Article:
Trodden, P. (2016) A one-step approach to computing a polytopic robust positively invariant set. IEEE Transactions on Automatic Control, 61 (12). pp. 4100-4105.

https://doi.org/10.1109/TAC.2016.2541300

A One-step Approach to Computing a Polytopic Robust Positively Invariant Set

Paul Trodden, *Member, IEEE*

***Abstract*—A procedure and theoretical results are presented for the problem of determining a minimal robust positively invariant (RPI) set for a linear discrete-time system subject to unknown, bounded disturbances. The procedure computes, via the solving of a single LP, a polytopic RPI set that is minimal with respect to the family of RPI sets generated from a finite number of inequalities with pre-defined normal vectors.**

***Index Terms*—Linear systems; Uncertain systems; Computational methods; Optimization; Invariant sets**

I. Introduction

We consider the problem of finding, for the discrete-time, linear time-invariant system,

$$x^+ = Ax + w, \tag{1}$$

a robust positively invariant (RPI) set. That is, a set $\mathcal{R} \subset \mathbb{R}^n$ with the property

$$Ax + w \in \mathcal{R}, \forall x \in \mathcal{R}, w \in \mathbb{W}. \tag{2}$$

In this problem, $x \in \mathbb{R}^n$ is the current state and x^+ its successor. The disturbance $w \in \mathbb{R}^n$ is unknown but lies in a polytopic (compact and convex) set $\mathbb{W}$ that contains the origin in its interior.

Robust or disturbance invariant sets are important in control, and their theory and computation have attracted significant attention; see, for example, [1]–[4] and references therein. One set that is of particular interest is the *minimal* RPI (mRPI) set—that is, the RPI set that is smallest in volume among all the RPI sets for a system—which is also the set of states reachable from the origin in the presence of a bounded disturbance. This set is an essential ingredient in many robust control algorithms. For example, in tube-based robust model predictive control (MPC) [5], an RPI set is used to guarantee robust stability and feasibility in the presence of bounded uncertainty; moreover, since the constraints in the MPC optimization problem are *tightened* according to the size of the RPI set, then the smallest RPI set (*i.e.*, the mRPI set) is desirable. However, computing an exact representation of the mRPI is generally impossible (except for special instances of A, as identified later), and instead an approximation is usually sought. A seminal contribution in this regard is [3], which proposes a method for computing an abitrarily close outer-approximation to the mRPI set, which is itself RPI.

The essence of the problem of computing exactly the mRPI set is that this set is, in general, not finitely determined. Methods for computing approximations to the mRPI set, including [3], rely on finding finite representations of the set. Recently, in the context of tube-based MPC, [6] introduced and studied the notion of a polytopic RPI set defined by a finite number, r, of *a-priori* selected linear inequalities. For a non-autonomous system $x^+ = Ax + Bu + w$ controlled by a continuous positively homogeneous control law, $u = \kappa(x)$, the authors showed that the RPI set dynamic condition (2) has an equivalent representation as r functional inequalities. It was established that a fixed-point solution to the functional equation corresponds to an RPI set that is *minimal*, in volume, with respect to the entire family of RPI sets defined by the pre-selected inequalities, and is an invariant outer-approximation to the mRPI set. To compute this set, the authors of [6] give an iterative procedure, based on solving a sequence of LPs, for which convergence is guaranteed.

P. A. Trodden is with the Department of Automatic Control & Systems Engineering, University of Sheffield, Mappin Street, Sheffield S1 3JD, UK (e-mail: p.trodden@shef.ac.uk).

In this note, we adopt the notions of [6] and specialize their results to the case of the linear autonomous system (1) (alternatively, the linear non-autonomous system with linear state feedback control law) in order to develop a one-step approach, based on solving a single LP, to the computation of the smallest RPI set defined by a pre-selected system of inequalities. Though simple, to the author's knowledge this has not appeared in the literature, although there are related results; for example, it is known that checking the invariance of an existing polytope is an LP [2]. On the other hand, the ability to synthesize a near-minimal RPI set by solving a single LP potentially paves the way for robust control methods that re-compute the disturbance invariant sets on-line, as done in, for example, the recently developed "plug-and-play" approach to distributed MPC [7].

The proposed approach differs to the one of [3] in one important assumption: the number and normal vectors of the inequalities that represent the RPI set are, as in [6], defined *a priori*, while in [3] both are unknown until termination of the algorithm. This *a-priori* definition, first proposed and studied by [6], has two consequences: firstly, the RPI set obtained is not necessarily the mRPI set, or even an abitrarily close outer-approximation (as it is in [3]); however, it is the smallest RPI set that can be represented by the finite number, r, of chosen inequalities with normal vectors $\{P_i^\top : i = 1 \ldots r\}$ [6]. To make a clear distinction, in this note we term this the (P, r)-mRPI set when the number of chosen inequalities is r and the matrix of normal vectors (the left-hand side of the defining system of inequalities) is P. Secondly, the method of [3] involves solving a sequence of LPs and then computing a Minkowski summation, but here only the solving of a single LP is required. The development of the procedure here comprises two steps, the enumeration of which also serves to clarify the contribution of this note with respect to [6]: first, we show that, for the studied linear autonomous system (1), the fixed-point solution to the functional equation, which [6] showed is guaranteed to exist, is in fact unique. Secondly, we show that the corresponding RPI set—which [6] proved to be minimal with respect to the family of RPI sets represented by (P, r)—can be computed via a single linear program (LP), as an alternative the iterative sequence of LPs proposed by [6].

Another method that uses a single LP to compute a disturbance invariant set is the optimized robust control invariance approach of [4], applicable to the linear non-autonomous system $x^+ = Ax + Bu + w$. Because a robust *control* invariant (RCI) set—and the associated control policy—is obtained, then this subsumes the robust positive invariance (where a *fixed* control law is assumed) considered here. However, that approach optimizes over only those control policies that guarantee a finitely determined set, achieved by employing a relaxed variation of the assumption, for (1), that $A^k \mathbb{W} \subseteq \alpha \mathbb{W}$ for some $\alpha \in [0, 1)$ and finite integer k. In this note, the assumption that A has eigenvalues inside the unit circle is required, which is different to the assumption used for finite determination of RCI sets in [4], but weaker than the assumption required for finite determination of the mRPI set for (1).

The organization of this note is as follows. First, in Section II, it is shown that for the system (1), the fixed-point solution is, under suitable assumptions, unique. Subsequently, in Section III, it is shown that the (P, r)-mRPI set for (1) may be computed via a single LP. Finally, examples are given in Section IV to illustrate the practicality of the proposed approach, before conclusions are made in Section V.

Notation: The sets of non-negative and positive reals are, respectively, $\mathbb{R}_{0+}$ and $\mathbb{R}_+$. For $a, b \in \mathbb{R}^n$, $a \leq b$ applies element by element. A matrix M is non-negative, denoted $M \geq 0$, if $M_{ij} \geq 0$ for all i and j. $\lambda \mathcal{X}$ is the scaling of a set $\mathcal{X}$ by $\lambda \in \mathbb{R}$, defined as $\{\lambda x : x \in \mathcal{X}\}$. $A\mathcal{X}$ denotes the image of a set $\mathcal{X} \subset \mathbb{R}^n$ under the linear map $A : \mathbb{R}^n \mapsto \mathbb{R}^p$, and is given by $\{Ax : x \in \mathcal{X}\}$. The support function of a set $\mathcal{X}$ is $h(\mathcal{X}, v) \triangleq \sup\{v^\top x : x \in \mathcal{X}\}$. A polyhedron is the convex intersection of a finite number of halfspaces, and a polytope is a closed and bounded (hence compact) polyhedron.

II. Existence and uniqueness of a (P, r)-mRPI set

For the system (1), we consider the case of a polytopic disturbance set

$$\mathbb{W} \triangleq \{w \in \mathbb{R}^n : Fw \leq g\}, \tag{3}$$

where $F \in \mathbb{R}^{p\times n}$, $g \in \mathbb{R}^p_{0+}$, and make the following two standing assumptions.

Assumption 1: The set $\mathbb{W}$ contains the origin in its interior.

Assumption 2: The eigenvalues of A are strictly within the unit circle.

The former assumption requires that $g \in \mathbb{R}^p_+$. The latter assumption implies, as shown in [1], that for a given compact disturbance set $\mathbb{W}$ there exists a compact RPI set, $\mathcal{R}$, for the system (1), satisfying (2).

Assumption 3: The RPI set $\mathcal{R}$ is a polytope that contains the origin in its interior.

Note that Assumptions 1 and 3 imply that the support functions to $\mathbb{W}$ and $\mathcal{R}$, respectively, are positive—a key technical property that will be used in this note to establish the existence and uniqueness of the RPI set that we aim to compute.

In this note, following [6], we consider the RPI set constructed from a finite number, r, of inequalities with pre-defined normal vectors. That is, $\mathcal{R} \triangleq \mathcal{R}(q)$, defined as

$$\mathcal{R}(q) \triangleq \{z \in \mathbb{R}^n : Pz \leq q\}, \tag{4}$$

where $P \in \mathbb{R}^{r\times n}$, $\{P_i^\top : i \in \{1, \dots, r\}\}$ spans $\mathbb{R}^n$, P_i is the ith row of matrix P, and $q \in \mathbb{R}^r_{0+}$. The left-hand side of the inequalities—the matrix P—is to be chosen *a priori* by the designer. The following result, which is an application of Farkas' Lemma, establishes basic conditions on the matrices A, P and F for the existence of an RPI set for the system (1) given the disturbance polytope (3).

Theorem 1 (Adapted from Hennet and Castellan [8]): Suppose Assumptions 1–3 hold. Then the set $\mathcal{R}(q)$ with some $q = \bar{q}$ is robust positive invariant for the system (1) if and only if there exist non-negative matrices $H \in \mathbb{R}^{r\times r}$ and $M \in \mathbb{R}^{r\times p}$ such that

$$HP = PA \tag{5a}$$
$$MF = P \tag{5b}$$
$$H\bar{q} + Mg \leq \bar{q} \tag{5c}$$

We will assume that P is chosen so that an RPI set exists:

Assumption 4: For the chosen P, and the system $(A, \mathbb{W})$, there exists a $\bar{q} \in \mathbb{R}^r_+$ such that (2) holds for all $x \in \mathcal{R}(\bar{q})$.

Remark 1: While Assumption 4 may appear strong, it is needed to narrow the class of matrices that we consider to those that admit an RPI set. However, the procedure presented in the next section includes a easy certification of existence of an RPI set for a chosen P: if an RPI set exists, the (P, r)-mRPI set is returned. If no RPI set exists, the optimization problem is unbounded.

The authors of [6] show—in the more general setting of a linear non-autonomous system controlled by a positively homogeneous state-feedback control law—that RPI condition (2) is equivalent to the functional inequality

$$c(q) + d \leq b(q), \tag{6}$$

where, for $i = 1 \dots r$, $b_i(q) \triangleq h(\mathcal{R}(q), P_i^\top)$, $c_i(q) \triangleq h(A\mathcal{R}(q), P_i^\top)$, $d_i \triangleq h(\mathbb{W}, P_i^\top)$. That is, the set inclusion requirement is replaced by support function inequalities, which is a standard technique [9]. Note that $b(q)$ may be different to q; for example, in the case of redundant inequalities defining $\mathcal{R}(q)$. The topological properties of these functions described in the following two lemmas are essential to establishing existence and uniqueness of the fixed-point solution to (6).

Lemma 1 (Adapted from Proposition 1 of [6]): Suppose that Assumptions 1–3 hold. Then the functions $b\colon \mathbb{R}^r_{0+} \mapsto \mathbb{R}^r_{0+}$, $c\colon \mathbb{R}^r_{0+} \mapsto \mathbb{R}^r_{0+}$ are continuous and monotonically non-decreasing; that is, $b(a_1) \leq b(a_2)$ for $a_1 \leq a_2$. Also, $d \in \mathbb{R}^r_+$.

Lemma 2 (Positive homegeneity of b, c): Suppose Assumptions 2 and 3 hold. Then the functions $b(\cdot)$ and $c(\cdot)$ are positively homogeneous; that is $b(\lambda a) = \lambda b(a)$ for $\lambda \geq 0$, with a similar expression for $c(\cdot)$.

Proof: Consider $b_i(\lambda a) = h(\mathcal{R}(\lambda a), P_i)$ for some $a \in \mathbb{R}^r_{0+}$, $\lambda \geq 0$ and $i \in \{1, \dots, r\}$. By definition of $\mathcal{R}(\cdot)$, $\mathcal{R}(\lambda a) = \lambda\mathcal{R}(a)$. Thus, $h(\mathcal{R}(\lambda a), P_i) = h(\lambda\mathcal{R}(a), P_i) = \lambda h(\mathcal{R}(a), P_i)$, for $\lambda \geq 0$, where the latter equality follows directly from the definition of the support function [9]. Hence, $b_i(\lambda a) = \lambda b_i(a)$, therefore $b(\lambda a) = \lambda b(a)$. Positive homogeneity of $c(\cdot)$ may be established by the same arguments. ∎

The next result, which concerns the existence of a fixed-point solution to (6), was established by [6] in the setting of a linear non-autonomous system controlled by positively homogeneous state-feedback control law, and hence immediately applies to the more specialized case considered in this note.

Theorem 2 (Theorem 1 of [6]): Suppose Assumptions 1–3 hold. Let $\mathcal{Q} \triangleq \{q \in \mathbb{R}^r_{0+} : 0 \leq q \leq \bar{q}\}$. Then, (i) for all $q \in \mathcal{Q}$, $c(q) + d \in \mathcal{Q}$ and (ii) there exists at least one $q^* \in \mathcal{Q}$ satisfying $c(q^*) + d = b(q^*) = q^*$ if and only if Assumption 4 holds.

Remark 2: The necessity and sufficiency of Assumption 4 follows by definition. In particular, if Assumption 4 does not hold, then there does not exist an RPI set for the system $(A, \mathbb{W})$ with the chosen P.

Remark 3: Note that, in view of the assumptions on g and the properties of $b(\cdot)$, $c(\cdot)$, and d, a fixed-point solution q^* must be strictly positive.

With respect to computing a fixed-point solution, the sequence generated by the iterative procedure $q^{[p+1]} = c(q^{[p]}) + d$, with $q^{[0]} = 0$, converges to the fixed-point solution q^* with the smallest 1-norm value, $\|q^*\|_1$ [6, Theorem 2]. As the following result states, the corresponding set $\mathcal{R}(q^*)$ is RPI, and, in fact, is the minimal (smallest volume) RPI set over the family of RPI sets defined by the r inequalities with left-hand side P.

Lemma 3 (Corollary 1 of [6]): $\mathcal{R}(q^*) = \bigcap_{X\in\mathcal{S}} X$ where

$$\mathcal{S} \triangleq \{\mathcal{R}(q) : q \in \mathcal{H}\}, \text{ and } \mathcal{H} \triangleq \{q \in \mathbb{R}^r_{0+} : c(q) + d \leq b(q)\}$$

For convenience, we define this set $\mathcal{R}(q^*)$ as the (P, r)-mRPI set.

Definition 1 ((P, r)-mRPI set): The (P, r)-mRPI set for system (1) is $\mathcal{R}(q^*)$ where $q^* = b(q^*) = c(q^*) + d$.

In this note, we propose an alernative to the iterative procedure of [6]. To this end, the next result shows that the fixed-point solution to (6) is, in fact, unique. This result is then exploited in Section III, wherein the problem of finding the fixed-point solution is cast as an LP.

Theorem 3 (Uniqueness of fixed-point solution): Suppose Assumptions 1–4 hold. Then there exists a unique $q^* \in \mathbb{R}^r_+$ satisfying $c(q^*) + d = b(q^*) = q^*$.

Proof: Existence is established by Theorem 2, so it remains to show that q^* is unique. Let $l(q) = c(q) + d - b(q)$ and $f(q) = b(q) - q$. Finding the fixed-point solution $c(q^*) + d = b(q^*) = q^*$ is equivalent to finding q^* such that $l(q^*) = f(q^*) = 0$. Suppose there exist $q^1 \in \mathbb{R}^r_+$ and $q^2 \in \mathbb{R}^r_+$ such that $l(q^1) = f(q^1) = 0$, $l(q^2) = f(q^2) = 0$, and $q^2 \neq q^1$, *i.e.*, $q^2 - q^1 \neq 0$. There are two possibilities:

(i) $q^2_i > q^1_i$ for at least one $i \in \{1, \dots, r\}$, with $q^2_j \leq q^1_j$ otherwise;
(ii) $q^2 \leq q^1$, with $q^2_i < q^1_i$ for at least one $i \in \{1, \dots, r\}$.

Consider case (i). Let

$$\alpha = \min_{i=1\ldots r} \left\{ \frac{q_i^1}{q_i^2} \right\} = \frac{q_p^1}{q_p^2} > 0$$

Strict positivity follows from the discussion in Remark 3. Since $q_i^2 > q_i^1$ for at least one i, then $\alpha < 1$. Let $s = \alpha q^2 < q^2$. It follows, from positive homogeneity of $b(\cdot)$ and the fact that $b(q^2) - q^2 = 0$, that $f(s) = b(s) - s = b(\alpha q^2) - \alpha q^2 = \alpha\big(b(q^2) - q^2\big) = 0$. Similarly,

$$\begin{aligned} l(s) &= c(s) + d - b(s) \\ &= c(\alpha q^2) + d - b(\alpha q^2) \\ &= \alpha c(q^2) + d - \alpha b(q^2) \\ &= \alpha\big(c(q^2) - b(q^2)\big) + d \\ &> 0 \end{aligned}$$

where the second line follows from the positive homegeneity of $c(\cdot)$ and $b(\cdot)$, and the strict inquality with zero follows from $c(q^2) - b(q^2) = -d$, $\alpha < 1$ and $d > 0$. Now, by definition of α, and since $\alpha < 1$, then $s \le q^1$ with $s_p = q_p^1$. For the same p, we have $f_p(q^1) = b_p(q^1) - q_p^1 = 0$, $f_p(s) = b_p(s) - s_p = 0$, and, since $s \le q^1$, then $b_p(s) \le b_p(q^1)$. In fact, $b_p(s) = b_p(q^1)$, as we have already shown that $b_p(s) = s_p = q_p^1$. We also have $l_p(q^1) = c_p(q^1) + d_p - b_p(q^1)$ and $l_p(s) = c_p(s) + d_p - b_p(s)$. Because $b_p(q^1) = b_p(s)$ and $c_p(s) \le c_p(q^1)$, it follows that $l_p(s) \le l_p(q^1)$. But then $0 = l_p(q^1) \ge l_p(s) > 0$, and we have a contradiction: therefore, we conclude that case (i) cannot hold, and either case (ii) holds or $q^2 = q^1$. Now consider case (ii), and its equivalent statement: $q_i^1 > q_i^2$ for at least one $i \in \{1, \ldots, r\}$, with $q_j^1 \ge q_j^2$ otherwise. Following the same set of arguments, starting with the opposite definitions of $\alpha = \min_{i=1\ldots r} \{q_i^2/q_i^1\}$ and $s = \alpha q^1$, we find that that case (ii) cannot hold either. Therefore, $q^1 = q^2 = q^*$, and the solution is unique. ■

III. Computing the (P, r)-mRPI set via a single LP

The problem of computing the (P, r)-mRPI set is that of finding the q that satisfies the functional inequality (RPI condition) (6) while attaining the smallest value of $\|q\|_1$. The results in the previous section show that this q in fact satisfies (6) with equality; it is the fixed-point solution q^*. Therefore, the problem of finding q^* is

$$q^* = \arg\min_q \left\{ \|q\|_1 : c(q) + d \le b(q) \right\} \tag{7}$$

This is not tractable, as, by the definitions of $b(\cdot)$ and $c(\cdot)$, the constraints are maximization problems involving the optimization variable:

$$\max\{P_i A x : P x \le q\} + \max\{P_i w : F w \le g\} \le \max\{P_i x : P x \le q\}$$

for $i = 1 \ldots r$. However, by noting that the fixed-point solution is unique, we may replace the problem of (7) with the maximization problem

$$q^* = \arg\max_q \left\{ \|q\|_1 : c(q) + d = b(q) \right\}$$

This problem then easily converts to a linear program, as shown by the following. Introduce auxiliary variables $\xi^i \in \mathbb{R}^n$ and $\omega^i \in \mathbb{R}^n$ for each RPI inequality $i \in \{1, \ldots, r\}$. Then, noting that $q = b(q) = c(q) + d$ at the desired fixed-point solution, eliminate q and $b(q)$, leading to the problem

$$\mathbb{P}\colon q^* = c^* + d^*, \text{ where } (c^*, d^*) = \arg \max_{\substack{\{c_i, d_i, \xi^i, \omega^i\} \\ \forall i \in \{1,\ldots,r\}}} \sum_{i=1}^r c_i + d_i \tag{8}$$

subject to, for all $i \in \{1, \ldots, r\}$,

$$c_i \le P_i A \xi^i, \tag{9a}$$
$$P \xi^i \le c + d, \tag{9b}$$
$$d_i \le P_i \omega^i, \tag{9c}$$
$$F \omega^i \le g. \tag{9d}$$

In this problem, maximizing each c_i subject to constraints (9a) and (9b) represents finding the vector of support functions to $A\mathcal{R}$. Constraint (9b) represents $Px \le b(q)$, with the condition $c(q) + d = b(q)$ enforced. Constraints (9c) and (9d) represent finding d, the vector of support functions to $\mathbb{W}$.

Remark 4: Note that, by definition, $d_i = h(\mathbb{W}, P_i^\top)$ is constant and does not depend on q. Therefore, d could be computed prior to solving $\mathbb{P}$, by solving a sequence of LPs, before entering the optimization as a parameter. However, our aim is to formulate a single LP (a one-step procedure) that computes, simultaneously, d, c and hence q.

Note that each d_i and ω_i is bounded, via (9c) and (9d) and the assumptions on $\mathbb{W}$. Further note that this problem always has a *feasible* solution, since one can choose, for example, $c_i = d_i = 0$ and $\xi^i = \omega^i = 0$. The question, then, is whether an optimal solution exists, or the problem is unbounded. To this end, we require the following result, which specializes Theorem 1 to the fixed-point solution.

Proposition 1: Suppose Assumptions 1–4 hold. A vector q^* satisfies the fixed-point relation $c(q^*) + d = b(q^*) = q^*$ if and only if there exist non-negative matrices $H \in \mathbb{R}^{r\times r}$ and $M \in \mathbb{R}^{r\times p}$ such that

$$HP = PA \tag{10a}$$
$$MF = P \tag{10b}$$
$$Hq^* + Mg = q^* \tag{10c}$$

Proof: Consider the ith element of each of $c(q^*)$, d and $b(q^*)$, defined by the (primal) LPs

$$c_i(q^*) = \max\{P_i A x : P x \le q^*\} \tag{11a}$$
$$d_i = \max\{P_i w : F w \le g\} \tag{11b}$$
$$b_i(q^*) = \max\{P_i x : P x \le q^*\} \tag{11c}$$

If Assumptions 1–4 hold, then by the previous results there exists a q^* satisfying the fixed-point equation. Moreover, each of the terms in (11) is well defined, which is the case if and only if each LP is feasible and attains an finite optimum. Therefore, by weak duality, the dual of each LP

$$\begin{aligned} c_i(q^*)&\colon \min\{h_i^\top q^* : h_i^\top P = P_i A, h_i \ge 0\}, \\ d_i&\colon \min\{m_i^\top g : m_i^\top F = P_i, m_i \ge 0\}, \\ b_i(q^*)&\colon \min\{y_i^\top q^* : y_i^\top P = P_i, y_i \ge 0\}, \end{aligned}$$

is feasible. Examining these dual problems, dual feasible solutions exist if and only if there exist non-negative $h_i \in \mathbb{R}^r$, $m_i \in \mathbb{R}^p$, $y_i \in \mathbb{R}^r$ such that

$$\begin{aligned} h_i^\top P &= P_i A, \\ m_i^\top F &= P_i, \\ y_i^\top P &= P_i. \end{aligned}$$

Applying strong duality, which holds in view of the previous arguments, to each of the three LPs

$$\begin{aligned} c_i(q^*) &= h_i^\top q^*, \\ d_i &= m_i^\top g, \\ b_i(q^*) &= y_i^\top q^*. \end{aligned}$$

Collecting all rows $i = 1 \dots r$,

$$\begin{aligned} c(q^*) &= Hq^* \\ d &= Mg, \\ b(q^*) &= Yq^*, \end{aligned}$$

where $HP = PA$, $MF = P$, $YP = P$. Therefore, it follows that if the fixed-point equation

$$c(q^*) + d = b(q^*) = q^*$$

is satisfied, then so are the conditions (10); conversely, if (10) are satisfied, then so is the fixed-point equation. ■

Then the main result of this section follows.

Theorem 4: Suppose Assumptions 1–3 hold. If P satisfies Assumption 4, then problem $\mathbb{P}$ admits an optimal solution corresponding to the fixed-point solution q^*. Otherwise, $\mathbb{P}$ is unbounded above.

Proof: We use duality to prove the theorem. Our goal is to prove that the optimal solution to $\mathbb{P}$ satisfies the conditions (10), for some non-negative H and M, if and only if Assumption 4 holds, and that $\mathbb{P}$ is otherwise unbounded. Since the primal LP problem $\mathbb{P}$ is known to be feasible, it suffices to show that the dual problem is feasible—and the solution is as claimed—if and only if P satisfies Assumption 4; on the other hand, if the dual is infeasible, then by weak duality the primal problem $\mathbb{P}$ is unbounded.

The dual problem is

$$\mathbb{D}\colon \min_{\substack{\{\lambda_i, \nu_i, \mu^i, \eta^i\} \\ \forall i \in \{1,\dots,r\}}} \sum_{k=1}^{r} (\eta^k)^\top g \tag{12}$$

subject to, for all $i \in \{1, \dots, r\}$,

$$\lambda - \sum_{k=1}^{r} \mu^k = \mathbf{1}, \tag{13a}$$

$$\nu - \sum_{k=1}^{r} \mu^k = \mathbf{1}, \tag{13b}$$

$$P^\top \mu^i - A^\top P_i^\top \lambda_i = 0, \tag{13c}$$

$$F^\top \eta^i - P_i^\top \nu_i = 0, \tag{13d}$$

$$\lambda_i, \nu_i \geq 0 \tag{13e}$$

$$\mu^i, \eta^i \geq 0 \tag{13f}$$

where $\lambda_i \in \mathbb{R}$, $\mu^i \in \mathbb{R}^r$, $\nu_i \in \mathbb{R}$, $\eta^i \in \mathbb{R}^p$ are the dual variables associated with constraints (9a)–(9d) respectively.

We first suppose the dual problem $\mathbb{D}$ is feasible. From (13a) and (13b), $\lambda_i = \nu_i = 1 + \sum_{k=1}^{r} \mu_i^k$, for all $i = 1, \dots, r$, where μ_i^k is the ith element of $\mu^k \in \mathbb{R}^r$. From this and (13c), (13d), it follows that

$$\begin{aligned} P_i A &= \frac{(\mu^i)^\top}{1 + \sum_{k=1}^{r} \mu_i^k} P, \\ P_i &= \frac{(\eta^i)^\top}{1 + \sum_{k=1}^{r} \mu_i^k} F, \end{aligned}$$

where the division is permitted since $\sum_{k=1}^{r} \mu_i^k \geq 0$. Collecting all rows $i = 1 \dots r$ of P, it follows that a *feasible* solution to $\mathbb{D}$ satisfies (10a) and (10b) with $H_{ij} = \mu_j^i / (1 + \sum_{k=1}^{r} \mu_i^k) \geq 0$, $j = 1 \dots r$, $M_{ij} = \eta_j^i / (1 + \sum_{k=1}^{r} \mu_i^k) \geq 0$, $j = 1 \dots p$; therefore, H and M are non-negative matrices.

Now we study the *optimal* solution to $\mathbb{D}$. Since the primal problem $\mathbb{P}$ is known to be feasible, and we assumed $\mathbb{D}$ to be feasible, then by strong duality (which holds regardless of the feasibility of $\mathbb{D}$) the optimal solutions to $\mathbb{P}$ and $\mathbb{D}$ are attained and equal in objective value. So, applying complementary slackness to (9a) and (9c),

$$\begin{aligned} \sum_{i=1}^{r} \lambda_i^* (c_i^* - P_i A \xi^{i*}) &= 0 \\ \sum_{i=1}^{r} \nu_i^* (d_i^* - P_i \omega^{i*}) &= 0. \end{aligned}$$

where * denotes a variable in the optimal solution. Since each term in these sums is non-positive,

$$\begin{aligned} \lambda_i^* (c_i^* - P_i A \xi^{i*}) &= 0, \\ \nu_i^* (d_i^* - P_i \omega^{i*}) &= 0, \end{aligned}$$

for $i = 1, \dots, r$. Morever, because (by (13a) and (13b)) $\lambda_i^* > 0$ and $\nu_i^* > 0$, then, at the optimum,

$$\begin{aligned} c_i^* &= P_i A \xi^{i*}, \\ d_i^* &= P_i \omega^{i*}. \end{aligned}$$

Hence,

$$\begin{aligned} c_i^* + d_i^* &= P_i A \xi^{i*} + P_i \omega^{i*} \\ &= \frac{(\mu^{i*})^\top}{1 + \sum_{k=1}^{r} \mu_i^{k*}} P \xi^{i*} + \frac{(\eta^{i*})^\top}{1 + \sum_{k=1}^{r} \mu_i^{k*}} F \omega^{i*} \\ &= \frac{1}{1 + \sum_{k=1}^{r} \mu_i^{k*}} \left(\sum_{j=1}^{r} \mu_j^{i*} P_j \xi^{i*} + \sum_{j=1}^{p} \eta_j^{i*} F_j \omega^{i*} \right). \end{aligned} \tag{14}$$

Now consider the inequality (9d). Suppose $F\omega^{i*} < g$ for some $i \in \{1, \dots, r\}$ (*i.e.*, $F_j \omega^{i*} < g_j$ for all $j = 1 \dots p$). Complementary slackness implies that $\eta^{i*} = 0$ which in turn implies (from (13d), assuming that P_i is not trivially all zeros) that $\nu_i^* = 0$; but $\nu_i^* \geq 1$ by (13b), which is a contradiction. Hence, there must exist a subset $\mathcal{K} \subset \{1, \dots, p\}$ of active constraints for which $F_k \omega^{i*} = g_k$ for $k \in \mathcal{K}$. But for any $j \notin \mathcal{K}$, $\eta_j^{i*} = 0$.

Similarly, consider the inequality (9b). By complementary slackness, if $P\xi^{i*} < c^* + d^*$ then $\mu^{i*} = 0$. By (13c), this implies that $A^\top P_i^\top \lambda_i^* = 0$. There are two cases to consider: (i) if any elements of $P_i A$ are non-zero then $\lambda_i^* = 0$; (ii) if $P_i A = 0$ then $\lambda_i^* > 0$ is permitted. We leave case (ii) for now and consider (i) first. $\lambda_i^* = 0$ contradicts (13a), which requires $\lambda_i^* \geq 1$. Hence, there must exist a subset $\mathcal{J} \subset \{1, \dots, r\}$ of active constraints for which $P_j \xi^{i*} = c_j^* + d_j^*$ for $j \in \mathcal{J}$. But for any $k \notin \mathcal{J}$, $\mu_k^{i*} = 0$. As a consequence of the preceding arguments, (14) may be re-written as

$$\begin{aligned} c_i^* + d_i^* &= \frac{1}{1 + \sum_{k=1}^{r} \mu_i^{k*}} \left(\sum_{j \in \mathcal{J}} \mu_j^{i*} P_j \xi^{i*} + \sum_{k \in \mathcal{K}} \eta_k^{i*} F_k \omega^{i*} \right) \\ &= H_i (c^* + d^*) + M_i g \end{aligned}$$

where H_i is the ith row of H and M_i is the ith row of M. The second line follows because $H_{ij} = 0$ for $j \notin \mathcal{J}$ and $M_{ik} = 0$ for $k \notin \mathcal{K}$, while $P_j \xi^{i*} = c_j^* + d_j^*$ for $j \in \mathcal{J}$ and $F_k \omega^{i*} = g_k$ for $k \in \mathcal{K}$.

Now case (ii). If $P_i A = 0$ then $c_i^* = 0$. Moreover, $\lambda_i^* \geq 1$ is permitted, so the same contradiction is not constructed. Then, however, either $P\xi^{i*} < d^*$, hence $\mu^{i*} = 0$, or $P\xi_j^{*i} = d_j^*$, with $\mu_j^{i*} \geq 0$, for $j \in \mathcal{J} \subset \{1, \dots, r\}$, and $\mu_k^{i*} = 0$ for all $k \notin \mathcal{J}$. Either way,

$$c_i^* + d_i^* = H_i (c^* + d^*) + M_i g$$

as before.

Finally, collecting all rows $i = 1 \dots r$,

$$H(c^* + d^*) + Mg = c^* + d^*$$

which is the third condition in (10). This establishes that the solution to $\mathbb{P}$, if it is attainable, satisfies the conditions (10) for it to be the fixed-point solution. It is attainable if and only if the dual problem $\mathbb{D}$ is feasible. Therefore, it remains to show that the $\mathbb{D}$ is feasible if and only if Assumption 4 holds.

First, necessity of Assumption 4. Suppose Assumption 4 is not satisfied, but the dual $\mathbb{D}$ is feasible. By definition, if Assumption 4 is not satisfied then for the chosen P and system $(A, \mathbb{W})$ there does not exist a q satisfying the functional inequality (6). Therefore, there exists no q^* satisfying the functional equation and, by Proposition 1, the conditions (10). However, the attainable optimal solution to $\mathbb{P}$ and $\mathbb{D}$ satisfies (10) with non-negative H and M, as has been shown. Therefore, we have a contradiction, and conclude the optimal solution is attainable, and $\mathbb{D}$ is feasible, only if Assumption 4 holds.

Second, sufficiency of Assumption 4. Writing the primal constraints (9) in the form $\mathbf{A}\mathbf{x} \le \mathbf{b}$, where $\mathbf{x}$ is the vector of primal decision variables, it follows that the dual constraints (13) may be written in the form $\mathbf{A}^\top \mathbf{y} = \mathbf{c}$, $\mathbf{y} \ge 0$, where $\mathbf{y}$ is the vector of dual variables and $\mathbf{c}$ is the coefficients vector in the vectorized form, $\mathbf{c}^\top \mathbf{x}$, of the objective function (8). By Farkas' Lemma, a feasible solution to $\mathbf{A}^\top \mathbf{y} = \mathbf{c}$, $\mathbf{y} \ge 0$ exists if and only if $\mathbf{A}\mathbf{x} \ge 0 \implies \mathbf{c}^\top \mathbf{x} \ge 0$. Hence, we aim to show that, if Assumption 4 holds, then for all $\mathbf{x}$ satisfying $\mathbf{A}\mathbf{x} \ge 0$ we also have $\mathbf{c}^\top \mathbf{x} \ge 0$. The system $\mathbf{A}\mathbf{x} \ge 0$ may be written in terms of the primal variables as

$$
\begin{aligned}
c_i &\ge P_i A \xi^i \\
P\xi^i &\ge c + d \\
d_i &\ge P_i \omega^i \\
F\omega^i &\ge 0
\end{aligned}
$$

for $i = 1 \ldots r$. If Assumption 4 holds, then $H_i P = P_i A$ and $M_i F = P_i$ for some non-negative H_i and M_i. Substituting into the system $\mathbf{A}\mathbf{x} \ge 0$,

$$
\begin{aligned}
c_i &\ge H_i P \xi^i \\
P\xi^i &\ge c + d \\
d_i &\ge M_i F \omega^i \\
F\omega^i &\ge 0,
\end{aligned}
$$

from which it follows that $d_i \ge 0$ and $c_i \ge H_i(c+d)$, hence $c \ge Hc$. But we also have that, if Assumption (4) holds, then there exists some $q \in \mathbb{R}^r_+$ for which $0 \le Hq \le Hq + Mg \le q$. Applying recursively, $0 \le H^n q \le Hq \le q$, $H^n \ge 0$ because $H \ge 0$, and therefore $\lim_{n\to\infty} H^n \ge 0$, if the limit exists. In fact, because $HP = PA$, the nullspace of P is A-invariant and P has rank n, then the eigenvalues of H are are subset of the eigenvalues of A; hence, $\lim_{n\to\infty} H^n = 0$ because $\rho(A) < 1$. Then $c \ge Hc \ge \lim_{n\to\infty} H^n c = 0$. Consequently, $\mathbf{c}^\top \mathbf{x} = \sum_{i=1}^r c_i + d_i \ge 0$. Therefore, $\mathbb{D}$ is feasible if Assumption 4 holds. ■

IV. Examples

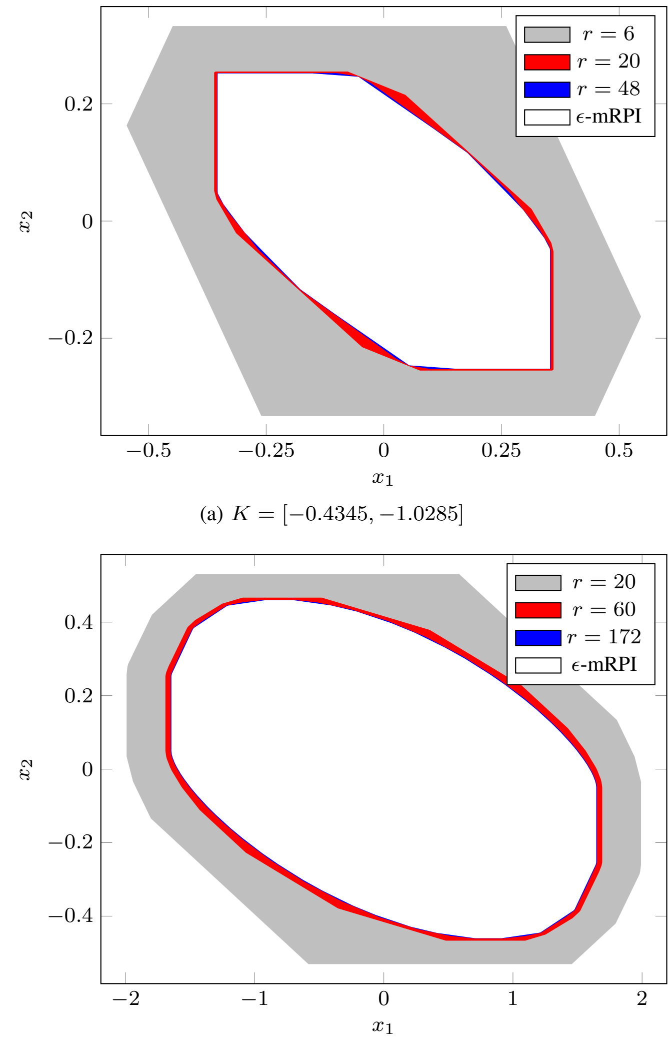
We consider the non-autonomous system

$$
x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u + w, \tag{15}
$$

with $w \in \mathbb{W} = \{w \in \mathbb{R}^2 : \|w\|_\infty \le 0.1\}$. This is converted to the linear autonomous system (1) by use of a state feedback control law $u = Kx$.

A. Computation of (P, r)-mRPI from selected inequalities

First, we use the feedback matrix $K = [-0.4345, -1.0285]$, corresponding to the infinite-horizon LQR solution with cost matrices $Q = I$ and $R = 1$. Note that in this example the mRPI set is not finitely determined, and therefore an approximation is required.



(a) $K = [-0.4345, -1.0285]$

(b) $K = [-0.0796, -0.4068]$

Fig. 1. Comparison of (P, r)-mRPI and ϵ-mRPI sets for the system (15) with different feedback matrices K.

Figure 1(a) shows the (P, r)-mRPI sets generated from $r = 6$, 20 and 48 inequalities, wherein the ith row of P is designed as

$$
P_i = \left[\sin\left(\tfrac{2\pi(i-1)}{r}\right) \quad \cos\left(\tfrac{2\pi(i-1)}{r}\right) \right], \tag{16}
$$

i.e., so that $Px \le 1$ is the r-sided regular polygon. Also shown is the outer approximation to the mRPI, which is itself RPI, computed using the algorithm of [3] and a tolerance $\epsilon = 10^{-4}$. This set, termed the ϵ-mRPI set, is defined by 48 non-redundant inequalities.

Figure 1(b) shows a similar comparison using $K = [-0.0796, -0.4068]$, obtained as the LQR solution with $Q = I$ and $R = 100$. Now the ϵ-mRPI ($\epsilon = 10^{-4}$) comprises 172 non-redundant inequalities, while the (P, r)-mRPI sets computed using the proposed method are shown for $r = 20$, 60 and 172, again using (16) for P.

Table I compares the computation times and number of operations for computing the (P, r)-mRPI with those for obtaining the ϵ-outer approximation using the algorithm of [3]. For the latter, the Multi-Parametric Toolbox v3.0 [10] was used for set operations, with CPLEX 12.6 as the LP solver for support function calculations. For the (P, r)-mRPI set computations (*i.e.*, solving the LP), CPLEX 12.6 was used as the LP solver. The platform was a 64-bit Intel Core i7-2600 at

3.40 GHz with 8 GB RAM. Times are reported as the mean elapsed time over 100 runs.

Comparison was also made with the iterative procedure of [6] for computing the (P, r)-mRPI. The iterative procedure is

$$q_{k+1} = c(q_k) + d \text{ with } q_0 = 0$$

for which $q_k \to q^*$ as $k \to \infty$. This was implemented in MATLAB using the MPT v3.0 [10] for support function calculations (with CPLEX 12.6 as underlying LP solver). The function $c(\cdot)$ was evaluated element by element at each iteration; that is, as r separate support function calculations. For the simplest case considered of $K = [-0.4345, -1.0285]$ and $r = 6$ (the first row of Table I), the number of iterations to convergence (of $|q_{k+1} - q_k|$ to within a chosen tolerance of 10^{-6}) was 34, which included the solving of 238 LPs and took a mean total time of 1.7 seconds. At the other end of the scale, for the most difficult problem considered ($K = [-0.0796, -0.4068]$ and $r = 172$), the iterative procedure required 70 iterations, the solving of over 12000 LPs, and took, on average, 90 seconds. While these times can, of course, be shortened by using optimized code, the intention here is merely to report the times obtained using standard computational tools.

B. Re-computing the (P, r)-mRPI set given P

An interesting use of the method is when an RPI set for the system is available, but is desired to be re-computed or modified; for example, if the disturbance set changes. Potential applications of this include "plug-and-play" tube-based approaches to distributed MPC, wherein a dynamic subsystems' disturbance set evolves over time as other subsystems are added to and removed from the system of coupled subsystems [7]; in such situations, one needs a new RPI set that takes into account the latest disturbance set. One could re-compute from scratch a new RPI set, but it may be advantageous, in the interests of computation time, to modify an existing RPI set instead. In the context of the approach proposed here, the P matrix of the known RPI set may be used as a basis for computing the new RPI set.

For the system (15) with $K = [-0.4345, -1.0285]$ and $\mathbb{W} = \{w \in \mathbb{R}^2 : |w|_\infty \leq 0.1\}$, the P matrix is obtained as that of the ϵ-mRPI set. For $\epsilon = 10^{-4}$, this comprises 48 inequalities. Now suppose the disturbance set enlarges to

$$\mathbb{W} = \left\{ w \in \mathbb{R}^2 : \begin{bmatrix} -0.3 \\ -0.4 \end{bmatrix} \leq w \leq \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} \right\}$$

Figure 2 shows the (P, r)-mRPI and ϵ-mRPI sets based on the new disturbance set, using for the former the P matrix from the old ϵ-mRPI set. The (P, r)-mRPI set, computed in 0.03 s using the proposed method, is visually indistinguishable from the new ϵ-mRPI set.

TABLE I
COMPARISON OF COMPUTATION TIMES AND OPERATIONS FOR (P, r)-MRPI AND ϵ-MRPI SETS.

	LPs solved	Minkowski sums	Mean time (s)
$K = [-0.4345, -1.0285]$			
$r = 6$	1	0	0.005
$r = 20$	1	0	0.007
$r = 48$	1	0	0.019
ϵ-mRPI [3] ($r = 48$)	369	11	2.9
$K = [-0.0796, -0.4068]$			
$r = 20$	1	0	0.008
$r = 60$	1	0	0.036
$r = 172$	1	0	0.30
ϵ-mRPI [3] ($r = 172$)	3250	42	25

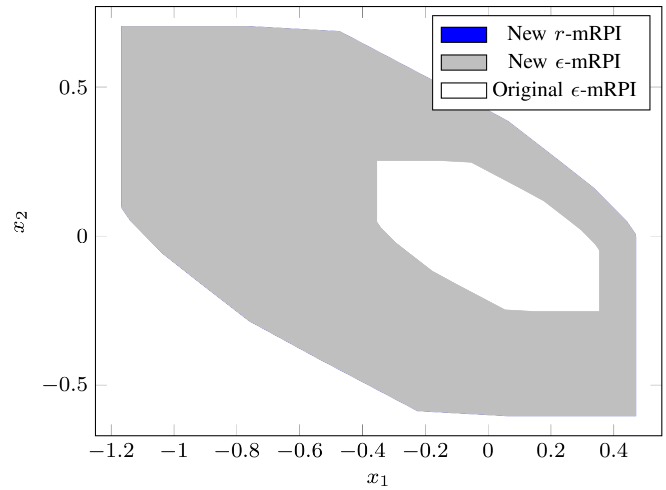


Fig. 2. Comparison of (P, r)-mRPI and ϵ-mRPI sets for the system (15) with $K = [-0.4345, -1.0285]$ and different disturbance sets.

V. CONCLUSIONS

A procedure for computing a polytopic robust positively invariant set for a linear uncertain system has been presented. The method, which requires the solution of a single LP, obtains the an RPI set that is the smallest among those represented by a finite number inequalities with pre-defined normal vectors, and offers an alternative method of computation to the iterative procedure of [6]. Existence and uniqueness of a solution has been established. The practicality of the approach has been demonstrated via examples.

ACKNOWLEDGMENTS

The author would like to thank the anonymous reviewers for their constructive and useful comments.

REFERENCES

[1] I. Kolmanovsky and E. G. Gilbert, "Theory and computation of disturbance invariant sets for discrete-time linear systems," *Mathematical Problems in Engineering*, vol. 4, pp. 317–367, 1998.
[2] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.
[3] S. V. Raković, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne, "Invariant approximations of the minimal robust positively invariant set," *IEEE Transactions on Automatic Control*, vol. 50, no. 4, pp. 406–410, April 2005.
[4] S. V. Raković, E. C. Kerrigan, D. Q. Mayne, and K. I. Kouramas, "Optimized robust control invariance for linear discrete-time systems: Theoretical foundations," *Automatica*, vol. 43, no. 5, pp. 831–841, May 2007.
[5] D. Q. Mayne, M. M. Seron, and S. V. Raković, "Robust model predictive control of constrained linear systems with bounded disturbances," *Automatica*, vol. 41, no. 2, pp. 219–224, 2005.
[6] S. V. Raković, B. Kouvaritakis, and M. Cannon, "Equi-normalization and exact scaling dynamics in homethetic tube model predictive control," *Systems & Control Letters*, vol. 62, no. 2, pp. 209–217, February 2013.
[7] S. Riverso, M. Farina, and G. Ferrari-Trecate, "Plug-and-play model predictive control based on robust control invariant sets," *Automatica*, vol. 50, pp. 2179–2186, 2014.
[8] J.-C. Hennet and E. B. Castelan, "Invariant polyhedra and control of large-scale systems," in *Proceedings of the 9th IFAC Symposium on Large Scale Systems: Theory and Applications*, 2001.
[9] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, ser. Encyclopedia of Mathematics and its Applications, G.-C. Rota, Ed. Cambridge University Press, 1993, vol. 44.
[10] M. Herceg, M. Kvasnica, C. N. Jones, and M. Morari, "Multi-Parametric Toolbox 3.0," in *Proc. of the European Control Conference*, Zürich, Switzerland, July 17–19 2013, pp. 502–510, http://control.ee.ethz.ch/~mpt.