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Friction Causing Unpredictability

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Abstract

We study the effect of friction on the dynamics of a classical point particle in a one-dimensional double-well potential. It is shown that finite uncertainty in the initial conditions of the particle may prevent us from reliably predicting the well in which the particle will come to rest. This difficulty – to make reliable long-term predictions – originates from the layered structure of phase-space regions sending the particle to the left and the right well, respectively. Similar structures are known to arise, for example, in models used to described the tossing of a coin where friction is, however, not the root cause of the phenomenon.

1 Introduction

The difficulty to reliably predict the behaviour of a classical dynamical system is usually related to the existence of fractal structures in the mathematical model describing the system. Conservative non-integrable systems such as three interacting planetary bodies [1] and chaotic dissipative systems such as Lorenz's model of the atmosphere [2] provide two well-known cases in point. Add, for example, a third body to the integrable system of two planetary bodies interacting through gravitation. The KAM theorem [3] describes how the original foliation of the system's phase-space into tori is being replaced by a highly intricate mixed phase space. Arbitrarily small balls of initial conditions may contain both *periodic* trajectories and *irregular* ones which separate at an exponential rate. The non-linearity present in the Lorenz model gives rise to a strange attractor [4]. Its properties dominate the long-term evolution of the system since trajectories with neighboring initial conditions are likely to visit different regions of phase space at comparable later times.

Repeatedly tossing a coin and recording the outcomes – "head" or "tail" – is commonly thought to create a random binary sequence. Attempts to justify this claim in terms of chaotic behaviour of the type just described have not been successful: simple mechanical models of a tossed coin neither lead to conservative non-integrable systems nor to chaotic dissipative systems. Typically, no fractal phase-space structures are found which would support the creation of randomness.

Keller [5], for example, models a coin by a massive line segment which moves in a two-dimensional plane under the influence of gravity. The orientation of the segment is described by an angle increasing at a constant rate, independent of the center-of-mass motion. Depending on the value of the angle when the line reaches the ground, a head or a tail is recorded. Clearly, small variations of the initial conditions will not change the outcome of the toss, except when the coin lands on its rim. Thus, no fractal structures will emerge but a layered pattern of "zebra-stripes" associated with initial conditions resulting in heads or tails, respectively. Finite balls of initial conditions correspond to imprecisely prepared initial states. Such balls will, when straddling across more than one stripe, contain pre-images of both heads and tails, effectively making it impossible to predict the final state of the coin.

A more realistic model [6] describes the coin as a massive three-dimensional object with rotational degrees of freedom, including air resistance as well. Ignoring inelastic bounces off the floor, the main conclusion does not change: the dynamics of the coin is not chaotic in a strict sense. Instead, any sufficiently large spread of initial conditions is compatible with both head and tails, making the outcome of an individual toss effectively unpredictable.

The purpose of this paper is to present a particularly simple scenario which also produces unpredictable final states if initial conditions are known only with finite accuracy. Starting from an *integrable* system with multiple stable equilibria, we will show that the addition of friction may *create* basins of attraction with intricate boundaries. The merits of the model are its simplicity – only a single degree of freedom is necessary – and in the fact that the unpredictability is solely due to the presence of friction.

Sec. 2 of this paper provides an initial, qualitative explanation of why adding friction to an integrable system with two or more stable equilibria can lead to the difficulty of predicting its final state. Then, in Sec. 3, we investigate the motion of a particle moving in a piece-wise constant double-well potential. For two different types of friction, we study the structure of the basins of attraction. We summarize and discuss our results in Sec. 4.

2 Basins of attraction in a double-well potential

To illustrate how friction can make the prediction of the long-term evolution of a system difficult, we consider a classical particle moving along a straight line in the presence of a symmetric double-well potential. The system is described by the Hamiltonian function

$$H(p,q) = \frac{p^2}{2m} + W(q)\,, \quad p,q \in \mathbb{R}\,, \tag{1}$$

where q and p denote position and the momentum of the particle, respectively. The minima of the potential $W(q)$ are located at $q = q_\pm$, separated by a barrier of height

$W_0 \equiv W(0)$ which defines the *critical* energy, $E_c \equiv W_0$. The phase-space diagram of the system is shown in Fig. 1, displaying the familiar types of trajectories. The minima L and

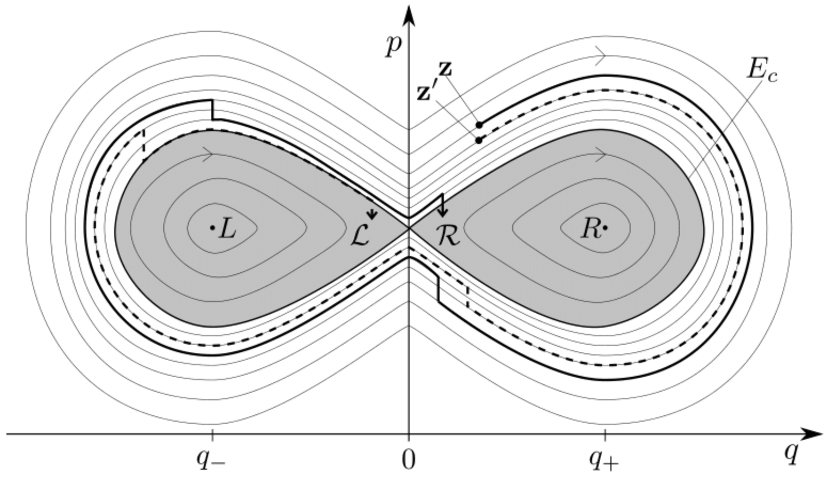

Figure 1: Phase-space structure of a double-well potential $W(q)$ with minima L and R, located within wells $\mathcal{L}$ and $\mathcal{R}$ (shaded areas), respectively; a particle with neighbouring initial conditions $\mathbf{z}, \mathbf{z}'$ (full and dashed lines, respectively) may end up in different wells when subjected to periodic, dissipative "kicks" which reduce its momentum and thus its energy.

R of the potential are *stable fixed points* each surrounded by *periodic orbits* with energies E not exceeding the critical value, $0 < E < E_c \equiv W_0$. For $E = E_c$, the particle may rest at the *unstable fixed point* at $q = 0$, or travel on one of the two *separatrices* connected to it. The trajectories with energy above the critical value, $E > E_c$ are *periodic*, encircling both minima on a single round trip. With a single degree of freedom and the energy $H(p, q)$ as a conserved quantity, the system is *integrable*, leading to the global foliation of its phase space into one-dimensional tori.

Adding friction will modify all trajectories except when then particle initially rests at one of the three fixed points. If located on a separatrix or on any periodic trajectory with energy less than E_c, dissipation will cause the particle to "spiral" into either the left or the right minimum of the potential $W(q)$, depending on its original position relative to the origin, $q = 0$. The particle cannot escape from a well once it has been trapped, and the fixed points L and R turn into *attractors*.

The destiny of a particle with initial energy $E > E_c$, however, it is not immediately obvious since it may end up in either well. Friction will inevitably "draw" the particle towards the location of the separatrices of the unperturbed system. At some time, the energy of the particle will drop below the critical value E_c. The position of the particle relative to the origin at the time of the drop will determine whether it becomes trapped in the left or in the right well.

For simplicity, let us assume that friction acts at discrete times only, repeatedly reducing the momentum of the particle by a constant factor. Suppose that for initial conditions $\mathbf{z} = (p, q)^T$, the particle will – after a possibly long time – settle in the *right* well as illustrated in Fig. 1. Intuitively, a slightly smaller initial momentum (see $\mathbf{z}'$ in the figure) could cause the particle to negotiate the barrier one less time and to settle in the *left* well instead. The slight change in the initial condition has thus altered the long-term behaviour of the system. Therefore, the finally state of a particle may become unpredictable from a *practical* point of view, i.e. whenever its initial conditions are known to lie within a small but finite volume of phase space only.

More formally, the non-Hamiltonian equations of motion map an initial state $\mathbf{z}(t_0)$ to a new value $\mathbf{z}(t)$ at time t,

$$\mathbf{z}(t_0) \mapsto \mathbf{z}(t), \quad \mathbf{z} \equiv \begin{pmatrix} p \\ q \end{pmatrix}, \tag{2}$$

leading to a *decrease* of the energy defined in (1): $E_0 \to E < E_0$. To ascertain whether a particle with initial state $\mathbf{z}(t_0)$ ends up near L or near R, one needs to determine the earliest time t such that its energy E falls below the critical value,

$$E < E_c \,. \tag{3}$$

Repeating this calculation for *all* initial conditions will divide the phase space into two disjoint sets known as *basins of attraction* which encode whether the particle ends up in well $\mathcal{L}$ or $\mathcal{R}$. Let us investigate the structure of their boundaries for two models of friction, using a particularly simple double-well potential.

3 Piece-wise constant double-well potential with friction

The double-well potential considered here is based on a "particle in a box" defined by two infinitely high potential walls at $q = \pm\ell$ which restrict motion to a line segment of length 2ℓ. The particle bounces off the walls elastically resulting in an instantaneous reversal of its momentum: $p \to -p$; its position $q = \pm\ell$ remains unchanged when hitting a wall. A piece-wise constant potential,

$$W(q) = \begin{cases} W_0\,, & |q| < \varepsilon\,, \\ 0\,, & \varepsilon < |q| < \ell\,, \end{cases} \tag{4}$$

models the smooth double-well. For simplicity, we take an arbitrarily thin potential barrier, corresponding to $\varepsilon \to 0$. The only impact of this "infinitesimal" barrier is to confine the particle in a well once its energy drops below the critical value W_0, thus creating the wells $\mathcal{L}$ and $\mathcal{R}$. Two continuous sets of potential minima exist because the bottom of the potential is flat.

A widespread method to investigate *non-integrable* systems is to start with an integrable system and add a perturbation, be it a time-independent potential term as in the KAM theorem or a deterministic time-dependent force [7]. To support our claim that friction generically causes intricate phase space structures, we will model it in two different ways which are inspired by the approaches just mentioned. In the first case, the elastic collisions of the particle with the boundary walls are made inelastic (cf. Sec. 3.1) while an impulsive friction force is applied periodically in the second case (cf. Sec. 3.2). The first model depends on a single parameter only, the *coefficient of restitution.* The second model depends on two parameters, the *frequency* and the *strength* of the dissipative "kick."

3.1 Inelastic collisions

The motion of the particle in the piece-wise constant double well (4) consists of free motion between the walls interspersed with momentum-reversing elastic collision at the walls. The dynamics changes fundamentally upon replacing the elastic collisions at the walls by inelastic ones, characterized by a *coefficient of restitution*, $r \in (0,1)$:

$$\mathbf{z} \mapsto \mathbf{z}' = \mathbf{R}\cdot\mathbf{z}\,, \quad \mathbf{R} \equiv \begin{pmatrix} -r & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{for} \quad q = \pm\ell\,. \tag{5}$$

This minor change turns the conservative system into a dissipative one and – as we will see – is sufficient to create an embryonic form of "zebra-stripes" in phase space.

The long-term dynamics of the particle does not depend on its initial *position*: all particles with fixed momentum p_0 but arbitrary position $q_0 \in (-\ell, \ell)$ will experience the same amount of friction, only to end up in same well. Thus, let us assume that the particle starts out with positive initial momentum $p_0 > p_c$, beings located at $q_0 = \ell_-$, i.e. just to left of the right wall. Then, the initial state $\mathbf{z}_0 = (p_0, \ell_-)$ at time t_0 evolves according to

$$\mathbf{z}(t_n) = \mathbf{R}^n \cdot \mathbf{z}_0 = \begin{pmatrix} (-r)^n p_0 \\ \ell_- \end{pmatrix}\,, \qquad n \in \mathbb{N}_0\,, \tag{6}$$

with the times t_n being defined by particle returning to its initial position ℓ_-. Monitoring the value of its momentum at the walls is sufficient to determine the well which will trap the particle. The particle will be trapped in well $\mathcal{R}$, for example, if its last collision at the right wall makes its energy drop below the critical value $E = E_c$ due to $p \mapsto (-r)p$.

For positive initial momentum p_0 the particle will hit the right wall first. The well to finally trap the particle is determined by the number of collisions $n(E_0)$ before it drops below E_c. Denoting the energy of the particle after n collisions by E_n, we need to find the number $n(E_0)$ such that the energy of the particle falls below E_c for the first time,

$$n(E_0) = \min_{n\in\mathbf{N}} \{E_n < E_c\}\,, \quad \text{with } E_0 > E_c\,. \tag{7}$$

Using Eq. (6) the number $n(E_0)$ is easily found to be

$$n(E_0) = \left\lceil \frac{1}{2}\frac{\ln(E_c/E_0)}{\ln r} \right\rceil = \left\lceil \frac{\ln(p_c/p_0)}{\ln r} \right\rceil\,, \tag{8}$$

where the initial moment defines the initial energy, $E_0 = p_0^2/2m$, and $\lceil x \rceil$ is the ceiling function extracting the smallest integer greater or equal to the number x. If n is odd (even), a particle with positive momentum p_0 will end up in the well on the right (left). The basins of attraction for the wells $\mathcal{L}$ and $\mathcal{R}$ are given by alternating horizontal bands in phase space shown in Fig. 2. The widths of the bands decrease with decreasing friction (and they increase with energy E which the figure does not show due to the limited momentum range).

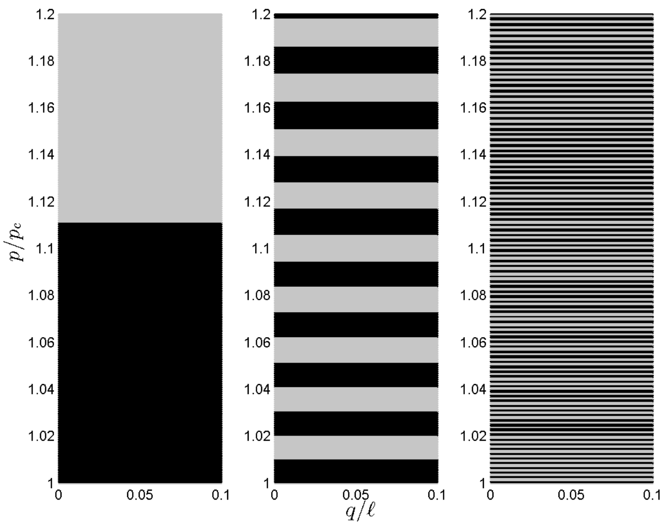

Figure 2: Basins of attraction for the phase-space region ($1 \leq p/p_c \leq 1.2$, $0 \leq q/\ell \leq 0.1$) of a particle of unit mass in a double-well, with friction arising from inelastic collisions at the boundary walls: initial conditions located in dark (light) regions will end up in the right (left) well of the potential. The vertical bars correspond to different values of the coefficient of restitution: $r = 0.9$, $r = 0.99$, $r = 0.999$ (left to right). Smaller values of friction lead to increasing "complexity" of the basin boundaries, in the sense of producing narrower bands.

If the initial conditions (p_0, q_0) of a particle are known *exactly*, then the deterministic dynamics leads to a unique and well-defined final state which can be predicted with certainty. However, limited precision of the initial conditions may results in a genuine indeterminacy of the final state. Assume that the initial state of the particle is only known to lie inside a rectangle with sides $\Delta q > 0$ and $\Delta p > 0$, centered about the point $\mathbf{z}_0$. Trajectories with initial momenta p_0 and $p_0' \equiv r p_0$ are bound to end up in different wells. Thus, if the inaccuracy in momentum exceeds this value,

$$\Delta p > p_0 - p_0' , \tag{9}$$

the uncertainty rectangle will cut across *at least two* adjacent basins of attraction. In other words, given the initial momentum p_0 and any finite uncertainty Δp about it, the prediction of the final state becomes impossible for a coefficient of restitution in the interval

$$1 - \frac{\Delta p}{p_0} < r < 1 , \tag{10}$$

since the rectangle with sides Δq and Δp will contain trajectories destined for the wells $\mathcal{L}$ and $\mathcal{R}$. We conclude that sufficiently weak inelasticity prevents the reliable prediction of the final state. In this well-defined sense, adding friction to an integrable system provides a mechanism which prevents accurate long-term predictions.

3.2 Periodic damping force

Now we turn to a model where friction is caused by a periodic, dissipative force which acts during a short time interval only. It will be convenient to consider the limit of an *instantaneous* action which multiplies the momentum of the particle by a constant factor $\gamma \in (0,1)$ at times $T_k = kT$, with $k \in \mathbb{N}$, and a free parameter T. This approach is analogous to periodically kicking a system with a deterministic force which, for a particle moving freely on a ring known as a "rotor," produces *deterministic chaotic* motion [3]. Since our model depends on *two* independent parameters, γ and T, we expect more complicated basins of attraction compared to the model with inelastic reflections.

To construct the basins of attraction of the wells $\mathcal{L}$ and $\mathcal{R}$, we need to determine when, for arbitrary initial conditions $(p_0, q_0)^T$, the energy of the particle falls below the critical value for the first time We then record whether, at that moment of time, it is located to the left or to the right of the origin, i.e. within $\mathcal{L}$ or $\mathcal{R}$. For simplicity, the particle is assumed to begin its journey at time $t = 0^+$, i.e. just after $t = 0$, with *positive* momentum $p_0 > p_c$ and arbitrary initial position $q_0 \in (-\ell, \ell)$.

The particle moves freely during intervals of length T, with perfectly elastic collisions occurring at the boundary walls which only change the sign of its momentum. An expression for its time evolution in closed form can be found if we "unfold" the trajectory by imagining identical copies of the double-well to be arranged along the position axis. Instead of being reflected at the right wall located at $q = \ell$, the particle enters the next double well, which occupies the range $(\ell, 3\ell)$, and continues to move to the right, etc. In this setting, the momentum does not change its sign when the particle moves from one double well to the adjacent one. The sign of its momentum in the original double well is negative (or positive) if the particle has reached the s^{th} copy of the double well, with $s \in \mathbb{N}$ being odd (or even).

To determine the dynamics of the system over one period of length T, we combine the free motion with the dissipative kicks:

1. during the motion of the particle from $t = 0^+$ to just before the first kick at time $t = T$, its phase-space coordinates are given by

$$\mathbf{z}(t) = \begin{pmatrix} p_0 \\ q_0 + p_0 t/m \end{pmatrix} \equiv \mathbf{F}(t) \cdot \mathbf{z}_0 \,, \qquad \mathbf{F}(t) = \begin{pmatrix} 1 & 0 \\ t/m & 1 \end{pmatrix} \,, \qquad t \in (0^+, T^-) \,, \tag{11}$$

   where $q \in (0, \infty)$ due to the unfolding;

2. the dissipative kick at time T reduces the momentum of the particle by the factor $\gamma \in (0,1)$,

$$\mathbf{z}(T^+) = \begin{pmatrix} \gamma p(T^-) \\ q(T^-) \end{pmatrix} \equiv \mathbf{D} \cdot \mathbf{z}(T^-) \,, \qquad \mathbf{D} = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \,, \qquad t \in (T^-, T^+). \tag{12}$$

To obtain the *actual* position and momentum of the particle inside the box at time t, we map (or "fold back") the expression $\mathbf{F}(t) \cdot \mathbf{z}$ to the interval $q \in (-\ell, \ell)$, by writing

$$\mathbf{z}(t) = \begin{pmatrix} (-)^{s(t)} p_0 \\ [(q_0 + p_0 t/m) \bmod 2\ell] - \ell \end{pmatrix}, \qquad t \in (0^+, T^+), \tag{13}$$

where the value of the integer $s(t)$ is determined by writing $q + pt/m = \overline{q}(t) + 2\ell s$, with $\overline{q}(t) \in (-\ell, +\ell)$. The momentum p changes sign whenever the "unfolded" coordinate passes through the values $\ell, 3\ell, 5\ell, \ldots$

The time evolution of the initial state $\mathbf{z}_0$ from time $t = 0^+$ to $t = T_k^+ \equiv (kT)^+$, i.e. just after the kick with label k, follows from concatenating Eqs. (11) and (12) k times,

$$\mathbf{z}(T_k^+) = \left(\mathbf{D} \cdot \mathbf{F}(T^-)\right)^k \cdot \mathbf{z}_0 \equiv \begin{pmatrix} \gamma & 0 \\ \gamma T/m & 1 \end{pmatrix}^k \cdot \mathbf{z}_0 = \begin{pmatrix} \gamma^k & 0 \\ \sigma_k(\gamma) T/m & 1 \end{pmatrix} \cdot \mathbf{z}_0, \tag{14}$$

where

$$\sigma_k(\gamma) = \frac{1 - \gamma^k}{1 - \gamma}, \qquad k \in \mathbb{N}. \tag{15}$$

In analogy to Eq. (13), the "true" coordinates of the particle inside the box are obtained as

$$\mathbf{z}(T_k^+) = \begin{pmatrix} (-)^{s(t)} \gamma^k p_0 \\ [(q_0 + \sigma_k(\gamma) p_0 T/m) \quad \bmod 2\ell] - \ell \end{pmatrix}, \tag{16}$$

assuming that, after k kicks, the energy $E_k = p_k^2/2m$ of the particle has not yet dropped below the critical value E_c.

We are now in the position to determine which initial conditions $\mathbf{z}_0$ will send the particle to the left and the right well, respectively. Using Eq. (16), we first determine the smallest value of k which reduces the energy of the particle below the critical value, $E_k < E_c$, or

$$k_c = \left\lceil \frac{1}{2} \frac{\ln(E_c/E_0)}{\ln \gamma} \right\rceil = \left\lceil \frac{\ln(p_c/p_0)}{\ln \gamma} \right\rceil, \qquad k_c \in \mathbb{N}, \tag{17}$$

assuming, of course, that $p_0 > p_c$. This relation structurally resembles the result (8), with the number k_c of dissipative kicks playing the role of the number of inelastic collisions n_c. The sign of the position coordinate after k_c kicks, $q(T_{k_c}^+)$, follows from Eq. (16) and determines whether the particle is trapped in $\mathcal{L}$ or $\mathcal{R}$. The explicit dependence of $\mathbf{z}(T_k^+)$ on the initial *position* implies that changes in q_0 may also produce different final states, in contrast to the model studied in Sec. 3.1.

Fig. 3, which has been generated numerically on the basis of Eq. (16), illustrates these conclusions. The first vertical bar visualizes the basins of attraction associated with the wells $\mathcal{L}$ and $\mathcal{R}$, respectively. The expected dependence on both initial momentum and position becomes clearly visible in the magnifications which also reveal that the boundaries of the apparently irregular basins of attractions are not fractal.

The boundaries of the basins can be found directly from Eq. (16): all initial conditions (p_0, q_0) mapped to a fixed value of position at time kT are located on lines of the form

$$p_0(q_0) = -\frac{m}{\sigma_k(\gamma) T} q_0 + \text{const} \simeq -\frac{1}{k}\frac{m}{T} q_0 + \text{const}, \qquad \gamma \lesssim 1, \tag{18}$$

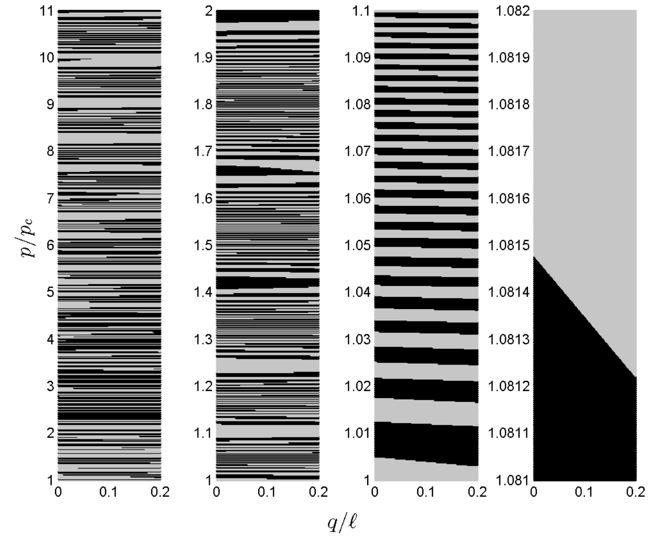

Figure 3: Basins of attraction for the phase-space region $(1 \leq p/p_c \leq 11, 0 \leq q/\ell \leq 0.2)$ of a particle with unit mass in a double-well, with friction arising from *periodic dissipative kicks* at times kT, $k \in \mathbb{N}$, with $\ell = 1$, $T = 100s$ and $\gamma = 0.99$: initial conditions located in dark (light) regions will end up in the right (left) well of the potential. Each of the three vertical bars on the right magnifies a horizontal strip of the bar to its left by a factor of ten. The basin boundaries clearly exhibit both a momentum *and* a position dependence. Each bar results from iterating 501×501 regularly spaced initial conditions inside the area shown.

using $\sigma_k(\gamma) \simeq (1/k) + \mathcal{O}(1-\gamma)$, which holds for weak damping, i.e. for γ approaching the value one from below. Consequently, the boundaries of the basins of attraction are *straight lines* in phase space just as for the model with inelastic reflections off the wall. The lines are no longer horizontal but their slope approaches the value zero if a large number of kicks is required for the particle to settle in a well.

Assume once again that the initial conditions of the particle can be prepared with *finite precision* only, i.e. they lie inside a phase-space rectangle with area $\Delta q \Delta p > 0$ and center $\mathbf{z}_0$. For any finite imprecision one can always find a damping strength γ such that at least one basin boundary crosses the rectangle; this is sufficient to prevent the prediction of the well to finally trap the particle. For large initial momenta p_0, the reasoning behind the derivation of the inequality (10) also applies here since the strips constituting the basins of attraction will, typically, have almost horizontal boundaries. Thus, for any initial conditions (p_0, q_0) and finite uncertainties, damping strengths within the interval

$$1 - \frac{\Delta p}{p_0} < \gamma < 1 \tag{19}$$

correspond to a situation with an unpredictable final state. Occasionally, the uncertainty rectangle with sides Δq and Δp may cover an area where a slight change in initial *position* only causes the particle to reach different wells, corresponding to a more intricate phase-space structure.

4 Summary and discussion

We have shown that adding friction to an integrable one-dimensional double-potential well causes its dynamics to turn from integrable to unpredictable – at least in the presence of imprecise initial conditions. For simplicity, the double well has been modeled as a piece-wise constant potential, i.e. a "box" divided into two regions by a thin wall. Two types of dissipative forces have been considered which, by reducing its initial energy, cause the particle to necessarily settle in one the wells after a finite, possibly long time. The main result of our study is that adding friction to an integrable system produces layered basins of attraction with smooth boundaries.

Damping introduced by *inelastic collisions* of the particle with the confining walls results in basins which foliate the phase space of the system into horizontal layers of variable width. The stripes get narrower for *decreasing* friction. Any ball of initial conditions which extends beyond more than one stripe prevents us from predicting with certainty the well in which the particle will finally settle. *Periodic dissipative kicks* create basins of attraction with slightly more intricate boundaries, due to the additional position dependence. Since the particle must settle in a well after *finite* time the observed structures cannot be fractal. In practice, however, it is crucial whether the initial conditions can be specified with sufficient accuracy to avoid a spread across basins which send the particle to different final states.

Both models clearly demonstrate that adding friction to an integrable system with two attractive wells can have a fundamental impact on long-term predictability. The motion is not "deterministically random" which would require fractal phase-space structures. However, if the accuracy of the initial conditions falls below a specific threshold, the final state of the system cannot be predicted reliably. Experimentally, the precision required for a reliable long-term prediction may well be out of reach.

We expect our conclusions to be structurally stable in the sense that they should not depend on the model of friction used. Any dissipative mechanism will, firstly, contract all initial conditions into a small phase-space region which is energetically just above the barrier of the double well; secondly, the energy of the particle will drop below E_c in a way which depends sensitively on the initial conditions. Continuous Stokes friction, for example, is thus likely to generate similar basins of attraction.

It is interesting to highlight the differences of the scenario studied here to model systems leading to similar conclusions. Keller [5] introduces a model of a massive rotating wheel which slows down due to the presence of a constant torque opposing its motion. Again, imprecise initial conditions lead to the difficulty of predicting in which position the wheel will come to rest. This result seems to justify the use of the wheel as a gambling device at fairs. However, the model predicts that, once at rest, the wheel would continue to rotate in the opposite direction which makes it unrealistic. In any case, friction does *not* cause foliation of phase space in this model but it is also based on a single degree of freedom only.

Isomäki et al. [8] study the motion of a particle moving along the positive real axis only, in the presence of two forces which result in more complicated behaviour including fractal structures. A linear force attracts the particle towards the origin from which it bounces off inelastically when hitting it. In addition, a periodic driving force acts on the particle. The combination of these forces gives rise to different types of limiting behaviour and associated basins of attraction with fractal boundaries. In the absence of the both friction and the driving force, the system exhibits only a single stable equilibrium, contrary to the double-well potential studied. Thus, adding only friction would not reproduce the phenomenon observed for the double well.

Basins of attraction with smooth boundaries may also arise by washing out existing fractal structures, through the addition of friction. This phenomenon has been described for a spherical pendulum with three stable equilibrium positions, in the presence of gravity [9]. In this scenario, however, friction is responsible for the smoothing of pre-existing fractal structures not for causing the basins in the first place.

The interest of layered phase-space structures with smooth boundaries is based on the fact no macroscopic physical system is actually capable of exhibiting fractal structures. With quantum mechanical properties of matter emerging on an atomic scale, the description of a physical system in terms of classical mechanics cannot hold on *arbitrary* length scales [10]. Experimentally observed structures may be highly intricate over many – but not all – orders of magnitude.

To conclude, we mention a natural application of our main result connecting it to work by Poincaré. Consider a large number of identical potential wells succeeding each other on a ring, (37 or 38 in number, say), mimicking a one-dimensional roulette wheel. Effectively, the final state of a particle moving in this potential will – in the presence of friction – become unpredictable given a finite spread in initial momenta and positions. This system provides an explicit and tractable dynamical realization of a model which was introduced by Poincaré in order to explain the emergence of probability. His roulette-like wheel, which slows down and stops under the influence of friction [11], continues to be relevant to philosophically inclined discussions of the notion of chance (e.g. in [12]).

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