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Optimal plans and timing under additive transformations to rewards

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Abstract

The nature and role of additive transformations to rewards are elucidated for a general class of deterministic, nonautonomous, optimal control problems with many state and control variables. Conditions relating to the optimal choice of initial and terminal times and initial and terminal values of state variables are identified such that additive transformations affect optimal plans. General comparative static results are derived and the framework is extended to cover two common classes of stochastic control problems. Three applications are presented: the canonical adjustment cost model of a firm, a stochastic extension of an irreversible pollution accumulation problem with regime switching and an extension of a lifecycle model of retirement in which an agent's retirement wealth evolves stochastically.

JEL classifications: C61, D92, J26

1 Introduction

Dynamic problems in economics in which an agent must choose an optimal initial, terminal or switching time in addition to the optimal paths of control variables – examples include lifecycle models of retirement, the optimal time at which to choose a new policy regime and the optimal time at which to terminate extraction of a nonrenewable resource – are commonplace. Also commonplace are problems in which an agent's rewards are subject to additive transformations, capturing such phenomena as a firm's fixed operating costs, the disutility of employment, the cost of a regime switch and the presence of shocks to future flows of wealth. Despite this, a general framework for studying deterministic control problems in the presence of additive transformations to rewards has yet to be established. Perhaps this is because of a mistaken belief acquired from static optimization theory that such transformations have no bearing on the solution. Surprisingly, such transformations do affect behaviour in dynamic optimization problems under conditions which are prevalent in economics.

Accordingly, this paper derives the conditions under which additive transformations to rewards affect optimal plans for a general class of deterministic, nonautonomous, optimal control problems with many state and control variables. The class varies according to the freedom given to the decision-maker to choose the initial and terminal times of the planning horizon and the initial and terminal values of the state variables. Salvage functions are included and a general and comprehensive set of comparative statics results is established.

Although the framework uses methods from deterministic optimal control theory, it is extended to cover two classes of stochastic control problems in which the variance of idiosyncratic shocks to a state equation appears additively in an agent's bequest function at the time of a regime switch. It is intended that the set of propositions contained herein may be referred to by researchers solving the aforementioned problems, circumventing the need for them to derive their own closed-form solutions or carry out analysis by simulation in cases where theoretical results are unambiguous, highlighting the importance of closed-form solutions and simulations when they are not. Researchers dealing with the latter scenario are referred to the general methods set out in Caputo and Wilen (1995); researchers dealing with the important case of comparative statics for discount rates, in both deterministic and stochastic settings, are referred to Quah and Strulovici (2013).

There is a wide range of literature to which the methods may be applied, as the three examples of section 5 illustrate. In the canonical adjustment cost model of a firm, it is shown how a flow of sunk fixed costs affects a firm's shut-down decision. In a stochastic extension of Tahvonen and Withagen's (1996) pollution accumulation problem with regime switching, it is shown how uncertainty about the critical threshold of the pollution stock affects the optimal timing of a move to irreversible pollution accumulation. In Prettner and Canning's (2014) lifecycle model of retirement, the evolution of retirement income is subjected to idiosyncratic shocks. Other areas for fruitful application of the methods include optimal regime switching and technology adoption decisions (such as in the models of Boucekkine et al. (2004), Boucekkine et al. (2013), Valente (2011) and Grass et al. (2012)) and optimal workplace reorganization (Valleé and Moreno-Galbis (2011)).

2 Background

One of the fundamental results in the atemporal theory of a firm states that, once a profit-maximizing firm has decided to produce, a change in any type of fixed cost does not affect the optimal mix of factors of production, nor the profit-maximizing rate of output. Avoidable fixed costs do, however, impact a firm's decision about when to shut-down, whereas sunk fixed costs do not (Besanko and Brauetigam 2013).

Contributions to various strands of the deterministic control literature have shown these conclusions to be in need of modification. For example, Farmer (1997) modelled environmental mandates as fixed and variable costs and showed how optimal production decisions and closure dates were affected by the nature of the particular mandate. In the natural resource literature, Schmalensee (1976) solved what is more or less the prototypical nonrenewable resource extraction problem with the addition of a flow of avoidable fixed costs. The cost is fixed because it is independent of the rate of production for positive rates; it is avoidable because it falls discontinuously to zero when the rate of production is zero. Schmalensee showed that the optimal length of a firm's planning horizon decreases as the flow of avoidable fixed costs increase. In a similar vein, Siebert (1983) demonstrated that an increase in the flow of sunk fixed costs – as opposed to avoidable fixed costs – decreases the optimal length of the planning horizon. Lewis et al. (1979) introduced a flow of avoidable fixed costs into a nonrenewable resource extraction problem and showed that, under a certain set of assumptions, a monopolist owner of a fixed nonrenewable resource stock extracts the stock at a faster rate than is socially optimal.

Models of the optimal time at which to switch policy regimes have also considered additive fixed costs. For example, Tomiyama (1985) and Makris (2001) derived necessary and sufficient conditions for the optimal switching time in finite and infinite horizon control problems, respectively, and Makris included a fixed switching cost which was a function of the state variable at the time of switching. Literature on the optimal retirement decision (Prettner and Canning 2014, Rogerson and Wallenius 2013) considered the disutility of employment (an additive fixed cost). Valente (2011) considered the role of fixed switching costs in a model of endogenous growth and backstop technology adoption.

Despite these developments, a general set of results for the effect of additive transformations on optimal decisions in deterministic control models has yet to be established. The framework presented herein applies to a general class of deterministic, nonautonomous, optimal control problems with many state and control variables and so nests all of the above models as special cases. The propositions are applicable to deterministic control models involving the choice of initial or terminal times and/or the choice of the initial or terminal values of the state variables under additive transformations to rewards. Results are not limited to problems in deterministic optimal control, however. In section 4 they are extended to cover two classes of stochastic optimal control problems, utilizing the method of backward induction to obtain expressions for stage two value functions which include the variance of idiosyncratic shocks from a second-stage state equation, and which then serve as the terminal salvage functions for the first-stage problem.

3 Theory

3.1 Additive transformations

Consider the class of deterministic optimal control problems with M control and N state variables defined by:

$$V(T,\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot),\mathbf{x}(0),\mathbf{x}(T)} \left\{ \int_0^T [f(t,\mathbf{x}(t),\mathbf{u}(t))+\varphi]\,\mathrm{e}^{-rt}\mathrm{d}t + \mathrm{e}^{-rT}[S^1(\mathbf{x}(T))+\varphi_T] + [S^0(\mathbf{x}(0))+\varphi_0] \right\} \tag{1}$$

$$\text{s. t. } \dot{\mathbf{x}}(t) = \mathbf{g}(t,\mathbf{x}(t),\mathbf{u}(t)), \tag{2}$$

where $T \in \mathbb{R}_{++}$ is the terminal time of the planning horizon, assumed fixed in problem (1), $\mathbf{u}(t) \in \mathbb{R}^M$ is the value of the control vector at time t, $\mathbf{x}(t) \in \mathbb{R}^N$ is the value of the state vector at time t, $\mathbf{g}(\cdot) \stackrel{\text{def}}{=} (g_1(\cdot), g_2(\cdot), \ldots, g_N(\cdot))'$, $r \in \mathbb{R}_{++}$ is a discount rate and $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\varphi, \varphi_0, \varphi_T, r)$. The parameters $\varphi \in \mathbb{R}$, $\varphi_0 \in \mathbb{R}$ and $\varphi_T \in \mathbb{R}$ represent additive transformations to the functions $f(\cdot)$, $S^0(\cdot)$ and $S^1(\cdot)$, respectively. In models of the firm, where $f(\cdot)$ can be thought of as profit flow and $S^0(\cdot)$ and $S^1(\cdot)$ as salvage value functions, $\varphi < 0$ could be a flow of sunk fixed costs incurred at every instant of the planning horizon, $\varphi_0 < 0$ and $\varphi_T < 0$ could be the one-time sunk fixed costs incurred at the initial time and terminal time, respectively. Positive values could be, respectively, flows of a subsidy, a start-up grant and a grant to incentivize cessation of trading. In models of the consumer, $f(\cdot)$ could be an instantaneous utility function and $\varphi < 0$ the instantaneous disutility of employment (as in Prettner and Canning 2014). In the lifecycle model of retirement contemplated in section 5.3, φ_T includes the variance of idiosyncratic shocks to income that occur once the agent is retired. Observe that the initial value of the state vector, $\mathbf{x}(0)$, as well its terminal value, $\mathbf{x}(T)$, are decision variables in the above control problem. Several common perturbations of this problem are considered in what follows.

As far as notation is concerned, the following more or less standard conventions are employed: (i) $\mathbf{x}(t)$, $\mathbf{u}(t)$ and the vector of costate variables $\boldsymbol{\lambda}(t)$ (defined below) are column vectors; (ii) the derivative of a scalar-valued function with respect to a column vector is a row vector; (iii) the derivative of a vector-valued function with respect to a vector is a Jacobian matrix, with number of rows equal to the number of functions being differentiated and number of columns equal to the number of elements in the vector that the derivative is taken with respect to; (iv) the Hessian matrix of a scalar-valued function is indicated by two subscripts on the said function, the order of which is $P \times Q$, where P is the order of the first subscript and Q the second, and (v) the symbol '$'$' denotes transposition.

The ensuing assumptions are imposed on the optimal control problem defined by Eqs. (1) and (2) and its variants, and are explained subsequently:

(A1) The functions $f(\cdot) : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ and $g^n(\cdot) : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$, $n = 1, 2, \ldots, N$, are $\mathcal{C}^{(0)}$ in t and $\mathcal{C}^{(1)}$ in $(\mathbf{x}, \mathbf{u})$ on their domains.

(A2) The functions $S^0(\cdot) : \mathbb{R}^N \to \mathbb{R}$ and $S^1(\cdot) : \mathbb{R}^N \to \mathbb{R}$ are $\mathcal{C}^{(2)}$ on their domains.

(A3) There exists a $\mathcal{C}^{(1)}$ optimal solution to each of the control problems below for all values of the parameters in some open set.

(A4) The optimal value functions in each of the control problems below are locally $\mathcal{C}^{(2)}$.

The assumed differentiability in assumptions (A1) and (A2) is useful in simplifying the exposition, as it permits the use of the differential calculus in stating the necessary conditions. These assumptions also help focus attention on the economic content of the results rather than on mathematical technicalities. In addition, the smoothness suppositions in (A2) and (A4) are necessary because a differential comparative statics analysis is carried out. Given that the class of optimal control problems under consideration is quite general, assumption (A3) is natural. Alternatively, one could assume that certain curvature conditions hold on the underlying functions in order to invoke a sufficiency theorem. In the important case when the terminal time is a decision variable, the sufficiency conditions are rather involved, as can be seen in Theorem 6.17 of Seierstad and Sydsæter (1987). In any case, the problem with such an approach is that it imposes conditions on the control problem that go beyond those needed for the discovery of intrinsic results, and hence is avoided by employing assumption (A3).

Define the present-value Hamiltonian as:

$$H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) \overset{\text{def}}{=} [\,f(\,t,\,\mathbf{x},\,\mathbf{u}\,)\,+\,\varphi\,]\,\mathrm{e}^{-rt}\,+\,\boldsymbol{\lambda}'\,\mathbf{g}(\,t,\,\mathbf{x},\,\mathbf{u}\,), \tag{3}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^N$ is the present-value costate vector. Given assumptions (A1)–(A3) and the absence of constraints in the control problem defined by Eqs. (1) and (2), it follows from Theorem 10.3 of Caputo (2005) that an optimal solution necessarily satisfies:

$$H_{\mathbf{u}}(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) = f_{\mathbf{u}}(t,\mathbf{x},\mathbf{u})\mathrm{e}^{-rt} + \boldsymbol{\lambda}'\mathbf{g}_{\mathbf{u}}(t,\mathbf{x},\mathbf{u}) = \mathbf{0}'_M, \tag{4a}$$

$$\dot{\boldsymbol{\lambda}}' = -H_{\mathbf{x}}(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) = -f_{\mathbf{x}}(t,\mathbf{x},\mathbf{u})\mathrm{e}^{-rt} - \boldsymbol{\lambda}'\mathbf{g}_{\mathbf{x}}(t,\mathbf{x},\mathbf{u}), \tag{4b}$$

$$\dot{\mathbf{x}} = H_{\boldsymbol{\lambda}}(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda})' = \mathbf{g}(t,\mathbf{x},\mathbf{u}), \tag{4c}$$

$$\boldsymbol{\lambda}(0)' = -S^0_{\mathbf{x}}(\mathbf{x}(0)), \tag{4d}$$

$$\boldsymbol{\lambda}(T)' = \mathrm{e}^{-rT}S^1_{\mathbf{x}}(\mathbf{x}(T)), \tag{4e}$$

where $\mathbf{0}_M$ is the null column vector in $\mathbb{R}^M$. Because $(\varphi,\varphi_0,\varphi_T)$ do not enter Eqs. (4a)–(4e), optimal time-paths for the state, control and costate variables in the problem defined by Eqs. (1) and (2) do not depend on $(\varphi,\varphi_0,\varphi_T)$. This conclusion can also be deduced by rewriting the objective functional in Eq. (1) in the equivalent form

$$\int_0^T [f(t,\mathbf{x}(t),\mathbf{u}(t))]\,\mathrm{e}^{-rt}\mathrm{d}t + \varphi r^{-1}[1-\mathrm{e}^{-rT}] + \mathrm{e}^{-rT}\left[S^1(\mathrm{x}(T))+\varphi_T\right] + \left[S^0(\mathbf{x}(0))+\varphi_0\right]. \tag{5}$$

Eq. (5) shows that $(\varphi,\varphi_0,\varphi_T)$ do not interact with the state vector, control vector, or the initial and terminal values of the state vector. Hence an optimal solution to the problem defined by Eqs. (1) and (2) cannot depend on $(\varphi,\varphi_0,\varphi_T)$. Indeed, all $(\varphi,\varphi_0,\varphi_T)$ do is change the value of $V(\cdot)$ by $\varphi r^{-1}[1-\mathrm{e}^{-rT}] + \varphi_0 + \mathrm{e}^{-rT}\varphi_T$. These results are summarized in the ensuing proposition.

Proposition 1 (Additive transformations do not matter) *Under assumptions (A1) – (A3), an optimal solution for* $(\mathbf{x}(\cdot),\mathbf{u}(\cdot))$ *and associated costate* $\boldsymbol{\lambda}(\cdot)$ *of the control problem defined by Eqs. (1) and (2) is independent of* $(\varphi,\varphi_0,\varphi_T)$*, while the value of the optimal value function* $V(\cdot)$ *is changed by the fixed amount* $\varphi r^{-1}[1-e^{-rT}] + \varphi_0 + e^{-rT}\varphi_T$*.*

The economic interpretation of Proposition 1 is straightforward. Consider it in the context of the theory of a firm, where the control vector consists of variable inputs and investment rates in the capital stocks, and the capital stocks are represented by the state variables. Sunk fixed costs are represented by $\varphi \in \mathbb{R}_{-}, \varphi_0 \in \mathbb{R}_{-}$ and $\varphi_T \in \mathbb{R}_{-}$, as noted earlier. Proposition 1 asserts that the optimal time-paths of the inputs, investment rates, the capital stocks and their shadow prices are not affected by any of these three forms of sunk fixed costs. Indeed, the only thing affected by sunk fixed costs is the firm's wealth. Note that these results are the intertemporal analogue to the prototypical result discussed in section 2, scilicet, an atemporal profit-maximizing firm's rate of output is independent of its sunk fixed costs once it has decided to produce.

Proposition 1 continues to hold under common alternative specifications. For example, if the initial value of the state vector $\mathbf{x}(0)$ is fixed at $\mathbf{x}_0$, that is, $\mathbf{x}(0) = \mathbf{x}_0$, and the terminal value of the state vector $\mathbf{x}(T)$ is similarly fixed at $\mathbf{x}_T$, that is, $\mathbf{x}(T) = \mathbf{x}_T$, then by Theorem 6.1 of Caputo (2005), the necessary transversality conditions (4d) and (4e) are replaced by $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(T) = \mathbf{x}_T$, respectively. As $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(T) = \mathbf{x}_T$ are independent of $(\varphi, \varphi_0, \varphi_T)$, just like the transversality condition in Eqs. (4d) and (4e), Proposition 1 is unaffected. This conclusion also holds if either $\mathbf{x}(0) = \mathbf{x}_0$ or $\mathbf{x}(T) = \mathbf{x}_T$ holds, for the reason just provided.

Now consider the case in which the planning horizon is infinite in length, that is, $T \to +\infty$, thereby implying that $S^1(\cdot) \equiv 0$ and $\varphi_T \equiv 0$. The terminal conditions on the state variables in this case are taken to be $\lim_{t\to+\infty} x_i(t) = x_i^s$, $i = 1, 2, \ldots, n_1$, $\lim_{t\to+\infty} x_i(t) \geq x_i^s$, $i = n_1 + 1, \ldots, n_2$, and no conditions on $x_i(t)$ as $t \to +\infty$, $i = n_2 + 1, \ldots, N$. By Theorem 14.3 of Caputo (2005), the necessary conditions are still given by Eqs. (4a)–(4d), while the necessary transversality condition (4e) no longer applies, nor do any, in general. Nonetheless, the applicable necessary conditions remain independent of (φ, φ_0) and hence Proposition 1 continues to hold. What is more, the same conclusion holds whether or not $\mathbf{x}(0)$ is free or fixed at $\mathbf{x}_0$, as noted in the preceding paragraph. As these results are sufficiently important, and will be referred to later, they are recorded in the following corollary.

Corollary 1 *Under assumptions (A1)–(A3), the conclusions of Proposition 1 continue to hold for the optimal control problem defined by Eqs. (1) and (2) if either of the following changes are made:*

1. *either or both of the initial and terminal values of the state vector are fixed, or,*

2. *the planning horizon is infinite in length, that is,* $T \to +\infty$, *the terminal conditions on the state variables are* $\lim_{t\to+\infty} x_i(t) = x_i^s$, $i = 1, 2, \ldots, n_1$, $\lim_{t\to+\infty} x_i(t) \geq x_i^s$, $i =$

$n_1+1,\ldots,n_2$, and no conditions on $x_i(t)$ as $t \to +\infty$, $i = n_2+1,\ldots,N$, and the initial value of the state vector is free or fixed.

Consider now the version of the control problem given by Eqs. (1) and (2) in which the terminal time of the planning horizon, T, is a decision variable

$$V^*(\boldsymbol{\beta}) \overset{\text{def}}{=} \max_{\mathbf{u}(\cdot),\mathbf{x}(0),\mathbf{x}(T),T} \left\{ \int_0^T [f(t,\mathbf{x}(t),\mathbf{u}(t)) + \varphi]\, \mathrm{e}^{-rt}\mathrm{d}t + \mathrm{e}^{-rT}[S^1(\mathbf{x}(T)) + \varphi_T] + [S^0(\mathbf{x}(0)) + \varphi_0] \right\}, \tag{6}$$

subject to Eq. (2). By Theorem 10.3 of Caputo (2005), assuming that $T > 0$, the necessary conditions for problem (6) subject to Eq. (2) comprise Eqs. (4a)–(4e) together with the transversality condition $H(T,\mathbf{x}(T),\mathbf{u}(T),\boldsymbol{\lambda}(T)) - r\mathrm{e}^{-rT}[S^1(\mathbf{x}(T)) + \varphi_T] = 0$. The latter may be equivalently written as

$$[f(T,\,\mathbf{x}(T),\,\mathbf{u}(T)) + \varphi]\,\mathrm{e}^{-rT} + \boldsymbol{\lambda}(T)'\,\mathbf{g}(T,\,\mathbf{x}(T),\,\mathbf{u}(T)) - r\mathrm{e}^{-rT}\left[S^1(\mathbf{x}(T)) + \varphi_T\right] = 0. \tag{7}$$

Because Eq. (7) is a function of (φ, φ_T), so too is an optimal solution for the state, control and corresponding costate variables, together with the terminal time, of problem (6). This conclusion also follows directly from Eq. (5) when T is a decision variable, seeing as (φ, φ_T) interact with T. The parameter φ_0, however, does not appear in the necessary conditions in this case, and so an optimal solution of problem (6) is not a function of φ_0. The ensuing proposition summarizes these facts.

Proposition 2 (Additive transformations matter - free terminal time) *Under assumptions (A1)–(A3), an optimal solution for $(\mathbf{x}(\cdot),\mathbf{u}(\cdot))$ and associated costate $\boldsymbol{\lambda}(\cdot)$ of the control problem defined by Eqs. (6) and (2) is a function of (φ,φ_T) but not φ_0, as is the optimal length of the planning horizon. The optimal value function $V^*(\cdot)$ is a function of $(\varphi,\varphi_0,\varphi_T)$.*

The economic interpretation of Proposition 2 is again straightforward. Continuing the example of the theory of a firm, it asserts that a flow of sunk fixed costs φ, and a one-time sunk termination cost φ_T, affect the optimal time-paths of the variable inputs, investment rates, the capital stocks, the present value shadow prices of the capital stocks, as well as the time at which the firm shuts down, assuming that it is optimal for the firm to be in business, i.e., that the optimal value of T is positive. These results stand in stark contrast to the archetypal result discussed in section 2, namely, that if a price-taking, atemporal, profit-maximizing firm has decided to operate, sunk fixed costs do not affect its production or shut-down decisions. Fully akin to the prototypical case, sunk start-up costs φ_0 do not affect the input or output decisions made by a wealth-maximizing firm, nor when to shut down.

As was the case for Proposition 1, Proposition 2 also holds under a common perturbation of problem (6), as summarized by the following corollary.

Corollary 2 *Under assumptions (A1)–(A3), the conclusions of Proposition 2 continue to hold for the control problem defined by Eqs. (6) and (2) whether or not the initial or terminal values of the state vector are fixed or free.*

To confirm the veracity of Corollary 2, note that the necessary transversality condition given in Eq. (7) continues to be a necessary condition whether or not the initial and terminal values of the state vector are fixed or free, seeing as T is still a decision variable. Because Eq. (7) is a function of (φ, φ_T), but not φ_0, the result follows.

In order to provide a comprehensive account of the conditions under which additive transformations matter, it is worthwhile to end this section with a compact discussion of another situation in which they do. Thus far it has been shown that φ and φ_T matter when the terminal time is a decision variable. It is therefore natural to consider a symmetric situation, viz., that in which the initial time, say t_0, is a decision variable but the terminal time T is fixed. In this case, t_0 replaces 0 as the lower limit of integration in Eq. (1), $\exp\left[-r(t-t_0)\right]$ becomes the discount factor and the terminal salvage value is given by $\exp\left[-r(T-t_0)\right]\left[S^1(\mathbf{x}(T))+\varphi_T\right]$. By Theorem 10.3 of Caputo (2005), a necessary transversality condition is that the present value Hamiltonian evaluated at t_0 equals $r\exp\left[-r(T-t_0)\right]\left[S^1(\mathbf{x}(T))+\varphi_T\right]$. But as (φ, φ_T) appear in this necessary condition whereas φ_0 does not, it follows that an optimal solution for $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ and associated costate $\boldsymbol{\lambda}(\cdot)$ of the corresponding optimal control problem are functions of (φ, φ_T) but not φ_0, as is the optimal value of t_0. Moreover, because the present value Hamiltonian and the terminal salvage value function are part of the necessary transversality condition when the initial time, terminal time, or both, are decision variables, and whether or not the initial and terminal values of the state vector are free or fixed, the ensuing result holds.

Proposition 3 (Additive transformations matter - free initial time) *Under assumptions (A1)–(A3), if the initial time, terminal time, or both, are decision variables in the optimal control problem defined by Eqs. (1) and (2), then an optimal solution for* $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ *and associated costate* $\boldsymbol{\lambda}(\cdot)$ *is a function of* (φ, φ_T) *but not* φ_0*, as are the optimal initial and terminal values of time, whether or not the initial and terminal values of the state vector are fixed or free.*

3.2 Comparative statics of additive transformations

This section derives the comparative statics of the optimal values of the terminal time and the initial and terminal values of the state vector of problem (6) using the two-stage approach of Caputo and Wilen (1995).

Denote the optimal values of T, $\mathbf{x}_0$ and $\mathbf{x}_T$ in problem (6) as $(T^*(\boldsymbol{\beta}), \mathbf{x}_0^*(\boldsymbol{\beta}), \mathbf{x}_T^*(\boldsymbol{\beta}))$. The fixed endpoints and fixed time horizon optimal control problem corresponding to problem (6) is

$$\hat{V}(T, \mathbf{x}_0, \mathbf{x}_T, \varphi, r) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_0^T [f(t, \mathbf{x}(t), \mathbf{u}(t)) + \varphi] \, \mathrm{e}^{-rt} \mathrm{d}t, \tag{8}$$

subject to Eq. (2), $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(T) = \mathbf{x}_T$. By Corollary 1, a solution of problem (8) is not a function of $(\varphi, \varphi_0, \varphi_T)$. Furthermore, $\hat{V}(\cdot)$ does not depend on (φ_0, φ_T), as is clear from inspection of problem (8). Note that problem (8) is identical to problem (6) – the problem of interest – save for the facts that: (i) $(T, \mathbf{x}_0, \mathbf{x}_T)$ are parameters in problem (8) but decision variables in problem (6) and (ii) the expressions $[S^0(\mathbf{x}_0) + \varphi_0]$ and $\mathrm{e}^{-rT}[S^1(\mathbf{x}_T) + \varphi_T]$ are absent in problem (8). As a result, the optimal value functions of problems (6) and (8) are related to each other by way of the second-stage static maximization problem:

$$V^*(\boldsymbol{\beta}) = \max_{T, \mathbf{x}_0, \mathbf{x}_T} \left\{ \hat{V}(T, \mathbf{x}_0, \mathbf{x}_T, \varphi, r) + \mathrm{e}^{-rT}[S^1(\mathbf{x}_T) + \varphi_T] + [S^0(\mathbf{x}_0) + \varphi_0] \right\}, \tag{9}$$

where a solution of problem (9) is denoted by $(T^*(\boldsymbol{\beta}), \mathbf{x}_0^*(\boldsymbol{\beta}), \mathbf{x}_T^*(\boldsymbol{\beta}))$, the aforementioned optimal values of the terminal time, initial state vector and terminal state vector in problem (6).

The first-order necessary conditions obeyed by $(T^*(\boldsymbol{\beta}), \mathbf{x}_0^*(\boldsymbol{\beta}), \mathbf{x}_T^*(\boldsymbol{\beta}))$ are:

$$\hat{V}_T(T, \mathbf{x}_0, \mathbf{x}_T, \varphi, r) - r\mathrm{e}^{-rT}[S^1(\mathbf{x}_T) + \varphi_T] = 0, \tag{10a}$$

$$\hat{V}_{\mathbf{x}_0}(T, \mathbf{x}_0, \mathbf{x}_T, \varphi, r) + S^0_{\mathbf{x}_0}(\mathbf{x}_0) = \mathbf{0}'_N, \tag{10b}$$

$$\hat{V}_{\mathbf{x}_T}(T, \mathbf{x}_0, \mathbf{x}_T, \varphi, r) + \mathrm{e}^{-rT} S^1_{\mathbf{x}_T}(\mathbf{x}_T) = \mathbf{0}'_N. \tag{10c}$$

The second-order sufficient condition requires that the $(2N+1) \times (2N+1)$ Hessian matrix

$$\mathbf{H} \stackrel{\text{def}}{=} \left[\begin{array}{c|c|c} \underset{1\times 1}{\hat{V}_{TT} + r^2 \mathrm{e}^{-rT}[S^1 + \varphi_T]} & \underset{1\times N}{\hat{V}_{T\mathbf{x}_0}} & \underset{1\times N}{\hat{V}_{T\mathbf{x}_T} - r\mathrm{e}^{-rT} S^1_{\mathbf{x}_T}} \\ \hdashline \underset{N\times 1}{\hat{V}_{\mathbf{x}_0 T}} & \underset{N\times N}{\hat{V}_{\mathbf{x}_0\mathbf{x}_0} + S^0_{\mathbf{x}_0\mathbf{x}_0}} & \underset{N\times N}{\hat{V}_{\mathbf{x}_0\mathbf{x}_T}} \\ \hdashline \underset{N\times 1}{\hat{V}_{\mathbf{x}_T T} - r\mathrm{e}^{-rT}(S^1_{\mathbf{x}_T})'} & \underset{N\times N}{\hat{V}_{\mathbf{x}_T\mathbf{x}_0}} & \underset{N\times N}{\hat{V}_{\mathbf{x}_T\mathbf{x}_T} + \mathrm{e}^{-rT} S^1_{\mathbf{x}_T\mathbf{x}_T}} \end{array}\right] \tag{11}$$

is negative definite when evaluated at $(T, \mathbf{x}_0, \mathbf{x}_T) = (T^*(\boldsymbol{\beta}), \mathbf{x}_0^*(\boldsymbol{\beta}), \mathbf{x}_T^*(\boldsymbol{\beta}))$.

To conduct the comparative statics analysis, substitute $(T^*(\boldsymbol{\beta}), \mathbf{x}_0^*(\boldsymbol{\beta}), \mathbf{x}_T^*(\boldsymbol{\beta}))$ into Eqs. (10a)–(10c) and differentiate the resulting identities with respect to, say, φ, to get

$$\mathbf{H}^* \begin{bmatrix} \underset{1\times 1}{\partial T^*(\boldsymbol{\beta})/\partial\varphi} \\ \underset{N\times 1}{\partial \mathbf{x}_0^*(\boldsymbol{\beta})/\partial\varphi} \\ \underset{N\times 1}{\partial \mathbf{x}_T^*(\boldsymbol{\beta})/\partial\varphi} \end{bmatrix} \equiv \begin{bmatrix} \underset{1\times 1}{-\hat{V}_{T\varphi}} \\ \underset{N\times 1}{-\hat{V}_{\mathbf{x}_0\varphi}} \\ \underset{N\times 1}{-\hat{V}_{\mathbf{x}_T\varphi}} \end{bmatrix}, \tag{12}$$

where $\mathbf{H}^*$ is the Hessian matrix in Eq. (11) evaluated at $(T^*(\boldsymbol{\beta}), \mathbf{x}_0^*(\boldsymbol{\beta}), \mathbf{x}_T^*(\boldsymbol{\beta}))$. By Theorem 9.1 of Caputo (2005), a dynamic envelope result,

$$\hat{V}_\varphi(T, \mathbf{x}_0, \mathbf{x}_T, \varphi, r) = \int_0^T \mathrm{e}^{-rt}\mathrm{d}t = \frac{1}{r}[1 - \mathrm{e}^{-rT}], \tag{13}$$

from which follow $\hat{V}_{\varphi T} = \mathrm{e}^{-rT} = \hat{V}_{T\varphi}$, $\hat{V}_{\varphi\mathbf{x}_0} \equiv \mathbf{0}'_N \equiv \hat{V}'_{\mathbf{x}_0\varphi}$ and $\hat{V}_{\varphi\mathbf{x}_T} \equiv \mathbf{0}'_N \equiv \hat{V}'_{\mathbf{x}_T\varphi}$, using assumption (A4). Using these results and Cramer's Rule in Eq. (12), one has

$$\frac{\partial T^*(\boldsymbol{\beta})}{\partial\varphi} \equiv -\mathrm{e}^{-rT} \frac{\left|\begin{array}{c|c} \hat{V}_{\mathbf{x}_0\mathbf{x}_0} + S^0_{\mathbf{x}_0\mathbf{x}_0} & \hat{V}_{\mathbf{x}_0\mathbf{x}_T} \\ \hdashline \hat{V}_{\mathbf{x}_T\mathbf{x}_0} & \hat{V}_{\mathbf{x}_T\mathbf{x}_T} + \mathrm{e}^{-rT} S^1_{\mathbf{x}_T\mathbf{x}_T} \end{array}\right|}{|\mathbf{H}^*|} > 0. \tag{14}$$

The sign of Eq. (14) follows from the facts that: (i) $\mathbf{H}^*$ is negative definite by the second-order sufficient condition; (ii) the determinant in the numerator is a leading principal minor, and (iii) the order of the leading principal minor is one less than that of $|\mathbf{H}^*|$, thereby implying that the leading principal minor and $|\mathbf{H}^*|$ have opposite signs.

The comparative statics result in Eq. (14) applies to the general class of optimal control problems defined by Eqs. (6) and (2), and is therefore independent of functional form, monotonicity and curvature assumptions made on $f(\cdot)$ and $\mathbf{g}(\cdot)$, seeing as none were made. Indeed, its sign follows solely from the second-order sufficient condition for problem (9) and the manner in which φ enters the control problem. As a result, the comparative statics result in Eq. (14) is intrinsic to the aforesaid class of problems. Eq. (14) demonstrates that, for the case of $\varphi > 0$, which could apply if the firm receives a flow of subsidy, the higher is φ, the later the firm shuts down. Similarly, for the case of sunk fixed costs ($\varphi < 0$), the greater they are, the sooner the firm shuts down.

At the present level of generality, there are no refutable comparative statics results for the initial or terminal values of the state vector. To see this, note that if one were to calculate, say, $\partial x_{Ti}^*(\boldsymbol{\beta})/\partial\varphi$, the numerator would include a cofactor which is not a principal minor, the sign of which is not prescribed by the second-order sufficient condition. Consequently, the effect of a change in the flow of fixed subsidies or sunk fixed costs on an optimal solution of problem (6) cannot be determined unambiguously either. Indeed, as is demonstrated in section 5, even in a stylized version of the adjustment cost model of a firm, one cannot obtain an unambiguous result for the effect of a change in φ on the terminal capital stock.

Consider now the case of φ_0. By Proposition 2, the solution $(T^*(\boldsymbol{\beta}), \mathbf{x}_0^*(\boldsymbol{\beta}), \mathbf{x}_T^*(\boldsymbol{\beta}))$ is not a function of φ_0. Hence it follows that $\partial T^*(\boldsymbol{\beta})/\partial\varphi_0 \equiv 0$, $\partial\mathbf{x}_0^*(\boldsymbol{\beta})/\partial\varphi_0 \equiv \mathbf{0}_N$, and $\partial\mathbf{x}_T^*(\boldsymbol{\beta})/\partial\varphi_0 \equiv \mathbf{0}_N$.

Finally, consider the comparative statics of φ_T. As before, differentiate the identity form of Eqs. (10a)–(10c) with respect to φ_T to get

$$\mathbf{H}^* \begin{bmatrix} \partial T^*(\boldsymbol{\beta})/\partial\varphi_T \\ \partial\mathbf{x}_0^*(\boldsymbol{\beta})/\partial\varphi_T \\ \partial\mathbf{x}_T^*(\boldsymbol{\beta})/\partial\varphi_T \end{bmatrix} \equiv \begin{bmatrix} r\mathrm{e}^{-rT} \\ \mathbf{0}_N \\ \mathbf{0}_N \end{bmatrix}. \tag{15}$$

Applying Cramer's rule to Eq. (15) yields:

$$\frac{\partial T^*(\boldsymbol{\beta})}{\partial\varphi_T} \equiv r\mathrm{e}^{-rT} \frac{\left| \begin{array}{c:c} \hat{V}_{\mathbf{x}_0\mathbf{x}_0} + S^0_{\mathbf{x}_0\mathbf{x}_0} & \hat{V}_{\mathbf{x}_0\mathbf{x}_T} \\ \hdashline \hat{V}_{\mathbf{x}_T\mathbf{x}_0} & \hat{V}_{\mathbf{x}_T\mathbf{x}_T} + \mathrm{e}^{-rT} S^1_{\mathbf{x}_T\mathbf{x}_T} \end{array} \right|}{|\mathbf{H}^*|} < 0, \tag{16}$$

where the inequality in Eq. (16) follows from the same considerations as those used to establish the inequality in Eq. (14). Eq. (16) asserts that, for an increase in φ_T when $\varphi_T > 0$, which would be the case of a payment to the firm upon shutting down, the firm shuts down sooner. In a similar manner, if $\varphi_T < 0$, which would be the case in which the firm pays, say, a decommissioning cost upon shutting down, an increase in this cost would cause the firm to shut-down later. These are exactly the opposite of the effects of changes in φ. Observe that an increase in sunk termination fixed costs occurs at the date the firm shuts down and is discounted. Hence it pays the firm to delay shutting down when such costs increase, because delaying lowers their present discounted value. Moreover, inspection of Eqs. (14) and (16) reveals that $\partial T^*(\boldsymbol{\beta})/\partial\varphi_T \equiv -r[\partial T^*(\boldsymbol{\beta})/\partial\varphi] < 0$, which is consistent with the fact that φ is a flow incurred at every point in time of the planning horizon and φ_T is a one-time incurred stock. Note, in passing, that the effect of an increase in φ_T on the initial or terminal state vectors is ambiguous, in general, for the reason given two paragraphs above.

The preceding results are summarized in the following proposition.

Proposition 4 *Under assumptions (A1)–(A4), and assuming that the second-order sufficient condition holds in problem (9), the following comparative statics results hold for the optimal control problem defined by Eqs. (6) and (2):*

1. $\partial T^*(\boldsymbol{\beta})/\partial\varphi > 0, \partial \mathbf{x}_0^*(\boldsymbol{\beta})/\partial\varphi \gtrless \mathbf{0}_N, \partial \mathbf{x}_T^*(\boldsymbol{\beta})/\partial\varphi \gtrless \mathbf{0}_N,$
2. $\partial T^*(\boldsymbol{\beta})/\partial\varphi_0 \equiv 0, \partial \mathbf{x}_0^*(\boldsymbol{\beta})/\partial\varphi_0 \equiv \mathbf{0}_N, \partial \mathbf{x}_T^*(\boldsymbol{\beta})/\partial\varphi_0 \equiv \mathbf{0}_N,$
3. $\partial T^*(\boldsymbol{\beta})/\partial\varphi_T \equiv -r[\partial T^*(\boldsymbol{\beta})/\partial\varphi] < 0, \partial \mathbf{x}_0^*(\boldsymbol{\beta})/\partial\varphi_T \gtrless \mathbf{0}_N, \partial \mathbf{x}_T^*(\boldsymbol{\beta})/\partial\varphi_T \gtrless \mathbf{0}_N.$

Note that Proposition 4, appropriately modified, continues to hold if either the initial value of the state vector, terminal value of the state vector, or both, are fixed in problem (6), as long as the second-order sufficient condition in the corresponding version of problem (9) holds. Finally, it is worth mentioning again that the preceding results can be applied to the literature cited in sections 1 and 2, seeing as the second-stage value function serves as the terminal salvage function in the regime-switching literature.

4 Stochastic extensions

Tomiyama (1985) and Makris (2001) showed how deterministic, two-stage optimal control problems with an endogenous switching time can be handled using standard optimal control techniques. This section shows how to extend the comparative statics results of section 3 to this two-stage framework for two common classes of stochastic control problems, thereby permitting the determination of the comparative statics effect of a change in the volatility of the second-stage process on the optimal switching time.

Begin by considering the following class of current value autonomous, infinite horizon, two-stage, stochastic optimal control problems:

$$\max_{\mathbf{u}(\cdot),x_T,T} \mathbb{E}_0 \left\{ \int_0^T f\left(t, x(t), \mathbf{u}(t)\right) \mathrm{e}^{-rt}\mathrm{d}t + \mathrm{e}^{-rT} \int_T^{\infty} [\alpha \ln x(t) + \beta \ln u(t)] \mathrm{e}^{-r(t-T)}\mathrm{d}t \right\} \tag{17}$$
$$\text{s. t. } \dot{x}(t) = g(t, x(t), \mathbf{u}(t)), x(0) = x_0, t \in [0, T],$$
$$\mathrm{d}x(t) = [ax(t) + bu(t)]\mathrm{d}t + \sigma x(t)\mathrm{d}Z(t),\ x(T) = x_T,\ t \in (T, +\infty),$$

where $\mathbb{E}_0$ is the conditional expectation operator at time zero, $(a, b, \alpha, \beta, \sigma)$ are parameters, with $b \neq 0$, $\beta \neq 0$ and $\sigma > 0$, $u(t)$ is any one of the control variables that comprise the control vector $\mathbf{u}(t) \in \mathbb{R}^M$, $T > 0$ is the switching time, $Z(t)$ is standard Brownian motion, and all other terms are as defined earlier.

Given admissible values of (x_T, T), the second-stage stochastic optimal control problem corresponding to problem (17) is

$$V_2(x_T) \stackrel{\text{def}}{=} \max_{u(\cdot)} \mathbb{E}_T \left\{ \int_T^{\infty} [\alpha \ln x(t) + \beta \ln u(t)] \mathrm{e}^{-r(t-T)}\mathrm{d}t \right\}, \tag{18}$$
$$\text{s. t. } \mathrm{d}x(t) = [ax(t) + bu(t)]\mathrm{d}t + \sigma x(t)\mathrm{d}Z(t), x(T) = x_T.$$

For notational simplicity, the stage-two current value function, $V_2(\cdot)$, is expressed only as a function of the state variable. As is well-known, the Hamilton-Jacobi-Bellman (HJB) equation corresponding to problem (18) is:

$$rV_2(x) = \max_u \left\{ \alpha \ln x + \beta \ln u + V_2'(x)[ax + bu] + \frac{1}{2}\sigma^2 x^2 V_2''(x) \right\}. \tag{19}$$

The determination of a function $V_2(\cdot)$ such that Eq. (19) holds is essential for extending a comparative statics result of section 3, seeing as $V_2(\cdot)$ serves as the terminal salvage function for the class of stochastic control problems defined by Eq. (17) et seq. by way of the backwards induction argument. It is therefore worthwhile to pause at this juncture in order to attain a better understanding of the functional form of $V_2(\cdot)$ that is necessary for the said extension.

In order to extend the comparative statics result of Proposition 4, part 3, to the class of stochastic optimal control problems defined by Eq. (17) et seq., the terminal salvage function must comprise a function of the state variable at the switching time, plus a constant. This follows from inspection of Eq. (6). But, as remarked above, the terminal salvage function for this class of problems is given by $V_2(\cdot)$. Moreover, for σ^2 to appear as part of the additive constant of

$V_2(\cdot)$ and not part of the function that depends on the state variable, Eq. (19) shows that the differential equation $x^2V_2''(x) = -A$ must be satisfied, where A is a constant that is independent of x. Integrating $x^2V_2''(x) = -A$ twice yields its general solution, namely

$$V_2(x) = A \ln x + Bx + C, \tag{20}$$

where B and C are also constants independent of x. As a result, in order to have σ^2 appear as part of the additive constant of the terminal salvage function, $V_2(\cdot)$ must be of the form given in Eq. (20). Finally, observe that Eq. (20) also suggests that the natural logarithms of the state and control variables should be included in the integrand of the second-stage problem, just as they are in Eq. (18).

Given the preceding deductions, it may be shown that there exist constants A, B and C such that $V_2(\cdot)$ as defined in Eq. (20) is a solution of Eq. (19). The proofs of the following two propositions are contained in the Appendix.

Proposition 5 *For the class of stochastic optimal control problems defined by Eq. (17) et seq., the second-stage current value function $V_2(\cdot)$ is as defined in Eq. (20), where $A = r^{-1}(\alpha + \beta)$, $B = 0$ and $C = r^{-1}\beta[\ln[-b^{-1}r\beta(\alpha+\beta)^{-1}] - 1] + r^{-2}(\alpha+\beta)(a - \frac{1}{2}\sigma^2)$, and where $\beta > 0$ and $b(\alpha+\beta) < 0$. Moreover, sign$[\partial T^*/\partial\sigma^2] =$ sign$[\alpha+\beta]$.*

The inequality $\beta > 0$ is an implication of the assumption $\beta \neq 0$ and the second-order necessary condition of the maximization problem in Eq. (19), while $b(\alpha+\beta) < 0$ is equivalent to the fact that the control variable must be positive for the integrand of problem (18) to be well defined. The latter inequality leads to two separate cases, which are taken up in turn.

For the first case, let $b > 0$. This implies that $\alpha + \beta < 0$, which can only hold if $\alpha < 0$, seeing as $\beta > 0$. But $\alpha + \beta < 0$ implies that $A < 0$, $V_2'(x) < 0$ and $V_2''(x) > 0$. The state variable is therefore a 'bad', as $V_2'(x) < 0$. This case therefore characterizes stochastic control problems in which the state variable is a stock of pollution, waste, or a bad habit or addiction. Notice too that the control variable contributes to the accumulation of the bad stock in this case ($b > 0$) and, as a result, it could represent a firm's rate of output or production, or an individual's or economy's rate of consumption.

The most important part of Proposition 5 is the comparative statics result for the switching time. In interpreting it, first note that the variance of the instantaneous change in the state variable is $\sigma^2[x(t)]^2$ in problem (17). Moreover, because $\alpha + \beta < 0$ under the present stipulation, $\partial T^*/\partial\sigma^2 < 0$. That is, the optimal time to switch to the second stage decreases as the variance of the instantaneous change in the state variable increases. In other words, as the variability of the change in the bad stock in the second stage increases, the decision maker finds it optimal to switch to the second stage sooner. The corresponding economic intuition is compelling. Because $V_2(\cdot)$ is strictly convex in the state variable, or equivalently, because the decision maker is a risk lover with respect to the state variable, the decision maker is willing to take their chances by switching to the second stage – the risky environment – sooner. Hence the seemingly counterintuitive comparative statics result is made intuitive by appealing to the implied risk preferences of the decision maker.

In the second case, let $b < 0$, which implies that $\alpha + \beta > 0$ and, in turn, that $A > 0$, $V_2'(x) > 0$, and $V_2''(x) < 0$. In contrast to the preceding case, the stock is now a 'good' because $V_2'(x) > 0$ and the decision maker is risk averse with respect to the state variable in view of the fact that $V_2''(x) < 0$. This case therefore characterizes stochastic control problems in which the state variable is a beneficial asset, such as a stock of wealth or a nonrenewable or renewable resource. Inasmuch as the control variable contributes to the reduction of the good stock, it could represent a firm's rate of consumption spending out of wealth, or a firm's rate of nonrenewable resource extraction or rate of harvest of a renewable resource. Also supporting the notion that the present case is the opposite of the previous is the fact that the optimal switching time increases as the variance of the instantaneous change in the state variable during the second stage increases, i.e., $\partial T^*/\partial\sigma^2 > 0$. The intuition here is essentially the same as in the prior case, in that the decision maker is now risk averse and finds it optimal to delay switching to the second stage, where accumulation of the good stock has a risky component.

Having addressed the first class of stochastic control problems in some detail, the second class will be given a more crisp treatment. The second class is given by the following problem:

$$\begin{aligned} \max_{\mathbf{u}(\cdot),x_T,T} \mathbb{E}_0 \Bigg\{ \int_0^T & f\left(t, x(t), \mathbf{u}(t)\right) \mathrm{e}^{-rt} \mathrm{d}t \\ & + \mathrm{e}^{-rT} \int_T^{+\infty} \left[\alpha_1 x(t) - \frac{1}{2}\alpha_2 [x(t)]^2 + \beta_1 u(t) - \frac{1}{2}\beta_2 [u(t)]^2 \right] \mathrm{e}^{-r(t-T)} \mathrm{d}t \Bigg\} \end{aligned} \tag{21}$$

$$\begin{aligned} &\text{s. t. } \dot{x}(t) = g(t, x(t), \mathbf{u}(t)), x(0) = x_0, t \in [0, T], \\ &\mathrm{d}x(t) = [ax(t) + bu(t)]\mathrm{d}t + \sigma \mathrm{d}Z(t), x(T) = x_T, t \in (T, +\infty), \end{aligned}$$

where $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ are parameters with $\beta_2 \neq 0$, and all the remaining terms are as defined in problem (17). There are two differences between problems (17) and (21). First, the integrand in the second stage of problem (21) is a linear-quadratic function of the state and control variables, whereas it is an additive and natural logarithmic function in problem (17). Second, the stochastic state equation in problem (21) assumes the variance of the diffusion is the constant σ^2, while in problem (17) it is a function of the square of the state variable, namely, $\sigma^2[x(t)]^2$.

Given admissible values of (x_T, T), the second-stage stochastic optimal control problem corresponding to problem (21) is:

$$V_2(x_T) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \mathbb{E}_T \left[\int_T^{+\infty} \left(\alpha_1 x(t) - \frac{1}{2}\alpha_2 [x(t)]^2 + \beta_1 u(t) - \frac{1}{2}\beta_2 [u(t)]^2 \right) \mathrm{e}^{-r(t-T)} \mathrm{d}t \right], \quad (22)$$
$$\text{s.t. } \mathrm{d}x(t) = [ax(t) + bu(t)]\mathrm{d}t + \sigma \mathrm{d}Z(t), x(T) = x_T,$$

and the corresponding HJB equation is

$$rV_2(x) = \max_u \left\{ \alpha_1 x - \frac{1}{2}\alpha_2 x^2 + \beta_1 u - \frac{1}{2}\beta_2 u^2 + V_2'(x)[ax + bu] + \frac{1}{2}\sigma^2 V_2''(x) \right\}. \quad (23)$$

Using the logic enunciated above in the conjecture for the second-stage current value function of problem (17), it follows that

$$V_2(x) = \frac{1}{2}Ax^2 + Bx + C \quad (24)$$

is the conjecture for $V_2(\cdot)$, where A, B and C are constants to be determined. The following result is the analogue of Proposition 5 for this class of problems.

Proposition 6 *For the class of stochastic optimal control problems defined by Eq. (21) et seq., the second-stage current value function $V_2(\cdot)$ is given by Eq. (24), where:*

$$A = \frac{-(2a - r)b^{-2}\beta_2 \pm \sqrt{(2a - r)^2 b^{-4}\beta_2^2 + 4\alpha_2 b^{-2}\beta_2}}{2}, \quad (25a)$$
$$B = \frac{\alpha_1 + b\beta_1\beta_2^{-1}A}{r - a - b^2\beta_2^{-1}A}, \quad (25b)$$
$$C = r^{-1}\left(\frac{1}{2}\beta_1^2\beta_2^{-1} + \frac{1}{2}b^2\beta_2^{-1}B^2 + b\beta_1\beta_2^{-1}B + \frac{1}{2}\sigma^2 A \right), \quad (25c)$$

and where $\beta_2 > 0$ and $\beta_1 + bB + bAx \geq 0$. Moreover, sign$[\partial T^*/\partial\sigma^2] = -$sign$[A]$.

Proposition 6 demonstrates that the linear-quadratic integrand in the second stage leads to a considerably more complex solution for the constants than did the first case. Nonetheless, the comparative statics result shows that the effect of an increase in the variance of the instantaneous change in the state variable on the optimal switching time is wholly determined by the sign of the constant A. But the value of A also completely determines the curvature of the second stage current value function, as $V_2''(x) = A$. Thus, as was the case in Proposition 5, the implied risk preferences of the decision maker fully determine the effect of an increase in the variance of the instantaneous change in the state variable on the optimal switching time – if the decision maker is risk averse, it is optimal to switch to the risky stage later; if risk loving, it is optimal to switch sooner.

This section is brought to a close by demonstrating the reach of Propositions 5 and 6. To begin, observe that the propositions are more general than they might appear at first glance. This is because the first-stage optimal control problems to which they pertain, defined in Eqs. (17) and (21), leave the integrands and state equations in a general form and account for multiple control variables. Moreover, the first-stage control problem can be further generalized to allow for multiple state variables without changing the content of Propositions 5 and 6. Only in the second stage do the integrands and state equations have to be of a specific functional form to make use of the propositions. Finally, Proposition 6 applies to the workhorse class of stochastic optimal control problems, namely, the linear-quadratic class. This fact speaks directly to the reach of Proposition 6.

5 Applications

This section presents three applications of the foregoing theory and considers some other fruitful areas for application in section 5.4.

5.1 The adjustment cost model of a firm

The adjustment cost model under consideration takes the form

$$\max_{u(\cdot),T,x(T)} \left\{ \int_0^T [\pi(x(t), u(t)) - \phi] \mathrm{e}^{-rt} \mathrm{d}t + \mathrm{e}^{-rT} S(x(T)) \right\}, \tag{26}$$
$$\text{s. t. } \dot{x}(t) = u(t) - \delta x(t), x(0) = x_0,$$

where $u(t)$ is the rate of investment in the firm's capital stock $x(t) > 0$ at time t, $r > 0$ is the discount rate, $\delta > 0$ is the rate of depreciation of the capital stock, and $\phi > 0$ is the flow of sunk fixed costs. Let $\pi(x, u) \stackrel{\text{def}}{=} x - 0.5u^2$ be the flow of total revenue less costs of adjustment, and $S(x) \stackrel{\text{def}}{=} \theta x$, $\theta > 0$, be the salvage value of the capital stock. These functional forms were chosen because they satisfy the typical assumptions employed in such models and because they contribute to maintaining the focus on matters of economics rather than on mathematical technicalities. Note that sunk start-up fixed costs can be ignored (by Proposition 3) and that sunk termination fixed costs have the opposite effect of the flow of sunk fixed costs on the shut down decision (by Proposition 4) and so can be ignored as well.

The necessary conditions are given by Eqs. (4a)-(4c) and (4e), with the transversality condition in Eq. (4d) replaced by the initial condition on the capital stock, together with Eq. (7). They can be reduced to the following pair of ordinary differential equations, initial and terminal conditions and transversality conditions by way of standard, and thus omitted, manipulations:

$$\dot{x}(t) = u(t) - \delta x(t),\; x(0) = x_0, \tag{27a}$$

$$\dot{u}(t) = u(t)[r+\delta] - 1,\; u(T) = \theta, \tag{27b}$$

$$x(T) - 0.5[u(T)]^2 - \phi + \theta u(T) - \theta x(T)[r+\delta] = 0. \tag{27c}$$

The phase diagram corresponding to Eqs. (27a) and (27b) is straightforward to derive. The difficulty lies in determining how Eq. (27c), which implicitly defines a curve relating $u(T)$ to $x(T)$, appears in the phase diagram. This determination is crucial, as an optimal solution to the adjustment cost model must terminate in the phase diagram where the horizontal line $u(T) = \theta$ intersects the curve implicitly defined by Eq. (27c). Denote the steady state value of the capital stock and investment rate as $(x^{ss}, u^{ss}) \stackrel{\text{def}}{=} (\delta^{-1}(r+\delta)^{-1}, [r+\delta]^{-1})$ and note that both are positive. Furthermore, define $F(x,u) \stackrel{\text{def}}{=} x - 0.5u^2 + \theta u - \theta x[r+\delta]$, so that Eq. (27c) can be written compactly as $F(x,u) = \phi$.

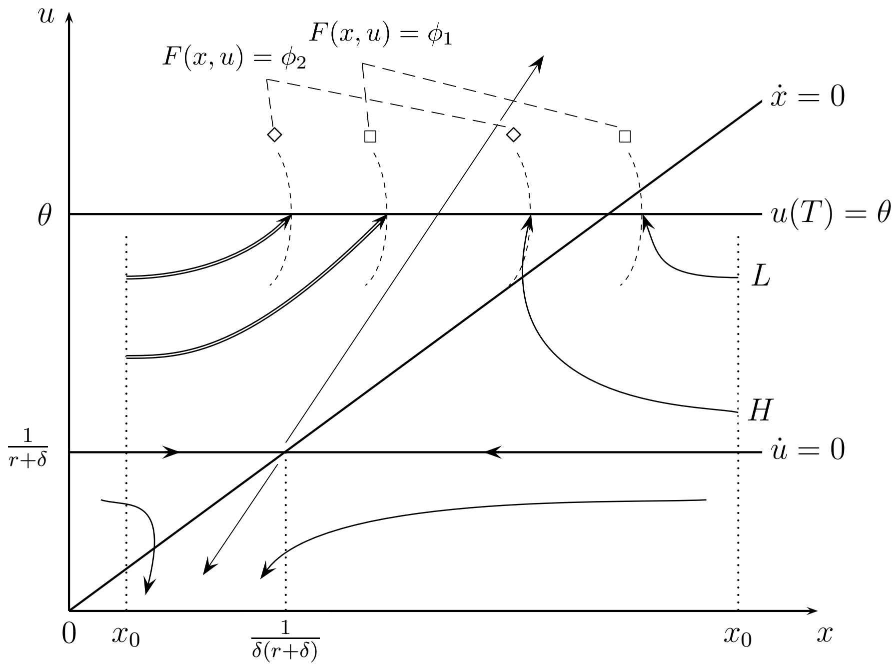
There are two possible configurations for the phase diagram corresponding to Eqs. (27a)-(27c). Figure 1(a) corresponds to the case $u(T) > u^{ss}$, that is, $\theta > [r+\delta]^{-1}$, and Figure 1(b) to the case $u(T) < u^{ss}$. In what follows, details pertaining to Figure 1(a) are given.

First, derive an explicit expression for the terminal value of the capital stock. Given that $u(T) = \theta$ from Eq. (27b), Eq. (27c) can be solved for $x(T)$ to get

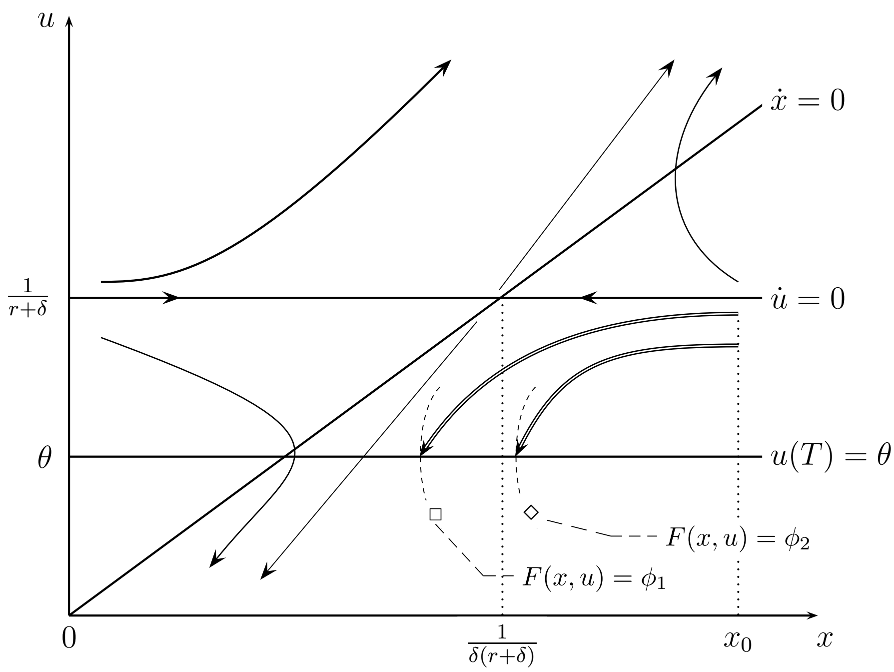
$$x(T) = \frac{\phi - 0.5\theta^2}{1 - \theta[r+\delta]}. \tag{28}$$

The case of interest is $x(T) > 0$, which is maintained in what ensues. Because $u(T) > u^{ss}$ in the present case, that is, $\theta > (r+\delta)^{-1}$, the denominator in Eq. (28) is negative, hence $\phi < 0.5\theta^2$. It then follows that $\partial x(T)/\partial\phi = 1/[1-\theta(r+\delta)] < 0$, that is, an increase in the flow of sunk fixed costs decreases the capital stock the firm has on hand when it shuts down. Note that the opposite is true in Figure 1(b), where $u(T) < u^{ss}$ holds, in view of the fact that $\theta < (r+\delta)^{-1}$ then holds.

Next, note that by the implicit function theorem, the slope of the curve implicitly defined by



(a) $\theta > 1/(r+\delta)$.



(b) $\theta < 1/(r+\delta)$

Figure 1: Phase diagrams for the adjustment cost model of the firm

$F(x,u)=\phi$ in the xu-plane is given by:

$$\left.\frac{\partial u}{\partial x}\right|_{\text{Eq.}(27c)} = \left.\frac{-F_x(x,u)}{F_u(x,u)}\right|_{F(x,u)=\phi} = \frac{\theta(r+\delta)-1}{\theta-u}\begin{cases} <0 \text{ iff } u>\theta \\ >0 \text{ iff } u<\theta \end{cases}, \tag{29}$$

as $\theta(r+\delta)-1>0$ in the case under consideration. Eq.(29) also demonstrates that, as $u\to\theta$, the slope of the curve implicitly defined by $F(x,u)=\phi$ becomes vertical. As a result, the curve implicitly defined by the transversality condition $F(x,u)=\phi$ in Eq. (27c) and the horizontal straight line determined by the endpoint condition $u(T)=\theta$ in Eq. (27b) intersect orthogonally at the optimal terminal time.

Finally, in order to complete the phase diagram, the curvature of the curve implicitly defined by $F(x,u)=\phi$ must also be determined. The said curvature is found by partially differentiating Eq. (29) with respect to u, remembering that u is a function of x along $F(x,u)=\phi$ via the implicit function theorem:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{F(x,u)=\phi} = \frac{\theta(r+\delta)-1}{[\theta-u]^2}\left.\frac{\partial u}{\partial x}\right|_{F(x,u)=\phi}\begin{cases} <0 \text{ iff } u>\theta \\ >0 \text{ iff } u<\theta \end{cases}. \tag{30}$$

Thus, under the present assumptions, Eqs. (29) and (30) show that the curve implicitly defined by $F(x,u)=\phi$ is increasing and strongly convex for all values of u below $u(T)=\theta$ and decreasing and strongly concave for all values of u above $u(T)=\theta$.

Putting the above information together yields Figure 1(a), where the curves $F(x,u)=\phi$ and $u(T)=\theta$ are shown intersecting in the northeast and northwest isosectors. Under the present stipulations, a solution to the adjustment cost problem must lie exclusively in the northwest isosector. To see this, recall that $\partial x(T)/\partial\phi = 1/[1-\theta(r+\delta)]<0$. Thus, as the flow of sunk fixed costs increases, the point where the curves $F(x,u)=\phi$ and $u(T)=\theta$ intersect moves to the left. Define two fixed costs, ϕ_1 and ϕ_2, such that $\phi_2>\phi_1$. The preceding implies that, for paths originating in the northeast isosector of Figure 1(a), the trajectory corresponding to ϕ_2, labeled H, must lie below the trajectory corresponding to ϕ_1, labeled L. Trajectory H therefore lies closer to the stable manifold of the saddlepoint steady state – the $\dot{u}=0$ isocline – implying that it has a larger value of terminal time than trajectory L. Trajectories originating in the northeast isosector therefore exhibit the property that, as the flow of sunk fixed costs increases, the terminal time increases, which contradicts Proposition 4. As a result, a solution to problem (26) cannot originate in the northeast isosector, that is, it must lie exclusively in the northwest isosector.

An optimal solution can now be qualitatively characterized using Figure 1(a). The doublelined trajectories correspond to the optimal time-paths of the capital stock and investment rate, and are monotonically increasing functions of time. They show that, the higher the flow of sunk fixed costs, the sooner the firm shuts down and the smaller is its capital stock when it does so.

Figure 1(b) shows the trajectories corresponding to a solution of the adjustment cost model when $u(T) < u^{ss}$. By Proposition 4, it remains the case that the optimal T is lower when the flow of sunk fixed costs are higher, but in this case the terminal stock of capital increases when the flow of sunk fixed costs increases, seeing as $\partial x(T)/\partial\phi = 1/[1 - \theta(r + \delta)] > 0$. In contrast to the case of Figure 1(a), the capital stock and investment trajectories lie exclusively in the southeast isosector and are monotonically decreasing functions of time. Moreover, the analysis shows that, in general, the terminal stock of capital may increase or decrease as the flow of sunk fixed costs increases, thereby confirming that no general comparative statics result is available in Proposition 4 for the terminal value of the state vector.

5.2 Optimal pollution accumulation with uncertainty over the critical pollution threshold

Tahvonen and Withagen (1996) developed a deterministic model of optimal pollution accumulation in which the pollution stock depreciates as long as it remains below a critical threshold. If the stock passes the threshold, it no longer depreciates and is therefore deemed to be 'irreversible'. The social planner is asserted to choose the rate of pollution-augmenting output over an infinite planning horizon that is divided into two distinct stages: during the first stage, the stock of pollution depreciates at a defined rate; during the second, upon reaching the threshold, the rate of depreciation falls to zero. One focus in the model is on the choice of the optimal timing of entry to the second stage.

A key assumption made by Tahvonen and Withagen is that the threshold at which the pollution stock becomes irreversible is known with certainty at the initial time of the planning horizon. This assumption is relaxed here by instead assuming that, at the start of the planning horizon, the social planner does not know the threshold at which the rate of depreciation of the pollution stock falls to zero. The model is further generalized by leaving the stage one instantaneous utility function, $U(\cdot)$, in a general form. As a result, the planner solves:

$$\max_{y(\cdot),T} \mathbb{E}_0 \left\{ \int_0^T U(y(t), z(t))\mathrm{e}^{-\rho t}\mathrm{d}t + \right. \tag{31a}$$

$$\left. \mathrm{e}^{-\rho T} \int_T^{+\infty} \left(-\frac{1}{2}\alpha_2[z(t)]^2 + \beta_1 y(t) - \frac{1}{2}\beta_2[y(t)]^2 \right) \mathrm{e}^{-\rho(t-T)}\mathrm{d}t \right\},$$

$$\text{s. t. } \dot{z}(t) = y(t) - \alpha z(t), \;\; z(0) = z_0, \;\; z(T) = \mu_Z + \gamma_Z X, \;\; t \in [0, T], \tag{31b}$$

$$\dot{z}(t) = y(t), \;\; t \in (T, +\infty), \tag{31c}$$

where $y(t)$ is the rate of pollution-augmenting output, $z(t)$ the stock of pollution and the Greek letters are parameters, all of which are positive and X is a random variable with mean zero and variance 1. The pollution stock at which irreversibility occurs is therefore a random variable $z(T)$ from the perspective of $t = 0$, with mean μ_Z and variance γ_Z^2.

As in Tahvonen and Withagen, it is assumed that the planner knows the moment at which irreversibility occurs. Hence, at the point at which the second stage is entered, the planner solves the following deterministic control problem:

$$V_2(z_T) = \max_{y(\cdot)} \int_T^{+\infty} \left(-\frac{1}{2}\alpha_2[z(t)]^2 + \beta_1 y(t) - \frac{1}{2}\beta_2[y(t)]^2 \right) \mathrm{e}^{-\rho(t-T)}\mathrm{d}t, \tag{32a}$$

$$\text{s. t. } \dot{z}(t) = y(t), \quad z(T) = z_T \tag{32b}$$

Noting that this is a deterministic version of problem (22), with $\alpha_1 = 0$, $a = 0$, $b = 1$ and $\sigma = 0$, it follows from Proposition 6 that

$$V_2(z_T) = A[z_T]^2 + Bz_T + C, \tag{33}$$

where A, B and C are as defined in Eq. (25).

From the perspective of $t = 0$, the planner solves:

$$\max_{y(\cdot),T} \left\{ \int_0^T U(y(t), z(t))\mathrm{e}^{-\rho t}\mathrm{d}t \right\} + \mathrm{e}^{-\rho T}\mathbb{E}_0[V_2(z(T))], \tag{34}$$

$$\text{s. t. } \dot{z}(t) = y(t) - \alpha z(t), \;\; z(0) = z_0, \;\; z(T) = \mu_Z + \gamma_Z X,$$

where, noting Eq. (33):

$$\mathbb{E}_0[V_2(z(T))] = \mathbb{E}_0[A(\mu_Z^2 + 2\mu_Z\gamma_Z X + \gamma_Z^2 X^2) + B(\mu_Z + \gamma_Z X) + C] = A(\mu_Z^2 + \gamma_Z^2) + B\mu_Z + C. \tag{35}$$

As the values of the stock of pollution are not decision variables at $t = 0$ and $t = T$, upon substituting $\mathbb{E}_0[V_2(Z(T))]$ for $[S^1(\mathbf{x}_T) + \psi_T]$ in Eq. (10a), it alone determines the optimal value of T. Hence, implicitly differentiating the resulting Eq. (10a) with respect to γ_Z^2 yields $\mathrm{sign}[\partial T^*/\partial\gamma_Z^2] = -\mathrm{sign}[A]$. Accordingly, for a risk averse planner, $A < 0$ and it is optimal to extend the first stage of the planning horizon, while if a planner is a risk lover, then $A > 0$ and it is optimal to enter the risky second stage earlier.

5.3 A lifecycle model of retirement with shocks to retirement income

This section applies Proposition 5 to extend Prettner and Canning's (2014) lifecycle model of retirement to establish the effect of idiosyncratic shocks to retirement income on the optimal timing of retirement. The defining characteristic of the analysis is that the solution of the retirement stage of the model yields a bequest function which is additive in the parameter governing the variance of the idiosyncratic shocks.

Consider, therefore, the following generalization of Prettner and Canning's control problem, in which shocks to retirement income and a general utility function during working life are postulated:

$$\max_{c(\cdot),l(\cdot),W_T,T} \mathbb{E}_0\left\{\int_0^T U(c(t),l(t))\mathrm{e}^{-\tilde{\rho}t}\mathrm{d}t + \mathrm{e}^{-\tilde{\rho}T}\int_T^{+\infty}\ln[c(t)]\mathrm{e}^{-\tilde{\rho}(t-T)}\mathrm{d}t\right\}, \tag{36a}$$

$$\text{s. t. } \dot{W}(t) = wl(t) + \tilde{r}W(t) - c(t),\ \ W(0) = W_0,\ \ t\in[0,T], \tag{36b}$$

$$\mathrm{d}W(t) = [rW(t) - c(t)]\mathrm{d}t + \sigma W(t)\mathrm{d}Z(t),\ \ W(T) = W_T,\ \ t\in(T,+\infty), \tag{36c}$$

where $W(t)$ is wealth, $U(\cdot)$ is a function of $c(t)$, the rate of consumption, and $l(t)$, the number of hours worked per unit time during the agent's working life, $w > 0$ the wage rate earned when working, $r > 0$ the rate of growth of wealth (assumed to be the same whether the individual is working or retired), $Z(t)$ is standard Brownian motion and $\sigma > 0$. The effective discount and interest rates faced by the individual, $\tilde{\rho} \stackrel{\text{def}}{=} \rho + \lambda$ and $\tilde{r} \stackrel{\text{def}}{=} r + \lambda$ include the constant mortality risk $\lambda > 0$, implying that the probability of being alive at t is $\exp(-\lambda t)$. This specification represents a considerable generalization of Prettner and Canning's problem, seeing as $U(\cdot)$ is left in a general form.

Inspection of problem (36a) shows that it is a special case of problem (17), where $\alpha = 0, \beta = 1, a = r$ and $b = -1$. It therefore follows from Proposition 5 that $\partial T^*/\partial\sigma^2 > 0$, as $\alpha + \beta > 0$. That is, the presence of idiosyncratic shocks to retirement income unambiguously increases the optimal retirement age. The second stage shocks have the effect of introducing risk into the evolution of wealth during an agent's retirement years, which the risk averse agent wishes to delay.

5.4 Further applications

In closing section 5, we briefly review several other papers in addition to those just discussed and those reviewed in sections 1 and 2, to which our results may be applied.

Consider first a pair of closely related papers by Caulkins et al. (2011, 2015). Each developed an optimal control model of conspicuous product pricing by a firm when an economy is in a recession that reduces demand and freezes credit markets, the latter extending the former by allowing the firm to develop an optimal cash management strategy. In the case when the recession lasts so long that the firm faces bankruptcy and therefore finds it optimal to shut down, the terminal time is a decision variable and hence Corollary 2 and Proposition 4 apply. As a result, in the case of an interior solution, a flow of sunk fixed costs affects the firm's optimal price trajectory and, moreover, it will shut down sooner if the flow of sunk fixed costs increases. These results are intrinsic to the models, as they do not rely upon the functional form assumptions made by Caulkins et al. (2011, 2015).

Proposition 4 can also be used to show that an increase in a fixed switching cost delays the adoption decision in the Grass et al. (2012) two-stage optimal control model of technology adoption with capital accumulation and technological progress. Then either Proposition 5 or 6 can be used to extend the model to determine the effect of an increase in the instantaneous variance of the change in the post-adoption capital stock on the adoption decision when the second-stage control problem has an infinite planning horizon. Indeed, these three propositions can be applied just as readily to the two-stage optimal control models of workplace reorganization of Valleé and Moreno-Galbis (2011) and closed- versus open-source software distribution of Caulkins et al. (2013), to draw similar qualitative conclusions.

In a different application of two-stage optimal control theory, Bultmann et al. (2008a, 2008b) modeled a country with a drug problem in which the drug's supply is significantly disrupted for some initial period of time, thereby causing price to be higher than usual. Later, price returns to its usual level. The time at which price switches to its usual level is treated as a parameter in the model, so Propositions 1-6 do not apply. But in discussing possible extensions of the model, Bultmann et al. (2008b) suggested that the switching time may be a decision variable as a result of a deliberate policy choice by a government, in which case Corollary 2 and Proposition 4 apply and Propositions 5 or 6 can be used to further extend the model and draw qualitative conclusions akin to those just mentioned.

With the growing use of two-stage optimal control problems to model all kinds of economic environments, as exemplified by Chapter 8 of Grass et al. (2008) and the applications contained therein, the number of papers which can take advantage of our basic results might be expected to increase in the coming years.

6 Concluding remarks

A framework for studying the impact of additive transformations to rewards for a general class of deterministic, nonautonomous optimal control problems has been established and a full set of comparative static results derived. The framework was then extended to two classes of stochastic control problems, one of which was the workhorse linear-quadratic class. The reach of the methods was demonstrated on three rather different optimal control problems and an economic interpretation of the comparative statics calculations was provided.

When the planning horizon is fixed, neither the optimal time-paths of the control or state variables, nor the latter's corresponding shadow prices, are functions of additive flow, start-up, or termination, sunk fixed costs or benefits. If, however, the initial or terminal time are decision variables – not uncommon in economic theory – then optimal time-paths of the state, control and costate variables, as well as the optimal initial and terminal times, are functions of additive flow and termination sunk fixed costs or benefits, but not such start-up costs and benefits.

It is important to emphasize that the conclusions reached in section 3 are not due to any special structure placed on the integrand and transition functions, and thus represent intrinsic behavior of optimizing agents. Similarly, even though the second-stage integrand and transition functions must be of certain forms in the stochastic case, their first-stage counterparts were left in general form and could accommodate multiple state and control variables, thereby pointing to the generality of the results, even in the stochastic setting.

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Appendix: Proofs of Propositions 5 and 6

Proof of Proposition 5. The first-order necessary condition of the maximization problem in Eq. (19) gives $u = -\beta b^{-1}[V_2'(x)]^{-1} > 0$, the strict inequality following from the fact that the domain of the natural logarithm function is $(0, +\infty)$. The second-order necessary condition is $-\beta u^{-2} \leq 0$, which is equivalent to $\beta \geq 0$. But, seeing as $\beta \neq 0$ by assumption, it follows that $\beta > 0$.

The supposition for $V_2(\cdot)$ is given in Eq. (20), where A, B, and C are constants to be determined, and where $V_2'(x) = Ax^{-1} + B$ and $V_2''(x) = -Ax^{-2}$. Substituting $u = -\beta b^{-1}[V_2'(x)]^{-1} > 0$ and the expressions for $V_2(x)$, $V_2'(x)$ and $V_2''(x)$ in Eq. (19) gives

$$rA\ln x + rBx + rC = \alpha \ln x + \beta \ln[-b^{-1}\beta[Ax^{-1} + B]^{-1}] + aBx + aA - \beta - \frac{1}{2}\sigma^2 A. \quad (37)$$

Equating coefficients on like terms in Eq. (37) results in

$$A = r^{-1}(\alpha + \beta), \quad (38a)$$
$$B = 0, \quad (38b)$$
$$C = r^{-1}\beta[\ln(-b^{-1}r\beta(\alpha+\beta)^{-1}) - 1] + r^{-2}(\alpha+\beta)\left(a - \frac{1}{2}\sigma^2\right), \quad (38c)$$

as the values of the three constants such that Eq. (20) is the stage two current value function.

Now observe that $u = -\beta b^{-1}[V_2'(x)]^{-1} = -\beta b^{-1} r(\alpha+\beta)^{-1}x > 0$, which is equivalent to $b(\alpha+\beta) < 0$, because none of the terms in the product can be zero and $x > 0$, $\beta > 0$ and $r > 0$.

Finally, in order to prove that $\mathbf{sign}[\partial T^*/\partial\sigma^2] = \mathbf{sign}[\alpha + \beta]$, first recall that the initial state variable is not a decision variable and the terminal state is a scalar. Upon setting $S^1(\mathbf{x}_T) + \psi_T = V_2(x_T)$ and using Eq. (38), it follows that Eqs. (10a) and (10c) implicitly yield the optimal values of (T, x_T). It then follows from differentiating the identity form of Eqs. (10a) and (10c) with respect to σ^2 that:

$$\mathbf{H}^* \begin{bmatrix} \partial T^*(\boldsymbol{\beta})/\partial\sigma^2 \\ \partial x_T^*(\boldsymbol{\beta})/\partial\sigma^2 \end{bmatrix} \equiv \begin{bmatrix} -\frac{1}{2}r^{-1}\mathrm{e}^{-rT}(\alpha+\beta) \\ 0 \end{bmatrix}, \quad (39)$$

and therefore that $\partial T^*(\boldsymbol{\beta})/\partial\sigma^2 \equiv -\frac{1}{2}r^{-1}\mathrm{e}^{-rT}(\alpha+\beta)(\hat{V}_{x_T x_T} + \mathrm{e}^{-rT}S^1_{x_T x_T}) \,/\, |\,\mathbf{H}^*|$. Given the second-order sufficient conditions, the result $\mathrm{sign}[\partial T^*/\partial\sigma^2] = \mathrm{sign}[\alpha+\beta]$ is immediate. *Q.E.D.*

Proof of Proposition 6. The first-order necessary condition of the maximization problem in Eq. (22) is equivalent to $u = \beta_2^{-1}[\beta_1 + bV_2'(x)] \geq 0$, the inequality following from the fact that the control variable is constrained to be nonnegative. The second-order necessary condition is $-\beta_2 \leq 0$, which is equivalent to $\beta_2 \geq 0$. But seeing as $\beta_2 \neq 0$ by assumption, it follows that $\beta_2 > 0$.

Substituting $u = \beta_2^{-1}(\beta_1 + bV_2'(x))$ and the expressions for $V_2(x)$, $V_2'(x)$ and $V_2''(x)$ into Eq. (23), yields:

$$\frac{1}{2}rAx^2 + rBx + rC = \left(aA + \frac{1}{2}b^2\beta_2^{-1}A^2 - \frac{1}{2}\alpha_2\right)x^2 + (\alpha_1 + aB + b^2\beta_2^{-1}AB + b\beta_1\beta_2^{-1}A)x \quad (40)$$
$$+ \left(\frac{1}{2}\beta_1^2\beta_2^{-1} + \frac{1}{2}b^2\beta_2^{-1}B^2 + b\beta_1\beta_2^{-1}B + \frac{1}{2}\sigma^2 A\right).$$

Equating coefficients on like terms in Eq. (40) yields the solutions for A, B, and C given in Proposition 6. Recalling that $u = \beta_2^{-1}(\beta_1 + bV_2'(x)) \geq 0$ which, because $\beta_2 > 0$, is equivalent to

$\beta_1 + bV_2'(x) = \beta_1 + bB + bAx \geq 0.$

Finally, repeating the steps that led to Eq. (39):

$$\mathbf{H}^* \begin{bmatrix} \partial T^*(\boldsymbol{\beta})/\partial\sigma^2 \\ \partial \mathbf{x}_T^*(\boldsymbol{\beta})/\partial\sigma^2 \end{bmatrix} \equiv \begin{bmatrix} \frac{1}{2}\mathrm{e}^{-rT}A \\ 0 \end{bmatrix},$$

and therefore that $\partial T^*(\boldsymbol{\beta})/\partial\sigma^2 \equiv \frac{1}{2}\mathrm{e}^{-rT}A(\hat{V}_{x_T x_T} + \mathrm{e}^{-rT}S^1_{x_T x_T}) \,/\, |\mathbf{H}^*|$. Given the second-order sufficient conditions, the result $\mathrm{sign}[\partial T^*/\partial\sigma^2] = -\mathrm{sign}[A]$ follows. *Q.E.D.*