with Theorem 4.9, this shows that a Dirac structure $\mathsf{D}_{G/H}$ on G/H is D-homogeneous if and only if $K^{\mathsf{s}} \subseteq (q^*\mathsf{D}_{G/H})|_M$ and $q^*\mathsf{D}_{G/H} = \mathsf{D} \cdot (q^*\mathsf{D}_{G/H})|_M$, that is, $(G/H, \mathsf{D}_{G/H})$ is $(G \rightrightarrows M, \mathsf{D})$-homogeneous if and only if $(G, q^*\mathsf{D}_{G/H})$ is.

We will need the following lemma for the proof of Theorem 4.12.

Lemma 4.13 *In the situation of Theorem 4.8, we have $\mathfrak{D} = \mathsf{L}|_M$.*

Proof: Choose $p \in M$ and $(v_p, \alpha_p) \in \mathfrak{D}(p)$. Then $\mathbb{T}\mathsf{t}(v_p, \alpha_p) \in U(p) \subseteq \mathsf{D}(p)$ and $(v_p, \alpha_p) = \mathbb{T}\mathsf{t}(v_p, \alpha_p) \star (v_p, \alpha_p) \in \mathsf{L}(p)$. This shows $\mathfrak{D} \subseteq \mathsf{L}|_M$ and we are done since both vector bundles have the same rank. $\square$

Proof (of Theorem 4.12): Assume first that $\mathfrak{D} = q^*(\mathsf{D}_{G/H})|_M$ for some D-homogeneous Dirac structure $\mathsf{D}_{G/H}$ on G/H and choose $\kappa \in \mathcal{B}(H)$ and $(v_p, \alpha_p) \in \mathfrak{D}(p)$, $p \in M$. Then there exists $\alpha_{pH} \in T^*_{pH}(G/H)$ such that $\alpha_p = (T_pq)^*\alpha_{pH}$ and $(T_pqv_p, \alpha_{pH}) \in \mathsf{D}_{G/H}(pH)$. If we set $\kappa(p) =: h \in H$ and write $\overline{(v_p, \alpha_p)}$ for $(v_p, \alpha_p) + K^{\mathsf{s}}(p) \in \bar{\mathfrak{D}}(p) \subseteq \mathsf{B}$, we have

$$\rho_\kappa\left(\overline{(v_p, \alpha_p)}\right) = \left(T_{h^{-1}}R_\kappa(v_{h^{-1}} \star v_p), (T_{\mathsf{s}(h)}R_\kappa^{-1})^*(\alpha_{h^{-1}} \star \alpha_p)\right) + K^{\mathsf{s}}(\mathsf{s}(h))$$

for any $(v_{h^{-1}}, \alpha_{h^{-1}}) \in \mathsf{D}(h^{-1})$ satisfying $\mathbb{T}\mathsf{s}(v_{h^{-1}}, \alpha_{h^{-1}}) = \mathbb{T}\mathsf{t}(v_p, \alpha_p)$. Since

$$\mathbb{T}\mathsf{J}(T_pqv_p, \alpha_{pH}) = \mathbb{T}\mathsf{t}(v_p, \alpha_p) = \mathbb{T}\mathsf{s}(v_{h^{-1}}, \alpha_{h^{-1}}),$$

the product $(v_{h^{-1}}, \alpha_{h^{-1}}) \cdot (T_pqv_p, \alpha_{pH})$ is defined and an element of $\mathsf{D}_{G/H}(\mathsf{s}(h)H)$. Note that since $\kappa \in \mathcal{B}(H)$, we have $q \circ R_\kappa = q$. The pair $(T_{h^{-1}}R_\kappa(v_{h^{-1}} \star v_p), (T_{\mathsf{s}(h)}R_\kappa^{-1})^*(\alpha_{h^{-1}} \star \alpha_p))$ satisfies

$$T_{\mathsf{s}(h)}q(T_{h^{-1}}R_\kappa(v_{h^{-1}} \star v_p)) = v_{h^{-1}} \cdot (T_pqv_p) \in T_{\mathsf{s}(h)H}(G/H)$$

and

$$(T_{\mathsf{s}(h)}R_\kappa^{-1})^*(\alpha_{h^{-1}} \star \alpha_p) = (T_{\mathsf{s}(h)}q)^*(\alpha_{h^{-1}} \cdot \alpha_{pH}).$$

Thus, $(T_{h^{-1}}R_\kappa(v_{h^{-1}} \star v_p), (T_{\mathsf{s}(h)}R_\kappa^{-1})^*(\alpha_{h^{-1}} \star \alpha_p))$ is an element of $\mathfrak{D}(\mathsf{s}(h))$ and $\rho_\kappa\left(\overline{(v_p, \alpha_p)}\right)$ is an element of $\bar{\mathfrak{D}}(\mathsf{s}(h))$. This shows (1) $\Rightarrow$ (2).

Assume now that $\bar{\mathfrak{D}}$ is invariant under the action of $\mathcal{B}(H)$ on B. Set $\mathcal{K} = \mathcal{H} \oplus 0_{T^*G}$, and hence $\mathcal{K}^\perp = TG \oplus \mathcal{H}^\circ$.

We have $AH \oplus \{0\} \subseteq \mathfrak{D}$ by hypothesis. By definition of L and $\mathcal{H}$, this yields immediately $\mathcal{K} = \mathcal{H} \oplus \{0\} \subseteq \mathsf{L}$, hence $\mathsf{L} \subseteq \mathcal{K}^\perp$ and $\mathsf{L} \cap \mathcal{K}^\perp = \mathsf{L}$ has constant rank on G. By (7), we have to show that L is invariant under the right action of $\mathcal{B}(H)$ on G. We will use the fact that L is spanned by the sections $\sigma^r \in \Gamma(\mathsf{D} \cap \ker \mathbb{T}\mathsf{s})$ for all $\sigma \in \Gamma(K^{\mathsf{s}})$ and $d + \tau^l$, $d \sim_{\mathsf{s}} u$, for all sections $u + \tau \in \Gamma(\mathfrak{D}) \subseteq \Gamma(U \oplus (\ker \mathbb{T}\mathsf{t})|_M)$.

Choose $\kappa \in \mathcal{B}(H)$. It is easy to verify that

$$R_\kappa^* \sigma^r = \sigma^r \quad \text{ for all } \quad \sigma \in \Gamma(\ker \mathbb{T}\mathsf{s}|_M).$$

Choose a section $d + \tau^l$ of L. We want to show that $R_\kappa^*(d + \tau^l)$ is then again a section of L. Choose $g \in G$ and set for simplicity $h = \kappa(\mathsf{s}(g)) \in H$, $p = \mathsf{s}(h)$, $q = \mathsf{t}(h) = \mathsf{s}(g)$, $d = (X, \alpha)$ and $(u + \tau)(p) = (v_p, \alpha_p)$. Then $\left(d + \tau^l\right)(g \cdot h) = d(g \cdot h) \star (u + \tau)(p)$ and we can compute

$$R_\kappa^*(d + \tau^l)(g) = \left(T_{g \cdot h} R_\kappa^{-1}, (T_g R_\kappa)^*\right) (d(gh) \star (u + \tau)(p)).$$

Choose $(v_g, \alpha_g) \in \mathsf{D}(g)$ such that $\mathbb{T}\mathsf{t}(v_g, \alpha_g) = \mathbb{T}\mathsf{t}(d(gh))$. Then the product $(w_h, \alpha_h) := (v_g, \alpha_g)^{-1} \star d(gh)$ is an element of $\mathsf{D}(h)$ such that $\mathbb{T}\mathsf{s}(w_h, \alpha_h) = u(p)$ and we have

$$\begin{aligned}
&R_\kappa^*(d + \tau^l)(g) \\
=&(v_g, \alpha_g) \star \left(T_h R_\kappa^{-1}\left(v_g^{-1} \star X(gh) \star v_p\right), (T_g R_\kappa)^* \left(\alpha_g^{-1} \star \alpha(gh) \star \alpha_p\right)\right) \\
=&(v_g, \alpha_g) \star \left(T_h R_\kappa^{-1}\left(w_h \star v_p\right), (T_q R_\kappa)^* \left(\beta_h \star \alpha_p\right)\right).
\end{aligned}$$

But since $\bar{\mathfrak{D}}$ is invariant under the action of $\mathcal{B}(H)$ on B and $(u \oplus \tau)(p)$ is an element of $\bar{\mathfrak{D}}(p)$, we have

$$\left(T_h R_\kappa^{-1}\left(w_h \star v_p\right), (T_q R_\kappa)^* \left(\beta_h \star \alpha_p\right)\right) + K^{\mathsf{s}}(q) = \rho_{\kappa^{-1}}\left((u \oplus \tau)(p)\right) \in \bar{\mathfrak{D}}(q).$$

Because $K^{\mathsf{s}}(q) \subseteq \mathfrak{D}(q)$, we have consequently

$$\left(T_h R_\kappa^{-1}\left(w_h \star v_p\right), (T_q R_\kappa)^* \left(\beta_h \star \alpha_p\right)\right) \in \mathfrak{D}(q)$$

and hence

$$R_\kappa^*(d + \tau^l)(g) = (v_g, \alpha_g) \star \left(T_h R_\kappa^{-1}\left(w_h \star v_p\right), (T_q R_\kappa)^* \left(\beta_h \star \alpha_p\right)\right) \in \mathsf{L}(g)$$

since $(v_g, \alpha_g) \in \mathsf{D}(g)$.

We show then that the push-forward $q(\mathsf{L})$ is a D-homogeneous Dirac structure on G/H. By definition of $\mathbb{T}\mathsf{J}$, we have $\mathbb{T}\mathsf{J}(q(\mathsf{L})) = \mathbb{T}\mathsf{t}(\mathsf{L}) \subseteq \mathbb{T}\mathsf{t}(\mathsf{D}) = U$. Choose $(v_{gH}, \alpha_{gH}) \in q(\mathsf{L})(gH)$ and $(w_{g'}, \beta_{g'}) \in \mathsf{D}(g')$ such that $\mathbb{T}\mathsf{s}(w_{g'}, \beta_{g'}) = \mathbb{T}\mathsf{J}(v_{gH}, \alpha_{gH})$. Then there exists $v_g \in T_g G$ such that $T_g q v_g = v_{gH}$ and $(v_g, (T_g q)^* \alpha_{gH}) \in \mathsf{L}(g)$. The pair $(v_g, (T_g q)^* \alpha_{gH})$ satisfies then $\mathbb{T}\mathsf{t}(v_g, (T_g q)^* \alpha_{gH}) = \mathbb{T}\mathsf{J}(v_{gH}, \alpha_{gH}) = \mathbb{T}\mathsf{s}(w_{g'}, \beta_{g'})$ and since (G, L) is a Dirac homogeneous space of $(G \rightrightarrows M, \mathsf{D})$, we have $(w_{g'}, \beta_{g'}) \star (v_g, (T_g q)^* \alpha_{gH}) \in \mathsf{L}(g' \cdot g)$ and the identities $(T_{g' \cdot g} q)^*(\beta_{g'} \cdot \alpha_{gH}) = \beta_{g'} \star (T_g q)^* \alpha_{gH}$ and $T_{g' \cdot g} q(w_{g'} \star v_g) = w_{g'} \cdot (T_g q v_g) =$

$w_{g'} \cdot v_{gH}$. Thus, the pair $(w_{g'}, \beta_{g'}) \cdot (v_{gH}, \alpha_{gH})$ is an element of $q(\mathsf{L})(gg'H)$ and $q(\mathsf{L})$ is shown to be D-homogeneous. Hence, we have shown (2) $\Rightarrow$ (3).

To show that (3) implies (1), we have just to show that the vector bundle $\mathfrak{D} \to M$ is the restriction to M of the pullback $q^*(q(\mathsf{L}))$. Since $\mathsf{L}|_M = \mathfrak{D}$ by Lemma 4.13, this follows from $\mathsf{L} = q^*(q(\mathsf{L}))$. But this is an easy consequence of the inclusion $\mathcal{H} \oplus 0_{T^*G} \subseteq \mathsf{L}$. $\square$

Theorem 4.14 *Let $(G \rightrightarrows M, \mathsf{D})$ be a closed Dirac groupoid. In the situation of the preceding theorem, the following are equivalent:*

1. *The Dirac structure $q(\mathsf{L}) = \mathsf{D}_{G/H}$ is closed.*
2. *The Dirac structure L is closed.*
3. *The set of sections of $\bar{\mathfrak{D}} \subseteq \mathsf{B}$ is closed under the bracket on the sections of the Courant algebroid B.*

Proof: If L is closed, then $q(\mathsf{L})$ is closed by a Theorem in [32] about Dirac reduction by foliations (see also Section 2.2). Conversely, assume that $q(\mathsf{L})$ is closed. By $\mathsf{L} \subseteq TG \oplus \mathcal{H}^\circ$ and the proof of Theorem 4.12, the Dirac structure L is spanned by q-descending sections, that is, sections (X, α) such that $\alpha \in \Gamma(\mathcal{H}^\circ)$ and $R_\kappa^*(X, \alpha) = (X, \alpha)$ for all $\kappa \in \mathcal{B}(H)$. Choose two descending sections $(X, \alpha), (Y, \beta)$ of L. Choose $(\bar{X}, \bar{\alpha}), (\bar{Y}, \bar{\beta}) \in \Gamma(q(\mathsf{L}))$ such that $(X, \alpha) \sim_q (\bar{X}, \bar{\alpha})$ and $(Y, \beta) \sim_q (\bar{Y}, \bar{\beta})$. Then the bracket $[\![(X, \alpha), (Y, \beta)]\!]$ descends to $[\![(\bar{X}, \bar{\alpha}), (\bar{Y}, \bar{\beta})]\!]$ which is a section of $q(\mathsf{L})$ since $(G/H, q(\mathsf{L}))$ is closed. But since $\mathcal{H} \oplus 0_{T^*G} \subseteq \mathsf{L}$, we have $\mathsf{L} = q^*(q(\mathsf{L}))$. Since $[\![(X, \alpha), (Y, \beta)]\!]$ is a section of $q^*(q(\mathsf{L}))$, we have shown that $[\![(X, \alpha), (Y, \beta)]\!] \in \Gamma(\mathsf{L})$. This proves (1) $\iff$ (2).

Assume that (G, L) is closed and choose two sections $u \oplus \tau, u' \oplus \tau'$ of $\bar{\mathfrak{D}} \subseteq \mathsf{B}$. Then, if $d \sim_{\mathsf{s}} u$ and $d' \sim_{\mathsf{s}} u'$, the two pairs $d + \tau^l, d' + \tau'^l$ are smooth sections of L by construction and since (G, L) is closed, we have $[\![d + \tau^l, d' + \tau'^l]\!] \in \Gamma(\mathsf{L})$. But since $\mathfrak{D} = \mathsf{L}|_M$ and $[u \oplus \tau, u' \oplus \tau'] = [\![d + \tau^l, d' + \tau'^l]\!]|_M + K^{\mathsf{s}}$, this yields $[u \oplus \tau, u' \oplus \tau'] \in \Gamma(\bar{\mathfrak{D}})$.

Conversely, assume that $\Gamma(\bar{\mathfrak{D}})$ is closed under the Courant bracket on sections of B and choose two spanning sections $d + \tau^l, d' + \tau'^l$ of L corresponding to $u + \tau$ and $u' + \tau' \in \Gamma(\mathfrak{D}) \subseteq \Gamma(U \oplus (\ker T\mathsf{t})|_M)$. Since $[u \oplus \tau, u' \oplus \tau']$ is then an element of $\Gamma(\bar{\mathfrak{D}})$ and $K^{\mathsf{s}} \subseteq \mathfrak{D}$, we have

$$[\![d + \tau^l, d' + \tau'^l]\!]|_M \in \Gamma(\mathfrak{D})$$

by definition of the bracket on the sections of B. We write $\tau^l = (a^l, \mathsf{s}^*\theta)$ and $\tau'^l = (b^l, \mathsf{s}^*\omega)$ with $a, b \in \Gamma(A)$ and $\theta, \omega \in \Omega^1(M)$. Recall that (13) shows that $[\![d + \tau^l, d' + \tau'^l]\!]$ equals the sum of $[\![d, d']\!] + \mathcal{L}_a d' - \mathcal{L}_b d$ with a left-invariant term σ^l. Hence, by Lemma 3.27, the value of $[\![d + \tau^l, d' + \tau'^l]\!]$ at $g \in G$ equals

$$\Big(\left([\![d, d']\!] + \mathcal{L}_a d' - \mathcal{L}_b d\right)(g)\Big) \star \left(\left[\!\!\left[d + \tau^l, d' + \tau'^l\right]\!\!\right](\mathsf{s}(g))\right)$$

and we find that $[\![d + \tau^l, d' + \tau'^l]\!]$ is a section of L, since the first factor is an element of $\mathsf{D}(g)$ and the second an element of $\mathfrak{D}(\mathsf{s}(g))$. A straightforward computation yields

$[\![d + \tau^l, \sigma^r]\!] \in \Gamma(\mathsf{D} \cap \ker \mathbb{T}\mathsf{s})$ for all $\sigma \in \Gamma(K^{\mathsf{s}})$. Finally, since D is closed, we know that $[\![\sigma_1^r, \sigma_2^r]\!] \in \Gamma(\mathsf{D})$ for all $\sigma_1, \sigma_2 \in \Gamma(\mathsf{D} \cap \ker \mathbb{T}\mathsf{s})$. Thus, by the Leibniz identity for the restriction to $\Gamma(\mathsf{L})$ of the Courant bracket on P_G, we have shown that (G, L) is closed.$\square$

Remark 4.15 Assume that $(G \rightrightarrows M, \mathsf{D})$ is a closed Dirac groupoid, $\mathfrak{D} \subseteq \mathsf{P}_G|_M$ a Dirac subspace satisfying (17) and $AH \oplus \{0\} \subseteq \mathfrak{D}$ for some t-connected wide Lie subgroupoid H of $G \rightrightarrows M$, and such that $\mathfrak{D}/K^{\mathsf{s}} \subseteq \mathsf{B}$ is closed under the bracket on B. It is easy to check (as in the proof of Theorem 4.14) that we have then $[\![(a^l, 0), d]\!] \in \Gamma(\mathsf{D} \cdot \mathfrak{D})$ for all $d \in \Gamma(\mathsf{D} \cdot \mathfrak{D})$ and $a \in \Gamma(AH)$. Since H is t-connected, we get then the fact that $R_\kappa^* d \in \Gamma(\mathsf{D} \cdot \mathfrak{D})$ for all bisections $\kappa \in \mathcal{B}(H)$ and the Dirac structure $\mathsf{D} \cdot \mathfrak{D}$ projects to a Dirac structure on G/H, that is D-homogeneous. The quotient $\mathfrak{D}/K^{\mathsf{s}}$ is then automatically invariant under the induced action of the bisections $\mathcal{B}(H)$ on B and this shows that the condition 2 of Theorem 4.12 is always satisfied if D is closed, $\mathfrak{D}/K^{\mathsf{s}}$ is closed under the Courant bracket on sections of B and H is t-connected. $\triangle$

Using this, we get our main result as a corollary of the Theorems 4.8, 4.9, 4.12, and 4.14. This theorem generalizes the correspondence theorems in [8], [19] and [10].

Theorem 4.16 *Let $(G \rightrightarrows M, \mathsf{D})$ be a Dirac groupoid. Let H be a wide Lie subgroupoid of G such that the quotient G/H is a smooth manifold and the map $q : G \to G/H$ a smooth surjective submersion.*

1. *There is a one-one correspondence between D-homogeneous Dirac structures on G/H and maximal isotropic subspaces $\mathfrak{D}$ of $\mathsf{P}_G|_M$ such that*
   a) *$AH \oplus \{0\} + K^{\mathsf{s}} \subseteq \mathfrak{D} \subseteq U \oplus (\ker \mathbb{T}\mathsf{t})|_M$ and*
   b) *$\bar{\mathfrak{D}} := \mathfrak{D}/K^{\mathsf{s}}$ is a $\mathcal{B}(H)$-invariant Dirac subspace of B.*
2. *If $(G \rightrightarrows M, \mathsf{D})$ is closed, then closed D-homogeneous Dirac structures on G/H are in one-one correspondence with closed Dirac structures $\bar{\mathfrak{D}} = \mathfrak{D}/K^{\mathsf{s}}$ in B such that $AH \oplus \{0\} + K^{\mathsf{s}} \subseteq \mathfrak{D} \subseteq U \oplus (\ker \mathbb{T}\mathsf{t})|_M$.*

Example 4.17 In [19], it is shown that for a Poisson groupoid $(G \rightrightarrows M, \pi_G)$, there is a one to one correspondence between π_G-homogeneous Poisson structures on smooth homogeneous spaces G/H and regular closed Dirac structures L of the Courant algebroid $A \oplus A^*$, such that H is the t-connected subgroupoid of G corresponding to the subalgebroid $L \cap (A \oplus 0_{A^*})$. Since pullbacks to G of Poisson structures on G/H correspond to closed Dirac structures on G with characteristic distribution $\mathcal{H}$, we recover this result as a special case of Theorem 4.16, using Remark 4.15 and the isomorphism in Example 3.24.

Note that in this particular situation of a Poisson groupoid, Theorem 4.16 classifies not only the Poisson homogeneous spaces of $(G \rightrightarrows M, \pi_G)$, but all its (not necessarily closed) Dirac homogeneous spaces. $\Diamond$

Example 4.18 Let $(G \rightrightarrows M, \pi_G)$ be a Poisson groupoid and H a wide subgroupoid of G. Assume that the Poisson structure descends to the quotient G/H, i.e. that π_G is invariant under the action of the bisections of H. Let π be the induced structure on G/H. We show that $(G, q^*\mathsf{D}_\pi)$ is a Dirac homogeneous space of $(G{\rightrightarrows}M, \pi_G)$. This is equivalent to the fact that $(G/H, \pi)$ is a Poisson homogeneous space of $(G{\rightrightarrows}M, \pi_G)$.

The Dirac structure $q^*\mathsf{D}_\pi$ is equal to $(\mathcal{H} \oplus 0_{T^*G}) \oplus \operatorname{Graph}\left(\pi_G^\sharp\big|_{\mathcal{H}^\circ} : \mathcal{H}^\circ \to TG\right)$. Since $\mathcal{H} \subseteq T^{\mathsf{t}}G$, the inclusion $\mathbb{T}\mathsf{t}\,(q^*\mathsf{D}_\pi) \subseteq U$ is obvious. Choose $(v_g, \alpha_g) \in (q^*\mathsf{D}_\pi)\,(g)$ and $\alpha_h \in T_h^*G$ such that $\mathbb{T}\mathsf{s}\left(\pi_G^\sharp(\alpha_h), \alpha_h\right) = \mathbb{T}\mathsf{t}(v_g, \alpha_g)$. Then we have $(v_g, \alpha_g) = \left(u_g + \pi_G^\sharp(\alpha_g), \alpha_g\right)$ with some $u_g \in \mathcal{H}(g)$ and the product $\left(\pi_G^\sharp(\alpha_h), \alpha_h\right) \star (v_g, \alpha_g)$ is equal to

$$\begin{aligned}(\pi_G^\sharp(\alpha_h), \alpha_h) \star (u_g + \pi_G^\sharp(\alpha_g), \alpha_g) &= \left(\pi_G^\sharp(\alpha_h) \star \pi_G^\sharp(\alpha_g) + 0_h \star u_g, \alpha_g \star \alpha_h\right) \\ &= \left(\pi_G^\sharp(\alpha_g \star \alpha_h) + T_gL_h u_g, \alpha_g \star \alpha_h\right)\end{aligned}$$

since π_G is multiplicative. The vector $T_gL_hu_g$ is an element of $\mathcal{H}$ by definition and consequently, $(\pi_G^\sharp(\alpha_h), \alpha_h) \star (u_g + \pi_G^\sharp(\alpha_g), \alpha_g)$ is an element of $q^*(\mathsf{D}_\pi)$, which is shown to be π_G-homogeneous. It corresponds to the closed Dirac structure $(AH \oplus 0_{T^*M}) \oplus \operatorname{Graph}\left(\pi_G^\sharp\big|_{AH^\circ} : AH^\circ \to TM\right) + K^{\mathsf{s}}$ of B, or more simply, to the closed Dirac structure $AH \oplus AH^\circ$ in the Courant algebroid $A \oplus A^*$.

Thus, Theorem 4.16 together with the isomorphism in Example 3.24 shows that the multiplicative Poisson structure on G descends to G/H if and only if the Lagrangian subspace $AH \oplus AH^\circ$ is a subalgebroid of the Courant algebroid $A \oplus A^*$.

The Poisson homogeneous space that corresponds in this way to the closed Dirac structure $A \oplus 0_{A^*}$ is the Poisson manifold (M, π_M), where π_M is the Poisson structure induced on M by π_G, see [30]. Note that the other trivial Dirac structure $0_A \oplus A^*$ corresponds to (G, π_G) seen as a Poisson homogeneous space of $(G{\rightrightarrows}M, \pi_G)$ (see Example 2.5).

In the same manner, we can show that if a Dirac groupoid $(G{\rightrightarrows}M, \mathsf{D})$ is invariant under the action of a wide subgroupoid H, and the Dirac structure descends to the quotient G/H, then $(G/H, q(\mathsf{D}))$ is $(G{\rightrightarrows}M, \mathsf{D})$-homogeneous. $\Diamond$

Example 4.19 Let (M, D_M) be a smooth Dirac manifold and $(M \times M \rightrightarrows M, \mathsf{D}_M \ominus \mathsf{D}_M)$ the pair Dirac groupoid associated to it.

The wide Lie subgroupoids of $M \times M \rightrightarrows M$ are the equivalence relations $R \subseteq M \times M$, and the corresponding homogeneous spaces are the products $M \times M/R$. For instance, if $\Phi : G \times M \to M$ is an action of a Lie group G (with Lie algebra $\mathfrak{g}$) on M, the subset $R_G = \{(m, \Phi_g(m)) \mid m \in M, g \in G\}$ is a wide subgroupoid of $M \times M$, and $(M \times M)/R_G$ is easily seen to equal $M \times M/G$. Hence, if the action is free and proper, the homogeneous space $(M \times M)/R_G$ has a smooth manifold structure such that the projection $q : M \times M \to M \times M/G$ is a smooth surjective submersion.

One finds easily that the $\mathsf{D}_M \ominus \mathsf{D}_M$-homogeneous Dirac structures on $M \times M/G$ are of the form $\mathsf{D}_M \oplus \bar{\mathsf{D}} := \mathsf{D}_M \oplus q_G(\mathsf{D})$, where $q_G : M \to M/G$ is the canonical projection and D a Dirac structure on M that is reducible to M/G. $\Diamond$

Example 4.20 The left invariant Dirac structures on a Lie group G are the homogeneous structures relative to the trivial Poisson bracket on G [10]. Hence, if we consider this example in the groupoid situation, we should recover the "right" definition for left invariant Dirac structures on a Lie groupoid. We say that a Dirac structure D on a Lie groupoid $G\rightrightarrows M$ is *left-invariant* if the action $\mathbb{T}\Phi$ of $TG \oplus T^*G$ on $\mathbb{T}\mathsf{t} : TG \oplus T^*G \to TM \oplus A^*$ restricts to an action of $0_{TG} \oplus T^*G$ on D, i.e.

$$(0_{TG} \oplus T^*G) \cdot \mathsf{D} = \mathsf{D}.$$

In [19], a Dirac structure on a Lie groupoid $G\rightrightarrows M$ is said to be left-invariant if it is the pullback under the map

$$\begin{array}{rcll} \Phi : T^{\mathsf{t}}G \oplus T^*G & \to & A \oplus A^* & \\ (v_g, \alpha_g) & \mapsto & (T_g L_{g^{-1}} v_g, \hat{\mathsf{s}}(\alpha_g)) & \in A_{\mathsf{s}(g)}G \times A^*_{\mathsf{s}(g)}G \end{array}$$

of a Dirac structure in $A \oplus A^*$ (where (A, A^*) is endowed with the trivial Lie bialgebroid structure, i.e. where the Lie algebroid structure on A^* is trivial). These two definitions are easily seen to be equivalent, the inclusion $0_{TG} \oplus (T^{\mathsf{t}}G)^\circ \subseteq \mathsf{D}$ is immediate and it is easy to check that D is invariant under the lifted right actions of the bisections if and only if the corresponding Dirac structure in $\mathsf{B} = \mathsf{B}(0 \oplus T^*G)$ is invariant under the induced action of $\mathcal{B}(G)$ on B (compare with Proposition 6.2 in [19]).

The result in Theorem 4.14 implies that a left-invariant Dirac structure D is closed if and only if the corresponding Dirac structure $\Phi(\mathsf{D}|_M) \subseteq A \oplus A^*$ is a subalgebroid. $\Diamond$

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M. Jotz Lean
School of Mathematics and Statistics,
The University of Sheffield,
Sheffield S37RH, UK.
M.Jotz-Lean@sheffield.ac.uk

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