Critical Review of Penelope Maddy, *Defending the Axioms: On the Philosophical Foundations of Set Theory* (Oxford: OUP, 2011)

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ABSTRACT

Penelope Maddy’s 2011 book, *Defending the Axioms*, argues that there may be an objectively correct answer to the question whether there are sets whose cardinality is strictly less than the real numbers, but strictly greater than the natural numbers, but that there is no objectively correct answer to the question of whether there are sets. This review examines Maddy’s reasons for these claims, and agrees with her that, once the position she calls ‘Robust Realism’ (which she herself defended in earlier work) is ruled out, the issue between realism and anti-realism in the philosophy of mathematics is a mere terminological dispute.

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A striking difference between mathematics and philosophy is the degree to which, in mathematics, we can say with a fair amount of confidence that an answer to a given question is right or wrong. While mathematicians and philosophers both trade in arguments (and often, even in the philosophical case, *deductive* arguments) from premises, in philosophy, unlike in mathematics, there is generally widespread disagreement over whether one should accept the conclusions reached via such arguments. This can sometimes be because of concerns over the validity of the argumentative steps. While concerns over the correctness of purported ‘proofs’ arise in mathematics as well as in philosophy, to the extent that the concepts in mathematics are more precisely defined, and the allowable steps more formally characterized, mathematics is less open to (though certainly not completely removed from) deductive fallacies. But the difference here is one of degree rather than of kind. Where the real difference lies is in the responses of mathematicians and philosophers to deductions that they accept to be valid. In these cases, while mathematicians will happily accept the conclusions of these deductions, an entirely appropriate philosophical move is to say, ‘OK, so I accept that *if* your premises are true, then so is your conclusion, but what’s with those premises? Why ought I to believe *those*?’ And whatever reasons we give the danger remains that we will be pushed back to assumptions so basic that perhaps nothing can be said to convince an opponent who does not accept them already. Whence the sorry predicament of philosophy: rather than finding answers to our questions that we can be happy with, our attempts to answer philosophical questions just throw up more questions in need of answers.

Why such a stark difference? One might think that it is because in mathematics the premises we begin with are *axioms*, and as such are *so obvious as to be indubitable* to all involved. But this doesn’t seem quite right – many axioms are far from indubitable. The standard ZFC axioms, for example, include some that are arguably obviously true of the concept of set (perhaps anyone with any grasp at all on the iterative conception of set will immediately agree that, if two sets exist then so does their pair). But others amongst these axioms are more contentious. Take the axiom of infinity, for example: does the basic idea of the iterated application of the ‘set of’ operation to generate levels in the hierarchy automatically require that we accept even the first limit level, containing the first set of size ℵ0? At best, it seems, this axiom’s justification is *extrinsic*: we accept it not because we see that it is intuitively true of our most basic conception of set, but rather, because of its fruitful consequences elsewhere in mathematics (without the axiom of infinity set theory just wouldn’t get off the ground in one of its main roles, for example, as providing a foundation for arithmetic). But such pragmatic justifications raise the possibility of sceptical questions: granted that an axiom is *useful* to adopt to meet certain mathematical aims, what reason (beyond mere wishful thinking) have we for concluding from that that the given axiom is likely to be *true*? What grounds our confidence in this axiom to the extent that we will be willing to accept it within mathematics as beyond reasonable doubt?

Nevertheless, despite this skeptical worry, for whatever reason (and we will come back to the question of what that reason may be), within mathematical practice, when doing ZFC set theory, it is simply not the done thing to question the ZFC axioms. So if we can find a convincing argument that the conclusion we are looking to establish follows from these axioms, we’re done. But what about those mathematical statements within the language of a theory that are known to be *independent* of the axioms of that theory? Statements such as the continuum hypothesis, CH, where it can be proved that neither it nor its negation is derivable from the axioms of ZFC? Can anything convincing be said in these cases in favour of, or against, the truth of such claims? Or do we find here a situation where mathematics presents us with questions as intractable as the most fundamental questions of philosophy?

In the case of the continuum hypothesis, the attitude we take to this question (the question of whether there are infinite subsets of the real numbers that are larger than the set of naturals but smaller than the set of reals) will depend on our attitude to sets themselves. On one view (associated with Kurt Gödel), the question as posed, about sizes of perfectly determinate entities (subsets of the real numbers) is perfectly meaningful and must have an answer, so if our current axioms do not answer it we’d better look for additional answers that will. On another view, this realist approach is hopelessly naïve – the concept of ‘set’ with which this apparently meaningful question is posed lacks sufficient determinacy to determine a correct answer. Indeed, on some views of mathematical theories,the axioms of our theories should be viewed as contextual definitions of their primitive terms. According to such an account, all there is to being a set is to be one of a system of objects satisfying the ZFC axioms (with the primitive terms ‘set’ and ‘member’ appropriately interpreted as about those objects). If so, then given that there are models of the axioms in which the continuum hypothesis is true, and models in which the continuum hypothesis is false, there is simply *no fact of the matter* as to whether CH is true of ‘the’ sets, the truths about the sets being those claims that are true in all models of the axioms. (Such an account, incidentally, provides a neat answer to the question of why mathematical axioms are treated as beyond question, whereas premises in philosophical arguments are not. If our axioms are implicit definitions, then there is no place for the skeptical question, ‘I see what you’re saying about the sets in these axioms, but *are sets really like that?*’. Sets, on this view, just are whatever (if anything) satisfies the axioms.) Less extreme versions of the ‘no fact of the matter’ thesis will hold that, while there is more to sets than what’s set out in the ZFC axioms (an intuitive concept of set as in the iterative conception that the axioms are intended to characterize), still our intuitive concept of set remains a vague one, indeterminate between various candidate concepts corresponding to different conceptions of the axioms. So insofar as mathematicians are involved in searching for new axioms to help to answer the continuum hypothesis, they should be thought of as engaged, not in answering a previously perfectly determinate question, but in developing a new concept of set within which the continuum hypothesis can be given meaning and a truth value (all the while realising that an alternative concept could have been developed (and indeed might still fruitfully be developed) that would provide a different answer to CH).

So what attitude *should* we take to open questions in set theory such as CH? Maddy’s naturalist, or second-philosophical (Maddy 2007), answer is that we should cut our metaphysical cloth to fit the practice of mathematicians. So if considerations internal to the practice speak in favour of mathematicians assuming that there is a single correct answer to CH, and seeking to uncover correct answer to this question, then we should provide a philosophical account of the sets that makes sense of this assumed objectivity. On the other hand, if the practice assumes that there is no single right way of going on, and that multiple different extensions of the concept of set are permissible, each providing their own answer to CH, then so too should we. Looking at that practice, Maddy first notes that, given that mathematicians use classical logic, ‘CH or not-CH’ is a theorem, and concludes from this that we should take either CH or its negation to be true of the set theoretic universe *V*. But this doesn’t seem quite enough: ‘CH or not-CH’ could be accepted as a theorem even if we adopted a supervaluationist semantics for set theory, holding that the truths about the sets are those that hold in all models of ZFC (or, perhaps, all acceptable extensions of the ZFC axioms). Then ‘CH or not-CH’ will be true even though neither CH nor its negation is. However, Maddy has more to say than this, which speaks against a supervaluationist semantics and in favour of seeing mathematicians as aiming to speak truly about a single universe *V* of sets. Examining the practice of searching for new axioms to establish CH and other open questions, Maddy finds that the considerations that are appealed to in justifying their choices to consider particular ways of extending ZFC seem to support an *objectivist* interpretation of mathematicians as engaged in looking for the *right* way to extend ZFC, not just picking one consistent extension amongst many options. In particular, in defending their proposals for new axioms, mathematicians argue that their choices are right on grounds of ‘mathematical depth’: axioms are adopted because they are fruitful; help to unify – particularly in set theory via its foundational goal; help to illuminate or explain phenomena; and a range of other extrinsic features that could be counted as showing that the tightening of our concept of set via a proposed new axiom produces something mathematically deep. So, Maddy holds, the practice (at least as things currently stand)[[1]](#footnote-1) supports the assumption that there is a right way to extend our axioms so as to refine our concept of set and produce an answer to the Continuum Hypothesis, a right way to answer the question of what is true in *V*, *the* universe of sets, this right way being determined on objective grounds of mathematical depth.

This naturally returns us to the skeptical worry we mentioned earlier regarding taking pragmatic virtues as justification for axioms. Let us grant that some axioms may be uncontroversially deep, having all the extrinsic virtues in spades. Nevertheless, what grounds have we for saying that the *deep* choices of axioms are the *true* ones? If our axioms are aiming to describe an independently existing universe *V* of sets, why should we think that the truths about those sets correspond to what we take to be mathematically interesting/deep? Couldn’t the truth about sets just be messy? It is here where Maddy’s second philosophy comes in. Noting that the norms guiding mathematical practice just do not seem to allow the possibility for the gap that a robust realism about sets would require between ‘mathematically fruitful’ and ‘really true’, Maddy concludes that it is the philosophy that should give here, rather than the mathematical practice. As such, she holds, Robust Realism must give way to an alternative version of realism according to which there is no such gap. Thus, Maddy advocates a position she calls ‘Thin Realism’, according to which sets are just the things that set theoretic methods are suited to tell us about. Probing this a little further, Maddy notes that our set theoretic methods aim to be responsive to objective facts about mathematical depth. So, Maddy proposes, we should conclude that mathematical objects – the objects that we find out about through the correct application of set theoretic methods – just are those objects described by mathematically deep theories. Thus:

the objective ‘something more’ that our set-theoretic methods track is these underlying contours of mathematical depth. Of course the simple answer—they track sets—is also true, so what we’ve learned here is that what sets are, most fundamentally, is markers for these contours, what they are, most fundamentally, is maximally effective trackers of certain strains of mathematical fruitfulness. … They mark off a mathematically rich vein within the indiscriminate network of logical possibilities. (82-3)

If we think of conceptual space as potentially including all logically possible axiom systems, then the mathematical objects just are those objects described by the axiom systems amongst those possibilities that track what Maddy calls ‘the topography of mathematical depth’ (80). To exist as a mathematical object, then, is to be quantified over in a mathematically deep theory . So once we have determined that a theory satisfies the requirements of mathematical depth, there is no further question to be answered about whether the objects of that theory *really exist*, or whether the axioms of that theory are *really true*.

By closing the gap between ‘mathematically fruitful’ and ‘really true’ in this way, Maddy ensures that her Thin Realism is not open to the standard objections to Platonism that plague ‘Robust Realism’ (and that she tried to respond to on behalf of the Robust Realist in her earlier work (particularly 1990’s *Realism in Mathematics*). But is this ‘Thin Realism’ realism enough? Maddy’s 1990 defence of Robust Realism was driven in part by the assumption that Quinean indispensability considerations required a robust view of mathematical objects as on a par with other theoretical posits, confirmed in their existence by their role in empirical thoeries. Part of the work of 1997’s *Naturalism in Mathematics*, however, was to throw the indispensability considerations into question and argue that mathematics neither has – nor needs – any external justification for its truth. If Maddy is right about this then the only reasons we have for believing our mathematical theories are internal mathematical reasons. As such, it seems, Maddy’s ‘Thin Realism’ is as much realism that is supported or required by the facts of mathematical and scientific practice.

On the other hand, though, one may wonder whether the facts of mathematical and scientific practice that Maddy wishes to respect support realism of any sort, even of the thin variety. Maddy herself has argued that the use of mathematics in science doesn’t support the truth of mathematical theories or the existence of mathematical objects. ‘Thin Realism’ provides a view of mathematical existence and truth that assumes that it is a feature of mathematical practice that mathematicians are engaged in finding out the truth about really existing mathematical objects. But must we see this concern with truth and existence as part of the practice of mathematics? Couldn’t we equally make sense of a practice that is driven by considerations of developing fruitful mathematical concepts and working out the consequences of the axioms that describe such concepts by holding that the truth of mathematical axioms is simply not something that mathematicians are particularly concerned with? If mathematics is about characterizing interesting/fruitful/deep concepts and characterizing those concepts axiomatically, and then working out the consequences of the axioms that one has developed, then rather than identify the true theories with the mathematically deep ones that mathematicians care about, why not simply conclude that mathematicians are not concerned with truth (following, perhaps, Bertrand Russell’s deductivist characterization of mathematics as ‘the subject in which we never know what we are talking about, nor whether what we are saying is true’ (Russell, 1901, p. 1577)). Shouldn’t Thin Realism give way to the view Maddy calls *Arealism*, according to which mathematical practice involves the development and probing of objectively fruitful/mathematically deep concepts, but incurs no ontological commitment whatsoever?

Perhaps the most interesting element of Maddy’s extremely stimulating discussion in *Defending the Axioms* is her response to the apparent standoff between a Thin Realist and an Arealist construal of mathematical practice. Rather than trying to defend Thin Realism, Maddy agrees that Arealism is equally consistent with what we know of mathematical practice. But rather than reverting to Arealism (on the grounds, perhaps, of Occam’s razor), or insisting on Thin Realism (on the grounds, perhaps, that one ought to interpret the utterances of practitioners as aiming at truth), or even concluding that the facts of the matter about the ontology of mathematics are simply *unknowable* in light of the standoff, Maddy concludes that there are no facts of the matter about the existence of mathematical objects out there to be found. This is not a new position: Mark Balaguer (1999), for example, argues for such a position in light of the standoff between Full Blooded Platonism and Fictionalism that he develops in his *Platonism and Anti-Platonism in Mathematics*, and more generally there has been in recent years a steady growth of neo-Carnapian approaches to metaontology that have tried to resurrect Carnap’s (1950) concern that ontological debates somehow rest on a mistake. What is interesting, though, is how Maddy’s ‘no fact of the matter’ diagnosis arises out of a second philosophical study of philosophical inquiry into truth and existence, and how this compares to her similar study of mathematical enquiry into sets.

We have to hand both a philosophical and a mathematical question. The philosophical question asks, do sets (or other mathematical objects) exist? The mathematical question asks, assuming that there are sets, are there sets whose cardinality lies properly between the cardinality of the natural numbers and the cardinality of the reals? It is a truism that the answers to each of these questions will depend, in part, on what is meant by their key terms – by words such as ‘set’, ‘cardinality, ‘exists’. In suggesting that there may be no fact of the matter concerning the latter question – the continuum hypothesis—those in the ‘no fact’ camp argue that the problem is with the meaning of the term set – that the concept of set is not sufficiently determinate to provide an answer to CH. There may be various ways of refining the concept of set to answer the question one way or the other, but, they hold, there is no one *right* way of developing the concept, so no one *right* answer to CH as it now stands. As we have noted, Maddy does not take this line in relation to CH, holding instead that the objective contours of mathematical depth may well be sufficient to determine that, even if our current conception of set is insufficient to provide an answer to CH, there is but one objectively correct way of developing the concept so as to answer the question. In the philosophical case, however, Maddy takes the opposite view. Her reason to think that there may be no fact of the matter about whether Robust Realism or Arealism is the correct ontological position is that the concept of ‘exists’, even when taken, as the naturalist advocates, from the use of ‘exists’ within natural science, is not sufficiently determinate to provide us with an answer.

Well why isn’t it? Wasn’t the point of the Quinean naturalism that motivates Maddy’s second philosopher exactly to move us away from the worry that there is something slippery about the *philosophical* concept of existence, by adopting instead the perfectly ordinary use of the word *exists* as utilized within natural science (as common sense extended)? Thus as naturalists we start our enquiry into what there is by looking at natural science and the paradigm cases of existents taken from there. We realise that the reasons we have to believe in the ordinary objects of observation also extend to theoretical objects such as electrons, also part of the best explanation of our experience, though further removed from it than tables and chairs. So we count such objects as existing also. And how about the objects posited by our mathematical theories? Should our concept of existence apply to those too? It is on this question, Maddy tells us, that the Thin Realist and Arealist diverge:

as the Second Philosopher conducts her inquiry into the way the world is, beginning with her ordinary methods of perception and observation, theory-formation and testing, she’s eventually faced with the effectiveness of pure mathematics and elects to add it to her ever-growing list of investigations; she also recognizes that the appropriate methods are different and that the objects studied are different; the point at issue hinges on what she concludes from this. If the new objects seem a bit odd—non-spatiotemporal, acausal, etc.—but still enough like the old—singular bearers of properties, etc.—, if the new methods seem a bit odd, but still of-a-piece with the old, then she concludes that she’s made a surprising discovery, that the world includes abstracta as well as concreta. If, on the other hand, she regards the new methods and would-be objects as sharply discontinuous with what came before, she has no grounds for thinking pure mathematics is true, so she concludes that this new practice—valuable as it is—isn’t in the business of developing a body of truths. So, which is it? Is pure mathematics just another inquiry among many or is it a different sort of thing that’s immensely helpful to the others? Are the grounds cited by Cantor, Dedekind, Zermelo, and the determinacy theorists just more evidence of an unexpected sort, or are they trademarks of a different sort of activity altogether? (101-2)

Now it might be that answering this question requires refinement or clarification of our concept of ‘exists’ (just as answering CH might require refinement or clarification of our concept of ‘set’). But this does not mean that there is no correct answer to be given – there may be, as Maddy suggests there is in the ‘set’ case – an objectively correct way of refining the concept. But in this example, Maddy suggests the contrary, comparing the case to an example of Mark Wilson’s (2006), of the empirical concept of ‘ice’. Ice is frozen water, and we become familiar with the concept through paradigmatic examples. But it turns out that there are many ways that water can freeze and become solidified, forming structures that do not neatly fit our paradigmatic examples of ice. Are these structures also ice, albeit of an amorphous kind? Or should we think of them as merely ‘ice-like’? Wilson’s point, which Maddy takes up, is that once we have described the structures and said in what sense they are, and in what sense they are not, like the paradigmatic examples of ice, at that point the question of whether to call these new structures ‘ice’ is not answerable to any substantial objective facts about ice, but simply depends on an arbitrary choice we can make about whether or not to refine our ordinary concept so as to apply it to these too. Either way we come to an acceptable way of describing the situation – either as a discovery of a new kind of ice, or a discovery that water can freeze solid without forming ice – ‘a matter of decision, rather than assertion’, as Carnap (1950, 6) would have it.

Thus, according to Maddy,

…our central questions—is pure mathematics of-a-piece with physics, astronomy, psychology, and the rest? is it a body of truths? do its methods confirm its claims?—these questions have no more determinate answers than ‘is amorphous ice really ice?’ Once we understand the various ways in which water can solidify, how these processes are affected by temperature, pressure, and other factors, how the various structures generated are similar and how they’re different, there’s nothing more to know; we can reflect these facts in either way of speaking, or, to put it the other way around, neither way of speaking comes into conflict with the facts. …

Likewise, once we understand how pure mathematics developed, how it now differs from empirical sciences, once we understand the many ways in which it remains intertwined with those sciences, how its methods work and what they are designed to track—once we understand all these things, what else do we need to know? Or better, what else is there to know? … Just as amorphous ice can be classified as ice or as ice-like, mathematics can be classified as science or as science-like—and nothing in the world makes one way of speaking right and the other wrong.

The proposal, then, comes to this: Thin Realism and Arealism are equally accurate, second-philosophical descriptions of the nature of pure mathematics. They are alternative ways of expressing the very same account of the objective facts that underlie mathematical practice. (111-112)

So whereas the contours of mathematical depth may well be, in Maddy’s view, sufficient to determine the objectively correct concept of set, the same cannot be said for the contours of empirical enquiry, when it comes to the concept of existence. Just as there is no uniquely correct way of extending the concept of ‘ice’ in light of nonparadigmatic examples of frozen water, there is, Maddy claims, no uniquely correct way of extending the concept of ‘exists’ in light of nonparadigmatic examples of theoretical posits. Once we have seen that mathematics is autonomous, that its theories do not receive confirmation from their role within science but answer instead to internal mathematical standards of correctness; once we have seen that these standards introduce objectivity, both in the development of appropriate mathematical concepts and the uncovering of their consequences; these are all the facts that matter. Whether or not we then choose to say that the theories confirmed according to internal mathematical standards are true and their objects exist is really a matter of taste on which nothing of substance depends.

Having endeavoured in my own work to plough the naturalist-arealist furrow (Leng (2010)), I am surprised to find myself in broad agreement with Maddy on this point. That is, I agree completely on the relation of mathematics to empirical science, on its autonomy and confirmational independence from empirical science. And I (perhaps surprisingly) am willing to concede that while my own taste has been to restrict ‘exists’ so that we take only those objects to exists whose existence is confirmed by the standards of empirical science, it would be equally coherent to extend the term to mathematical existents – those objects that exist according to our best mathematical theories – and state the relevant contrast between mathematical and physical entities in terms of the different contexts of confirmation (where mathematical objects are confirmed as existing when they can be shown to be part of a mathematically deep theory, rather than as part of our best explanation of observed phenomena). Insofar as I disagree at all it is with Maddy’s apparent confidence that the concept of set is likely to be different in this regard, in determining just one objectively correct way of proceeding. Mathematicians working on multiverse conceptions of set theory will certainly beg to differ on this point, and though Maddy does acknowledge the possibility that considerations of depth will speak in favour of multiple developments of the concept of set rather than a single unique set theoretic universe, it is less clear to me that we should be so optimistic about the prospects for a single right answer, though perhaps it is compatible with the ‘striking difference’ between mathematics and philosophy noted at the start to hold that conceptual space in mathematics is at least a little bit neater, with more obviously correct ways to proceed. Insofar as Maddy’s main philosophical point stands, though, concerning the freedom we have in deciding how to extend our concept ‘exists’, does this mean that the long debate over Platonism and anti-Platonism has simply been a disagreement over terms all along? I don’t think so. Note that peace has broken out, to the extent that it has, between Thin Realism and Arealism. The real work, to which Maddy and, I hope, I too have contributed in previous work has been in resisting the Robust Realism (of Maddy 1990) that many took to be supported by Quinean considerations, and which gives rise to the Benacerrafian worries that make Platonism so problematic in the first place. Only with Robust Realism set aside can the question of whether one adopts a Thin Realism or Arealism come down to a mere terminological dispute, but with Robust Realism set aside, perhaps we can be happy that what remains is the mere question of how to define our terms.

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1. The qualifier is important here. Maddy notes (p. 63 n.7) that future developments in mathematics may change this assessment if, for example, we were to come up with equally attractive but mutually incompatible ways of extending the concept of set, and as a result drop set theory’s foundational goal. [↑](#footnote-ref-1)