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Compactness, differentiability and similarity to isometry of composition semigroups

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Dedicated to the memory of James Jamison

Abstract

This paper provides sufficient conditions for eventual compactness and differentiability of C_0-semigroups on the Hardy and Dirichlet spaces on the unit disc with a prescribed generator of the form $Af = Gf'$. Moreover, the isometric semigroups (or isometric up to a similarity) of composition operators on the Hardy space are characterized in terms of G.

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1 Introduction

The analysis of semigroups of composition operators acting on the Hardy space $H^2(\mathbb{D})$ was initiated in [5], and since then, it has been extensively studied, considering also other spaces of analytic functions such as the Dirichlet space $\mathcal{D}$ or the Bergman space A^2 (see, for example, [1, 2, 3, 5, 8, 13, 14]).

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Semigroups of composition operators on spaces of analytic functions on the open unit disc $\mathbb{D}$ are associated with the notion of semiflow (φ_t) of analytic functions mapping $\mathbb{D}$ to itself, and satisfying $\varphi_{s+t} = \varphi_s \circ \varphi_t$; here s and t lie either in $\mathbb{R}_+$ or in a sector of the complex plane. By definition of semiflow, it is always assumed that the mapping $(t, z) \mapsto \varphi_t(z)$ is jointly continuous. Berkson and Porta [5] proved that there exists an analytic function G on $\mathbb{D}$ such that

$$\frac{\partial \varphi_t}{\partial t} = G \circ \varphi_t.$$

A semiflow induces a semigroup of composition operators C_{φ_t} on $H^2(\mathbb{D})$, $\mathcal{D}$, or A^2, where $C_{\varphi_t} f = f \circ \varphi_t$ for $t \geq 0$. Since it is strongly continuous, it has a densely-defined generator A given by

$$Af = \lim_{t \to 0+} \frac{C_{\varphi_t} f - f}{t} \qquad (f \in D(A)),$$

where $D(A)$, the domain of A, is the subspace consisting of all f for which the above limit exists. In this case, A has the explicit form $Af = Gf'$, with G as above.

For example, taking $\varphi_t(z) = e^{-t}z + 1 - e^{-t}$, it is easily verified that $G(z) = 1 - z$.

First recall that a C_0-semigroup T will be called analytic (or holomorphic) if there exists a sector $\Sigma_\theta = \{re^{i\alpha}, r \in \mathbb{R}_+, |\alpha| < \theta\}$ with $\theta \in (0, \frac{\pi}{2}]$ and an analytic mapping $\widetilde{T} : \Sigma_\theta \to \mathcal{L}(X)$ such that $\widetilde{T}$ is a semigroup extending T and

$$\sup_{\xi \in \Sigma_\theta \cap \mathbb{D}} \|\widetilde{T}(\xi)\| < \infty.$$

Recall also that T being immediately compact means that $T(t)$ is compact for all $t > 0$, whereas T being eventually compact means that there exists $t_0 > 0$ such that the compactness of $T(t)$ holds for all $t > t_0$.

For analytic semigroups, the compactness (eventual or immediate) is completely characterized in terms of G by the following result.

Theorem 1.1 (Thm. 3.13 in [3]). *Let $G : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that the operator A defined by $Af(z) = G(z)f'(z)$ with dense domain $D(A) \subset H^2(\mathbb{D})$ generates an analytic semigroup $(T(t))_{t\geq 0}$ of composition operators. Then the following assertions are equivalent:*

1. *$(T(t))_{t\geq 0}$ is immediately compact;*

2. $(T(t))_{t\geq 0}$ *is eventually compact;*

3. $\forall \xi \in \mathbb{T}$, $\lim_{z\in\mathbb{D}, z\to\xi} \left|\frac{G(z)}{z-\xi}\right| = \infty$.

This chain of equivalences holds because for analytic semigroups immediate (eventual) compactness is equivalent to the compactness of the resolvent operator $R(\lambda, A)$ for $\lambda \in \rho(A)$, which is always characterized by the third condition.

There exist examples of non-analytic semigroups for which the resolvent is compact but no $T(t)$ is compact. For example (see [14, Sec. 3]), let h be the Riemann map from $\mathbb{D}$ onto the starlike region

$$\Omega := \mathbb{D} \cup \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) \text{ and } 0 < \mathrm{Im}(z) < 1\},$$

with $h(0) = 0$. Since $\partial\Omega$ is a Jordan curve, the Carathéodory theorem implies that h extends continuously to $\partial\mathbb{D}$.

Let $\varphi_t(z) = h^{-1}(e^{-t}h(z))$. Note that for $t > 0$, $\varphi_t(\mathbb{T})$ intersects $\mathbb{T}$ on a set of positive measure, and thus, C_{φ_t} is not compact by the following proposition.

Proposition 1.2 (Prop.3.1 in [3])**.** *Suppose that for some* $t_0 > 0$ *one has* $|\varphi_{t_0}(\xi)| = 1$ *on a set of positive measure; then* $C_{\varphi_{t_0}}$ *is not compact on* $H^2(\mathbb{D})$ *or* $\mathcal{D}$*, and so the semigroup* $(C_{\varphi_t})_{t\geq 0}$ *is not immediately compact.*

In general, it is a challenging question to give a complete characterization of compact semigroups in terms of the infinitesimal generator. It is known (see for example [3, Rem. 3.8]) that for eventually compact semigroups the Denjoy–Wolff point of each φ_t must lie in $\mathbb{D}$.

If G generates a semiflow of analytic functions on $\mathbb{D}$, then G has an expression of the form $G(z) = (\alpha - z)(1 - \overline{\alpha}z)F(z)$, where $\alpha \in \overline{\mathbb{D}}$ is the Denjoy–Wolff point, and $F : \mathbb{D} \to \mathbb{C}_+$ is holomorphic (see [5]). Note that $\alpha \in \mathbb{D}$ if and only if G has a zero in $\mathbb{D}$. We remark also that G has radial limits almost everywhere on $\mathbb{T}$, since F is the composition of a Möbius mapping and a function in $H^\infty(\mathbb{D})$.

Recall that when the Denjoy–Wolff point is 0, then there is a model available for the semigroup, namely,

$$\varphi_t(z) = h^{-1}(e^{-ct}h(z)), \tag{1}$$

where $c \in \mathbb{C}$ with $\operatorname{Re} c \geq 0$, and $h : \mathbb{D} \to \Omega$ is a conformal bijection between $\mathbb{D}$ and a domain $\Omega \subset \mathbb{C}$, with $h(0) = 0$ and Ω is spiral-like or star-like (if c is real), in the sense that

$$e^{-ct} w \in \Omega \quad \text{for all} \quad w \in \Omega \quad \text{and} \quad t \geq 0.$$

For more details we refer to [14].

Since φ_t is injective on $\mathbb{D}$, we may use a standard characterization of compactness, namely the following result.

Theorem 1.3 (pp. 132, 139 in [6]). *For $\varphi : \mathbb{D} \to \mathbb{D}$ analytic and injective, the composition operator C_φ is compact on $H^2(\mathbb{D})$ if and only if*

$$\lim_{|z| \to 1} \frac{1 - |\varphi(z)|}{1 - |z|} = \infty.$$

Likewise, the Hilbert–Schmidt property may be characterised as follows.

Theorem 1.4 (p. 26 in [12]). *For $\varphi : \mathbb{D} \to \mathbb{D}$ analytic the composition operator C_φ is Hilbert–Schmidt on $H^2(\mathbb{D})$ if and only if*

$$\int_0^{2\pi} \frac{1}{1 - |\varphi(e^{i\theta})|^2} \, d\theta < \infty.$$

We also have the following:

Proposition 1.5 (p. 149 in [6] and Cor. 6.3.3 in [7]). *For $\varphi : \mathbb{D} \to \mathbb{D}$ analytic with $\|\varphi\|_\infty < 1$, the composition operator C_φ is trace-class on $H^2(\mathbb{D})$. If in addition $\varphi \in \mathcal{D}$, then C_φ is Hilbert–Schmidt on $\mathcal{D}$*

This is clearly linked to the following result.

Lemma 1.6 (Lem. 3.9 in [3]). *Let $(\varphi_t)_{t \geq 0}$ be a semiflow on $\mathbb{D}$ with Denjoy–Wolff point 0. Then the following are equivalent:*
1. There is a $t_0 > 0$ with $\|\varphi_{t_0}\|_\infty < 1$;
2. There is a $t_0 > 0$ with $\|\varphi_t\|_\infty < 1$ for all $t \geq t_0$;
3. In the semiflow model for $(\varphi_t)_{t \geq 0}$, $\operatorname{Re} c > 0$, and the domain Ω is bounded.

The aim of this note is, first, to give a characterization of (immediate/eventual) compactness of semigroups in terms of the generating function G, as well as the stronger condition that $\|\varphi_t\|_\infty < 1$ for t sufficiently large.

This will imply a complete characterization of eventually compact semigroups of composition operators on $H^\infty(\mathbb{D})$.

Second, we will give a sufficient condition in terms of G, which implies the differentiability of the semigroup of composition operators on the Hardy space, when t is large enough.

Finally we will study the possible forms of the generators of isometric or similar to isometric semigroups of composition operators on $H^2(\mathbb{D})$.

2 Main results

2.1 Compactness

We first give a necessary and sufficient condition for the property $\|\varphi_t\|_\infty < 1$ for all $t > t_0$; note that this is easily implied by the stronger sufficient condition, that for some $\delta, \epsilon > 0$ we have $\operatorname{Re} \overline{z} G(z) \le -\delta$ for all z with $1 - \epsilon < |z| < 1$, which was considered in [3].

Theorem 2.1. *For a composition semigroup $(C_{\varphi_t})_{t \ge 0}$ on $H^2(\mathbb{D})$ with generator $A : f \mapsto Gf'$, a necessary and sufficient condition for the existence of a $t_0 \ge 0$ such that $\|\varphi_t\|_\infty < 1$ for all $t > t_0$ is that there exists an $\alpha \in \mathbb{D}$ with $G(\alpha) = 0$ such that (i) $\operatorname{Re} G'(\alpha) < 0$, and (ii) $\sup_{w \in \mathbb{D}} \operatorname{Re} \int_\beta^w \frac{G'(\alpha)}{G(z)}\, dz < \infty$ for one (or, equivalently, every) $\beta \ne \alpha$.*

Proof. We have already mentioned that the Denjoy–Wolff point α of an eventually compact semigroup must lie in $\mathbb{D}$.

We begin with the case $\alpha = 0$. With the model (1), we have that there exists $t_0 \ge 0$ with $\|\varphi_t\|_\infty < 1$ for all $t > t_0$ if and only if $h(\mathbb{D})$ is bounded and $\operatorname{Re} c > 0$ (as in Lemma 1.6).

Moreover, we have the expression $G(z) = -c\dfrac{h(z)}{h'(z)}$ and $c = -G'(0)$ (this was also noted by Siskakis [14]). Therefore, for each $\beta \ne \alpha$, there is a constant C_1 such that

$$h(w) = C_1 \exp\left(\int_\beta^w \frac{G'(0)}{G(z)}\, dz \right).$$

Note that changing the value of C_1 does not change the expression for φ_t, so we may take $C_1 = 1$ if we choose.

It follows that, in the case $\alpha = 0$, we have h bounded and $\operatorname{Re} c > 0$ if and only if $\operatorname{Re} G'(0) < 0$, and

$$\sup_{w \in \mathbb{D}} \operatorname{Re} \int_\beta^w \frac{G'(0)}{G(z)}\, dz < \infty \qquad \text{for one (or, equivalently, every) } \beta \neq 0.$$

For a general $\alpha \in \mathbb{D}$ we write b_α for the involutive disc automorphism $z \mapsto \dfrac{\alpha - z}{1 - \overline{\alpha} z}$, and for $G : \mathbb{D} \to \mathbb{C}$ analytic write

$$G_\alpha(z) = G(b_\alpha(z)) \frac{(1 - \overline{\alpha} z)^2}{|\alpha|^2 - 1}. \tag{2}$$

Note that $G_\alpha(0) = 0$, and G_α is the generator of the semigroup $(C_{\psi_t})_{t \geq 0}$, where $\psi_t = b_\alpha \circ \varphi_t \circ b_\alpha$.

Also

$$G'_\alpha(z) = G'(b_\alpha(z)) + 2\overline{\alpha} \frac{1 - \overline{\alpha} z}{1 - |\alpha|^2} G(b_\alpha(z)),$$

and so $G'(\alpha) = G'_\alpha(0)$. Also

$$\sup_{w \in \mathbb{D}} \operatorname{Re} \int_\beta^w \frac{G'_\alpha(0)}{G_\alpha(z)}\, dz = \sup_{w \in \mathbb{D}} \operatorname{Re} \int_{b_\alpha(\beta)}^{b_\alpha(w)} \frac{G'(\alpha)}{G(s)}\, ds.$$

The result now follows for G using the result for G_α. □

Example 2.2. *The following example was discussed in [3], but a simpler analysis can now be given using Theorem 2.1. Take $G(z) = 2z/(z-1)$. Then it is easily verified that $G(0) = 0$ and $G'(0) = -2$. Also, since $G'(0)/G(z) = -1 + 1/z$, we see that condition (ii) is also satisfied. As observed in [3], the associated semigroup is not analytic, although Theorem 2.1 implies that it is eventually compact.*

Although most of this work is concerned with composition operators on Hilbert function spaces, the above theorem is particularly relevant for operators acting on $H^\infty(\mathbb{D})$. However, it is well-known that semigroups of composition operators acting on $H^\infty(\mathbb{D})$ cannot be strongly continuous (see, for example, [14]).

Corollary 2.3. *A composition semigroup $(C_{\varphi_t})_{t \geq 0}$ with generator $A : f \mapsto Gf'$ is eventually compact on $H^\infty(\mathbb{D})$ if and only if the conditions of Theorem 2.1 hold.*

Proof. This follows from the result of Schwartz [11] that a composition operator C_φ is compact on $H^\infty(\mathbb{D})$ if and only if $\|\varphi\|_\infty < 1$. $\square$

The following result is an easy translation of Theorems 1.3 and 1.4. It gives necessary and sufficient conditions for compactness or the Hilbert–Schmidt property in terms of G.

Proposition 2.4. *For a composition semigroup $(C_{\varphi_t})_{t\geq 0}$ on $H^2(\mathbb{D})$ with generator $A : f \mapsto Gf'$ we have:*
(i) $(C_{\varphi_t})_{t\geq 0}$ is compact for $t \geq t_0$ if and only if there is an $\alpha \in \mathbb{D}$ with $G(\alpha) = 0$ such that

$$\lim_{|z|\to 1} \frac{1 - |h^{-1}(e^{G'(\alpha)t_0} h(z))|}{1 - |z|} = \infty,$$

where

$$h(w) = \exp\left(\int_\beta^w \frac{G'(\alpha)}{G(z)}\, dz\right) \tag{3}$$

for some (any) $\beta \neq \alpha$.
(ii) $(C_{\varphi_t})_{t\geq 0}$ is Hilbert–Schmidt for $t \geq t_0$ if and only if there is an $\alpha \in \mathbb{D}$ with $G(\alpha) = 0$ such that

$$\int_0^{2\pi} \frac{1}{1 - |h^{-1}(e^{G'(\alpha)t_0} h(e^{i\theta}))|^2}\, d\theta < \infty,$$

where h is defined in (3).

Without the hypothesis of analyticity of the semigroup (as used in [3]), we have provided necessary and sufficient conditions in terms of the generator, although the formulation is not elegant. This is not surprising since there is no intrinsic characterization of continuity in norm for the semigroup $(C_{\varphi_t})_{t\geq t_0}$, a consequence of the fact that estimation of quantities such as $\|C_{\varphi_{t_1}} - C_{\varphi_{t_2}}\|$ is a difficult problem.

2.2 Differentiability

We recall from [10, Sec. 2.2.4] that a C_0 semigroup $(T(t)_{t\geq 0}$ is differentiable for $t > t_0$ if for every $x \in X$ the mapping $t \to T(t)x$ is differentiable for $t > t_0$. Since, for $x \in D(A)$ we have $\dfrac{d}{dt}T(t)x = T(t)Ax$, the semigroup is differentiable as soon as $T(t)X \subset D(A)$.

Theorem 2.5. *For a composition semigroup* $(C_{\varphi_t})_{t\geq 0}$ *on* $H^2(\mathbb{D})$ *with generator* $A : f \mapsto Gf'$*, a sufficient condition for the existence of* $t_0 \geq 0$ *such that the semigroup is differentiable for* $t > t_0$ *is that properties (i) and (ii) of Theorem 2.1 hold.*

Proof. By Theorem 2.1, the conditions imply that $\|\varphi_t\|_\infty < 1$ for all $t > t_0$. Take $f \in H^2(\mathbb{D})$ and $t > t_0$; then

$$G(f \circ \varphi_t)'(z) = G(z)\varphi_t'(z)f'(\varphi_t(z)) = G(\varphi_t(z))f'(\varphi_t(z)).$$

Since G is analytic in $\mathbb{D}$ and $\|\varphi_t\|_\infty < 1$, we have that $z \mapsto G(\varphi_t(z))$ lies in $H^\infty(\mathbb{D})$. The remaining factor $f'(\varphi_t(z))$ is also bounded independently of z. Hence $G(f \circ \varphi_t)' \in H^\infty$ and so certainly $f \circ \varphi_t \in D(A)$ for $t > t_0$. □

Example 2.6. *The function* $G(z) = 2z/(z-1)$ *given in Example 2.2 gives an immediately compact semigroup that is eventually differentiable but not analytic.*

Another example, discussed in [3, 14], is defined using the model (1) with h *the Riemann mapping from* $\mathbb{D}$ *onto the domain*

$$\mathbb{D} \cup \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 2 \text{ and } 0 < \operatorname{Im} z < 1\}$$

with $h(0) = 0$. *In this case, the semigroup is differentiable for* $t > \ln 2$, *but is not analytic.*

2.3 Isometric semigroups

In general it is possible for a semigroup of composition operators on a Hilbert function space to contain an isometry C_{φ_t} with $t \neq 0$, while at the same time not consisting entirely of isometries: for example for a weighted L^2 space $L^2_w(\mathbb{T})$ with non-constant weight w and $\varphi_t(z) = e^{it}z$, we have that C_{φ_t} is an isometry if and only if t is a multiple of 2π.

However, in the context of composition semigroups of $H^2(\mathbb{D})$ the situation is rather different.

Definition 2.7. *A semigroup* $(T(t))_{t\geq 0}$ *is said to be isometric if every operator* $T(t)$ *is isometric; it is similar to an isometric semigroup if there is an isomorphism* V *such that each* $V^{-1}T(t)V$ *is isometric.*

Theorem 2.8. *For a composition semigroup $(C_{\varphi_t})_{t\geq 0}$ on $H^2(\mathbb{D})$,*
(i) $(C_{\varphi_t})_{t\geq 0}$ is isometric if and only there is a $t_0 > 0$ with $C_{\varphi_{t_0}}$ isometric;
(ii) $(C_{\varphi_t})_{t\geq 0}$ is similar to an isometric semigroup if and only if there is a $t_0 > 0$ with $C_{\varphi_{t_0}}$ similar to an isometry.

Proof. (i) If $C_{\varphi_{t_0}}$ is an isometry, then by [9], φ_{t_0} is inner and $\varphi_{t_0}(0) = 0$; indeed, since φ_{t_0} is injective, we have $\varphi_{t_0}(z) = e^{i\theta t_0}z$ for some $\theta \in \mathbb{R}$. Hence the Denjoy–Wolff point of the semigroup is 0, and so each $\varphi_t(0) = 0$, which means that every C_{φ_t} is a contraction. From this fact we see that if $C_{\varphi_{t_0}}$ is an isometry, then every C_{φ_t} must also be an isometry.
(ii) We now use the result of [4] that a composition operator C_φ on H^2 is similar to an isometry if and only if φ is inner with a fixed point in $\mathbb{D}$. Suppose that $\varphi_{t_0}(\alpha) = \alpha$. Then, conjugating by the involutive automorphism b_α, we see from (i) that the semigroup $(C_{b_\alpha} C_{\varphi_t} C_{b_\alpha})_{t\geq 0}$ is isometric. □

Corollary 2.9. *(i) A composition semigroup $(C_{\varphi_t})_{t\geq 0}$ on $H^2(\mathbb{D})$ is isometric if and only if there is a $\theta \in \mathbb{R}$ with $G(z) = i\theta z$. Moreover, the semigroup extends to a C_0 group $(C_{\varphi_t})_{t\in\mathbb{R}}$.*
(ii) $(C_{\varphi_t})_{t\geq 0}$ is similar to an isometric semigroup if and only if there is an $\alpha \in \mathbb{D}$ such that

$$G(z) = \frac{i\theta b_\alpha(z)}{(1-\overline{\alpha} b_\alpha z)^2}(|\alpha|^2 - 1).$$

Proof. Part (i) follows directly from Theorem 2.8; for (ii) we also use expression for G_α given in (2). □

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