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## Poincaré Extension of Möbius Transformations

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Given sphere preserving (Möbius) transformations in  $n$ -dimensional Euclidean space one can use the Poincaré extension to obtain sphere preserving transformations in a half-space of  $n + 1$  dimensions. The Poincaré extension is usually provided either by an explicit formula or by some geometric construction. We investigate its algebraic background and describe all available options. The solution is given in terms of one-parameter subgroups of Möbius transformations acting on triples of quadratic forms. To focus on the concepts, this paper deals with the Möbius transformations of the real line only.

**Keywords:** Möbius transformation, Poincaré extension, quadratic forms, Cayley–Klein geometries.

*Dedicated to John Ryan on the occasion of his 60th birthday*

### 1. Introduction

It is known, that Möbius transformations on  $\mathbb{R}^n$  can be expanded to the “upper” half-space in  $\mathbb{R}^{n+1}$  using the Poincaré extension [1, § 3.3] [2, § 5.2]. An explicit formula is usually presented without a discussion of its origin. In particular, one may get an impression that the solution is unique. This paper considers various aspects of such extension and describes different possible realisations. Our consideration is restricted to the case of extension from the real line to the upper half-plane. However, we made an effort to present it in a way, which allows numerous further generalisations.

### 2. Geometric construction

We start from the geometric procedure in the standard situation. The group  $SL_2(\mathbb{R})$  consists of real  $2 \times 2$  matrices with the unit determinant.  $SL_2(\mathbb{R})$  acts on the real line by linear-fractional maps:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : x \mapsto \frac{ax + b}{cx + d}, \quad \text{where } x \in \mathbb{R} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}). \quad (1)$$

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A pair of (possibly equal) real numbers  $x$  and  $y$  uniquely determines a semicircle  $C_{xy}$  in the upper half-plane with the diameter  $[x, y]$ . For a linear-fractional transformation  $M$  (1), the images  $M(x)$  and  $M(y)$  define the semicircle with the diameter  $[M(x), M(y)]$ , thus, we can define the action of  $M$  on semicircles:  $M(C_{xy}) = C_{M(x)M(y)}$ . Geometrically, the Poincaré extension is based on the following lemma, see Fig. 1(a) and more general Lem. 18 below:

**LEMMA 1** *If a pencil of semicircles in the upper half-plane has a common point, then the images of these semicircles under a transformation (1) have a common point as well.*

Elementary geometry of right triangles tells that a pair of intersecting intervals  $[x, y]$ ,  $[x', y']$ , where  $x < x' < y < y'$ , defines the point

$$\left( \frac{xy - x'y'}{x + y - x' - y'}, \frac{\sqrt{(x - y')(x - x')(x' - y)(y - y')}}{x + y - x' - y'} \right) \in \mathbb{R}_+^2. \quad (2)$$

An alternative demonstration uses three observations:

- (1) the scaling  $x \mapsto ax$ ,  $a > 0$  on the real line produces the scaling  $(u, v) \mapsto (au, av)$  on pairs (2);
- (2) the horizontal shift  $x \mapsto x + b$  on the real line produces the horizontal shift  $(u, v) \mapsto (u + b, v)$  on pairs (2);
- (3) for the special case  $y = -x^{-1}$  and  $y' = -x'^{-1}$  the pair (2) is  $(0, 1)$ .

Finally, expression (2), as well as (3)–(4) below, can be calculated by the specialised CAS for Möbius invariant geometries [3, 4].

This standard approach can be widened as follows. The above semicircle can be equivalently described through the unique circle passing  $x$  and  $y$  and orthogonal to the real axis. Similarly, an interval  $[x, y]$  uniquely defines a right-angle hyperbola in  $\mathbb{R}^2$  orthogonal to the real line and passing (actually, having her vertices at)  $(x, 0)$  and  $(y, 0)$ . An intersection with the second such hyperbola having vertices  $(x', 0)$  and  $(y', 0)$  defines a point with coordinates, see Fig. 1(f):

$$\left( \frac{xy - x'y'}{x + y - x' - y'}, \frac{\sqrt{(x - y')(x - x')(x' - y)(y' - y)}}{x + y - x' - y'} \right), \quad (3)$$

where  $x < y < x' < y'$ . Note, the opposite sign of the product under the square roots in (2) and (3).

If we wish to consider the third type of conic sections—parabolas—we cannot use the unaltered procedure: there is no a non-degenerate parabola orthogonal to the real line and intersecting the real line in two points. We may recall, that a circle (or hyperbola) is orthogonal to the real line if its centre belongs to the real line. Analogously, a parabola is *focally orthogonal* (see [5, § 6.6] for a general consideration) to the real line if its focus belongs to the real line. Then, an interval  $[x, y]$  uniquely defines a downward-opened parabola with the real roots  $x$  and  $y$  and focally orthogonal to the real line. Two such parabolas defined by intervals  $[x, y]$  and  $[x', y']$  have a common point, see Fig. 1(c):

$$\left( \frac{xy' - yx' + D}{x - y - x' + y'}, \frac{(x' - x)(y' - y)(y - x + y' - x') + (x + y - x' - y')D}{(x - y - x' + y')^2} \right), \quad (4)$$

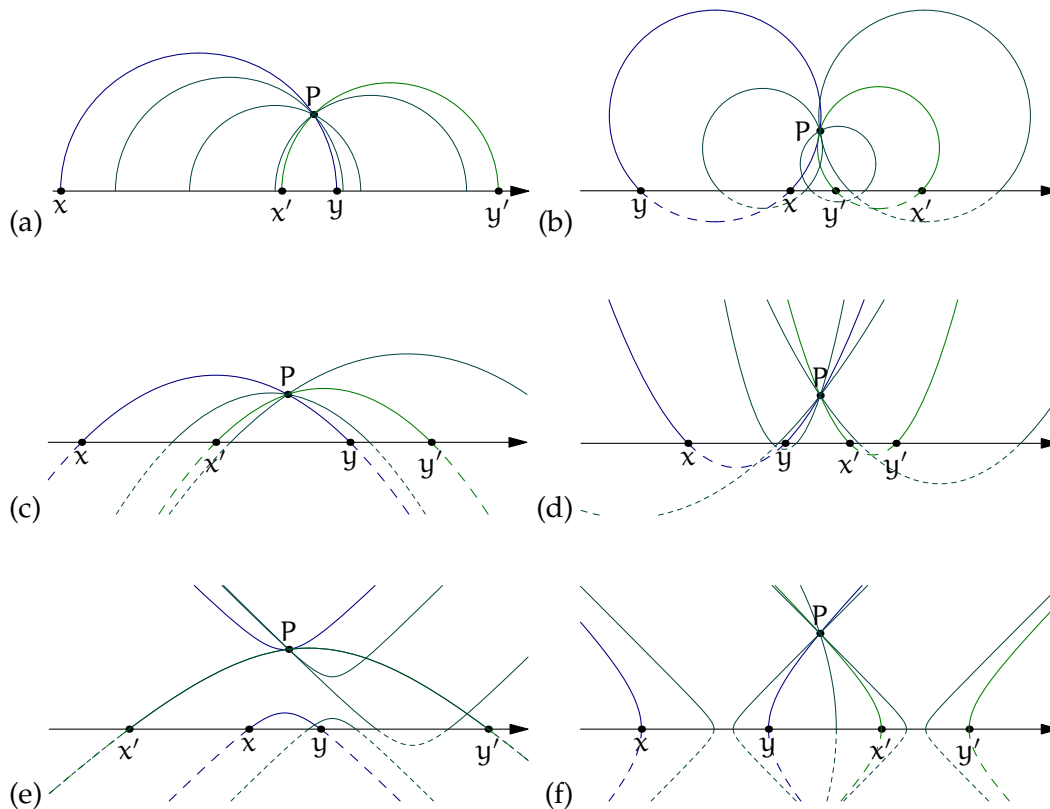


Figure 1. Poincaré extensions: first column presents points defined by the intersecting intervals  $[x, y]$  and  $[x', y']$ , the second column—by disjoint intervals. Each row uses the same type of conic sections—circles, parabolas and hyperbolas respectively. Pictures are produced by the software [4].

where  $D = \pm\sqrt{(x - x')(y - y')(y - x)(y' - x')}$ . For pencils of such hyperbolas and parabolas respective variants of Lem. 1 hold.

Focally orthogonal parabolas make the angle  $45^\circ$  with the real line. This suggests to replace orthogonal circles and hyperbolas by conic sections with a fixed angle to the real line, see Fig. 1(b)–(e). Of course, to be consistent this procedure requires a suitable modification of Lem. 1, we will obtain it as a byproduct of our study, see Lem. 18. However, the respective alterations of the above formulae (2)–(4) become more complicated in the general case.

The considered geometric construction is elementary and visually appealing. Now we turn to respective algebraic consideration.

### 3. Möbius transformations and Cycles

The group  $SL_2(\mathbb{R})$  acts on  $\mathbb{R}^2$  by matrix multiplication on column vectors:

$$\mathcal{L}_g : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}). \quad (5)$$

A linear action respects the equivalence relation  $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$ ,  $\lambda \neq 0$  on  $\mathbb{R}^2$ . The collection of all cosets for non-zero vectors in  $\mathbb{R}^2$  is the *projective line*  $\mathbb{P}\mathbb{R}^1$ . Explic-

itly, a non-zero vector  $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$  corresponds to the point with homogeneous coordinates  $[x_1 : x_2] \in \mathbb{P}\mathbb{R}^1$ . If  $x_2 \neq 0$  then this point is represented by  $[\frac{x_1}{x_2} : 1]$  as well. The embedding  $\mathbb{R} \rightarrow \mathbb{P}\mathbb{R}^1$  defined by  $x \mapsto [x : 1]$ ,  $x \in \mathbb{R}$  covers the all but one of points in  $\mathbb{P}\mathbb{R}^1$ . The exceptional point  $[1 : 0]$  is naturally identified with the infinity.

The linear action (5) induces a morphism of the projective line  $\mathbb{P}\mathbb{R}^1$ , which is called a Möbius transformation. Considered on the real line within  $\mathbb{P}\mathbb{R}^1$ , Möbius transformations takes fraction-linear form:

$$g : [x : 1] \mapsto \left[ \frac{ax + b}{cx + d} : 1 \right], \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \text{ and } cx + d \neq 0.$$

This  $\text{SL}_2(\mathbb{R})$ -action on  $\mathbb{P}\mathbb{R}^1$  is denoted as  $g : x \mapsto g \cdot x$ . We note that the correspondence of column vectors and row vectors  $\mathbf{i} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto (x_2, -x_1)$  intertwines the left multiplication  $\mathcal{L}_g$  (5) and the right multiplication  $\mathcal{R}_{g^{-1}}$  by the inverse matrix:

$$\mathcal{R}_{g^{-1}} : (x_2, -x_1) \mapsto (cx_1 + dx_2, -ax_1 - bx_2) = (x_2, -x_1) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \quad (6)$$

We extended the map  $\mathbf{i}$  to  $2 \times 2$ -matrices by the rule:

$$\mathbf{i} : \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_2 & -y_1 \\ x_2 & -x_1 \end{pmatrix}. \quad (7)$$

Two columns  $\begin{pmatrix} x \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} y \\ 1 \end{pmatrix}$  form the  $2 \times 2$  matrix  $M_{xy} = \begin{pmatrix} x & y \\ 1 & 1 \end{pmatrix}$ . For geometrical reasons appearing in Cor. 3, we call a *cycle* the  $2 \times 2$ -matrix  $C_{xy}$  defined by

$$C_{xy} = \frac{1}{2} M_{xy} \cdot \mathbf{i}(M_{xy}) = \frac{1}{2} M_{yx} \cdot \mathbf{i}(M_{yx}) = \begin{pmatrix} \frac{x+y}{2} & -xy \\ 1 & -\frac{x+y}{2} \end{pmatrix}. \quad (8)$$

We note that  $\det C_{xy} = -(x - y)^2/4$ , thus  $\det C_{xy} = 0$  if and only if  $x = y$ . Also, we can consider the Möbius transformation produced by the  $2 \times 2$ -matrix  $C_{xy}$  and calculate:

$$C_{xy} \begin{pmatrix} x \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} x \\ 1 \end{pmatrix} \quad \text{and} \quad C_{xy} \begin{pmatrix} y \\ 1 \end{pmatrix} = -\lambda \begin{pmatrix} y \\ 1 \end{pmatrix} \quad \text{where } \lambda = \frac{x - y}{2}. \quad (9)$$

Thus, points  $[x : 1]$ ,  $[y : 1] \in \mathbb{P}\mathbb{R}^1$  are fixed by  $C_{xy}$ . Also,  $C_{xy}$  swaps the interval  $[x, y]$  and its complement.

Due to their structure, matrices  $C_{xy}$  can be parametrised by points of  $\mathbb{R}^3$ . Furthermore, the map from  $\mathbb{R}^2 \rightarrow \mathbb{R}^3$  given by  $(x, y) \mapsto C_{xy}$  naturally induces the projective map  $(\mathbb{P}\mathbb{R}^1)^2 \rightarrow \mathbb{P}\mathbb{R}^2$  due to the identity:

$$\frac{1}{2} \begin{pmatrix} \lambda x & \mu y \\ \lambda & \mu \end{pmatrix} \begin{pmatrix} \mu & -\mu y \\ \lambda & -\lambda x \end{pmatrix} = \lambda \mu \begin{pmatrix} \frac{x+y}{2} & -xy \\ 1 & -\frac{x+y}{2} \end{pmatrix} = \lambda \mu C_{xy}.$$

Conversely, a zero-trace matrix  $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$  with a non-positive determinant is projectively equivalent to a product  $C_{xy}$  (8) with  $x, y = \frac{a \pm \sqrt{a^2 + bc}}{c}$ . In particular, we can embed a point  $[x : 1] \in \mathbb{P}\mathbb{R}^1$  to  $2 \times 2$ -matrix  $C_{xx}$  with zero determinant.

The combination of (5)–(8) implies that the correspondence  $(x, y) \mapsto C_{xy}$  is  $SL_2(\mathbb{R})$ -covariant in the following sense:

$$gC_{xy}g^{-1} = C_{x'y'}, \quad \text{where } x' = g \cdot x \text{ and } y' = g \cdot y. \quad (10)$$

To achieve a geometric interpretation of all matrices, we consider the bilinear form  $Q : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$  generated by a  $2 \times 2$ -matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ :

$$Q(x, y) = (x_1 \ x_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \text{where } x = (x_1, x_2), \ y = (y_1, y_2). \quad (11)$$

Due to linearity of  $Q$ , the null set

$$\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid Q(x, y) = 0\} \quad (12)$$

factors to a subset of  $\mathbb{P}\mathbb{R}^1 \times \mathbb{P}\mathbb{R}^1$ . Furthermore, for the matrices  $C_{xy}$  (8), a direct calculation shows that:

**LEMMA 2** *The following identity holds:*

$$C_{xy}(\mathbf{i}(x'), y') = \text{tr}(C_{xy}C_{x'y'}) = \frac{1}{2}(x + y)(x' + y') - (xy + x'y'). \quad (13)$$

*In particular, the above expression is a symmetric function of the pairs  $(x, y)$  and  $(x', y')$ .*

The map  $\mathbf{i}$  appearance in (13) is justified once more by the following result.

**COROLLARY 3** *The null set of the quadratic form  $C_{xy}(x') = C_{xy}(\mathbf{i}(x'), x')$  consists of two points  $x$  and  $y$ .*

Alternatively, the identities  $C_{xy}(x) = C_{xy}(y) = 0$  follows from (9) and the fact that  $\mathbf{i}(z)$  is orthogonal to  $z$  for all  $z \in \mathbb{R}^2$ . Also, we note that:

$$\mathbf{i}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (x_1 \ x_2) \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Motivated by Lem. 2, we call  $\langle C_{xy}, C_{x'y'} \rangle := -\text{tr}(C_{xy}C_{x'y'})$  the *pairing* of two cycles. It shall be noted that the pairing is *not* positively defined, this follows from the explicit expression (13). The sign is chosen in such way, that

$$\langle C_{xy}, C_{xy} \rangle = -2 \det(C_{xy}) = \frac{1}{2}(x - y)^2 \geq 0.$$

Also, an immediate consequence of Lem. 2 or identity (11) is

**COROLLARY 4** *The pairing of cycles is invariant under the action (10) of  $SL_2(\mathbb{R})$ :*

$$\langle g \cdot C_{xy} \cdot g^{-1}, g \cdot C_{x'y'} \cdot g^{-1} \rangle = \langle C_{xy}, C_{x'y'} \rangle.$$

From (13), the null set (12) of the form  $Q = C_{xy}$  can be associated to the family of cycles  $\{C_{x'y'} \mid \langle C_{xy}, C_{x'y'} \rangle = 0, (x', y') \in \mathbb{R}^2 \times \mathbb{R}^2\}$  which we will call *orthogonal* to  $C_{xy}$ .

#### 4. Extending cycles

Since bilinear forms with matrices  $C_{xy}$  have numerous geometric connections with  $\mathbb{P}\mathbb{R}^1$ , we are looking for a similar interpretation of the generic matrices. The previous discussion identified the key ingredient of the recipe:  $SL_2(\mathbb{R})$ -invariant pairing (13) of two forms. Keeping in mind the structure of  $C_{xy}$ , we will parametrise<sup>1</sup> a generic  $2 \times 2$  matrix as  $\begin{pmatrix} l+n & -m \\ k & -l+n \end{pmatrix}$  and consider the corresponding four dimensional vector  $(n, l, k, m)$ . Then, the similarity with  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ :

$$\begin{pmatrix} l'+n' & -m' \\ k' & -l'+n' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} l+n & -m \\ k & -l+n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}, \quad (14)$$

corresponds to the linear transformation of  $\mathbb{R}^4$ , cf. [5, Ex. 4.15]:

$$\begin{pmatrix} n' \\ l' \\ k' \\ m' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cb+ad & bd & ca \\ 0 & 2cd & d^2 & c^2 \\ 0 & 2ab & b^2 & a^2 \end{pmatrix} \begin{pmatrix} n \\ l \\ k \\ m \end{pmatrix}. \quad (15)$$

In particular, this action commutes with the scaling of the first component:

$$\lambda : (n, l, k, m) \mapsto (\lambda n, l, k, m). \quad (16)$$

This expression is helpful in proving the following statement.

**LEMMA 5** *Any  $SL_2(\mathbb{R})$ -invariant (in the sense of the action (15)) pairing in  $\mathbb{R}^4$  is isomorphic to*

$$2\tau\tilde{n}n - 2\tilde{l}l + \tilde{k}m + \tilde{m}k = (\tilde{n} \quad \tilde{l} \quad \tilde{k} \quad \tilde{m}) \begin{pmatrix} 2\tau & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} n \\ l \\ k \\ m \end{pmatrix},$$

where  $\tau = -1, 0$  or  $1$  and  $(n, l, k, m), (\tilde{n}, \tilde{l}, \tilde{k}, \tilde{m}) \in \mathbb{R}^4$ .

*Proof.* Let  $T$  be  $4 \times 4$  a matrix from (15), if a  $SL_2(\mathbb{R})$ -invariant pairing is defined by a  $4 \times 4$  matrix  $J = (j_{fg})$ , then  $T'JT = J$ , where  $T'$  is transpose of  $T$ . The equivalent

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<sup>1</sup>Further justification of this parametrisation will be obtained from the equation of a quadratic curve (18).

identity  $T'J = JT^{-1}$  produces a system of homogeneous linear equations which has the generic solution:

$$\begin{aligned} j_{12} = j_{13} = j_{14} = j_{21} = j_{31} = j_{41} &= 0, \\ j_{22} &= \frac{(d-a)j_{42} - 2bj_{44}}{c} - 2j_{43}, & j_{23} &= -\frac{b}{c}j_{42}, & j_{24} &= -\frac{cj_{42} + 2(a-d)j_{44}}{c}, \\ j_{34} &= \frac{c(a-d)j_{42} + (a-d)^2j_{44}}{c^2} + j_{43}, & j_{33} &= \frac{b^2}{c^2}j_{44}, & j_{32} &= \frac{b}{c}\frac{cj_{42} + 2(a-d)j_{44}}{c}, \end{aligned}$$

with four free variables  $j_{11}$ ,  $j_{42}$ ,  $j_{43}$  and  $j_{44}$ . Since a solution shall not depend on  $a$ ,  $b$ ,  $c$ ,  $d$ , we have to put  $j_{42} = j_{44} = 0$ . Then by the homogeneity of the identity  $T'J = JT^{-1}$ , we can scale  $j_{43}$  to 1. Thereafter, an independent (sign-preserving) scaling (16) of  $n$  leaves only three non-isomorphic values  $-1, 0, 1$  of  $j_{11}$ . ■

The appearance of the three essential different cases  $\tau = -1, 0$  or  $1$  in Lem. 5 is a manifestation of the common division of mathematical objects into elliptic, parabolic and hyperbolic cases [6] [5, Ch. 1]. Thus, we will use letters “e”, “p”, “h” to encode the corresponding three values of  $\tau$ .

Now we may describe all  $SL_2(\mathbb{R})$ -invariant pairings of bilinear forms.

**COROLLARY 6** Any  $SL_2(\mathbb{R})$ -invariant (in the sense of the similarity (14)) pairing between two bilinear forms  $Q = \begin{pmatrix} l+n & -m \\ k & -l+n \end{pmatrix}$  and  $\tilde{Q} = \begin{pmatrix} \tilde{l}+\tilde{n} & -\tilde{m} \\ \tilde{k} & -\tilde{l}+\tilde{n} \end{pmatrix}$  is isomorphic to:

$$\begin{aligned} \langle Q, \tilde{Q} \rangle_\tau &= -\text{tr}(Q_\tau \tilde{Q}) & (17) \\ &= 2\tau\tilde{n}n - 2\tilde{l}l + \tilde{k}m + \tilde{m}k, & \text{where } Q_\tau &= \begin{pmatrix} l-\tau n & -m \\ k & -l-\tau n \end{pmatrix} \end{aligned}$$

and  $\tau = -1, 0$  or  $1$ .

Note that we can explicitly write  $Q_\tau$  for  $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  as follows:

$$Q_e = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad Q_p = \begin{pmatrix} \frac{1}{2}(a-d) & b \\ c & -\frac{1}{2}(a-d) \end{pmatrix}, \quad Q_h = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}.$$

In particular,  $Q_h = -Q^{-1}$  and  $Q_p = \frac{1}{2}(Q_e + Q_h)$ . Furthermore,  $Q_p$  has the same structure as  $C_{xy}$ . Now, we are ready to extend the projective line  $P\mathbb{R}^1$  to two dimensions using the analogy with properties of cycles  $C_{xy}$ .

**Definition 7** (1) Two bilinear forms  $Q$  and  $\tilde{Q}$  are  $\tau$ -orthogonal if  $\langle Q, \tilde{Q} \rangle_\tau = 0$ .  
 (2) A form is  $\tau$ -isotropic if it is  $\tau$ -orthogonal to itself.

If a form  $Q = \begin{pmatrix} l+n & -m \\ k & -l+n \end{pmatrix}$  has  $k \neq 0$  then we can scale it to obtain  $k = 1$ , this form of  $Q$  is called *normalised*. A normalised  $\tau$ -isotropic form is completely determined by its diagonal values:  $\begin{pmatrix} u+v & -u^2 + \tau v^2 \\ 1 & -u+v \end{pmatrix}$ . Thus, the set of such forms is in a one-to-one correspondence with points of  $\mathbb{R}^2$ . Finally, a form  $Q =$

$\begin{pmatrix} l+n & -m \\ k & -l+n \end{pmatrix}$  is  $e$ -orthogonal to the  $\tau$ -isotropic form  $\begin{pmatrix} u+v & -u^2+\tau v^2 \\ 1 & -u+v \end{pmatrix}$  if:

$$k(u^2 - \tau v^2) - 2lu - 2nv + m = 0, \quad (18)$$

that is the point  $(u, v) \in \mathbb{R}^2$  belongs to the quadratic curve with coefficients  $(k, l, n, m)$ .

## 5. Homogeneous spaces of cycles

Obviously, the group  $SL_2(\mathbb{R})$  acts on  $\mathbb{P}\mathbb{R}^1$  transitively, in fact it is even 3-transitive in the following sense. We say that a triple  $\{x_1, x_2, x_3\} \subset \mathbb{P}\mathbb{R}^1$  of distinct points is *positively oriented* if

$$\text{either } x_1 < x_2 < x_3, \quad \text{or } x_3 < x_1 < x_2,$$

where we agree that the ideal point  $\infty \in \mathbb{P}\mathbb{R}^1$  is greater than any  $x \in \mathbb{R}$ . Equivalently, a triple  $\{x_1, x_2, x_3\}$  of reals is positively oriented if:

$$(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) > 0.$$

Also, a triple of distinct points, which is not positively oriented, is *negatively oriented*. A simple calculation based on the resolvent-type identity:

$$\frac{ax+b}{cx+d} - \frac{ay+b}{cy+d} = \frac{(x-y)(ad-bc)}{(cx+b)(cy+d)}$$

shows that both the positive and negative orientations of triples are  $SL_2(\mathbb{R})$ -invariant. On the other hand, the reflection  $x \mapsto -x$  swaps orientations of triples. Note, that the reflection is a Moebius transformation associated to the cycle

$$C_{0\infty} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{with } \det C_{0\infty} = -1. \quad (19)$$

A significant amount of information about Möbius transformations follows from the fact, that any continuous one-parametric subgroup of  $SL_2(\mathbb{R})$  is conjugated to one of the three following subgroups<sup>2</sup>:

$$A = \left\{ \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\}, \quad K = \left\{ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \right\}, \quad (20)$$

where  $t \in \mathbb{R}$ . Also, it is useful to introduce subgroups  $A'$  and  $N'$  conjugated to  $A$  and  $N$  respectively:

$$A' = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \mid t \in \mathbb{R} \right\}, \quad N' = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}. \quad (21)$$

<sup>2</sup>A reader may know that  $A$ ,  $N$  and  $K$  are factors in the Iwasawa decomposition  $SL_2(\mathbb{R}) = ANK$  (cf. Cor. 10, however this important result does not play any rôle in our consideration.

Thereafter, all three one-parameter subgroups  $A'$ ,  $N'$  and  $K$  consist of all matrices with the universal structure

$$\begin{pmatrix} a & \tau b \\ b & a \end{pmatrix} \quad \text{where } \tau = 1, 0, -1 \text{ for } A', N' \text{ and } K \text{ respectively.} \quad (22)$$

We use the notation  $H_\tau$  for these subgroups. Again, any continuous one-dimensional subgroup of  $SL_2(\mathbb{R})$  is conjugated to  $H_\tau$  for an appropriate  $\tau$ .

We note, that matrices from  $A$ ,  $N$  and  $K$  with  $t \neq 0$  have two, one and none different real eigenvalues respectively. Eigenvectors in  $\mathbb{R}^2$  correspond to fixed points of Möbius transformations on  $\mathbb{P}\mathbb{R}^1$ . Clearly, the number of eigenvectors (and thus fixed points) is limited by the dimensionality of the space, that is two. For this reason, if  $g_1$  and  $g_2$  take equal values on three different points of  $\mathbb{P}\mathbb{R}^1$ , then  $g_1 = g_2$ .

Also, eigenvectors provide an effective classification tool:  $g \in SL_2(\mathbb{R})$  belongs to a one-dimensional continuous subgroup conjugated to  $A$ ,  $N$  or  $K$  if and only if the characteristic polynomial  $\det(g - \lambda I)$  has two, one and none different real root(s) respectively. We will illustrate an application of fixed points techniques through the following well-known result, which will be used later.

**LEMMA 8** *Let  $\{x_1, x_2, x_3\}$  and  $\{y_1, y_2, y_3\}$  be positively oriented triples of points in  $\mathbb{R}$ . Then, there is a unique (computable!) Möbius map  $\phi \in SL_2(\mathbb{R})$  with  $\phi(x_j) = y_j$  for  $j = 1, 2, 3$ .*

*Proof.* Often, the statement is quickly demonstrated through an explicit expression for  $\phi$ , cf. [7, Thm. 13.2.1]. We will use properties of the subgroups  $A$ ,  $N$  and  $K$  to describe an algorithm to find such a map. First, we notice that it is sufficient to show the Lemma for the particular case  $y_1 = 0$ ,  $y_2 = 1$ ,  $y_3 = \infty$ . The general case can be obtained from composition of two such maps. Another useful observation is that the fixed point for  $N$ , that is  $\infty$ , is also a fixed point of  $A$ .

Now, we will use subgroups  $K$ ,  $N$  and  $A$  in order of increasing number of their fixed points. First, for any  $x_3$  the matrix  $g' = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in K$  such that  $\cot t = x_3$  maps  $x_3$  to  $y_3 = \infty$ . Let  $x'_1 = g'x_1$  and  $x'_2 = g'x_2$ . Then the matrix  $g'' = \begin{pmatrix} 1 & -x'_1 \\ 0 & 1 \end{pmatrix} \in N$ , fixes  $\infty = g'x_3$  and sends  $x'_1$  to  $y_1 = 0$ . Let  $x''_2 = g''x'_2$ , from positive orientation of triples we have  $0 < x''_2 < \infty$ . Next, the matrix  $g''' = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \in A$  with  $a = \sqrt{x''_2}$  sends  $x''_2$  to 1 and fixes both  $\infty = g''g'x_3$  and  $0 = g''g'x_1$ . Thus,  $g = g'''g''g'$  makes the required transformation  $(x_1, x_2, x_3) \mapsto (0, 1, \infty)$ . ■

**COROLLARY 9** *Let  $\{x_1, x_2, x_3\}$  and  $\{y_1, y_2, y_3\}$  be two triples with the opposite orientations. Then, there is a unique Möbius map  $\phi \in SL_2(\mathbb{R})$  with  $\phi \circ C_{0\infty}(x_j) = y_j$  for  $j = 1, 2, 3$ .*

We will denote by  $\phi_{XY}$  the unique map from Lem. 8 defined by triples  $X = \{x_1, x_2, x_3\}$  and  $Y = \{y_1, y_2, y_3\}$ .

Although we are not going to use it in this paper, we note that the following important result [8, § III.1] is an immediate consequence of our *proof* of Lem. 8.

**COROLLARY 10 (Iwasawa decomposition)** *Any element of  $g \in \mathrm{SL}_2(\mathbb{R})$  is a product  $g = g_A g_N g_K$ , where  $g_A$ ,  $g_N$  and  $g_K$  belong to subgroups  $A$ ,  $N$ ,  $K$  respectively and those factors are uniquely defined.*

In particular, we note that it is not a coincidence that the subgroups appear in the Iwasawa decomposition  $\mathrm{SL}_2(\mathbb{R}) = \mathrm{ANK}$  in order of decreasing number of their fixed points.

## 6. Triples of intervals

We change our point of view and instead of two ordered triples of points consider three ordered pairs, that is—three intervals. For them we will need the following definition.

*Definition 11* We say that a triple of intervals  $\{[x_1, y_1], [x_2, y_2], [x_3, y_3]\}$  is *aligned* if the triples  $X = \{x_1, x_2, x_3\}$  and  $Y = \{y_1, y_2, y_3\}$  of their endpoints have the same orientation.

Aligned triples determine certain one-parameter subgroups of Möbius transformations as follows:

**PROPOSITION 12** *Let  $\{[x_1, y_1], [x_2, y_2], [x_3, y_3]\}$  be an aligned triple of intervals.*

- (1) *If  $\phi_{XY}$  has at most one fixed point, then there is a unique (up to a parametrisation) one-parameter semigroup of Möbius map  $\psi(t) \in \mathrm{SL}_2(\mathbb{R})$ , which maps  $[x_1, y_1]$  to  $[x_2, y_2]$  and  $[x_3, y_3]$ :*

$$\psi(t_j)(x_1) = x_j, \quad \psi(t_j)(y_1) = y_j, \quad \text{for some } t_j \in \mathbb{R} \text{ and } j = 2, 3.$$

- (2) *Let  $\phi_{XY}$  have two fixed points  $x < y$  and  $C_{xy}$  be the orientation inverting Möbius transformation with the matrix (8). For  $j = 1, 2, 3$ , we define:*

$$\begin{aligned} x'_j &= x_j, & y'_j &= y_j, & x''_j &= C_{xy}x_j, & y''_j &= C_{xy}y_j & \text{if } x < x_j < y; \\ x'_j &= C_{xy}x_j, & y'_j &= C_{xy}y_j, & x''_j &= x_j, & y''_j &= y_j, & \text{otherwise.} \end{aligned}$$

*Then, there is a one-parameter semigroup of Möbius map  $\psi(t) \in \mathrm{SL}_2(\mathbb{R})$ , and  $t_2, t_3 \in \mathbb{R}$  such that:*

$$\psi(t_j)(x'_1) = x'_j, \quad \psi(t_j)(x''_1) = x''_j, \quad \psi(t_j)(y'_1) = y'_j, \quad \psi(t_j)(y''_1) = y''_j,$$

*where  $j = 2, 3$ .*

*Proof.* Consider the one-parameter subgroup of  $\psi(t) \in \mathrm{SL}_2(\mathbb{R})$  such that  $\phi_{XY} = \psi(1)$ . Note, that  $\psi(t)$  and  $\phi_{XY}$  have the same fixed points (if any) and no point  $x_j$  is fixed since  $x_j \neq y_j$ . If the number of fixed points is less than 2, then  $\psi(t)x_1$ ,  $t \in \mathbb{R}$  produces the entire real line except a possible single fixed point. Therefore, there are  $t_2$  and  $t_3$  such that  $\psi(t_2)x_1 = x_2$  and  $\psi(t_3)x_1 = x_3$ . Since  $\psi(t)$  and  $\phi_{XY}$  commute for all  $t$  we also have:

$$\psi(t_j)y_1 = \psi(t_j)\phi_{XY}x_1 = \phi_{XY}\psi(t_j)x_1 = \phi_{XY}x_j = y_j, \quad \text{for } j = 2, 3.$$

If there are two fixed points  $x < y$ , then the open interval  $(x, y)$  is an orbit for the subgroup  $\psi(t)$ . Since all  $x'_1, x'_2$  and  $x'_3$  belong to this orbit and  $C_{xy}$  commutes with  $\phi_{XY}$  we may repeat the above reasoning for the dashed intervals  $[x'_j, y'_j]$ . Finally,  $x''_j = C_{xy}x'_j$  and  $y''_j = C_{xy}y'_j$ , where  $C_{xy}$  commutes with  $\phi$  and  $\psi(t_j)$ ,  $j = 2, 3$ . Uniqueness of the subgroup follows from Lemma 13. ■

The group  $SL_2(\mathbb{R})$  acts transitively on collection of all cycles  $C_{xy}$ , thus this is a  $SL_2(\mathbb{R})$ -homogeneous space. It is easy to see that the fix-group of the cycle  $C_{-1,1}$  is  $A'$  (21). Thus the homogeneous space of cycles is isomorphic to  $SL_2(\mathbb{R})/A'$ .

**LEMMA 13** *Let  $H$  be a one-parameter continuous subgroup of  $SL_2(\mathbb{R})$  and  $X = SL_2(\mathbb{R})/H$  be the corresponding homogeneous space. If two orbits of one-parameter continuous subgroups on  $X$  have at least three common points then these orbits coincide.*

*Proof.* Since  $H$  is conjugated either to  $A'$ ,  $N'$  or  $K$ , the homogeneous space  $X = SL_2(\mathbb{R})/H$  is isomorphic to the upper half-plane in double, dual or complex numbers [5, § 3.3.4]. Orbits of one-parameter continuous subgroups in  $X$  are conic sections, which are circles, parabolas (with vertical axis) or equilateral hyperbolas (with vertical axis) for the respective type of geometry. Any two different orbits of the same type intersect at most at two points, since an analytic solution reduces to a quadratic equation. ■

Alternatively, we can reformulate Prop. 12 as follows: three different cycles  $C_{x_1y_1}, C_{x_2y_2}, C_{x_3y_3}$  define a one-parameter subgroup, which generate either one orbit or two related orbits passing the three cycles.

We have seen that the number of fixed points is the key characteristics for the map  $\phi_{XY}$ . The next result gives an explicit expression for it.

**PROPOSITION 14** *The map  $\phi_{XY}$  has zero, one or two fixed points if the expression*

$$\det \begin{pmatrix} 1 & x_1y_1 & y_1 - x_1 \\ 1 & x_2y_2 & y_2 - x_2 \\ 1 & x_3y_3 & y_3 - x_3 \end{pmatrix}^2 - 4 \det \begin{pmatrix} x_1 & 1 & y_1 \\ x_2 & 1 & y_2 \\ x_3 & 1 & y_3 \end{pmatrix} \cdot \det \begin{pmatrix} x_1 & -x_1y_1 & y_1 \\ x_2 & -x_2y_2 & y_2 \\ x_3 & -x_3y_3 & y_3 \end{pmatrix} \quad (23)$$

*is negative, zero or positive respectively.*

*Proof.* If a Möbius transformation  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  maps  $x_1 \mapsto y_1, x_2 \mapsto y_2, x_3 \mapsto y_3$  and  $s \mapsto s$ , then we have a homogeneous linear system, cf. [7, Ex. 13.2.4]:

$$\begin{pmatrix} x_1 & 1 & -x_1y_1 & -y_1 \\ x_2 & 1 & -x_2y_2 & -y_2 \\ x_3 & 1 & -x_3y_3 & -y_3 \\ s & 1 & -s^2 & -s \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (24)$$

A non-zero solution exists if the determinant of the  $4 \times 4$  matrix is zero. Expanding it over the last row and rearranging terms we obtain the quadratic equation for the fixed point  $s$ :

$$s^2 \det \begin{pmatrix} x_1 & 1 & y_1 \\ x_2 & 1 & y_2 \\ x_3 & 1 & y_3 \end{pmatrix} + s \det \begin{pmatrix} 1 & x_1y_1 & y_1 - x_1 \\ 1 & x_2y_2 & y_2 - x_2 \\ 1 & x_3y_3 & y_3 - x_3 \end{pmatrix} + \det \begin{pmatrix} x_1 & -x_1y_1 & y_1 \\ x_2 & -x_2y_2 & y_2 \\ x_3 & -x_3y_3 & y_3 \end{pmatrix} = 0.$$

The value (23) is the discriminant of this equation. ■

*Remark 15* It is interesting to note, that the relation  $ax + b - cxy - dy = 0$  used in (24) can be stated as e-orthogonality of the cycle  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and the isotropic bilinear form  $\begin{pmatrix} x & -xy \\ 1 & -y \end{pmatrix}$ .

If  $y = g_0 \cdot x$  for some  $g_0 \in H_\tau$ , then for any  $g \in H_\tau$  we also have  $y_g = g_0 \cdot x_g$ , where  $x_g = g \cdot x$  and  $y_g = g \cdot y$ . Thus, we demonstrated the first part of the following result.

**LEMMA 16** *Let  $\tau = 1, 0$  or  $-1$  and a real constant  $t \neq 0$  be such that  $1 - \tau t^2 > 0$ .*

(1) *The collections of intervals:*

$$I_{\tau,t} = \left\{ \left[ x, \frac{x+\tau t}{t x+1} \right] \mid x \in \mathbb{R} \right\} \quad (25)$$

*is preserved by the actions of subgroup  $H_\tau$ . Any three different intervals from  $I_{\tau,t}$  define the subgroup  $H_\tau$  in the sense of Prop. 12.*

(2) *All  $H_\tau$ -invariant bilinear forms compose the family  $P_{\tau,t} = \left\{ \begin{pmatrix} a & \tau b \\ b & a \end{pmatrix} \right\}$ .*

The family  $P_{\tau,t}$  consists of the eigenvectors of the  $4 \times 4$  matrix from (15) with suitably substituted entries. There is (up to a factor) exactly one  $\tau$ -isotropic form in  $P_{\tau,t}$ , namely  $\begin{pmatrix} 1 & \tau \\ 1 & 1 \end{pmatrix}$ . We denote this form by  $\iota$ . It corresponds to the point  $(0, 1) \in \mathbb{R}^2$  as discussed after Defn. 7. We may say that the subgroup  $H_\tau$  fixes the point  $\iota$ , this will play an important rôle below.

## 7. Geometrisation of cycles

We return to the geometric version of the Poincaré extension considered in Sec. 2 in terms of cycles. Cycles of the form  $\begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix}$  are  $\tau$ -isotropic for any  $\tau$  and are parametrised by the point  $x$  of the real line. For a fixed  $\tau$ , the collection of all  $\tau$ -isotropic cycles is a larger set containing the image of the real line from the previous sentence. Geometrisation of this embedding is described in the following result.

**LEMMA 17** (1) *The transformation  $x \mapsto \frac{x+\tau t}{t x+1}$  from the subgroup  $H_\tau$ , which maps  $x \mapsto y$ , corresponds to the value  $t = \frac{x-y}{x y-\tau}$ .*

(2) *The unique (up to a factor) bilinear form  $Q$  orthogonal to  $C_{xx}$ ,  $C_{yy}$  and  $\iota$  is*

$$Q = \begin{pmatrix} \frac{1}{2}(x+y+xy-\tau) & -xy \\ 1 & \frac{1}{2}(-x-y+xy-\tau) \end{pmatrix}.$$

(3) The defined above  $t$  and  $Q$  are connected by the identity:

$$\frac{\langle Q, \mathbb{R} \rangle_\tau}{\sqrt{|\langle Q, Q \rangle_\tau|}} = \frac{\tau}{\sqrt{|t^2 - \tau|}}. \quad (26)$$

Here, the real line is represented by the bilinear form  $\mathbb{R} = \begin{pmatrix} 2^{-1/2} & 0 \\ 0 & 2^{-1/2} \end{pmatrix}$  normalised such that  $\langle \mathbb{R}, \mathbb{R} \rangle_\tau = \pm 1$ .

(4) For a cycle  $Q = \begin{pmatrix} l+n & -m \\ k & -l+n \end{pmatrix}$ , the value  $\frac{\langle Q, \mathbb{R} \rangle_e}{\sqrt{|\langle Q, Q \rangle_e|}} = -\frac{n}{\sqrt{|l^2+n^2-km|}}$  is equal to the cosine of the angle between the curve  $k(u^2 + \tau v^2) - 2lu - 2nv + m = 0$  (18) and the real line, cf. [5, Ex. 5.23].

*Proof.* The first statement is verified by a short calculation. A form  $Q = \begin{pmatrix} l+n & -m \\ k & -l+n \end{pmatrix}$  in the second statement may be calculated from the homogeneous system:

$$\begin{pmatrix} 0 & -2x & x^2 & 1 \\ 0 & -2y & y^2 & 1 \\ -2 & 0 & -\tau & 1 \end{pmatrix} \begin{pmatrix} n \\ l \\ k \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which has the rank 3 if  $x \neq y$ . The third statement can be checked by a calculation as well. Finally, the last item is a particular case of the more general statement as indicated. Yet, we can derive it here from the implicit derivative  $\frac{dv}{du} = \frac{ku-l}{n}$  of the function  $k(u^2 + \tau v^2) - 2lu - 2nv + m = 0$  (18) at the point  $(u, 0)$ . Note that this value is independent from  $\tau$ . Since this is the tangent of the intersection angle with the real line, the square of the cosine of this angle is:

$$\frac{1}{1 + \left(\frac{dv}{du}\right)^2} = \frac{n^2}{l^2 + k^2u^2 + n^2 - 2kul} = \frac{n^2}{l^2 + n^2 - km} = \frac{\langle Q, \mathbb{R} \rangle_e^2}{\langle Q, Q \rangle_e},$$

if  $ku^2 - 2ul + m = 0$ . ■

Also, we note that, the independence of the left-hand side of (26) from  $x$  can be shown from basic principles. Indeed, for a fixed  $t$  the subgroup  $H_\tau$  acts transitively on the family of triples  $\{x, \frac{x+\tau t}{t x+1}, t\}$ , thus  $H_\tau$  acts transitively on all bilinear forms orthogonal to such triples. However, the left-hand side of (26) is  $SL_2(\mathbb{R})$ -invariant, thus may not depend on  $x$ . This simple reasoning cannot provide the exact expression in the right-hand side of (26), which is essential for the geometric interpretation of the Poincaré extension.

To restore a cycle from its intersection points with the real line we need also to know its cycle product with the real line. If this product is non-zero then the sign of the parameter  $n$  is additionally required. At the cycles' language, a common point of cycles  $C$  and  $\tilde{C}$  is encoded by a cycle  $\hat{C}$  such that:

$$\langle \hat{C}, C \rangle_e = \langle \hat{C}, \tilde{C} \rangle_e = \langle \hat{C}, \hat{C} \rangle_\tau = 0. \quad (27)$$

For a given value of  $\tau$ , this produces two linear and one quadratic equation for parameters of  $\hat{C}$ . Thus, a pair of cycles may not have a common point or have up to two such points. Furthermore, Möbius-invariance of the above conditions (26) and (27) supports the geometrical construction of Poincaré extension, cf. Lem. 1:

**LEMMA 18** *Let a family consist of cycles, which are  $e$ -orthogonal to a given  $\tau$ -isotropic cycle  $\hat{C}$  and have the fixed value of the fraction in the left-hand side of (26). Then, for a given Möbius transformation  $g$  and any cycle  $C$  from the family,  $gC$  is  $e$ -orthogonal to the  $\tau$ -isotropic cycle  $g\hat{C}$  and has the same fixed value of the fraction in the left-hand side of (26) as  $C$ .*

Summarising the geometrical construction, the Poincaré extension based on two intervals and the additional data produces two situations:

- (1) If the cycles  $C$  and  $\tilde{C}$  are orthogonal to the real line, then a pair of overlapping cycles produces a point of the elliptic upper half-plane, a pair of disjoint cycles defines a point of the hyperbolic. However, there is no orthogonal cycles uniquely defining a parabolic extension.
- (2) If we admit cycles, which are not orthogonal to the real line, then the same pair of cycles may define any of the three different types (EPH) of extension.

These peculiarities make the extension based on three intervals, described above, a bit more preferable.

## 8. Concluding remarks

Based on the consideration in Sections 3–7 we describe the following steps to carry out the generalised extension procedure:

- (1) Points of the extended space are equivalence classes of aligned triples of cycles in  $\mathbb{P}\mathbb{R}^1$ , see Defn. 11. The equivalence relation between triples will emerge at step 3.
- (2) A triple  $T$  of different cycles defines the unique one-parameter continuous subgroup  $S(t)$  of Möbius transformations as defined in Prop. 12.
- (3) Two triples of cycles are equivalent if and only if the subgroups defined in step 2 coincide (up to a parametrisation).
- (4) The geometry of the extended space, defined by the equivalence class of a triple  $T$ , is elliptic, parabolic or hyperbolic depending on the subgroup  $S(t)$  being similar  $S(t) = gH_\tau(t)g^{-1}$ ,  $g \in \text{SL}_2(\mathbb{R})$  (up to parametrisation) to  $H_\tau$  (22) with  $\tau = -1, 0$  or  $1$  respectively. The value of  $\tau$  may be identified from the triple using Prop. 14.
- (5) For the above  $\tau$  and  $g \in \text{SL}_2(\mathbb{R})$ , the point of the extended space, defined by the the equivalence class of a triple  $T$ , is represented by  $\tau$ -isotropic (see Defn. 7(2)) bilinear form  $g^{-1} \begin{pmatrix} 1 & \tau \\ 1 & 1 \end{pmatrix} g$ , which is  $S$ -invariant, see the end of Section 6.

Obviously, the above procedure is more complicated than the geometric construction from Section 2. There are reasons for this, as discussed in Section 7: our procedure is uniform and we are avoiding consideration of numerous sub-cases created by an incompatible selection of parameters. Furthermore, our pre-

sensation is aimed for generalisations to Möbius transformations of moduli over other rings. This can be considered as an analog of Cayley–Klein geometries [9, Apps. A–B] [10].

It shall be rather straightforward to adopt the extension for  $\mathbb{R}^n$ . Möbius transformations in  $\mathbb{R}^n$  are naturally expressed as linear-fractional transformations in Clifford algebras [11] with a similar classification of subgroups based on fixed points [12, 13]. The Möbius invariant matrix presentation of cycles  $\mathbb{R}^n$  is already known [11, (4.12)] [14] [15, § 5]. Of course, it is necessary to enlarge the number of defining cycles from 3 to, say,  $n + 2$ . This shall have a connection with Cauchy–Kovalevskaya extension considered in Clifford analysis [16, 17]. Naturally, a consideration of other moduli and rings may require some more serious adjustments in our scheme.

Our construction is based on the matrix presentations of cycles. This techniques is effective in many different cases [5, 15]. Thus, it is not surprising that such ideas (with some technical variation) appeared independently on many occasions [11, (4.12)] [14] [18, § 1.1] [19, § 4.2]. The interesting feature of the present work is the complete absence of any (hyper)complex numbers. It deemed to be unavoidable [5, § 3.3.4] to employ complex, dual and double numbers to represent three different types of Möbius transformations extended from the real line to a plane. Also (hyper)complex numbers were essential in [5, 6] to define three possible types of cycle product (17), and now we managed without them.

Apart from having real entries, our matrices for cycles share the structure of matrices from [11, (4.12)] [14] [5] [6]. To obtain another variant, one replaces the map  $\mathbf{i}$  (7) by

$$\mathbf{t} : \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 & y_2 \\ x_1 & x_2 \end{pmatrix}.$$

Then, we may define symmetric matrices in a manner similar to (8):

$$C_{xy}^t = \frac{1}{2} M_{xy} \cdot \mathbf{t}(M_{xy}) = \begin{pmatrix} xy & \frac{x+y}{2} \\ \frac{x+y}{2} & 1 \end{pmatrix}.$$

This is the form of matrices for cycles similar to [18, § 1.1] [19, § 4.2]. The property (10) with matrix similarity shall be replaced by the respective one with matrix congruence:  $g \cdot C_{xy}^t \cdot g^t = C_{x'y'}^t$ . The rest of our construction may be adjusted for these matrices accordingly.

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