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# Inhomogeneous theory of dual Diophantine approximation on manifolds

*Dedicated to Bob Vaughan on his 65th birthday*

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**Abstract**

The theory of inhomogeneous Diophantine approximation on manifolds is developed. In particular, the notion of nice manifolds is introduced and the divergence part of the Groshev type theory is established for all such manifolds. Our results naturally incorporate and generalize the homogeneous measure and dimension theorems for non-degenerate manifolds established to date. The results have natural applications beyond the standard inhomogeneous theory such as Diophantine approximation by algebraic integers.

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## 1. Introduction

### *1.1. Extremality, the Khintchine-Groshev theory and beyond*

Throughout $\mathbb{R}^+ = (0, +\infty)$, $|\cdot|$ denotes the supremum norm, $\|\cdot\|$ is the distance to the nearest integer and $\mathbf{a}\cdot\mathbf{b} := a_1b_1 + \ldots + a_nb_n$ is the standard inner product of vectors $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$.

The point $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is called *very well approximable* (abbr. *VWA*) if there exists $\varepsilon > 0$ such that

$$\|\mathbf{a}\cdot\mathbf{y}\| < |\mathbf{a}|^{-(1+\varepsilon)n} \tag{1}$$

holds for infinitely many $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. By Dirichlet's theorem, when $\varepsilon = 0$ for all $\mathbf{y} \in \mathbb{R}^n$ inequality (1) holds for infinitely many $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Thus, the essence of the definition of very well approximable points is that for these points the exponent within (1) can be improved beyond the trivial.

A relatively straightforward application of the Borel-Cantelli Lemma yields that almost every point $\mathbf{y} \in \mathbb{R}^n$ is not VWA. However, restricting $\mathbf{y}$ to a proper submanifold $\mathcal{M}$ of $\mathbb{R}^n$ introduces major difficulties in attempting to describe the measure theoretic structure of the VWA points $\mathbf{y} \in \mathcal{M}$. Essentially, it is this investigation that has given rise to the now flourishing area of 'Diophantine approximation on manifolds' within metric number theory.

Diophantine approximation on manifolds dates back to the 1930's with a conjecture of K. Mahler [53] in transcendence theory. Using the above terminology, the conjecture states that almost all points on the Veronese curve

$$\mathcal{V}_n := \{(x, \ldots, x^n) : x \in \mathbb{R}\}$$

are not VWA. Mahler's conjecture remained a key open problem in metric number theory for over thirty years and was eventually solved by Sprindžuk [57]. Moreover, its solution led Sprindžuk [58] to make an important general conjecture. He claimed that any analytic non-degenerate[4] submanifold of $\mathbb{R}^n$ satisfies a similar property which we now make precise. A differentiable manifold $\mathcal{M}$ in $\mathbb{R}^n$ is said to be *extremal* if almost all points of $\mathcal{M}$ (with respect to the natural Riemannian measure on $\mathcal{M}$) are not VWA.

[4]The notion of non-degeneracy will be formally introduced below.

Related, but far more delicate problems arise when, instead of (1), one considers the inequality

$$\|\mathbf{a}\cdot\mathbf{y}\| < \Pi_+(\mathbf{a})^{-1-\varepsilon}, \tag{2}$$

where

$$\Pi_+(\mathbf{a}) = \prod_{i=1}^{n} \max\{1, |a_i|\}.$$

The point $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is called *very well multiplicatively approximable* (abbr. *VWMA*) if there exists $\varepsilon > 0$ such that (2) holds for infinitely many $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. A differentiable manifold $\mathcal{M}$ in $\mathbb{R}^n$ is said to be *strongly extremal* if almost all points of $\mathcal{M}$ are not VWMA. It is easily verified that any VWA point $\mathbf{y}$ is VWMA and so any strongly extremal manifold is extremal. Baker [3] suggested the far-reaching generalisation of Mahler's problem that Veronese curves are strongly extremal. This was later extended to manifolds by Sprindžuk [58]:

BAKER-SPRINDŽUK CONJECTURE: *Any analytic non-degenerate submanifold of* $\mathbb{R}^n$ *is strongly extremal.*

This fundamental conjecture was proved in 1998 by Kleinbock & Margulis in their landmark paper [50] for arbitrary (not necessarily analytic) non-degenerate manifolds. Essentially, non-degenerate manifolds are smooth sub-manifolds of $\mathbb{R}^n$ which are sufficiently curved so as to deviate from any hyperplane. Formally, a manifold $\mathcal{M}$ of dimension $m$ embedded in $\mathbb{R}^n$ is said to be *non-degenerate* if it arises from a non-degenerate map $\mathbf{f} : U \to \mathbb{R}^n$ where $U$ is an open subset of $\mathbb{R}^m$ and $\mathcal{M} := \mathbf{f}(U)$. The map $\mathbf{f} : U \to \mathbb{R}^n : \mathbf{u} \mapsto \mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \ldots, f_n(\mathbf{u}))$ is said to be *$l$-non-degenerate at* $\mathbf{u} \in U$ if $\mathbf{f}$ is $l$ times continuously differentiable on some sufficiently small ball centred at $\mathbf{u}$ and the partial derivatives of $\mathbf{f}$ at $\mathbf{u}$ of orders up to $l$ span $\mathbb{R}^n$. The map $\mathbf{f}$ is *$l$-non-degenerate* if it is $l$-non-degenerate at almost every (in terms of $m$-dimensional Lebesgue measure) point in $U$; in turn the manifold $\mathcal{M} = \mathbf{f}(U)$ is also said to be $l$-non-degenerate. Finally, we say that $\mathbf{f}$ is *non-degenerate* if it is $l$-non-degenerate for some $l$; in turn the manifold $\mathcal{M} = \mathbf{f}(U)$ is also said to be non-degenerate. It is well known that any real connected analytic manifold which is not contained in any hyperplane of $\mathbb{R}^n$ is non-degenerate.

Without a doubt, the proof of the Baker-Sprindžuk conjecture has acted as the catalyst for the subsequent development of the homogeneous theory of Diophantine approximation on manifolds. In particular, the notion of extremality has been generalised to and established for other classes of manifolds including complex analytic manifolds [48], support of measures [49], $p$-adic and more generally the $S$-arithmetic framework [52] and for systems of linear forms [16, 51].

The Khintchine-Groshev theory is a delicate refinement of the theory of extremal manifolds obtained by replacing the right hand side of (1) with a monotonic function of $|\mathbf{a}|$ or $\Pi_+(\mathbf{a})$, or more generally with a multivariable approximating function $\Psi(\mathbf{a})$. Formally, the function $\Psi : \mathbb{R}^n \to \mathbb{R}^+$ such that

$$\Psi(a_1, \dots, a_n) \geqslant \Psi(b_1, \dots, b_n) \quad \text{if } |a_i| \leqslant |b_i| \text{ for all } i = 1, \dots, n\,, \tag{3}$$

is referred to as a *multivariable approximating function.* In the special case when $\Psi(\mathbf{a}) = \psi(|\mathbf{a}|)$ or $\Psi(\mathbf{a}) = \psi(\Pi_+(\mathbf{a}))$ for a monotonic function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ we simply refer to $\psi$ as an *approximating function.*

Given a multivariable approximating function $\Psi$, let

$$\mathcal{W}_n(\Psi) \;:=\; \left\{ \mathbf{y} \in \mathbb{R}^n : \begin{array}{l} \|\mathbf{a} \cdot \mathbf{y}\| < \Psi(\mathbf{a}) \\ \text{for infinitely many } \mathbf{a} \in \mathbb{Z}^n \setminus \{0\} \end{array} \right\}. \tag{4}$$

For obvious reasons points $\mathbf{y}$ in $\mathcal{W}_n(\Psi)$ are referred to as $\Psi$-*approximable.* When $\Psi(\mathbf{a}) = \psi(|\mathbf{a}|)$ we naturally write $\mathcal{W}_n(\psi)$ for $\mathcal{W}_n(\Psi)$. The Khintchine-Groshev fundamental theorem [28, §2.3] in the theory of metric Diophantine approximation provides a beautiful and simple criterion for the 'size' of $\mathcal{W}_n(\psi)$ expressed in terms of $n$-dimensional Lebesgue measure $|\ .\ |_n$. Essentially, for any approximating function $\psi$

$$|\mathcal{W}_n(\psi)|_n = \begin{cases} \text{Zero} & \text{if} \quad \sum_{t=1}^{\infty} t^{n-1}\psi(t) < \infty, \\ \text{Full} & \text{if} \quad \sum_{t=1}^{\infty} t^{n-1}\psi(t) = \infty. \end{cases} \tag{5}$$

Here 'Full' simply means that the complement of the set under consideration is of measure zero. Many years later, and building upon the work of Jarník, this criterion was generalized to incorporate Hausdorff measures [38]. For background, precise statements and generalizations the reader is refereed to [12, 14, 18, 22] and references within.

As with extremality, the starting point for developing the Khintchine-Groshev type theory for manifolds $\mathcal{M}$ was to study the case of Veronese curves $\mathcal{V}_n$. The following analogue of (5) for $\mathcal{W}_n(\psi) \cap \mathcal{M}$ with $\mathcal{M} = \mathcal{V}_n$ was formally conjectured by Baker in [3] and took nearly twenty fives years to establish:

$$|\mathcal{W}_n(\psi) \cap \mathcal{M}|_{\mathcal{M}} = \begin{cases} \text{Zero} & \text{if} \quad \sum_{t=1}^{\infty} t^{n-1}\psi(t) < \infty, \\ \text{Full} & \text{if} \quad \sum_{t=1}^{\infty} t^{n-1}\psi(t) = \infty. \end{cases} \tag{6}$$

Here and elsewhere $|\ .\ |_{\mathcal{M}}$ denotes the induced Lebesgue measure on $\mathcal{M}$. By definition, $|X|_{\mathcal{M}} = \text{Full}$ means that the measure of the complement of $X$ on $\mathcal{M}$ is zero. The convergence case of the above statement was proved in [24] and the divergence case was proved in [6]. More generally, the analogue of (6) have been established for non-degenerate manifolds $\mathcal{M}$ in [7, 29] for convergence and in [13] for divergence. See also [10, 42, 43, 44, 45] for the analogues statements in the case of affine subspaces and their submanifolds. It is worth emphasizing that [29] deals with the multiplicative aspects of the Khintchine-Groshev theory for convergence. Namely, the authors show that for any non-degenerate manifold $\mathcal{M} \subset \mathbb{R}^n$ and any multivariable approximating function $\Psi$

$$|\mathcal{W}_n(\Psi) \cap \mathcal{M}|_{\mathcal{M}} = \text{Zero} \qquad \text{if} \qquad \sum_{\mathbf{a} \in \mathbb{Z}^n \setminus \{0\}} \Psi(\mathbf{a}) < \infty. \tag{7}$$

In particular, when $\Psi(\mathbf{a}) = \psi(\Pi_+(\mathbf{a}))$ for some approximating function $\psi$ the left hand side of (7) holds whenever $\sum_{t=1}^{\infty} t^{n-1}\psi(t)\log^{n-1} t < \infty$.

Beyond the Khintchine-Groshev theory for manifolds, in which the size of the sets $\mathcal{W}_n(\Psi) \cap \mathcal{M}$ is measured in terms of Lebesgue measure, it is natural to develop the 'deeper' Hausdorff theory in which size is measured in terms of Hausdorff measures and dimension. Once again, investigating the case of Veronese curves $\mathcal{V}_n$ laid the foundations for the Hausdorff theory. In particular, for $v \geqslant 0$ consider the approximating function $\psi_v(t) = t^{-v}$ and write $\mathcal{W}_n(v)$ for $\mathcal{W}_n(\psi_v)$. In 1970, Baker & Schmidt [5] proved that

$$\frac{n+1}{v+1} \leqslant \dim(\mathcal{W}_n(v) \cap \mathcal{V}_n) \leqslant 2 \times \frac{n+1}{v+1} \qquad \text{for any } v > n. \tag{8}$$

In the same work, Baker & Schmidt claimed that the left hand side of (8) is the precise value for $\dim(\mathcal{W}_n(v) \cap \mathcal{V}_n)$. This challenging Baker-Schmidt problem was eventually solved in 1983 by Bernik [25]. A few years prior to this, R.C. Baker [4] had proved an analogue of the Baker-Schmidt problem for non-degenerate curves in $\mathbb{R}^2$. More recently, Dickinson & Dodson [37] have proved that for any extremal submanifold $\mathcal{M}$ of $\mathbb{R}^n$ one has the lower bound

$$\dim\left(\mathcal{W}_n(v)\cap\mathcal{M}\right) \geqslant \frac{n+1}{v+1} \qquad \text{for any } v>n. \tag{9}$$

In the case that $\mathcal{M}$ is a non-degenerate curve in $\mathbb{R}^n$, Beresnevich, Bernik & Dodson [11] have proved equality in (9) under the assumption that $n \leqslant v < n+\frac{1}{4n}$. Verifying equality in (9) for arbitrary non-degenerate manifolds and any $v>n$ represents a major open problem.

Generalised Baker-Schmidt problem (GBSP) : *Prove (or less likely disprove) that for any non-degenerate manifold $\mathcal{M}$ in $\mathbb{R}^n$ and any $v>n$ one has that*

$$\dim\left(\mathcal{W}_n(v)\cap\mathcal{M}\right) = \frac{n+1}{v+1} + \dim\mathcal{M} - 1. \tag{10}$$

This statement is know to be true for manifolds $\mathcal{M}$ satisfying certain geometric conditions that impose 'strong' constraints on the dimension and co-dimension of $\mathcal{M}$ which in turn totally excludes the situation that $\mathcal{M}$ is a non-degenerate curve– see [28]. Indeed, the key to establishing GBSP is to verify it for non-degenerate curves and arbitrary $v$. Foliation techniques can then be used to deal with general situation of manifolds of arbitrary dimension. The following is the generalised Baker-Schmidt problem for Hausdorff measures and naturally incorporates the Khintchine-Groshev theory; i.e. the Lebesgue theory.

GBSP for Hausdorff Measures : *Prove that for any non-degenerate manifold $\mathcal{M}$ in $\mathbb{R}^n$, any approximating function $\psi$ and any $s>m-1$, where $m=\dim\mathcal{M}$, one has that*

$$\mathcal{H}^s(\mathcal{W}_n(\psi)\cap\mathcal{M}) = \begin{cases} 0 & \text{if} \quad \displaystyle\sum_{t=1}^{\infty} t^n\left(\frac{\psi(t)}{t}\right)^{s+1-m} < \infty, \\[2ex] \mathcal{H}^s(\mathcal{M}) & \text{if} \quad \displaystyle\sum_{t=1}^{\infty} t^n\left(\frac{\psi(t)}{t}\right)^{s+1-m} = \infty. \end{cases} \tag{11}$$

The case $s = m$ reduces to (6) and is thus known. The case $s > m$ is trivial. For $s < m$, the divergence case of (11) has been established in [14, Theorem 18]. However, for $s < m$, the convergence case represents completely unexplored territory. Indeed, unlike the dimension statement given by (10), the convergence case of (11) is not known for either Veronese curves $\mathcal{V}_n$ or non-degenerate curves in $\mathbb{R}^2$.

To complete the overview of recent developments in the homogeneous theory of Diophantine approximation on manifolds we direct the reader to [2, 9, 14, 15, 16, 19, 23, 51, 60] and references within.

*1.2. Inhomogeneous approximation and main results*

This paper constitutes part of an ongoing programme to develop a coherent inhomogeneous theory of Diophantine approximation on manifolds in line with the homogeneous theory. In the case of simultaneous approximation on planar curves, the programme has successfully been carried out in [17]. Here we deal with the dual approximation aspect of the programme.

Given a multivariable approximating function $\Psi$ and a function $\theta : \mathbb{R}^n \to \mathbb{R}$, define the set

$$\mathcal{W}_n^\theta(\Psi) := \left\{ \mathbf{y} \in \mathbb{R}^n : \begin{array}{l} \|\mathbf{a}\cdot\mathbf{y} + \theta(\mathbf{y})\| < \Psi(\mathbf{a}) \\ \text{for infinitely many } \mathbf{a} \in \mathbb{Z}^n \setminus \{0\} \end{array} \right\}. \tag{12}$$

For obvious reasons points $\mathbf{y}$ in $\mathcal{W}_n^\theta(\Psi)$ are referred to as $(\Psi, \theta)$-*approximable* and when $\Psi(\mathbf{a}) = \psi(|\mathbf{a}|)$ we naturally write $\mathcal{W}_n^\theta(\psi)$ for $\mathcal{W}_n^\theta(\Psi)$. In the case the function $\theta$ is constant, the set $\mathcal{W}_n^\theta(\Psi)$ corresponds to the familiar inhomogeneous setting within the general theory of dual Diophantine approximation. In turn, with $\theta \equiv 0$ the corresponding set reduces to the homogeneous discussed above.

Until the recent proof of the inhomogeneous Baker-Sprindžuk conjecture [20, 21], the theory of inhomogeneous Diophantine approximation on manifolds had remained essentially non-existent and ad-hoc – see [1, 27, 30, 61, 62]. As a consequence of the measure results in [21] we now know that for any non-degenerate manifold $\mathcal{M}$ and $\theta \equiv$ constant,

$$|\mathcal{M} \cap \mathcal{W}_n^\theta(\Psi_\epsilon)|_{\mathcal{M}} = 0 \qquad \forall \quad \epsilon > 0 \ , \tag{13}$$

where $\Psi_\varepsilon(\mathbf{a}) = \Pi_+(\mathbf{a})^{-1-\varepsilon}$. The primary goals of this paper are (i) to develop a metric theory for the sets $\mathcal{M}\cap\mathcal{W}_n^\theta(\Psi)$ akin to the Khintchine-Groshev theorem, and (ii) to obtain the lower bounds for the Hausdorff dimension/measure in the inhomogeneous setting akin to (9) and (11).

Our first result provides a zero Lebesgue measure criterion for $\mathcal{M}\cap\mathcal{W}_n^\theta(\Psi)$. It represents the complete inhomogeneous version of the main result of [29] and it implies (13) without imposing the condition that the 'inhomogeneous' function $\theta : \mathbb{R}^n \to \mathbb{R}$ is constant. Throughout, $\theta|_{\mathcal{M}}$ will denote the restriction of the inhomogeneous function $\theta$ to $\mathcal{M}$ and as usual, $C^{(n)}$ will denote the set of $n$-times continuously differentiable functions.

**Theorem 1.** *Let $\mathcal{M}$ be an $l$-non-degenerate manifold in $\mathbb{R}^n$ $(n \geqslant 2)$ and $\theta : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\theta|_{\mathcal{M}} \in C^{(l)}$. Let $\Psi$ be a multivariable approximating function. Then*

$$\left|\mathcal{W}_n^\theta(\Psi) \cap \mathcal{M}\right|_{\mathcal{M}} = 0 \qquad \textit{if} \qquad \sum_{\mathbf{a}\in\mathbb{Z}^n\setminus\{0\}} \Psi(\mathbf{a}) < \infty.$$

For the divergence counterpart, we are able to prove the more general statement in terms of $s$-dimensional Hausdorff measure $\mathcal{H}^s$. However, there is a downside in that we impose a 'convexity' condition on $\Psi$ which we refer to as property $\mathbf{P}$. For an $n$-tuple $\mathbf{v} = (v_1, \ldots, v_n)$ of positive numbers satisfying $v_1 + \ldots + v_n = n$, define the $\mathbf{v}$-*quasinorm* $|\cdot|_{\mathbf{v}}$ on $\mathbb{R}^n$ by setting

$$|\mathbf{y}|_{\mathbf{v}} := \max_{1\leqslant i\leqslant n} |y_i|^{1/v_i} .$$

*A multivariable approximating function $\Psi$ is said to satisfy property $\mathbf{P}$ if $\Psi(\mathbf{a}) = \psi(|\mathbf{a}|_{\mathbf{v}})$ for some approximating function $\psi$ and $\mathbf{v}$ as above.* Trivially, with $\mathbf{v} = (1, \ldots, 1)$ we have that $|\mathbf{a}|_{\mathbf{v}} = |\mathbf{a}|$ and we see that any approximating function $\psi$ satisfies property $\mathbf{P}$, where $\psi$ is regarded as the function $\mathbf{a} \mapsto \psi(|\mathbf{a}|)$.

**Theorem 2.** *Let $\mathcal{M}$ be a non-degenerate manifold in $\mathbb{R}^n$ of dimension $m$ and let $s > m - 1$. Let $\theta : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\theta|_{\mathcal{M}} \in C^{(2)}$ and $\Psi$ be a multivariable approximating function satisfying property $\mathbf{P}$. Then*

$$\mathcal{H}^s(\mathcal{W}_n^\theta(\Psi) \cap \mathcal{M}) = \mathcal{H}^s(\mathcal{M}) \qquad \textit{if} \qquad \sum_{\mathbf{a}\in\mathbb{Z}^n\setminus\{0\}} |\mathbf{a}| \left(\frac{\Psi(\mathbf{a})}{|\mathbf{a}|}\right)^{s+1-m} = \infty.$$

The above theorem will be derived from a general statement which significantly broadens the scope of potential applications and is of independent interest. Given a manifold $\mathcal{M} \subset \mathbb{R}^n$, an $n$-tuple $\mathbf{v} = (v_1, \ldots, v_n)$ of positive numbers satisfying $v_1 + \ldots + v_n = n$, $\delta > 0$ and $Q > 1$, let

$$\Phi_{\mathbf{v}}(Q, \delta) = \Big\{ \mathbf{y} \in \mathcal{M} : \exists\, \mathbf{a} \in \mathbb{Z}^n \setminus \{0\} \text{ such that } \|\mathbf{a} \cdot \mathbf{y}\| < \delta Q^{-n} \ \&\ |\mathbf{a}|_{\mathbf{v}} \leqslant Q \Big\}.$$

As a consequence of Dirichlet's theorem, $\Phi_{\mathbf{v}}(Q, \delta) = \mathcal{M}$ if $\delta \geq 1$. We say that the manifold $\mathcal{M}$ is $\mathbf{v}$-*nice at* $\mathbf{y}_0 \in \mathcal{M}$ if there is a neighborhood $\Omega \subset \mathcal{M}$ of $\mathbf{y}_0$ and constants $0 < \delta, \omega < 1$ such that for any ball $B \subset \Omega$ we have that

$$\limsup_{Q \to \infty} |\Phi_{\mathbf{v}}(Q, \delta) \cap B|_{\mathcal{M}} \leqslant \omega |B|_{\mathcal{M}} \, .$$

The manifold is said to be $\mathbf{v}$-*nice* if it is $\mathbf{v}$-nice at almost every point in $\mathcal{M}$. Furthermore, the manifold is said to be *nice* if it is $\mathbf{v}$-nice for all choices of $\mathbf{v}$.

**Theorem 3.** *Let $\mathcal{M}$ be a $\mathbf{v}$-nice $C^{(2)}$ manifold in $\mathbb{R}^n$ of dimension $m$ and let $s > m - 1$. Let $\theta : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\theta|_{\mathcal{M}} \in C^{(2)}$ and $\Psi(\mathbf{a}) = \psi(|\mathbf{a}|_{\mathbf{v}})$ for some approximating function $\psi$. Then*

$$\mathcal{H}^s(\mathcal{W}_n^\theta(\Psi) \cap \mathcal{M}) = \mathcal{H}^s(\mathcal{M}) \qquad \text{if} \qquad \sum_{\mathbf{a} \in \mathbb{Z}^n \setminus \{0\}} |\mathbf{a}| \left( \frac{\Psi(\mathbf{a})}{|\mathbf{a}|} \right)^{s+1-m} = \infty.$$

A consequence of Lemma 4 in §3.1 is that non-degenerate manifolds are nice. Thus

$$\text{Theorem 3} \quad \Longrightarrow \quad \text{Theorem 2.}$$

*1.3. Remarks and Corollaries*

*Remark 1.* For $s < m$, the non-degeneracy of $\mathcal{M}$ in Theorem 2 can be relaxed to the condition that there exists at least one non-degeneracy point on $\mathcal{M}$. Also, note that $\mathcal{H}^s(\mathcal{M}) = \infty$ when $s < m$.

*Remark 2.* It follows from the definition of Hausdorff measure that

$$\mathcal{H}^s(\mathcal{W}_n^\theta(\psi) \cap \mathcal{M}) \leq \mathcal{H}^s(\mathcal{M}) = 0$$

for any $s > m$ irrespective of $\Psi$. Thus the meat of Theorem 2 is when $s \leq m$.

*Remark 3.* To the best of our knowledge, Theorem 2 with $\mathcal{M} = \mathbb{R}^n$ and $\theta \not\equiv$ constant is new. In other words, the theorem makes a new contribution even to the classical theory of Diophantine approximation of independent variables.

*Remark 4.* Theorem 1 holds for arbitrary multivariable approximating function and so represents a genuine strengthening of the inhomogeneous Baker-Sprindžuk conjecture established in [21]. Unfortunately, Property P does not hold for arbitrary multivariable approximating function. Indeed, it excludes any 'multiplicative' approximating function $\Psi(\mathbf{a}) = \psi(\Pi_+(\mathbf{a}))$, where $\psi : \mathbb{N} \to \mathbb{R}^+$ is monotonic. We emphasize that removing Property P from the statement of Theorem 2 is an open challenging problem even in the homogeneous case and planar curves.

*Remark 5.* Consider the problem of Diophantine approximation on the Veronese curves $\mathcal{M} := \{(x, x^2, \ldots, x^n) : x \in \mathbb{R}\}$, where $n \geqslant 2$. Take $\theta(x, \ldots, x^n) = x^{n+1}$. Then the inequality in (12) becomes

$$|x^{n+1} + a_n x^n + \ldots + a_1 x + a_0| < \Psi(\mathbf{a})\,.$$

Clearly the function $\theta$ as defined above is $C^{(\infty)}$. In the case when $\Psi(\mathbf{a}) = \psi(|\mathbf{a}|)$ the corresponding divergence results have been proved by Bugeaud [35] and the corresponding convergence results by Bernik & Shamukova [31, 59]. Theorems 1 and 2 naturally extend their results to the case of multivariable approximating functions $\Psi$.

We now discuss various corollaries of our main theorems which are of independent interest. The following statement is a direct consequence of Theorem 2 and the fact that any approximating function $\psi$ satisfies property **P**.

**Corollary 1.** *Let $\mathcal{M}$ be a non-degenerate manifold in $\mathbb{R}^n$ of dimension $m$ and $s > m-1$. Let $\theta : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\theta|_{\mathcal{M}} \in C^{(2)}$ and $\psi$ be an approximating function. Then*

$$\mathcal{H}^s(\mathcal{W}_n^\theta(\psi) \cap \mathcal{M}) = \mathcal{H}^s(\mathcal{M}) \qquad \textit{if} \qquad \sum_{t=1}^{\infty} t^n \left(\frac{\psi(t)}{t}\right)^{s+1-m} = \infty.$$

In the case of curves this corollary was first established in [1]. In the case $s = m$, the Hausdorff measure $\mathcal{H}^s$ is comparable to the induced $m$-dimensional Lebesgue measure $|\ .\ |_{\mathcal{M}}$ on $\mathcal{M}$ and Corollary 1 represents the complete inhomogeneous version of the main result of [13]. Furthermore, Theorem 1 together with Corollary 1 provides a simple criterion for the 'size' of $\mathcal{W}_n^\theta(\psi) \cap \mathcal{M}$ expressed in terms of the induced measure; i.e. the desired inhomogeneous Groshev type theorem for manifolds. More precisely and more generally, under the hypotheses of Theorem 1 we have that for any $\Psi$ satisfying property **P**

$$|\mathcal{W}_n^\theta(\Psi) \cap \mathcal{M}|_{\mathcal{M}} = \begin{cases} 0 & \text{if} \quad \sum_{\mathbf{a}\in\mathbb{Z}^n\setminus\{0\}} \Psi(\mathbf{a}) < \infty \\ |\mathcal{M}|_{\mathcal{M}} & \text{if} \quad \sum_{\mathbf{a}\in\mathbb{Z}^n\setminus\{0\}} \Psi(\mathbf{a}) = \infty \, . \end{cases}$$

In the case $s < m$, Corollary 1 naturally generalizes the homogeneous result of [14, Theorem 18].

Given an approximating function $\psi$, the *lower order* $\tau_\psi$ of $1/\psi$ is defined by

$$\tau_\psi := \liminf_{t\to\infty} \frac{-\log\psi(t)}{\log t}$$

and indicates the growth of the function $1/\psi$ 'near' infinity. With this definition at hand, it is relatively easy to verify that the divergent sum condition of Corollary 1 is satisfied whenever $s < m-1+(n+1)/(\tau_\psi+1)$. It follows from the definition of Hausdorff measure and dimension that $\dim(\mathcal{W}_n^\theta(\psi)\cap\mathcal{M}) \geqslant s$ if $\mathcal{H}^s(\mathcal{W}_n^\theta(\psi) \cap \mathcal{M}) > 0$ and $\mathcal{H}^s(\mathcal{M}) > 0$ if $s \leq \dim \mathcal{M}$. Thus, Corollary 1 readily yields the following inhomogeneous version of the dimension result of [37].

**Corollary 2.** *Let $\mathcal{M}$ be a non-degenerate manifold in $\mathbb{R}^n$ of dimension $m$ and $\theta : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\theta|_{\mathcal{M}} \in C^{(2)}$. Let $\psi$ be an approximating function such that $n \leq \tau_\psi < \infty$. Then*

$$\dim \mathcal{W}_n^\theta(\psi) \cap \mathcal{M} \ \geqslant\ m - 1 + \frac{n+1}{\tau_\psi + 1} \, . \tag{14}$$

In the case that $\theta \equiv$ constant and $\psi(t) := t^{-\tau}$ with $\tau > n$, this dimension statement corresponds to the main result of [32]. However, Corollary 1 implies the stronger measure statement that $\mathcal{H}^s(\mathcal{W}_n^\theta(\psi) \cap \mathcal{M}) = \infty$ at $s = m-1+(n+1)/(\tau+1)$ which in all likelihood is the critical exponent. In a wider context, it would not be unreasonable to expect that the above lower bound for $\dim \mathcal{W}_n^\theta(\psi) \cap \mathcal{M}$ is in fact sharp. Even within the homogenous setting, establishing equality in (14) represents a key open problem. To date the homogeneous problem has been settled by Bernik [25] for Veronese curves and by R.C. Baker [4] for non-degenerate planar curves. For non-degenerate curves in $\mathbb{R}^n$ the current results are limited to situation that $\tau_\psi \le n + \frac{1}{4n}$ – see [11]. Most recently, the inhomogeneous version of Baker's result has been established in [1]. In other words, if $\mathcal{M}$ is a non-degenerate planar curve then in (14) we have equality.

*1.4. Possible developments*

*Affine subspaces and their submanifolds.* By definition, any manifold contained in a proper affine subspace of $\mathbb{R}^n$, in particular any affine subspace of $\mathbb{R}^n$, is degenerate everywhere and so Theorems 1 & 2 are not applicable. Nevertheless the 'extremal' theory of homogeneous Diophantine approximation for such manifolds has been developed in [46, 47]. Furthermore, the homogeneous Groshev type theorems for planes in $\mathbb{R}^n$ and their submanifolds have been established in [10, 42, 43, 44, 45]. A natural problem is to develop the analogous inhomogeneous theory.

*The p-adic setting.* The homogeneous Groshev type theorems have recently been established in [54, 55] for the '$S$-arithmetic' setting. This builds upon the 'extremality' results of Kleinbock and Tomanov [52] and includes the more familiar $p$-adic case. In all likelihood the techniques developed in this paper can be used to extend the homogeneous $S$-arithmetic results to the inhomogeneous setting. For inhomogeneous $p$-adic results restricted to Veronese curves see [27, 34, 61].

*The non-monotonic setting.* By definition, any approximating function $\psi$ is monotonic. Thus, monotonicity is implicitly assumed within the context of the classical Groshev theorem as stated in §1.2. Recently in [22], this classical result has been freed from all unnecessary monotonicity constraints. Naturally, it would be highly desirable to obtain analogous statements for Diophantine approximation on manifolds. This in full generality is a difficult problem. Even in the case $\Psi(\mathbf{a}) = \psi(|\mathbf{a}|)$, to remove the implicit monotonicity assumption from Theorems 1 & 2 is believed to be currently out of reach. For homogeneous convergent Groshev type results without monotonicity but restricted to non-degenerate curves in $\mathbb{R}^n$ see [8, 33]. In the first instance it would be interesting to extend these homogeneous results for curves to the inhomogeneous setting.

*1.5. Global assumptions and useful conventions*

In the course of proving our results we will conveniently and without loss of generality assume that the manifold $\mathcal{M}$ under consideration is immersed in $\mathbb{R}^n$ via a smooth map $\mathbf{f} = (f_1, \ldots, f_n) : U \to \mathbb{R}^n$ defined on a ball $U \subset \mathbb{R}^m$. Thus, $\mathcal{M} = \{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in U\}$. Furthermore, in view of the Implicit Function Theorem we can assume that

$$f_i(\mathbf{x}) = x_i \qquad \text{for} \quad i = 1, \ldots, m\,.$$

In other words, $\mathbf{f}$ is a Monge parameterisation of $\mathcal{M}$. Note that this implies that $\mathbf{f}$ is locally bi-Lipschitz.

Let $\mathcal{A}_{\mathbf{f}}(\Psi, \theta)$ denote the projection of $\mathcal{W}_n^\theta(\Psi) \cap \mathcal{M}$ onto $U$; that is

$$\mathcal{A}_{\mathbf{f}}(\Psi, \theta) := \{\mathbf{x} \in U \,:\, \mathbf{f}(\mathbf{x}) \in \mathcal{W}_n^\theta(\Psi)\}\,.$$

Thus, a point $\mathbf{x} \in \mathcal{A}_{\mathbf{f}}(\Psi, \theta)$ if and only if the point $\mathbf{f}(\mathbf{x}) \in \mathcal{M}$ is $(\Psi, \theta_{\mathbf{f}}(\mathbf{x}))$-approximable with $\theta_{\mathbf{f}}(\mathbf{x}) := \theta(\mathbf{f}(\mathbf{x}))$. For convenience and clarity we will drop the subscript from $\theta_{\mathbf{f}}$. In the case when $\Psi(\mathbf{a}) = \psi(|\mathbf{a}|)$ for some approximating function $\psi$ we write $\mathcal{A}_{\mathbf{f}}(\psi, \theta)$ for $\mathcal{A}_{\mathbf{f}}(\Psi, \theta)$. A consequence of the fact that $\mathbf{f}$ is locally bi-Lipschitz is that Theorems 1–3 can be equally stated in terms of $\mathcal{A}_{\mathbf{f}}(\Psi, \theta)$. Indeed the proof of the theorems will make use of this alternative formulation.

In the case of Theorem 1 the functions $\mathbf{f}$ and $\theta$ are $C^{(l)}$. Thus we can assume without loss of generality that there is a constant $C_0 > 0$ depending only on $U$, $\mathbf{f}$ and $\theta$ such that

$$\max_{0 \leqslant i \leqslant l} \sup_{\mathbf{x} \in U} |\mathbf{f}^{(i)}(\mathbf{x})| \leqslant C_0 \qquad \text{and} \qquad \max_{0 \leqslant i \leqslant l} \sup_{\mathbf{x} \in U} |\theta^{(i)}(\mathbf{x})| \leqslant C_0. \tag{15}$$

In the case of Theorems 2 & 3 the functions $\mathbf{f}$ and $\theta$ are $C^{(2)}$ and therefore without loss of generality we can assume (15) with $l = 2$.

*Notation.* The Vinogradov symbols $\ll$ and $\gg$ will be used to indicate an inequality with an unspecified positive multiplicative constant. If $a \ll b$ and $a \gg b$ we write $a \asymp b$, and say that the quantities $a$ and $b$ are comparable. We denote by $B = B(\mathbf{x}, r)$ the ball centred at $\mathbf{x} \in \mathbb{R}^m$ with radius $r$. For any real number $\lambda > 0$, we let $\lambda B$ denote the ball $B$ scaled by a factor $\lambda$; i.e. $\lambda B(\mathbf{x}, r) := B(\mathbf{x}, \lambda r)$.

## 2. The convergence theory

The goal is to prove Theorem 1. Thus, throughout $\Psi$ is a multivariable approximating function satisfying the convergent sum condition

$$\sum_{\mathbf{a}\in\mathbb{Z}^n\setminus\{0\}} \Psi(\mathbf{a}) < \infty\,. \tag{16}$$

In view of the discussion of §1.5 the goal is equivalent to establishing that $|\mathcal{A}_{\mathbf{f}}(\Psi,\theta)|_m = 0$. Note that the set $\mathcal{A}_{\mathbf{f}}(\Psi,\theta)$ can be written as

$$\mathcal{A}_{\mathbf{f}}(\Psi,\theta) = \limsup_{|\mathbf{a}|\to\infty} A_{\mathbf{f}}(\mathbf{a},\Psi,\theta) := \bigcap_{h=1}^{\infty} \bigcup_{|\mathbf{a}|\geqslant h} A_{\mathbf{f}}(\mathbf{a},\Psi,\theta),$$

where

$$A_{\mathbf{f}}(\mathbf{a},\Psi,\theta) := \{\mathbf{x}\in U : \|\mathbf{a}\cdot\mathbf{f}(\mathbf{x}) + \theta(\mathbf{x})\| < \Psi(\mathbf{a})\}\,.$$

For each $\mathbf{a}\in\mathbb{Z}^n\setminus\{0\}$ it is convenient to decompose the set $A_{\mathbf{f}}(\mathbf{a},\Psi,\theta)$ into the following two subsets

$$A^1_{\mathbf{f}}(\mathbf{a},\Psi,\theta) := \{\mathbf{x}\in A(\mathbf{a},\Psi,\theta)\ :\ |\nabla(\mathbf{f}\cdot\mathbf{a}+\theta)(\mathbf{x})| \geqslant C_1\,|\mathbf{a}|^{1/2}\,\} \tag{17}$$

and

$$A^2_{\mathbf{f}}(\mathbf{a},\Psi,\theta) := \{\mathbf{x}\in A(\mathbf{a},\Psi,\theta)\ :\ |\nabla(\mathbf{f}\cdot\mathbf{a}+\theta)(\mathbf{x})| < C_1\,|\mathbf{a}|^{1/2}\,\}\,.$$

Here $\nabla$ as usual denotes the gradient operator and

$$C_1 := \sqrt{(n+1)mC_0} \tag{18}$$

where $C_0$ is as in (15). Obviously

$$\mathcal{A}_{\mathbf{f}}(\Psi,\theta) = \mathcal{A}^1_{\mathbf{f}}(\Psi,\theta) \cup \mathcal{A}^2_{\mathbf{f}}(\Psi,\theta),$$

where

$$\mathcal{A}^i_{\mathbf{f}}(\Psi,\theta) = \limsup_{|\mathbf{a}|\to\infty} A^i_{\mathbf{f}}(\mathbf{a},\Psi,\theta) := \bigcap_{h=1}^{\infty} \bigcup_{|\mathbf{a}|\geqslant h} A^i_{\mathbf{f}}(\mathbf{a},\Psi,\theta) \qquad (i=1,2)\,.$$

The desired statement that $|\mathcal{A}_{\mathbf{f}}(\Psi,\theta)|_m = 0$ will follow by establishing the separate cases:

**Case A** $\quad |\mathcal{A}^1_{\mathbf{f}}(\Psi,\theta)|_m = 0$

**Case B** $\quad |\mathcal{A}^2_{\mathbf{f}}(\Psi,\theta)|_m = 0.$

*2.1. Establishing Case A*

The aim is to show that $|\mathcal{A}^1_{\mathbf{f}}(\Psi,\theta)|_m = 0$. This will follow as a consequence of Theorem 1.3 from [29] which is now explicitly stated using slightly different notation.

**Theorem 4** (Bernik, Kleinbock & Margulis)**.** *Let $B \subset \mathbb{R}^m$ be a ball of radius $r > 0$ and let $\mathbf{g} = (g_1, g_2, \ldots, g_{n+1}) \in C^{(2)}(2B)$. Fix $\delta > 0$ and suppose that*

$$L := \max_{1\leqslant i,j\leqslant m} \sup_{\mathbf{x}\in 2B} \left|\frac{\partial^2 \mathbf{g}(\mathbf{x})}{\partial x_i \partial x_j}\right| \ < \ \infty\,. \tag{19}$$

*Then for every $\mathbf{q} \in \mathbb{Z}^{n+1}$ such that*

$$|\mathbf{q}| \geqslant \frac{1}{4(n+1)Lr^2} \tag{20}$$

*the set of $\mathbf{x} \in B$ satisfying the system of inequalities*

$$\left\{\begin{array}{lcl} \|\mathbf{g}(\mathbf{x})\cdot\mathbf{q}\| & < & \delta \\[2ex] |\nabla \mathbf{g}(\mathbf{x})\cdot\mathbf{q}| & \geqslant & ((n+1)mL\,|\mathbf{q}|)^{1/2} \end{array}\right. \tag{21}$$

*has measure at most $K\,\delta|B|_m$, where $K$ is a constant depending only on $m$.*

With the above theorem at our disposal, consider any non-empty open ball $B$ such that $2B \subset U$. Let $\mathbf{g} = (f_1, f_2, \ldots, f_n, \theta)$ and $\mathbf{q} = (a_1, \ldots a_n, 1)$ where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n \setminus \{0\}$. Then, in view of (15), we have that (19) is automatically satisfied. Furthermore, (20) holds for all except finitely many $\mathbf{a} \in \mathbb{Z}^n \setminus \{0\}$. In view of (15) and (18), the lower bound inequality of (21) is implied by the inequality associated with (17). Therefore, $A^1_{\mathbf{f}}(\mathbf{a}, \Psi, \theta) \cap B$ is contained in the set defined by (21) with $\delta := \Psi(\mathbf{a})$. It now follows via Theorem 4, that

$$|A^1_{\mathbf{f}}(\mathbf{a}, \Psi, \theta) \cap B|_m \ll \Psi(\mathbf{a})$$

where the implied constant is independent of $\mathbf{a}$. This together with (16) and the Borel-Cantelli lemma readily implies that $|\mathcal{A}^1_{\mathbf{f}}(\Psi,\theta) \cap B|_m = 0$. Now simply observe that the open balls $B$ such that $2B \subset U$ cover the whole of $U$. The upshot is that $|\mathcal{A}^1_{\mathbf{f}}(\Psi,\theta)|_m = 0$ as required.

*2.2. Preliminaries for establishing Case B*

Establishing Case B relies upon the recent transference technique introduced in [21] and the properties of $(C, \alpha)$-good functions introduced by Kleinbock & Margulis in [50].

*2.2.1. Good functions*

The following formal definition can be found in [50].

**Definition 1.** Let $C$ and $\alpha$ be positive numbers and $f : V \to \mathbb{R}$ be a function defined on an open subset $V$ of $\mathbb{R}^m$. Then $f$ is called $(C, \alpha)$-*good on* $V$ if for any open ball $B \subset V$ and any $\varepsilon > 0$ one has that

$$\left| \left\{ x \in B \ : \ |f(x)| < \varepsilon \sup_{x \in B} |f(x)| \right\} \right|_m \leqslant C\varepsilon^{\alpha} |B|_m. \tag{22}$$

We now recall various useful properties of $(C, \alpha)$-good functions.

**Lemma 1. ([29, Lemma 3.1])**

(a) *If $f$ is $(C, \alpha)$-good on $V$ then so is $\gamma f$ for any $\gamma \in \mathbb{R}$.*

(b) *If $f$ and $g$ are $(C, \alpha)$-good on $V$ then so is* $\max\{|f|, |g|\}$.

(c) *If $f$ is $(C, \alpha)$-good on $V$ then $f$ is $(C', \alpha')$-good on $V'$ for every $C' \geqslant C, \alpha' \leqslant \alpha$ and $V' \subseteq V$.*

(d) *If $f$ is $(C, \alpha)$-good on $V$ and $c_1 \leq \frac{|f(x)|}{|g(x)|} \leq c_2$ for all $x \in V$, then $g$ is $(C(c_2/c_1)^{\alpha}, \alpha)$-good on $V$.*

The next lemma is the key tool for establishing that a given function is $(C, \alpha)$-good. The following notation is needed to state the lemma. An $m$-tuple $\beta = (\beta_1, \dots, \beta_m)$ of non-negative integers will be referred to as a multiindex and we let $|\beta|_* := \beta_1 + \ldots + \beta_m$. Given a multiindex $\beta$, let

$$\partial_\beta := \frac{\partial^{|\beta|_*}}{\partial x_m^{\beta_1} \cdots \partial x_m^{\beta_m}} \quad \text{and} \quad \partial_i^k := \frac{\partial^k}{\partial x_i^k}.$$

**Lemma 2.** **([29, Lemma 3.3])** *Let $U$ be an open subset of $\mathbb{R}^m$ and let $g \in C^{(k)}(U)$ be such that for some constants $A_1, A_2 > 0$*

$$|\partial_\beta g(\mathbf{x})| \leqslant A_1 \quad \forall\, \beta \text{ with } |\beta|_* \leqslant k, \tag{23}$$

*and*

$$|\partial_i^k g(\mathbf{x})| \geqslant A_2 \quad \forall\, i = 1, \dots, m \tag{24}$$

*for all $\mathbf{x} \in U$. Also let $V$ be a subset of $U$ such that whenever a ball $B$ lies in $V$ any cube circumscribed around $B$ is contained in $U$. Then $g$ is $(C, \frac{1}{mk})$-good on $V$ for some explicit positive constant $C$ depending on $A_1$, $A_2$, $m$ and $k$ only.*

The following proposition[5] is a generalization of Proposition 3.4 from [29].

**Proposition 1.** *Let $U$ be an open subset of $\mathbb{R}^m$, $\mathbf{x}_0 \in U$ and let $\mathcal{F} \subset C^{(l)}(U)$ be a compact family of functions $f : U \to \mathbb{R}$ for some $l \geqslant 2$ . Assume also that*

$$\inf_{f \in \mathcal{F}} \max_{0 < |\beta|_* \leqslant l} |\, \partial_\beta f(\mathbf{x}_0)| > 0\,. \tag{25}$$

*Then there exists a neighborhood $V \subset U$ of $\mathbf{x}_0$ and positive constants $C$ and $\delta$ satisfying the following property. For any $\Theta \in C^{(l)}(U)$ such that*

$$\sup_{\mathbf{x} \in U} \max_{|\beta|_* \leqslant l} |\partial_\beta \Theta(\mathbf{x})| \;\leqslant\; \delta \tag{26}$$

*and any $f \in \mathcal{F}$ we have that*

(a) *$f + \Theta$ is $(C, \frac{1}{ml})$-good on $V$,*

(b) *$|\nabla(f + \Theta)|$ is $(C, \frac{1}{m(l-1)})$-good on $V$.*

[5] In Proposition 1 we assume that $\mathcal{F}$ is compact. This assumption is not made in Proposition 3.4 of [29] although it is used in its proof. Note that the compactness of $\mathcal{F}$ does not follow from the assumption that $\{\nabla f : f \in \mathcal{F}\}$ is compact. In fact, the family $\mathcal{F}$ defined in Corollary 3.5 of [29], which is the main application of [29, Proposition 3.4], is not compact. The proof of the corollary as given in [29] is therefore incomplete. Nevertheless, the corollary as stated is correct. These issues are carried over unaddressed into Theorem 4.5 of [54]. In this paper the issues are addressed by our Proposition 1 and Corollary 3.

*Proof.* The proof is a modification of the ideas used to establish Proposition 3.4 in [29]. First of all note that in view of (25), there exists a constant $C_1 > 0$ such that for any $f \in \mathcal{F}$ one can find a multiindex $\beta$ with $0 < |\beta|_* = k \leqslant l$, where $k = k(f)$, such that

$$|\partial_\beta f(\mathbf{x}_0)| \geqslant C_1. \tag{27}$$

Since the number of different $\beta$'s is finite, without loss of generality we can assume that $\beta$ appearing in (27) is the same for all $f \in \mathcal{F}$. By an appropriate rotation of the coordinate system one can ensure that

$$|\tilde{\partial}_i^k f(\mathbf{x}_0)| \geqslant C_2 \tag{28}$$

for all $i = 1, \ldots, m$ and some positive $C_2$ independent of $f$. Here $\tilde{\partial}$ denotes differentiation with respect to the rotated coordinate system. Also, by (26) there exists a constant $c = c(l) > 1$ such that

$$\sup_{\mathbf{x}\in U} \max_{|\beta|_*\leqslant l} |\tilde{\partial}_\beta \Theta(\mathbf{x})| \leqslant c\delta. \tag{29}$$

Now take $\delta := C_2/(2c)$. Then, by (28) and (29), for any $f \in \mathcal{F}$ we have that

$$|\tilde{\partial}_i^k (f + \Theta)(\mathbf{x}_0)| \geqslant \delta \quad \text{ for all } i = 1, \ldots, m.$$

Then, by the continuity of derivatives of $f + \Theta$ and the compactness of $\mathcal{F}$, we can choose a neighborhood $V' \subset U$ of $\mathbf{x}_0$ and positive constants $A_1, A_2$ independent of $f$ such that (23) and (24) with $\partial$ replaced by $\tilde{\partial}$ hold for all $\mathbf{x} \in V'$ and all $g = f + \Theta$. Finally, let $V$ be a smaller neighborhood of $\mathbf{x}_0$ such that whenever a ball $B$ lies in $V$, the cube $\widetilde{B}$ circumscribed around $B$ is contained in $V'$. Then, on applying Lemma 2 establishes part (a) of Proposition 1.

Regarding part (b), first assume that $k$ appearing in (28) is at least 2. Since $\mathcal{F}$ is compact and differentiation is a continuous map from $C^{(l)}(U)$ to $C^{(l-1)}(U)$, we have that for every $i = 1, \ldots, m$

$$\mathcal{F}_i := \{\tilde{\partial}_i f \ : \ f \in \mathcal{F}\} \quad \text{ is compact in } C^{(l-1)}(U).$$

In view of the definition of $\mathcal{F}$ condition (25) holds when $l$ is replaced by $l - 1$ and $\mathcal{F}$ is replaced by $\mathcal{F}_i$. Therefore, the arguments used to prove part (a) apply to $\mathcal{F}_i$ and we conclude that for every $f \in \mathcal{F}_i$ the function

$\tilde{\partial}_i(f+\Theta)$ is $(C_i, \frac{1}{m(l-1)})$-good on some neighborhood $V_i$ of $\mathbf{x}_0$. It follows via Lemma 1, that $|\tilde{\nabla}(f+\Theta)|$ is $(\tilde{C}, \frac{1}{m(l-1)})$-good with $\tilde{C} = \max_i C_i$, $V = \cap_i V_i$ and $f \in \mathcal{F}$. Naturally, $\tilde{\nabla}$ denotes the gradient operator with respect to the rotated coordinate system. Now simply notice that the quantity

$$\frac{|\nabla(f+\Theta)(\mathbf{x})|}{|\tilde{\nabla}(f+\Theta)(\mathbf{x})|}$$

for all $\mathbf{x} \in V$ is bounded between two positive constants. Hence, by making use of part (d) of Lemma 1 we obtain the statement of part (b) of Proposition 1.

It remains to consider the case when $k$ appearing in (28) is equal to 1. Let $A_1, A_2$ and $V$ be defined as in the proof of part (a) above. Then,

$$A_2 \leqslant |\tilde{\nabla}(f+\Theta)(\mathbf{x})| \leqslant A_1 \quad \text{for all } \mathbf{x} \in V. \tag{30}$$

In view of part (d) of Lemma 1 and the definition of $(C, \alpha)$-good functions, to complete the proof it suffices to verify that

$$\left| \{\mathbf{x} \in B \ : \ |\tilde{\nabla}(f(\mathbf{x}) + \Theta(\mathbf{x}))| < \varepsilon \sup_{\mathbf{y} \in B} |\tilde{\nabla}(f(\mathbf{y}) + \Theta(\mathbf{y}))|\} \right|_m \leqslant \left(\frac{A_1}{A_2}\right)^{\frac{1}{l-1}} \varepsilon^{\frac{1}{l-1}} \, |B|_m \tag{31}$$

for any positive $\varepsilon$ and any $B \subset V$. Firstly, note that if $\varepsilon \geqslant A_2/A_1$ then the r.h.s. of (31) is at least $|B|_m$ and so (31) is obviously true. Thus, suppose that $\varepsilon < A_2/A_1$, Then in view of (30), the set on the l.h.s. of (31) is empty and again (31) is trivially satisfied. This thereby completes the proof of the proposition. $\square$

**Corollary 3.** *Let $U$ be an open subset of $\mathbb{R}^m$, $\mathbf{x}_0 \in U$ be fixed and let $\mathbf{f} = (f_1, \ldots, f_n) : U \to \mathbb{R}^n$ be $l$-nondegenerate at $\mathbf{x}_0$ for some $l \geqslant 2$. Let $\theta \in C^{(l)}(U)$. Then there exists a neighborhood $V \subset U$ of $\mathbf{x}_0$ and positive constants $C$ and $H_0$ such that for any $\mathbf{a} \in \mathbb{R}^n$ satisfying $|\mathbf{a}| \geqslant H_0$*

(*a*) *$a_0 + \mathbf{a} \cdot \mathbf{f} + \theta$ is $(C, \frac{1}{ml})$-good on $V$ for every $a_0 \in \mathbb{R}$, and*

(*b*) *$|\nabla(\mathbf{a} \cdot \mathbf{f} + \theta)|$ is $(C, \frac{1}{m(l-1)})$-good on $V$.*

*Proof.* To start with choose the neighborhood $V \subset U$ of $\mathbf{x}_0$ so that $\mathbf{f}$ and $\theta$ are bounded on $V$. Then there exists a positive constant $K$ such that

$$\sup_{\mathbf{x}\in V} |\mathbf{f}(\mathbf{x})| \leqslant K/(n+1) \qquad \text{and} \qquad \sup_{\mathbf{x}\in V} |\theta(\mathbf{x})| \leqslant K/(n+1). \tag{32}$$

Let $f$ be the function given by $f(\mathbf{x}) := a_0 + \mathbf{a}\cdot\mathbf{f}(\mathbf{x}) + \theta(\mathbf{x})$. Assume for the moment that $|a_0| \geqslant 2K|\mathbf{a}|$. Then, on using (32) we find that

$$\sup_{\mathbf{x}\in B} |f(\mathbf{x})| \ \leqslant\ 3 \inf_{\mathbf{x}\in B} |f(\mathbf{x})|$$

for any ball $B \subset V$. Therefore, if $\varepsilon < 1/3$ then the set on the l.h.s. of (22) is empty and (22) is trivially satisfied with any positive $C$ and $\alpha$. On the other hand, if $\varepsilon \geqslant 1/3$, then (22) is obviously true for any $C \geqslant 3$ and any positive $\alpha \leqslant 1$. The upshot is that part (a) of the corollary holds for any $C \geqslant 3$ and $0 < \alpha \leqslant 1$ whenever $|a_0| \geqslant 2K|\mathbf{a}|$. Thus, without loss of generality we will assume that $|a_0| \leqslant 2K|\mathbf{a}|$.

Let $\mathcal{F}$ be the collection of functions of the form $\mathbf{c}\cdot\mathbf{f}(\mathbf{x})+c_0$, where $\mathbf{c}\in\mathbb{R}^n$ such that $|\mathbf{c}| = 1$ and $|c_0| \leqslant 2K$. Using the compactness of the set

$$\{\mathbf{c}\in\mathbb{R}^n : |\mathbf{c}| = 1\} \times \{c_0 \in \mathbb{R} : |c_0| \leqslant 2K\}\,,$$

one readily verifies that $\mathcal{F}$ is compact in $C^{(l)}(U)$. This together with the fact that $\mathbf{f}$ is non-degenerate at $\mathbf{x}_0$ ensures that $\mathcal{F}$ satisfies (25). Next note that by shrinking the neighborhood $V$ of $\mathbf{x}_0$ if necessary, we have that

$$\sup_{\mathbf{x}\in V} \max_{|\beta|_*\leqslant l} |\partial_\beta \theta(\mathbf{x})| \leqslant M$$

for some positive constant $M$. Now let $C$ and $\delta$ be the constants associated with Proposition 1 and let

$$H_0 \ :=\ M/\delta\,.$$

Consider an arbitrary vector $\mathbf{a}\in\mathbb{R}^n$ with $|\mathbf{a}| \geqslant H_0$ and any real number $a_0$ such that $|a_0| \leqslant 2K|\mathbf{a}|$. Then, $\Theta$ given by $\Theta(\mathbf{x}) := \theta(\mathbf{x})/|\mathbf{a}|$ satisfies (26) and

$$f : \mathbf{x} \to f(\mathbf{x}) := |\mathbf{a}|^{-1}(a_0 + \mathbf{f}(\mathbf{x})\cdot\mathbf{a})$$

belongs to the compact family $\mathcal{F}$. In view of Proposition 1, the function $f+\Theta$ given by $f(\mathbf{x})+\Theta(\mathbf{x}) = |\mathbf{a}|^{-1}(a_0+\mathbf{f}(\mathbf{x})\cdot\mathbf{a}+\theta(\mathbf{x}))$ satisfies the desired conclusions of the corollary. The assertions for the function without the $|\mathbf{a}|^{-1}$ multiplier are a simple consequence of part (a) of Lemma 1. $\square$

**Proposition 2.** *Let $U$, $\mathbf{x}_0$ and $\mathcal{F}$ be as in Proposition 1 and suppose that (25) is valid. Then for any neighborhood $V \subset U$ of $\mathbf{x}_0$, we have that*

$$\inf_{f\in\mathcal{F}} \sup_{\mathbf{x}\in V} |f(\mathbf{x})| > 0\,.$$

*Proof.* In view of (25) it follows that $\|f\|_V := \sup_{\mathbf{x}\in V} |f(\mathbf{x})| > 0$ for every $f \in \mathcal{F}$ and any neighborhood $V \subset U$ of $\mathbf{x}_0$. The map $f \mapsto \|f\|_V$ is continuous with respect to the $C^{(0)}$ norm. By the compactness of $\mathcal{F}$, we have that $\inf_{f\in\mathcal{F}} \|f\|_V = \|f_0\|_V$ for some $f_0 \in \mathcal{F}$. The claim of the proposition now follows on combining these facts. $\square$

**Corollary 4.** *Let $U$, $\mathbf{x}_0$, $\mathbf{f}$ and $\theta$ be as in Corollary 3. Then for every sufficiently small neighborhood $V \subset U$ of $\mathbf{x}_0$, there exists $H_0 > 1$ such that*

$$\inf_{\substack{(\mathbf{a},a_0)\in\mathbb{R}^{n+1}\\ |\mathbf{a}|\geqslant H_0}} \sup_{\mathbf{x}\in V} |a_0 + \mathbf{a}\cdot\mathbf{f}(\mathbf{x}) + \theta(\mathbf{x})| > 0.$$

*Proof.* Consider any neighborhood $V \subset U$ of $\mathbf{x}_0$ for which the inequalities given by (32) are satisfied for some $K > 0$. Let $f$ denote the function given by $f(\mathbf{x}) := a_0 + \mathbf{a}\cdot\mathbf{f}(\mathbf{x}) + \theta(\mathbf{x})$. Notice that if $|a_0| \geqslant 2K|\mathbf{a}|$, then in view of (32) it follows that

$$\sup_{\mathbf{x}\in V} |f(\mathbf{x})| \;\geqslant\; K\,H_0 > K > 0$$

for any $(\mathbf{a}, a_0) \in \mathbb{R}^{n+1}$ with $|\mathbf{a}| \geqslant H_0 > 1$ and $|a_0| \geqslant 2K|\mathbf{a}|$. Thus for the rest of the proof we may assume that $|a_0| \leqslant 2K|\mathbf{a}|$.

As in the proof of Corollary 3, let $\mathcal{F}$ be the collection of functions of the form $\mathbf{c}\cdot\mathbf{f}(\mathbf{x}) + c_0$, where $\mathbf{c} \in \mathbb{R}^n$ such that $|\mathbf{c}| = 1$ and $|c_0| \leqslant 2K$. Then $\mathcal{F}$ is a compact subset of $C^{(l)}(U)$ and since $\mathbf{f}$ is non-degenerate at $\mathbf{x}_0$, we have that $\mathcal{F}$ satisfies (25). Thus, Proposition 2 implies that $M := \inf_{f\in\mathcal{F}} \sup_{\mathbf{x}\in V} |f(\mathbf{x})| > 0$. Therefore, for any $(\mathbf{a}, a_0) \in \mathbb{R}^{n+1}$ with $|\mathbf{a}| \geqslant H_0 > 1$ and $|a_0| \leqslant 2K|\mathbf{a}|$ we have that

$$\sup_{\mathbf{x}\in V} |a_0 + \mathbf{a}\cdot\mathbf{f}(\mathbf{x})| \geqslant MH_0. \tag{33}$$

Now take $H_0 > \max\{1, K/M\}$. Then, by (32) and (33) it follows that

$$\sup_{\mathbf{x}\in V} |a_0 + \mathbf{a}\cdot\mathbf{f}(\mathbf{x}) + \theta(\mathbf{x})| \geqslant MH_0/2$$

and this completes the proof of the corollary. $\square$

*2.2.2. Inhomogeneous Transference Principle*

In this section we describe a simplified version of the Inhomogeneous Transference Principle introduced in [21, Section 5]. The simplified version takes into consideration the specific applications that we have in mind. Throughout, $V$ denotes a finite open ball in $\mathbb{R}^m$ and $\mu$ is $m$-dimensional Lebesgue measure restricted to $V$. Clearly the support of $\mu$ is the closure $\overline{V}$ of $V$. For consistency with the notation used in [21], will be write $\mathbf{S}$ for $\overline{V}$.

Let $\mathbf{T}$ and $\mathcal{A}$ be two countable 'indexing' sets and let H and I be two maps from $\mathbf{T} \times \mathcal{A} \times \mathbb{R}^+$ into the set of open subsets of $\mathbb{R}^m$ such that

$$\mathrm{H} \,:\, (\mathbf{t}, \alpha, \varepsilon) \mapsto \mathrm{H}_{\mathbf{t}}(\alpha, \varepsilon) \qquad \text{and} \qquad \mathrm{I} \,:\, (\mathbf{t}, \alpha, \varepsilon) \mapsto \mathrm{I}_{\mathbf{t}}(\alpha, \varepsilon). \tag{34}$$

Let $\Phi$ denote a set of functions $\phi \,:\, \mathbf{T} \to \mathbb{R}^+$. For $\phi \in \Phi$, consider the lim sup sets

$$\Lambda_{\mathrm{I}}(\phi) := \limsup_{\mathbf{t} \in \mathbf{T}} \bigcup_{\alpha \in \mathcal{A}} \mathrm{I}_{\mathbf{t}}(\alpha, \phi(\mathbf{t})) \quad \text{and} \quad \Lambda_{\mathrm{H}}(\phi) := \limsup_{\mathbf{t} \in \mathbf{T}} \bigcup_{\alpha \in \mathcal{A}} \mathrm{H}_{\mathbf{t}}(\alpha, \phi(\mathbf{t})). \tag{35}$$

The following two key properties enables us to transfer zero $\mu$-measure statements for the 'homogenous' lim sup sets $\Lambda_{\mathrm{H}}(\phi)$ to the 'inhomogeneous' lim sup sets $\Lambda_{\mathrm{I}}(\phi)$.

**Intersection Property:** The triple $(\mathrm{H}, \mathrm{I}, \Phi)$ is said to satisfy the intersection property if for any $\phi \in \Phi$ there exists $\phi^* \in \Phi$ such that for all but finitely many $\mathbf{t} \in \mathbf{T}$ and all distinct $\alpha, \alpha' \in \mathcal{A}$

$$\mathrm{I}_{\mathbf{t}}(\alpha, \phi(\mathbf{t})) \cap \mathrm{I}_{\mathbf{t}}(\alpha', \phi(\mathbf{t})) \subset \bigcup_{\alpha'' \in \mathcal{A}} \mathrm{H}_{\mathbf{t}}(\alpha'', \phi^*(\mathbf{t})). \tag{36}$$

**Contracting Property:** We say that $\mu$ is contracting with respect to $(\mathrm{I}, \Phi)$ if for any $\phi \in \Phi$ there exists $\phi^+ \in \Phi$ and a sequence of positive numbers $\{k_{\mathbf{t}}\}_{\mathbf{t} \in \mathbf{T}}$ such that

$$\sum_{\mathbf{t} \in \mathbf{T}} k_{\mathbf{t}} < \infty \tag{37}$$

and for all but finitely many $\mathbf{t} \in \mathbf{T}$ and all $\alpha \in \mathcal{A}$ there exists a collection $C_{\mathbf{t},\alpha}$ of balls $B$ centred in $\mathbf{S}$ satisfying the following three conditions:

$$\mathbf{S} \cap \mathrm{I}_{\mathbf{t}}(\alpha, \phi(\mathbf{t})) \subset \bigcup_{B \in C_{\mathbf{t},\alpha}} B, \tag{38}$$

$$\mathbf{S} \cap \bigcup_{B \in C_{\mathbf{t},\alpha}} B \subset \mathrm{I}_{\mathbf{t}}(\alpha, \phi^{+}(\mathbf{t})) \tag{39}$$

and

$$\mu(5B \cap \mathrm{I}_{\mathbf{t}}(\alpha, \phi(\mathbf{t}))) \leqslant k_{\mathbf{t}}\, \mu(5B). \tag{40}$$

The following transference theorem is an immediate consequence of [21, Theorem 5].

**Theorem 5.** *Suppose that* $(\mathrm{H}, \mathrm{I}, \Phi)$ *satisfies the intersection property and* $\mu$ *is contracting with respect to* $(\mathrm{I}, \Phi)$*. Then*

$$\forall\, \phi \in \Phi \quad \mu(\Lambda_{\mathrm{H}}(\phi)) = 0 \qquad \Longrightarrow \qquad \forall\, \phi \in \Phi \quad \mu(\Lambda_{\mathrm{I}}(\phi)) = 0.$$

*2.3. Establishing Case B*

Recall that our aim is to show that $|\mathcal{A}_{\mathbf{f}}^2(\Psi, \theta)|_m = 0$, where $\Psi$ satisfies (3) and (16). Using (3) and (16) one readily verifies that

$$\Psi(\mathbf{a}) < \Psi_0(\mathbf{a}) := \prod_{\substack{i=1\\ a_i \neq 0}}^{n} |a_i|^{-1} \tag{41}$$

for all but finitely many $\mathbf{a} \in \mathbb{Z}^n$. Therefore,

$$\mathcal{A}_{\mathbf{f}}^2(\Psi, \theta) \subset \mathcal{A}_{\mathbf{f}}^2(\Psi_0, \theta) \tag{42}$$

and so it suffices to show that $|\mathcal{A}_{\mathbf{f}}^2(\Psi_0, \theta)|_m = 0$. With reference to the inhomogeneous transference framework of §2.2.2, let $\mathbf{T} := (\mathbb{Z}_{\geqslant 0})^n$ and $\mathcal{A} := \mathbb{Z}^n \backslash \{0\} \times \mathbb{Z}$. Define the auxiliary function $r : \mathbf{T} \to \mathbb{R}^+$ by setting

$$r(\mathbf{t}) := \sqrt{2(n+1)mC_0} \cdot 2^{|\mathbf{t}|/2} \tag{43}$$

where $C_0$ is as in (15). Then, given $\varepsilon > 0$, $\mathbf{t} \in \mathbf{T}$ and $\alpha = (\mathbf{a}, a_0) \in \mathcal{A}$, let

$$\mathrm{I}_{\mathbf{t}}(\alpha, \varepsilon) := \left\{ \mathbf{x} \in U \ : \begin{array}{rcl} |a_0 + \mathbf{a}\cdot\mathbf{f}(\mathbf{x}) + \theta(\mathbf{x})| & < & \varepsilon\, \Psi_0(2^{\mathbf{t}}) \\ |\nabla(\mathbf{a}\cdot\mathbf{f}(\mathbf{x}) + \theta(\mathbf{x}))| & < & \varepsilon\, r(\mathbf{t}) \\ 2^{t_i} \leqslant \max\{1, |a_i|\} & < & 2^{t_i+1} \quad (1 \leqslant i \leqslant n) \end{array} \right\} \tag{44}$$

and

$$\mathrm{H}_{\mathbf{t}}(\alpha,\varepsilon) := \left\{ \mathbf{x}\in U \ : \ \begin{array}{rcl} |a_0+\mathbf{a}\cdot\mathbf{f}(\mathbf{x})| & < & 2\,\varepsilon\,\Psi_0(2^{\mathbf{t}}) \\ |\nabla(\mathbf{a}\cdot\mathbf{f}(\mathbf{x}))| & < & 2\,\varepsilon\, r(\mathbf{t}) \\ |a_i| & < & 2^{t_i+2} \quad (1\leqslant i\leqslant n) \end{array} \right\} \tag{45}$$

where $2^{\mathbf{t}} := (2^{t_1},\dots,2^{t_n})$. This defines the maps H and I – see (34). Furthermore, given $\delta\in\mathbb{R}$, let $\phi_\delta:\mathbf{T}\to\mathbb{R}^+$ be given by

$$\phi_\delta(\mathbf{t}) := 2^{\delta|\mathbf{t}|}\,, \tag{46}$$

and let

$$\Phi := \left\{\phi_\delta \ : \ 0\leqslant\delta<\tfrac{1}{4}\right\}\,.$$

For any $\delta\in[0,1/4)$, it follows that

$$\mathcal{A}^2_{\mathbf{f}}(\Psi_0,\theta)\subset\Lambda_{\mathrm{I}}(\phi_\delta)$$

where $\Lambda_{\mathrm{I}}(\phi_\delta)$ is the 'inhomogeneous' lim sup set as defined by (35). Therefore, in view of (42), to establish Case B it suffices to show that

$$|\Lambda_{\mathrm{I}}(\phi_\delta)|_m=0 \quad \text{ for some } \delta\in[0,\tfrac14). \tag{47}$$

With this in mind, let $\mathbf{x}_0$ be any point in $U$ at which $\mathbf{f}$ is $l$-non-degenerate and let $V$ be a sufficiently small open ball centred at $\mathbf{x}_0$ such that Corollary 3 and the following statement are valid on $V$.

**Theorem 6. ([29, Theorem 1.4])** *Let $\mathbf{x}_0\in U$ and $\mathbf{f}:U\to\mathbb{R}^n$ be $l$-nondegenerate at $\mathbf{x}_0$. Then there exists a neighborhood $V\subset U$, of $\mathbf{x}_0$ satisfying the following property. For any ball $B\subset V$ there exists $E>0$ such that for any choice of real numbers $\omega,K,T_1,\dots,T_n$ satisfying the inequalities*

$$0<\omega\leqslant 1,\quad T_1,\dots,T_n\geqslant 1,\quad K>0 \quad \textit{and} \quad \frac{\omega K T_1\cdots T_n}{\max_i T_i}\leqslant 1$$

*the set*

$$S(\omega,K,T_1,\dots,T_n) :=$$

$$\left\{ x\in B \ : \ \exists\ \mathbf{q}\in\mathbb{Z}^n\backslash\{0\} \ \textit{such that} \ \begin{array}{l} \|\mathbf{f}(\mathbf{x})\cdot\mathbf{q}\|<\omega \\ |\nabla\mathbf{f}(\mathbf{x})\cdot\mathbf{q}|<K \\ |q_i|<T_i \quad (1\leqslant i\leqslant n) \end{array} \right\}$$

*has $m$-dimensional Lebesgue measure at most $E\,\varepsilon^{\frac{1}{m(2l-1)}}|B|_m$, where*

$$\varepsilon := \max\left(\omega, \left(\frac{\omega K T_1\cdots T_n}{\max_i T_i}\right)^{\frac{1}{n+1}}\right). \tag{48}$$

Furthermore, let $\mu$ be $m$-dimensional Lebesgue measure restricted to $V$. Since $\mathbf{f}$ is $l$-non-degenerate almost everywhere, the desired statement (47) follows on showing that

$$\mu(\Lambda_{\mathrm{I}}(\phi_\delta)) = 0 \quad \text{for some } \delta \in [0, \tfrac{1}{4}). \tag{49}$$

For this, we make use of the Inhomogeneous Transference Principle. Indeed, suppose for the moment that $(\mathrm{H}, \mathrm{I}, \Phi)$ satisfies the intersection property and $\mu$ is contracting with respect to $(\mathrm{I}, \Phi)$. Then, in view of Theorem 5, to establish (49) it suffices to show that

$$\mu(\Lambda_{\mathrm{H}}(\phi_\delta)) = 0 \quad \text{for some } \ \delta \in [0, \tfrac{1}{4}). \tag{50}$$

Armed with Theorem 6, it is relatively painless to establish (50). Fix any $\delta \in [0, 1/4)$ and notice that in view of (45) it follows that

$$\bigcup_{\alpha\in\mathcal{A}} \mathrm{H}_{\mathbf{t}}(\alpha, \phi_\delta(\mathbf{t})) = S(\omega, K, T_1, \ldots, T_n)$$

with

$$\omega = 2\,\phi_\delta(\mathbf{t})\,\Psi_0(2^{\mathbf{t}}), \quad K = 2\,\phi_\delta(\mathbf{t})\,r(\mathbf{t}) \quad \text{and} \quad T_i = 2^{t_i+2} \ \ (1 \leqslant i \leqslant n).$$

Using the explicit values of $\Psi_0(2^{\mathbf{t}})$, $r(\mathbf{t})$ and $\phi_\delta(\mathbf{t})$ given by (41), (43) and (46) respectively, we find that the quantity $\varepsilon$ defined by (48) satisfies

$$\varepsilon \ll 2^{-\frac{(1/2-2\delta)}{n+1}|\mathbf{t}|}.$$

Therefore, Theorem 6 implies that

$$\left|\bigcup_{\alpha\in\mathcal{A}} \mathrm{H}_{\mathbf{t}}(\alpha, \phi_\delta(\mathbf{t}))\right|_m \ll 2^{-\gamma|\mathbf{t}|}$$

where $\gamma := \frac{(1/2-2\delta)}{m(n+1)(2l-1)}$ is a positive constant. The upshot is that

$$\sum_{\mathbf{t}\in\mathbf{T}} \left|\cup_{\alpha\in\mathcal{A}} \mathrm{H}_{\mathbf{t}}(\alpha, \phi_\delta(\mathbf{t}))\right|_m \ll \sum_{\mathbf{t}\in\mathbb{Z}^n} 2^{-\gamma|\mathbf{t}|} < \infty\,,$$

which together with the Borel-Cantelli lemma implies the desired zero measure statement

$$\mu(\Lambda_{\mathrm{H}}(\phi_\delta)) = 0\,.$$

It remains to verify the intersection and contracting properties.

*2.3.1. Verifying the intersection property*

Let $\mathbf{t} \in \mathbf{T}$ with $|\mathbf{t}| \geq 2$ and suppose that

$$\mathbf{x} \in \mathrm{I}_{\mathbf{t}}(\alpha, \phi_\delta(\mathbf{t})) \cap \mathrm{I}_{\mathbf{t}}(\alpha', \phi_\delta(\mathbf{t}))$$

for some distinct $\alpha = (\mathbf{a}, a_0)$ and $\alpha' = (\mathbf{a}', a_0')$ in $\mathcal{A}$. Then, by (44) and (45) we have that

$$\begin{cases} |a_0 + \mathbf{a}\cdot\mathbf{f}(\mathbf{x}) + \theta(\mathbf{x})| & < & \phi_\delta(\mathbf{t})\ \Psi_0(2^{\mathbf{t}}) \\ |a_0' + \mathbf{a}'\cdot\mathbf{f}(\mathbf{x}) + \theta(\mathbf{x})| & < & \phi_\delta(\mathbf{t})\ \Psi_0(2^{\mathbf{t}}) \end{cases}$$

$$\begin{cases} |\nabla(\mathbf{a}\cdot\mathbf{f}(\mathbf{x}) + \theta(\mathbf{x}))| & < & \phi_\delta(\mathbf{t})\ r(\mathbf{t}) \\ |\nabla(\mathbf{a}'\cdot\mathbf{f}(\mathbf{x}) + \theta(\mathbf{x}))| & < & \phi_\delta(\mathbf{t})\ r(\mathbf{t}) \end{cases}$$

and

$$\begin{cases} |a_i| & < & 2^{t_i+1} & (1 \leqslant i \leqslant n) \\ |a_i'| & < & 2^{t_i+1} & (1 \leqslant i \leqslant n)\,, \end{cases}$$

where $(a_1, \ldots, a_n) = \mathbf{a}$ and $(a_1', \ldots, a_n') = \mathbf{a}'$. Subtracting the first inequality from the second within each of the above three systems gives

$$\begin{cases} |a_0'' + \mathbf{a}''\cdot\mathbf{f}(\mathbf{x})| & < & 2\phi_\delta(\mathbf{t})\ \Psi_0(2^{\mathbf{t}}) \\ |\nabla(\mathbf{a}''\cdot\mathbf{f}(\mathbf{x}))| & < & 2\phi_\delta(\mathbf{t})\ r(\mathbf{t}) \\ |a_i''| & < & 2^{t_i+2} \qquad (1 \leqslant i \leqslant n)\,, \end{cases} \tag{51}$$

where $\mathbf{a}'' = (a_1'', \ldots, a_n'') := \mathbf{a}' - \mathbf{a}$ and $a_0'' := a_0' - a_0$. Regarding the first of the above inequalities, by (41) and the definition of $\Phi$, we have that $\phi_\delta(\mathbf{t})\ \Psi_0(2^{\mathbf{t}}) < 2^{-\frac{3}{4}|\mathbf{t}|}$. Suppose for the moment that $\mathbf{a}'' = 0$. Since $\alpha, \alpha' \in \mathcal{A}$ are distinct, we must have that $a_0' \neq a_0$ and so

$$|a_0'' + \mathbf{a}''\cdot\mathbf{f}(\mathbf{x})| = |a_0''| \geqslant 1\,.$$

However, for any $\mathbf{t}$ with $|\mathbf{t}| \geq 2$, this contradicts the first inequality of (51). Hence $\mathbf{a}'' \neq 0$ and it follows that $\alpha'' \in \mathcal{A}$. The upshot is that $\mathbf{x} \in \mathrm{H}_{\mathbf{t}}(\alpha'', \phi_\delta(\mathbf{t}))$ and therefore (36) is satisfied with $\phi^* = \phi_\delta$. This verifies the intersection property.

*2.3.2. Verifying the contracting property*

To start with recall that $V$ is a sufficiently small open ball such that Corollary 3 is valid on $5V$. Thus, there exist positive numbers $H_0$ and $C$ such that for any $\mathbf{t} \in \mathbf{T}$ and $\alpha = (\mathbf{a}, a_0) \in \mathcal{A}$ satisfying $|\mathbf{a}| \geqslant H_0$ both $a_0 + \mathbf{a} \cdot \mathbf{f} + \theta$ and $|\nabla(\mathbf{a} \cdot \mathbf{f} + \theta|$ are $(C, \frac{1}{ml})$-good on $5V$. In turn, by Lemma 1, for any $\mathbf{t} \in \mathbf{T}$ and $\alpha = (\mathbf{a}, a_0) \in \mathcal{A}$ satisfying $|\mathbf{a}| \geqslant H_0$ we have that

$$\mathbf{F}_{\mathbf{t},\alpha} \text{ is } (C, \tfrac{1}{ml})\text{-good on } 5V, \tag{52}$$

where $\mathbf{F}_{\mathbf{t},\alpha} : U \to \mathbb{R}$ is the function given by

$$\mathbf{F}_{\mathbf{t},\alpha}(\mathbf{x}) := \max\Big\{\Psi_0^{-1}(2^{\mathbf{t}})r(\mathbf{t})|a_0 + \mathbf{a} \cdot \mathbf{f}(\mathbf{x}) + \theta(\mathbf{x})|,\ |\nabla(\mathbf{a} \cdot \mathbf{f}(\mathbf{x}) + \theta(\mathbf{x}))|\Big\}.$$

Notice that the first two inequalities of (44) are equivalent to the single inequality

$$\mathbf{F}_{\mathbf{t},\alpha}(\mathbf{x}) < \varepsilon\, r(\mathbf{t})\,.$$

Therefore, by definition

$$\mathrm{I}_{\mathbf{t}}(\alpha, \varepsilon) = \{\mathbf{x} \in U \ :\ \mathbf{F}_{\mathbf{t},\alpha}(\mathbf{x}) < \varepsilon\, r(\mathbf{t})\} \tag{53}$$

if

$$2^{t_i} \leqslant \max\{1, |a_i|\} < 2^{t_i+1} \qquad (1 \leqslant i \leqslant n). \tag{54}$$

Obviously, if (54) is not fulfilled then $\mathrm{I}_{\mathbf{t}}(\alpha, \varepsilon) = \emptyset$ irrespective of $\varepsilon$.

Next, given $\phi_\delta \in \Phi$ let

$$\phi_\delta^+ := \phi_{\frac{1}{2}(\delta + \frac{1}{4})}.$$

Clearly, $\phi_\delta^+$ also lies in $\Phi$. It is easily seen that $\phi_\delta(\mathbf{t}) \leqslant \phi_\delta^+(\mathbf{t})$ for all $\mathbf{t} \in \mathbf{T}$ and therefore

$$\mathrm{I}_{\mathbf{t}}(\alpha, \phi_\delta(\mathbf{t})) \subset \mathrm{I}_{\mathbf{t}}(\alpha, \phi_\delta^+(\mathbf{t})). \tag{55}$$

We now construct the collection $C_{\mathbf{t},\alpha}$ of balls centred in $V$ that satisfy the conditions (38)–(40) for an appropriate sequence $k_{\mathbf{t}}$. If $\mathrm{I}_{\mathbf{t}}(\alpha, \phi_\delta(\mathbf{t})) = \emptyset$, the collection $C_{\mathbf{t},\alpha} = \emptyset$ obviously suffices. Thus, we can assume that (54) is satisfied and so $\mathrm{I}_{\mathbf{t}}(\alpha, \varepsilon)$ is defined by (53). By (41) and the definition of $\Phi$, it follows that

$$\mathrm{I}_{\mathbf{t}}(\alpha, \phi_\delta^+(\mathbf{t})) \subset \{\mathbf{x} \in U : |a_0 + \mathbf{a} \cdot \mathbf{f}(\mathbf{x}) + \theta(\mathbf{x})| < 2^{-\frac{3}{4}|\mathbf{t}|}\}.$$

As already pointed out above, $a_0 + \mathbf{a} \cdot \mathbf{f} + \theta$ is $(C, \frac{1}{ml})$-good on $5V$ for all sufficiently large $|\mathbf{a}|$. Therefore, by the definition of $(C, \alpha)$-good (Definition 1) and Corollary 4 we have that

$$\begin{aligned} |\mathrm{I}_{\mathbf{t}}(\alpha, \phi_\delta^+(\mathbf{t})) \cap V|_m &\leqslant |\{\mathbf{x} \in V : |a_0 + \mathbf{a} \cdot \mathbf{f}(\mathbf{x}) + \theta(\mathbf{x})| < 2^{-\frac{3}{4}|\mathbf{t}|}\}|_m \\ &\ll 2^{-\frac{3|\mathbf{t}|}{4ml}} |V|_m \,, \end{aligned}$$

whenever $|\mathbf{t}|$ is sufficiently large. Hence,

$$\mathrm{I}_{\mathbf{t}}(\alpha, \phi_\delta^+(\mathbf{t})) \not\subset V \quad \text{for all sufficiently large } |\mathbf{t}|. \tag{56}$$

By (55) and the fact that $\mathrm{I}_{\mathbf{t}}(\alpha, \phi_\delta^+(\mathbf{t}))$ is open, for every $\mathbf{x} \in \mathbf{S} \cap \mathrm{I}_{\mathbf{t}}(\alpha, \phi_\delta(\mathbf{t}))$ there is a ball $B'(\mathbf{x})$ centred at $\mathbf{x}$ such that

$$B'(\mathbf{x}) \subset \mathrm{I}_{\mathbf{t}}(\alpha, \phi_\delta^+(\mathbf{t})). \tag{57}$$

On combining (56), (57) and the fact that $V$ is bounded, we find that there exists a scaling factor $\tau \geqslant 1$ such that the ball $B = B(\mathbf{x}) := \tau B'(\mathbf{x})$ satisfies

$$\mathbf{S} \cap B(\mathbf{x}) \subset \mathrm{I}_{\mathbf{t}}(\alpha, \phi_\delta^+(\mathbf{t})) \not\supset \mathbf{S} \cap 5B(\mathbf{x}) \tag{58}$$

and

$$5B(\mathbf{x}) \subset 5V. \tag{59}$$

We now let

$$C_{\mathbf{t},\alpha} := \{B(\mathbf{x}) \ : \ \mathbf{x} \in \mathbf{S} \cap \mathrm{I}_{\mathbf{t}}(\alpha, \phi_\delta(\mathbf{t}))\}\,.$$

Then, by construction and the l.h.s. of (58), conditions (38) and (39) are automatically satisfied. Regarding condition (40), consider any ball $B \in C_{\mathbf{t},\alpha}$. By (53) and the r.h.s. of (58), we have that

$$\sup_{\mathbf{x} \in 5B} \mathbf{F}_{\mathbf{t},\alpha}(\mathbf{x}) \ \geqslant \ \sup_{\mathbf{x} \in 5B \cap \mathbf{S}} \mathbf{F}_{\mathbf{t},\alpha}(\mathbf{x}) \ \geqslant \ \phi_\delta^+(\mathbf{t})\, r(\mathbf{t}). \tag{60}$$

On the other hand,

$$\sup_{\mathbf{x} \in 5B \cap \mathrm{I}_{\mathbf{t}}(\alpha, \phi_\delta(\mathbf{t}))} \mathbf{F}_{\mathbf{t},\alpha}(\mathbf{x}) \ \leqslant \ \phi_\delta(\mathbf{t})\, r(\mathbf{t}). \tag{61}$$

Then, in view of the definitions of $\phi_\delta$, $\phi_\delta^+$ and $r(\mathbf{t})$, we obtain via (60) and (61) that

$$\sup_{\mathbf{x}\in 5B\cap \mathrm{I}_{\mathbf{t}}(\alpha,\phi_\delta(\mathbf{t}))} \mathbf{F}_{\mathbf{t},\alpha}(\mathbf{x}) \ \leqslant\ 2^{-\frac{1}{2}(\frac{1}{4}-\delta)|\mathbf{t}|} \sup_{\mathbf{x}\in 5B} \mathbf{F}_{\mathbf{t},\alpha}(\mathbf{x}). \tag{62}$$

Now notice that since (54) holds, we have that $|\mathbf{a}| > H_0$ for all $\mathbf{t}\in\mathbf{T}$ with $|\mathbf{t}|$ sufficiently large. Thus, whenever $|\mathbf{t}|$ is sufficiently large, (52) is valid which together with (59) and (62) implies that

$$\begin{aligned}
&\Big|5B\cap \mathrm{I}_{\mathbf{t}}(\alpha,\phi_\delta(\mathbf{t}))\Big|_m \ \leqslant \\
&\qquad \leqslant\ \Big|\{\mathbf{x}\in 5B : |\mathbf{F}_{\mathbf{t},\alpha}(\mathbf{x})| \leqslant 2^{-\frac{1}{2}(\frac{1}{4}-\delta)|\mathbf{t}|} \sup\nolimits_{\mathbf{x}\in 5B} \mathbf{F}_{\mathbf{t},\alpha}(\mathbf{x})\}\Big|_m \\
&\qquad \leqslant\ C2^{-\delta^*|\mathbf{t}|}|5B|_m
\end{aligned} \tag{63}$$

where $\delta^* := \frac{1}{2}(\frac{1}{4}-\delta)\,\frac{1}{lm} > 0$. On using the fact that $B$ is centred in $V\subset\mathbf{S}$, we have that $|5B|_m \leqslant c_m\mu(5B)$ for some constant $c_m$ depending on $m$ only. Hence (63) implies that for all but finitely many $\mathbf{t}\in\mathbf{T}$

$$\mu(5B\cap \mathrm{I}_{\mathbf{t}}(\alpha,\phi_\delta(\mathbf{t}))) \ \leqslant\ |5B\cap \mathrm{I}_{\mathbf{t}}(\alpha,\phi_\delta(\mathbf{t}))|_m \ \leqslant\ c_mC2^{-\delta^*|\mathbf{t}|}\mu(5B).$$

This verifies (40) with

$$k_{\mathbf{t}} := c_mC2^{-\delta^*|\mathbf{t}|}\,.$$

Furthermore, it is easily seen that the convergence condition (37) is fulfilled. The upshot is that all the conditions of the contracting property are satisfied for the collection $C_{\mathbf{t},\alpha}$ as defined above.

## 3. The divergence theory

The goal is to prove Theorems 2 & 3. Thus, throughout $s > m-1$ and $\Psi$ is a multivariable approximating function satisfying property $\mathbf{P}$ and the divergent sum condition

$$\sum_{\mathbf{a}\in\mathbb{Z}^n\setminus\{0\}} |\mathbf{a}| \left(\frac{\Psi(\mathbf{a})}{|\mathbf{a}|}\right)^{s+1-m} = \infty. \tag{64}$$

Without loss of generality, we will assume that the vector $\mathbf{v} = (v_1,\dots,v_n)$ appearing in the definition of property $\mathbf{P}$ satisfies

$$v_1 = |\mathbf{v}| = \max_{1\leqslant i\leqslant n} |v_i|\,. \tag{65}$$

*3.1. Theorem 3 $\implies$ Theorem 2*

We will need the following technical lemma.

**Lemma 3.** *Let $\mu$ be a finite doubling Borel regular measure on a metric space $(X,d)$ such that $X$ can be covered by a countable collection of arbitrarily small balls. Let $f : X \to \mathbb{R}^+$ be a uniformly continuous bounded function and let $\nu$ be a measure on $X$ given by*

$$\nu(A) := \int_A f(x)d\mu(x) \tag{66}$$

*for every measurable set $A \subset X$. Let $\{S_Q\}_{Q\in\mathbb{N}}$ be a sequence of measurable subsets of $X$ and $0 < \omega < 1$ be a constant. Suppose that for every sufficiently small closed ball $B \subset X$*

$$\limsup_{Q\to\infty} \ \mu(S_Q \cap B) \ \leqslant \ \omega\, \mu(B)\,. \tag{67}$$

*Then for every measurable set $W \subset X$*

$$\limsup_{Q\to\infty} \ \nu(S_Q \cap W) \ \leqslant \ \omega\, \nu(W)\,. \tag{68}$$

*Proof.* Let $W$ be any measurable subset of $X$. For every $\varepsilon > 0$ and $\delta > 0$ there is a finite collection $\mathcal{C}_{\varepsilon,\delta}$ of disjoint closed balls with radii $< \delta$ such that

$$\mu(W \triangle W_{\varepsilon,\delta}) < \varepsilon, \tag{69}$$

where $E\triangle F := (E\backslash F)\cup(F\backslash E)$ and $W_{\varepsilon,\delta} := \bigcup_{B\in\mathcal{C}_{\varepsilon,\delta}} B$. This is a consequence of [41, Theorem 2.2.2] and the discussion of [14, p.28]. Since $f$ is bounded, there is a constant $C > 0$ such that $\nu(A) \leqslant C\mu(A)$ for every measurable set $A$. Therefore, (69) implies that

$$\nu(W \triangle W_{\varepsilon,\delta}) < C\varepsilon. \tag{70}$$

For every $B \in \mathcal{C}_{\varepsilon,\delta}$ let $s_B := \sup_{x\in B} f(x)$. Since $f$ is bounded, the quantity $s_B$ is finite. Next, since $f$ is uniformly continuous, for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $B \in \mathcal{C}_{\varepsilon,\delta}$ we have that

$$0 \leqslant s_B - f(x) < \varepsilon \qquad \text{for all } x \in B\,. \tag{71}$$

Since $\mathcal{C}_{\varepsilon,\delta}$ is finite, property (67) implies that there is a sufficiently large $Q_0$ such that for all $Q \geqslant Q_0$ and any $B \in \mathcal{C}_{\varepsilon,\delta}$ we have that

$$\mu(S_Q \cap B) \ \leqslant \ (\omega+\varepsilon)\,\mu(B)\,. \tag{72}$$

Then, for $Q \geqslant Q_0$ it follows that

$$\begin{aligned}
\nu(S_Q\cap W) \quad & \stackrel{(70)}{\leqslant} \quad C\varepsilon + \textstyle\sum_{B\in\mathcal{C}_{\varepsilon,\delta}} \nu(S_Q\cap B)\\
& \stackrel{(66)}{=} \quad C\varepsilon + \textstyle\sum_{B\in\mathcal{C}_{\varepsilon,\delta}} \int_{S_Q\cap B} f(x)d\mu(x)\\
& \stackrel{(71)}{\leqslant} \quad C\varepsilon + \textstyle\sum_{B\in\mathcal{C}_{\varepsilon,\delta}} s_B \int_{S_Q\cap B} d\mu(x)\\
& = \quad C\varepsilon + \textstyle\sum_{B\in\mathcal{C}_{\varepsilon,\delta}} s_B\, \mu(S_Q\cap B)\\
& \stackrel{(72)}{\leqslant} \quad C\varepsilon + (\omega+\varepsilon)\textstyle\sum_{B\in\mathcal{C}_{\varepsilon,\delta}} s_B\mu(B)\\
& \stackrel{(71)}{\leqslant} \quad C\varepsilon + (\omega+\varepsilon)\textstyle\sum_{B\in\mathcal{C}_{\varepsilon,\delta}} \int_B (f(x)+\varepsilon)d\mu(x)\\
& = \quad C\varepsilon + (\omega+\varepsilon)\textstyle\int_{W_{\delta,\varepsilon}} (f(x)+\varepsilon)d\mu(x)\\
& = \quad C\varepsilon + (\omega+\varepsilon)(\nu(W_{\delta,\varepsilon})+\varepsilon\mu(W_{\delta,\varepsilon}))\\
& \stackrel{(69)\&(70)}{\leqslant} \quad C\varepsilon + (\omega+\varepsilon)(\nu(W)+C\varepsilon+\varepsilon(\mu(W)+\varepsilon))\,.
\end{aligned}$$

The latter expression tends to $\omega\nu(W)$ as $\varepsilon \to 0$. Since $\nu(S_Q\cap W)$ is independent of $\varepsilon$, we obtain (68) as required. ☐

Let $\mathbf{f} : U \to \mathbb{R}^n$ be a map defined on an open set $U \subset \mathbb{R}^m$. Given an $n$-tuple $\mathbf{v} = (v_1,\dots,v_n)$ of positive numbers satisfying $v_1+\ldots+v_n = n$, $\delta > 0$ and $Q > 1$, let

$$\Phi^{\mathbf{f}}_{\mathbf{v}}(Q,\delta) \ = \ \Big\{\mathbf{x}\in U : \exists\ \mathbf{a}\in\mathbb{Z}^n\setminus\{0\} \text{ such that } \|\mathbf{a}\cdot\mathbf{f}(\mathbf{x})\| < \delta Q^{-n} \ \&\ |\mathbf{a}|_{\mathbf{v}} \leqslant Q\Big\}.$$

**Definition 2.** We will say that $\mathbf{f}$ is $\mathbf{v}$-*nice at* $\mathbf{x}_0 \in U$ if there is a neighborhood $U_0 \subset U$ of $\mathbf{x}_0$ and constants $0 < \delta, \omega < 1$ such that for any sufficiently small ball $B \subset U_0$ we have that

$$\limsup_{Q \to \infty} |\Phi_{\mathbf{v}}^{\mathbf{f}}(Q, \delta) \cap B|_m \leqslant \omega |B|_m \,.$$

The map $\mathbf{f}$ is said to be $\mathbf{v}$-*nice* if it is $\mathbf{v}$-nice at almost every point in $U$. Furthermore, $\mathbf{f}$ is said to be *nice* if it is $\mathbf{v}$-nice for all choices of $\mathbf{v}$.

Let $A$ be any Lebesgue measurable subset of $U$. Consider the measure $\nu$ given by

$$\nu(A) := \int_A \det G(\mathbf{x})^{1/2} d\mathbf{x},$$

where $G(\mathbf{x}) := (g_{i,j}(x))_{1 \leqslant i,j \leqslant m}$ with $\mathbf{g}_{i,j} := \partial \mathbf{f}/\partial x_i \cdot \partial \mathbf{f}/\partial x_j$. It is well known that the induced measure of a set $S$ on the manifold $\mathcal{M}$ parameterised by $\mathbf{f}$ is given by $\nu(A)$ with $A = \mathbf{f}^{-1}(S)$. It is easily verified that

$$|A|_m = \int_A \det G(\mathbf{x})^{-1/2} d\nu(\mathbf{x}).$$

Since $\mathbf{f}$ is a Monge parameterisation, $\det G(\mathbf{x})$ is bounded away from both zero and infinity on a sufficiently small neighborhood of any point $\mathbf{x}$. Hence, together with Lemma 3 we deduce the following statement.

**Proposition 3.** *Let $\mathbf{f} : U \to \mathbb{R}^n$ be a $C^2$ parameterisation of a $C^2$ manifold $\mathcal{M} \subset \mathbb{R}^n$. Let $\mathbf{x}_0 \in U$ and $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$. Then $\mathbf{f}$ is $\mathbf{v}$-nice at $\mathbf{x}_0$ if and only if $\mathcal{M}$ is $\mathbf{v}$-nice at $\mathbf{y}_0$.*

In turn this proposition together with the following lemma implies that non-degenerate manifolds are nice and so Theorem 2 is a consequence of Theorem 3.

**Lemma 4.** *Let $\mathbf{f}$ be non-degenerate at $\mathbf{x}_0 \in U$. Then there is a ball $B_0 \subset U$ centred at $\mathbf{x}_0$ and a constant $C > 0$ such that for any ball $B \subset B_0$ we have $|\Phi_{\mathbf{v}}^{\mathbf{f}}(Q, \delta) \cap B|_m \leqslant C\delta |B|_m$ for all sufficiently large $Q$.*

In the case $\mathbf{v} = (1, \ldots, 1)$, the lemma coincides with Theorem 2.1 in [13]. For arbitrary $\mathbf{v}$, on replacing the supremum norm by the $\mathbf{v}$-quasinorm, the arguments in [13] can be naturally adapted to establish Lemma 4. The details are left to the energetic reader.

*3.2. Ubiquitous systems in* $\mathbb{R}^m$

The proof of Theorem 3 will make use of the ubiquity framework developed in [14]. The framework introduced below is a much simplified version of that in [14] and takes into consideration the specific application that we have in mind.

Throughout, balls in $\mathbb{R}^m$ are assumed to be defined in terms of the supremum norm $|\cdot|$. Let $U$ be a ball in $\mathbb{R}^m$ and $\mathcal{R} = (R_\alpha)_{\alpha\in J}$ be a family of subsets $R_\alpha \subset \mathbb{R}^m$ indexed by a countable set $J$. The sets $R_\alpha$ are referred to as *resonant sets*. Throughout, $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ will denote a function such that $\rho(r) \to 0$ as $r \to \infty$. Given a set $A \subset U$, let

$$\Delta(A, r) := \{\mathbf{x} \in U \ : \ \mathrm{dist}(\mathbf{x}, A) < r\}$$

where $\mathrm{dist}(\mathbf{x}, A) := \inf\{|\mathbf{x} - \mathbf{a}| : \mathbf{a} \in A\}$. Next, let $\beta : J \to \mathbb{R}^+ : \alpha \mapsto \beta_\alpha$ be a positive function on $J$. Thus the function $\beta$ attaches a 'weight' $\beta_\alpha$ to the set $R_\alpha$. We will assume that for every $t \in \mathbb{N}$ the set $J_t = \{\alpha \in J : \beta_\alpha \leqslant 2^t\}$ is finite.

**The intersection conditions:** There exists a constant $\gamma$ with $0 \leq \gamma \leq m$ such that for any sufficiently large $t$ and for any $\alpha \in J_t$, $c \in R_\alpha$ and $0 < \lambda \leqslant \rho(2^t)$ the following conditions are satisfied:

$$|B(c, \tfrac{1}{2}\rho(2^t)) \cap \Delta(R_\alpha, \lambda)|_m \geq c_1 \, |B(c, \lambda)|_m \left(\frac{\rho(2^t)}{\lambda}\right)^\gamma \tag{73}$$

$$|B \cap B(c, 3\rho(2^t)) \cap \Delta(R_\alpha, 3\lambda)|_m \leq c_2 \, |B(c, \lambda)|_m \left(\frac{r(B)}{\lambda}\right)^\gamma \tag{74}$$

where $B$ is an arbitrary ball centred on a resonant set with radius $r(B) \leqslant 3\,\rho(2^t)$. The constants $c_1$ and $c_2$ are positive and absolute. The constant $\gamma$ is referred to as the *common dimension* of $\mathcal{R}$.

**Definition 3.** Suppose that there exists a ubiquitous function $\rho$ and an absolute constant $k > 0$ such that for any ball $B \subseteq U$

$$\liminf_{t\to\infty} \left| \bigcup_{\alpha\in J_t} \Delta(R_\alpha, \rho(2^t)) \cap B \right|_m \ \geqslant \ k\,|B|_m\,. \tag{75}$$

Furthermore, suppose that the intersection conditions (73) and 74 are satisfied. Then the system $(\mathcal{R}, \beta)$ is called *locally ubiquitous in* $U$ *relative to* $\rho$.

Let $(\mathcal{R}, \beta)$ be a ubiquitous system in $U$ relative to $\rho$ and $\phi$ be an approximating function. Let $\Lambda(\phi)$ be the set of points $\mathbf{x} \in U$ such that the inequality

$$\operatorname{dist}(\mathbf{x}, R_\alpha) < \phi(\beta_\alpha) \tag{76}$$

holds for infinitely many $\alpha \in J$.

**Lemma 5** (Ubiquity Lemma)**.** *Let $\phi$ be an approximating function and $(\mathcal{R}, \beta)$ be a locally ubiquitous system in U relative to $\rho$. Suppose that there is a $\lambda \in \mathbb{R}$, $0 < \lambda < 1$ such that $\rho(2^{t+1}) < \lambda\rho(2^t)$ for all $t \in \mathbb{N}$. Then for any $s > \gamma$*

$$\mathcal{H}^s(\Lambda(\phi)) = \mathcal{H}^s(U) \qquad \textit{if} \qquad \sum_{t=1}^{\infty} \frac{\phi(2^t)^{s-\gamma}}{\rho(2^t)^{m-\gamma}} = \infty. \tag{77}$$

*Remark.* When $s > m$, we have that $\mathcal{H}^s(\Lambda(\phi)) = \mathcal{H}^s(U) = 0$ and the lemma is trivial. In the case $s = m$ it is a consequence of [14, Corollary 2] and in the case $s < m$ it is a consequence of [14, Corollary 4 ].

*3.3. The appropriate ubiquitous system for Theorem 3*

Recall that $\mathbf{f} = (f_1, \dots, f_n) : U \to \mathbb{R}^n$ is a $\mathbf{v}$-nice $C^2$ map satisfying (15), where $U$ is a ball in $\mathbb{R}^m$. Also recall that $\theta : U \to \mathbb{R}$ is a $C^{(2)}$ function. Let $\mathcal{F}_n$ denote the set of all functions $F : U \to \mathbb{R}$ given by

$$F(\mathbf{x}) = a_0 + a_1 f_1(\mathbf{x}) + a_2 f_2(\mathbf{x}) + \dots + a_n f_n(\mathbf{x})\,,$$

where $a_0, \dots, a_n$ are integer coefficients not all zero. Given $F \in \mathcal{F}_n$, let

$$\tilde{R}_F := \{\mathbf{x} \in U : F(\mathbf{x}) + \theta(\mathbf{x}) = 0\} \qquad \text{and} \qquad H_{\mathbf{v}}(F) := \max_{1 \leqslant i \leqslant n} |a_i|^{1/v_i}. \tag{78}$$

The key to establishing Theorem 3 is the following ubiquity statement. With reference to the abstract setup of §3.2, the indexing set $J = \mathcal{F}_n$ and so $F$ plays the role of $\alpha \in J$.

**Proposition 4.** *Let $\mathbf{x}_0 \in U$ be such that $\mathbf{f}$ is $\mathbf{v}$-nice at $\mathbf{x}_0$. Then there is a neighborhood $U_0$ of $\mathbf{x}_0$, constants $\kappa_0 > 0$ and $\kappa_1 > 1$ and a collection $\mathcal{R} := (R_F)_{F \in \mathcal{F}_n}$ of sets $R_F \subset \tilde{R}_F \cap U_0$ such that the system $(\mathcal{R}, \beta)$, where*

$$\beta \,:\, \mathcal{F}_n \to \mathbb{R}^+ \,:\, F \mapsto \beta_F := \kappa_0 H_{\mathbf{v}}(F),$$

*is locally ubiquitous in $U_0$ relative to $\rho(r) := \kappa_1 r^{-n-v_1}$ with common dimension $\gamma := m - 1$.*

The sets $\tilde{R}_F$ are essentially the appropriate resonant sets. However, to ensure that the intersection conditions associated with ubiquity are satisfied, in particular, the lower bound condition (73), we cannot in general work with the sets $\tilde{R}_F$ directly[6]. To illustrate this, consider the following explicit examples.

*Example 1.* Let $m = 2$, $n = 3$, $U = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ and $f(x_1, x_2) = \sqrt{1 - x_1^2 - x_2^2}$. It is easily seen that for most choices of $F$ the intersection conditions are satisfied with $\gamma = 1$. However, when $\mathbf{a} = (-1, 0, 0, 1)$ and so $F = f - 1$, we have that $\tilde{R}_F = \{(0, 0)\}$. Then the l.h.s. of (73) is comparable to $\lambda^2$, while the r.h.s. of (73) is comparable to $\lambda\rho(2^t)$. Thus (73) is violated.

*Example 2.* Let $m = 2$, $n = 3$, $U = (\alpha, \alpha + 1)^2$ with $\alpha$ a Liouville number and $f(x_1, x_2) = x_1^2 + x_2^2$. As in the above example it is easily seen that for most choices of $F$ the intersection conditions are satisfied with $\gamma = 1$. Since $\alpha$ is Liouville, for any real $v$ we have that $|\alpha - p/q| < q^{-v}$ for infinitely many rationals $p/q$ $(q > 0)$. Consider $\mathbf{a} = (-2p, q, q, 0)$ if $\alpha - p/q < 0$ and $\mathbf{a} = (-2(p+q), q, q, 0)$ if $\alpha - p/q > 0$. It is a simple matter to verify that $\tilde{R}_F$ is a line segment of length comparable to $|\alpha - p/q| < q^{-v}$. Then the l.h.s. of (73) is comparable to $\lambda(\lambda + q^{-v})$, while the r.h.s. of (73) is comparable to $\lambda\rho(2^t)$. For large enough $v$, the upshot is that (73) is violated.

The upshot is that the sets $\tilde{R}_F$ need to be modified in an appropriate manner to yield the resonant sets $R_F$ – namely via the 'trimming' procedure described in §3.3.2 below[7].

### *3.3.1. Proof of Theorem 3 modulo Proposition 4*

Fix $\mathbf{x}_0 \in U$ such that $\mathbf{f}$ is $\mathbf{v}$-nice at $\mathbf{x}_0$ and let $U_0$ be as in Proposition 4. Since $\mathbf{f}$ is $\mathbf{v}$-nice (i.e. $\mathbf{f}$ is $\mathbf{v}$-nice at almost every point in $U$), it suffices to

[6]In various previous applications of ubiquity to approximation problems on manifolds the intersection conditions have not always been explicitly addressed. Indeed, it is not clear in some instances whether or not the authors have defined $\tilde{R}_F$ to be the resonant sets.

[7]The trimming procedure can be replicated to address the oversights alluded in the previous footnote.

prove that

$$\mathcal{H}^s(\mathcal{A}_{\mathbf{f}}(\Psi,\theta)\cap U_0)=\mathcal{H}^s(U_0)\,. \tag{79}$$

With reference to §3.2, let $U=U_0$ and

$$\phi : r\to\phi(r):=(2nC_0)^{-1}(\kappa_0^{-1}r)^{-v_1}\,\psi(\kappa_0^{-1}r)\,.$$

Here the approximating function $\psi$ and the vector $\mathbf{v}=(v_1,\ldots,v_n)$ are associated with the fact that $\Psi$ is a multivariable approximation function satisfiing property $\mathbf{P}$. Our first goal is to show that

$$\Lambda(\phi)\subset\mathcal{A}_{\mathbf{f}}(\Psi,\theta). \tag{80}$$

Let $\mathbf{x}=(x_1,\ldots,x_m)\in\Lambda(\phi)$. By definition, $\Lambda(\phi)$ is a subset of $U_0$ and inequality (76) is satisfied for infinitely many $F=a_0+a_1f_1+\ldots+a_nf_n\in\mathcal{F}_n$ – recall that we have identified $\alpha$ with $F$ and $J$ with $\mathcal{F}_n$. Now fix such a function $F$. Then, by the definition of $\beta$ and the properties of $R_F$ within Proposition 4, there exists a point $\mathbf{z}=(z_1,\ldots,z_m)\in U_0$ such that $F(\mathbf{z})+\theta(\mathbf{z})=0$ and

$$|\mathbf{x}-\mathbf{z}|<\phi(\kappa_0H_{\mathbf{v}}(F)). \tag{81}$$

Thus, by the Mean Value Theorem it follows that there exists some $\tilde{\mathbf{x}}\in U_0$ such that

$$\begin{array}{rcl}
|F(\mathbf{x})+\theta(\mathbf{x})| & = & \left|\sum_{i=1}^m\frac{\partial}{\partial x_i}(F+\theta)(\tilde{\mathbf{x}})(x_i-z_i)\right| \\[2ex]
 & \leqslant & |\mathbf{x}-\mathbf{z}|\,\sum_{i=1}^m\left|\frac{\partial}{\partial x_i}(\,\sum_{j=1}^n a_jf_j+\theta)(\tilde{\mathbf{x}})\right| \\[2ex]
 & \stackrel{(15)}{\leqslant} & 2nC_0\,|\mathbf{x}-\mathbf{z}|\,\max_{1\leqslant j\leqslant n}|a_j| \\[2ex]
 & \stackrel{(81)}{\leqslant} & 2nC_0\,\phi(\kappa_0H_{\mathbf{v}}(F))\,\max_{1\leqslant j\leqslant n}|a_j| \\[2ex]
 & \stackrel{(65)+(78)}{\leqslant} & 2nC_0\,\phi(\kappa_0H_{\mathbf{v}}(F))\,H_{\mathbf{v}}(F)^{v_1} \\[2ex]
 & \leqslant & \psi(H_{\mathbf{v}}(F))\ \ =\ \ \Psi(\mathbf{a})\,.
\end{array}$$

The upshot is that there are infinitely many $F\in\mathcal{F}_n$ satisfying the above inequalities. This verifies (80) and together with Lemma 5 implies (79) as long as the sum in (77) diverges. We now verify this divergent condition. Recall that $\gamma := m-1$ and so

$$\sum_{t=1}^{\infty} \frac{\phi(2^t)^{s-m+1}}{\rho(2^t)} \asymp \sum_{t=1}^{\infty} \frac{(2^{-v_1 t}\psi(\kappa_0^{-1}2^t))^{s-m+1}}{2^{-(n+v_1)t}}. \tag{82}$$

On using the fact that $v_1+\ldots+v_n=n$, it follows that for any $t\in\mathbb{N}$ the number of $\mathbf{a}\in\mathbb{Z}^n$ such that $\kappa_0 2^t<|\mathbf{a}|_{\mathbf{v}}\leqslant\kappa_0 2^{t+1}$ is comparable to $2^{nt}$. Also, by (65) we have that $|\mathbf{a}|\asymp 2^{v_1 t}$ whenever $\kappa_0 2^t<|\mathbf{a}|_{\mathbf{v}}\leqslant\kappa_0 2^{t+1}$. Therefore,

$$\text{r.h.s. of (82)} \asymp \sum_{t=1}^{\infty} \sum_{\kappa_0 2^t<|\mathbf{a}|_{\mathbf{v}}\leqslant\kappa_0 2^{t+1}} |\mathbf{a}| \left(\frac{\psi(\kappa_0^{-1}2^t)}{|\mathbf{a}|}\right)^{s-m+1}. \tag{83}$$

Next, since $\psi$ is decreasing, it follows that $\psi(\kappa_0^{-1}2^t)\geqslant\psi(|\mathbf{a}|_{\mathbf{v}})=\Psi(\mathbf{a})$ whenever $\kappa_0 2^t<|\mathbf{a}|_{\mathbf{v}}\leqslant\kappa_0 2^{t+1}$. Therefore,

$$\begin{aligned}\text{r.h.s. of (83)} &\gg \sum_{t=1}^{\infty} \sum_{\kappa_0 2^t<|\mathbf{a}|_{\mathbf{v}}\leqslant\kappa_0 2^{t+1}} |\mathbf{a}| \left(\frac{\Psi(\mathbf{a})}{|\mathbf{a}|}\right)^{s-m+1} \\ &\asymp \sum_{\mathbf{a}\in\mathbb{Z}\setminus\{0\}} |\mathbf{a}| \left(\frac{\Psi(\mathbf{a})}{|\mathbf{a}|}\right)^{s-m+1} \overset{(64)}{=} \infty.\end{aligned}$$

This completes the proof of Theorem 3 modulo Proposition 4.

*3.3.2. The resonant sets*

As already mentioned, the sets $\widetilde{R}_F$ given by (78) are essentially the appropriate resonant sets. However, to ensure that the intersection conditions associated with ubiquity are satisfied, these sets require modification. Essentially, we impose the condition that

$$|\tfrac{\partial}{\partial x_1}(F+\theta)(\mathbf{x})| > p\,|\nabla(F+\theta)(\mathbf{x})| \qquad \text{for all } \mathbf{x}\in U_0 \tag{84}$$

for some fixed $p\in(0,1)$. In what follows the projection map $\pi:\mathbb{R}^m\to\mathbb{R}^{m-1}$ will be given by

$$\pi(x_1,x_2,\ldots,x_m)=(x_2,\ldots,x_m). \tag{85}$$

**Proposition 5.** *Let $\rho$ and $\beta$ be as in Proposition 4. Let $U_0$ be any open subset of $U$ and $p\in(0,1)$. For $F\in\mathcal{F}_n$ let*

$$\widetilde{V} := \pi(\widetilde{R}_F\cap U_0), \qquad V := \bigcup_{3\rho(\beta_F)\text{-balls } B\subset\widetilde{V}} \tfrac{1}{2}B \tag{86}$$

*and*

$$R_F := \begin{cases} \pi^{-1}(V) \cap \widetilde{R}_F & \text{if } F \text{ satisfies (84)} \\ \emptyset & \text{otherwise} \end{cases} \tag{87}$$

*where $3\rho(\beta_F)$-balls are open balls in $\mathbb{R}^{m-1}$ of radius $3\rho(\beta_F)$. Then, $R_F$ satisfies the intersection conditions (73) and (74) with*

$$c_1 := 2^{-2m+3} v_m^{-1} \qquad \text{and} \qquad c_2 := 3m2^m (p\, v_m)^{-1}\,,$$

*where $v_m$ is the volume of an $m$-dimensional ball of unit radius.*

*Proof.* Let $t \in \mathbb{N}$, $F \in \mathcal{F}_n$ and $\beta_F \leqslant 2^t$. In view of (84) the gradient of $F + \theta$ never vanishes on $U_0$ and therefore the set $\widetilde{R}_F \cap U_0 := \{\mathbf{x} \in U_0 : F(\mathbf{x}) + \theta(\mathbf{x}) = 0\}$ is a regular $C^{(2)}$ submanifold of $U_0$ of dimension $(m - 1)$. This is a well known fact from differential geometry – see, for example [56, Theorem 1.13]. Furthermore, (84) together with the Implicit Function Theorem implies that $R_F \cap U_0$ can be defined as the graph $G_g(\widetilde{V})$ of a $C^{(2)}$ function $g : \widetilde{V} \to \mathbb{R}$, where

$$G_g(S) := \{(g(x_2, \ldots, x_m), x_2, \ldots, x_m) : (x_2, \ldots, x_m) \in S\} \tag{88}$$

for $S \subseteq \widetilde{V}$. Then, by the definition of $R_F$, we have that $R_F = G_g(V)$. If $R_F$ happens to be empty, the intersection conditions (73) and (74) are trivially satisfied. Otherwise, $R_F \neq \emptyset$ and we proceed as follows.

Given $r > 0$ and a set $A \subset \mathbb{R}^m$, let

$$\Delta_1(A, r) := \{\lambda \mathbf{e}_1 + \mathbf{x} : |\lambda| \leqslant r,\ \mathbf{x} \in A\},$$

where $\mathbf{e}_1 := (1, 0, \ldots, 0) \in \mathbb{R}^m$. By the definition of $g$,

$$(F + \theta)(g(x_2, \ldots, x_m), x_2, \ldots, x_m) = 0 \qquad \text{for all} \quad (x_2, \ldots, x_m) \in \tilde{V}\,.$$

Then differentiating this identity and using (84), we obtain that

$$|\nabla g(x_2, \ldots, x_m)| \leqslant p^{-1} \quad \text{for all} \;\; (x_2, \ldots, x_m) \in \widetilde{V}. \tag{89}$$

We now show that

$$\Delta_1(R_F, \eta) \subset \Delta(R_F, \eta) \subset \Delta_1(\widetilde{R}_F \cap U_0, \eta m p^{-1}) \quad \text{for any } \eta \leqslant 3\rho(\beta_F). \tag{90}$$

Indeed, the l.h.s. of (90) is a straightforward consequence of the definitions of $\Delta(A,r)$ and $\Delta_1(A,r)$. To prove the r.h.s. of (90) take any $\mathbf{z}\in\Delta(R_F,\eta)$. Then there exists $\mathbf{x}\in R_F$ such that $\mathrm{dist}(\mathbf{z},\mathbf{x})<\eta$. By the definition of $R_F$ and $V$, we have that $\pi\mathbf{x}\in\frac{1}{2}B$ for some $3\rho(\beta_F)$-ball $B\subset\widetilde{V}$. Hence, $B(\pi\mathbf{x},3\rho(\beta_F))\subset B\subset\widetilde{V}$. Since $\mathrm{dist}(\pi\mathbf{z},\pi\mathbf{x})\leqslant\mathrm{dist}(\mathbf{z},\mathbf{x})<\eta\leqslant3\rho(\beta_F)$, we have that $\pi\mathbf{z}\in\widetilde{V}$. Then, on making use of the Triangle Inequality and the Mean Value Theorem we find that

$$|z_1-g(\pi\mathbf{z})|\stackrel{(88)}{=}|z_1-x_1+g(\pi\mathbf{x})-g(\pi\mathbf{z})|\leqslant\eta+|g(\pi\mathbf{x})-g(\pi\mathbf{z})|\stackrel{(89)}{\leqslant}\eta mp^{-1}\,.$$

This verifies the r.h.s. of (90). We are now in the position to establish the intersection conditions (73) and (74).

*The lower bound condition.* Let $\mathbf{c}\in R_F$ and $0<\lambda\leqslant\rho(2^t)$. Since $\rho$ is decreasing, we have that $\rho(2^t)\leqslant\rho(\beta_F)$. Then, by (90), we find that

$$B(\mathbf{c},\tfrac{1}{2}\rho(2^t))\cap\Delta(R_F,\lambda)\ \supset\ B(\mathbf{c},\tfrac{1}{2}\rho(2^t))\cap\Delta_1(R_F,\lambda)\ \supset\ \Delta_1(G_g(W),\lambda),\quad(91)$$

where $W:=\pi(B(\mathbf{c},\frac{1}{2}\rho(2^t)))\cap V$. Since $\mathbf{c}\in R_F$, we have that $\pi\mathbf{c}\in V$ and therefore there exists a $3\rho(\beta_F)$-ball $B\subset\widetilde{V}$ such that $\pi\mathbf{c}\in\frac{1}{2}B$. Hence, since $3\rho(\beta_F)\geqslant\rho(2^t)$ and $\pi\mathbf{c}\in\frac{1}{2}B\subset V$, the set $\pi(B(\mathbf{c},\frac{1}{2}\rho(2^t)))\cap\frac{1}{2}B$ contains a ball of radius $\frac{1}{4}\rho(2^t)$ and therefore

$$|\pi(B(\mathbf{c},\tfrac{1}{2}\rho(2^t)))\cap\tfrac{1}{2}B|_{m-1}\geqslant(\tfrac{1}{4}\rho(2^t))^{m-1}v_{m-1}\geqslant(\tfrac{1}{4}\rho(2^t))^{m-1}.$$

Consequently, $|W|_{m-1}\geqslant(\frac{1}{4}\rho(2^t))^{m-1}$. Finally using (91) and Fubini's theorem gives

$$\begin{aligned}|B(\mathbf{c},\tfrac{1}{2}\rho(2^t))\cap\Delta(R_F,\lambda)|_m &\geqslant |W|_{m-1}\,2\lambda \geqslant (\tfrac{1}{4}\rho(2^t))^{m-1}\,2\lambda\\ &= c_1\,|B(\mathbf{c},\lambda)|_m\left(\frac{\rho(2^t)}{\lambda}\right)^{m-1}.\end{aligned}$$

*The upper bound condition.* Take any $\mathbf{c}\in R_F$, any positive $\lambda\leqslant\rho(2^t)$ and any ball $B$ with radius $r(B)\leqslant3\rho(2^t)$. Since $\rho$ is decreasing, we also have that $\rho(2^t)\leqslant\rho(\beta_F)$. Then, by (90), we find that

$$\begin{aligned}B\cap B(\mathbf{c},3\rho(2^t))\cap\Delta(R_F,3\lambda)&\subset B\cap B(\mathbf{c},3\rho(2^t))\cap\Delta_1(\widetilde{R}_F\cap U_0,3\lambda mp^{-1})\\ &\subset\Delta_1(G_g(W'),3\lambda mp^{-1}),\end{aligned}\qquad(92)$$

where $W' := \pi(B \cap B(\mathbf{c}, 3\rho(2^t)) \cap \widetilde{R}_F \cap U_0)$. Obviously, $\operatorname{diam} W' \leqslant 2r(B)$. Therefore, using (92) and Fubini's theorem gives

$$\begin{aligned} |B \cap B(\mathbf{c}, 3\rho(2^t)) \cap \Delta(R_F, 3\lambda)|_m &\leqslant |W'|_{m-1}\, 6\lambda m p^{-1} \\ &\leqslant (2r(B))^{m-1}\, 6\lambda m p^{-1} \\ &= c_2\, |B(\mathbf{c}, \lambda)|_m \left(\frac{r(B)}{\lambda}\right)^{m-1}. \end{aligned}$$

□

*3.4. Proof of Proposition 4*

Let $\mathbf{x}_0 \in U$ be such that $\mathbf{f}$ is $\mathbf{v}$-nice at $\mathbf{x}_0$ and let $U_0$ be the neighborhood of $\mathbf{x}_0$ that arises from Definition 2. Without loss of generality, we will assume that $U_0$ is a ball satisfying

$$\operatorname{diam} U_0 \leqslant \left(2nm(n+1)C_0\delta^{-n}\right)^{-1}, \tag{93}$$

where $\delta$ is as in Definition 2 and $C_0$ is as in (15). We shall show that there are constants $\kappa_0 > 0$ and $\kappa_1 > 1$ and a value for $p$ associated with (84) such that the collection $(R_F)_{F\in\mathcal{F}_n}$ given by (87) satisfies the statement of Proposition 4. In view of Proposition 5, the intersection conditions (73) and (74) are then automatically satisfied. Thus, to establish ubiquity all that remains is to verify the measure theoretic 'covering' condition (75).

Let $B \subset U_0$ be an arbitrary ball and $t$ be a sufficiently large integer. Let

$$Q = 2^t.$$

By Definition 2, for some fixed $\delta, \omega \in (0,1)$ we have that

$$\limsup_{Q\to\infty} |\Phi^{\mathbf{f}}_{\mathbf{v}}(Q,\delta) \cap \tfrac{1}{2}B|_m \leqslant \omega |\tfrac{1}{2}B|_m\,.$$

Therefore, for sufficiently large $Q$ we have that

$$|\tfrac{1}{2}B \setminus \Phi^{\mathbf{f}}_{\mathbf{v}}(Q,\delta)|_m \geqslant \tfrac{1}{2}(1-\omega)|\tfrac{1}{2}B|_m = 2^{-m-1}(1-\omega)|B|_m.$$

Therefore, if we can show that

$$\tfrac{1}{2}B \setminus \Phi^{\mathbf{f}}_{\mathbf{v}}(Q,\delta) \subset \bigcup_{\substack{F\in\mathcal{F}_n \\ \beta_F \leqslant Q}} \Delta(R_F, \rho(Q)) \cap B \tag{94}$$

then (75) would follow as required. With this in mind, let

$$\mathbf{x} \in \tfrac{1}{2}B \setminus \Phi^{\mathbf{f}}_{\mathbf{v}}(Q,\delta)$$

and consider the system of inequalities

$$\left\{\begin{array}{rcll} |a_n f_n(\mathbf{x}) + \ldots + a_1 f_1(\mathbf{x}) + a_0| & < & Q^{-n} & \\ |a_i| & \leqslant & Q^{v_i} & (1 \leqslant i \leqslant n). \end{array}\right. \tag{95}$$

The set of $(a_0, \ldots, a_n) \in \mathbb{R}^{n+1}$ satisfying (95) gives rise to a convex body $D$ in $\mathbb{R}^{n+1}$ which is symmetric about the origin. Let $\tau_0, \ldots, \tau_{n+1}$ be the successive minima of $D$. By definition, $\tau_1 \leqslant \tau_2 \leqslant \ldots \leqslant \tau_{n+1}$. Since $\mathbf{x} \notin \Phi^{\mathbf{f}}_{\mathbf{v}}(Q,\delta)$, we have that $\tau_1 \geqslant \delta$. By Minkowski's theorem on successive minima [36], we have that

$$\tau_1 \cdots \tau_{n+1} \mathrm{Vol}(D) \leqslant 2^{n+1}\,.$$

In view of the fact that $v_1 + \ldots + v_n = n$ we find that $\mathrm{Vol}(D) = 2^{n+1}$. Therefore, $\tau_1 \cdots \tau_{n+1} \leqslant 1$, whence

$$\tau_{n+1} \leqslant (\tau_1 \cdot \tau_2 \cdots \tau_n)^{-1} < \delta^{-n}\,.$$

By the definition of $\tau_{n+1}$, there are linearly independent integer vectors $\mathbf{a}_j = (a_{j,0}, \ldots, a_{j,n}) \in \mathbb{Z}^{n+1}$ $(0 \leqslant j \leqslant n)$ such that the functions $F_j$ given by

$$F_j(\mathbf{x}) := a_{j,n} f_n(\mathbf{x}) + \ldots + a_{j,1} f_1(\mathbf{x}) + a_{j,0}$$

satisfy

$$\left\{\begin{array}{rcll} |F_j(\mathbf{x})| & \leqslant & C_2 Q^{-n} & \\ |a_{j,i}| & \leqslant & C_2 Q^{v_i} & (1 \leqslant i \leqslant n), \end{array}\right. \tag{96}$$

where

$$C_2 := \delta^{-n}. \tag{97}$$

The next step is to construct a linear combination of $F_j$ which gives rise to a resonant set $R_F$ with $\mathbf{x}$ lying within a sufficiently small neighborhood of $R_F$. With this in mind, consider the following system of linear equations

$$\left\{\begin{array}{rcll} \eta_0 F_0(\mathbf{x}) + \ldots + \eta_n F_n(\mathbf{x}) + \theta(\mathbf{x}) & = & 0 & \\ \eta_0 \frac{\partial}{\partial x_1} F_0(\mathbf{x}) + \ldots + \eta_n \frac{\partial}{\partial x_1} F_n(\mathbf{x}) + \frac{\partial}{\partial x_1}\theta(\mathbf{x}) & = & Q^{v_1} + \sum_{i=0}^n |\frac{\partial}{\partial x_1} F_i(\mathbf{x})| & \\ \eta_0 a_{0,j} + \ldots + \eta_n a_{n,j} & = & 0 & (2 \leqslant j \leqslant n). \end{array}\right. \tag{98}$$

Using the fact that $f_1(\mathbf{x}) = x_1$, it is readily verified that the determinant of this system is equal to $\det(a_i^{(j)})_{0\leqslant i,j\leqslant n}$. The latter is non-zero since $\mathbf{a}_0, \ldots, \mathbf{a}_n$ are linearly independent. Therefore, the system (98) has a unique solution $\eta_0, \ldots, \eta_n$. For the integers $t_i := \lfloor \eta_i \rfloor$ we have that

$$|t_i - \eta_i| < 1 \qquad (0 \leqslant i \leqslant n). \tag{99}$$

Let

$$F(\mathbf{x}) := t_0F_0(\mathbf{x}) + \ldots + t_nF_n(\mathbf{x}) = a_nf_n(\mathbf{x}) + \ldots + a_1f_1(\mathbf{x}) + a_0,$$

where $a_i := t_0a_{0,i} + \ldots + t_na_{n,i}$. We claim that $F$ satisfies (84), the height condition $\beta_F \leqslant Q$ and moreover $\mathbf{x} \in \Delta(R_F, \rho(Q))$. Thus (94) follows and we are done.

**Verifying the height condition:** By making use of (96), (98) and (99), we find that

$$|a_j| \leqslant (n+1)C_2Q^{v_i} \qquad (2 \leqslant j \leqslant n) \tag{100}$$

and

$$|F(\mathbf{x}) + \theta(\mathbf{x})| \leqslant (n+1)C_2Q^{-n}. \tag{101}$$

Using the second equation of (98), we find that

$$\left|\tfrac{\partial}{\partial x_1}(F+\theta)(\mathbf{x})\right| \geqslant Q^{v_1}. \tag{102}$$

In particular, this means that $F$ is not identically zero and so $F \in \mathcal{F}_n$. Next, using (15), (96) and the assumption that $v_1 = |\mathbf{v}|$ we find that

$$\left|\frac{\partial}{\partial x_1}F_i(\mathbf{x})\right| \leqslant nC_0Q^{v_1} \qquad \text{for all } i = 0, \ldots, n.$$

Together with (98) and (99), this implies that

$$\left|\tfrac{\partial}{\partial x_1}(F+\theta)(\mathbf{x})\right| \leqslant (2nC_0+1)Q^{v_1}. \tag{103}$$

Furthermore, since $\mathbf{f}$ is a Monge parameterisation we have that

$$a_1 = \tfrac{\partial}{\partial x_1}(F+\theta)(\mathbf{x}) - \tfrac{\partial}{\partial x_1}\theta(\mathbf{x}) - \sum_{j=2}^{n} a_j \tfrac{\partial}{\partial x_1}f_j(\mathbf{x})\,.$$

Then, on using (15), (100) and (103) we obtain that

$$|a_1| \leqslant C_3 Q^{v_1}, \qquad \text{where} \quad C_3 := (n+3)^2 C_0 C_2 \,. \tag{104}$$

This together with (100) and (102) gives that

$$\kappa_0^* Q \leqslant \beta_F := \kappa_0 H_{\mathbf{v}}(F) \leqslant Q \,,$$

for some explicitly computable constant $\kappa_0, \kappa_0^* > 0$ depending only on $\mathbf{v}$, $n$, $C_0$ and $C_2$.

**Verifying condition (84):** In view of Taylor's formula, for any $\mathbf{y} \in U_0$ we have that

$$\left|\tfrac{\partial}{\partial x_1}(F+\theta)(\mathbf{y})\right| \ \geqslant \ \left|\tfrac{\partial}{\partial x_1}(F+\theta)(\mathbf{x})\right| - \sum_{i=1}^{m} \left|\tfrac{\partial^2}{\partial x_1 \partial x_i}(F+\theta)(\tilde{\mathbf{y}})(y_i - x_i)\right| . \tag{105}$$

By making use of (15), (100) and (104) we find that the second term of the r.h.s. of (105) is bounded above by $mnC_0(n+1)C_2 \operatorname{diam} U_0\, Q^{v_1}$. In view of (93) and (97) the latter is no larger than $\frac{1}{2}Q^{v_1}$. On the other hand, by (102) the first term in the r.h.s. of (105) is $\geqslant Q^{v_1}$. Thus, (105) implies that

$$\left|\tfrac{\partial}{\partial x_1}(F+\theta)(\mathbf{y})\right| \geqslant \tfrac{1}{2}Q^{v_1} \,.$$

On the other hand, by using (15), (100) and (104) we find that

$$\left|\tfrac{\partial}{\partial x_i}(F+\theta)(\mathbf{y})\right| \ \leqslant \ C_4\, Q^{v_1}$$

for any $i = 1, \ldots, m$ and $\mathbf{y} \in U_0$, where

$$C_4 := (n+1)C_0 \max\{C_3,\, (n+1)C_2\} \,.$$

This together with the above lower bound inequality implies (84) with $p := (2mC_4)^{-1}$.

**Verifying that $\mathbf{x} \in \Delta(R_F, \rho(Q))$:** We will makes use of the following easy consequence of the Mean Value Theorem.

**Lemma 6.** *Let $f : I \to \mathbb{R}$ be a $C^1$ function on an interval $I$ such that $|f'(x)| \geqslant d > 0$ for all $x \in I$. Let $x_1 \in I$ and suppose that $B(x_1, |f(x_1)|d^{-1}) \subset I$. Then, there is an $x_0 \in B(x_1, |f(x_1)|d^{-1})$ such that $f(x_0) = 0$.*

Let $\mathbf{x} = (x_1, \ldots, x_m)$. Consider the interval

$$I := \{x \in \mathbb{R} : (x, x_2, \ldots, x_m) \in B\}$$

and the function $f : I \to \mathbb{R}$ given by $f(x) = (F + \theta)(x, x_2, \ldots, x_m)$. In view of (101) and (102) and the fact that $\mathbf{x} \in \frac{1}{2}B$, Lemma 6 is applicable and implies that there exists some $x_0 \in I$ such that $f(x_0) = 0$ and $|x_1 - x_0| \leqslant (n+1)C_2Q^{-n-v_1}$. Then $\mathbf{x}' := (x_0, x_2, \ldots, x_m) \in B$ satisfies $F(\mathbf{x}') + \theta(\mathbf{x}') = 0$ and

$$|\mathbf{x} - \mathbf{x}'| \leqslant (n+1)C_2Q^{-n-v_1}\,. \tag{106}$$

On making use of (15) and the Mean Value Theorem, we find that $|(F + \theta)(\mathbf{y})| \ll Q^{-n}$ for any $\mathbf{y}$ satisfying $|\mathbf{y} - \mathbf{x}'| \ll Q^{-n-v_1}$. Then, on using the above argument for determining $\mathbf{x}'$, enables us to conclude that for sufficiently large $Q$ the ball of radius $3\rho(\beta_F)$ centred at $\pi\mathbf{x}'$ is contained in $\tilde{V}$, where $\pi$ is the projection map given by (85) and $\tilde{V}$ is as in (86). The details are pretty straightforward and are left to the reader. The upshot is that $\mathbf{x}' \in R_F$ which together with (106) implies that $\mathbf{x} \in \Delta(R_F, \rho(Q))$ as required, where

$$\rho(Q) = \kappa_1 Q^{-n-v_1} \qquad \text{with} \qquad \kappa_1 := (n+1)C_2\,.$$

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