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Mechanisms that Govern how the Price of Anarchy varies with Travel Demand

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Abstract

Selfish routing, represented by the User-Equilibrium (UE) model, is known to be inefficient when compared to the System Optimum (SO) model. However, there is currently little understanding of how the magnitude of this inefficiency, which can be measured by the Price of Anarchy (PoA), varies across different structures of demand and supply. Such understanding would be useful for both transport policy and network design, as it could help to identify circumstances in which policy interventions that are designed to induce more efficient use of a traffic network, are worth their costs of implementation.

This paper identifies four mechanisms that govern how the PoA varies with travel demand in traffic networks with separable and strictly increasing cost functions. For each OD movement, these are expansions and contractions in the sets of routes that are of minimum cost under UE and minimum marginal total cost under SO. The effects of these mechanisms on the PoA are established via a combination of theoretical proofs and conjectures supported by numerical evidence. In addition, for the special case of traffic networks with BPR-like cost functions having common power, it is proven that there is a systematic relationship between link flows under UE and SO, and hence between the levels of demand at which expansions and contractions occur. For this case, numerical evidence also suggests that the PoA has power law decay for large demand.

Keywords

Selfish Routing, Price of Anarchy, User-Equilibrium, System Optimum

1 Introduction

Experimental studies have shown that when choosing routes in traffic networks, network users tend to act selfishly and select routes that minimise their individual travel costs (Rapoport et al., 2009, Selten et al., 2007). This is the basis for the User Equilibrium (UE) routing principle (Wardrop, 1952). It is well established that selfish routing does not, in general, result in an optimal use of the supply network (Pigou, 1920), such as that which is achieved under System Optimal (SO) routing. The best known illustration of this inefficiency is Pigou's Example; in which it is shown, for a network of two parallel routes, that a UE assignment of traffic flow produces a total network travel cost that, whilst optimal for individuals, is suboptimal for society overall (Roughgarden, 2005). However, it is unknown how or why the *extent* of this inefficiency varies across different demand and supply structures; such as those that are known to exist in the traffic networks of cities from across the world (Barthelemy, 2011). Such understanding would be useful for transport policy and network design; for example, it could be used to help identify circumstances in which policy interventions, which are designed to induce a more efficient use of a traffic network, are worth their costs of implementation (Mak and Rapoport, 2013).

The extent of the inefficiency of selfish routing can be quantified by the *Price of Anarchy* (Papadimitriou, 2001). This measure is defined as the ratio of the total network travel cost under UE to the total network travel cost under SO (Roughgarden, 2005). The Price of Anarchy was first proposed by Koutsoupias and Papadimitriou (1999), and it has been the subject of considerable study since its inception. The main focus of such studies has been on producing upper bounds for the *worst-case* value of the measure across broad families of traffic networks. The earliest result, in this respect, was by Roughgarden and Tardos (2002), who demonstrated that the Price of Anarchy has an upper bound of $4/3$ in traffic networks with affine link cost functions. Generalisations and extensions of this result have since followed to families of traffic networks with separable, polynomial link costs (Roughgarden, 2003, Dumrauf and Gairing, 2006); non-separable, symmetric costs (Chau and Sim, 2003); and non-separable, asymmetric costs (Perakis, 2007). Upper bounds have also been produced in the context of elastic demand assignment for traffic networks with non-separable, symmetric cost maps (Chau and Sim, 2003); and non-separable, asymmetric and non-linear costs (Han et al., 2008). In each instance, the upper bounds that have been presented depend only on characteristics of the cost functions, such as the value of the highest power across all network links or the degree of link cost asymmetry. For example, for traffic networks with separable, polynomial link costs, Roughgarden (2003) showed that the Price of Anarchy is bounded above by equation (1), where p is the value of the highest power across all network links.

$$\left[1 - p(p+1)^{-(p+1)/p}\right]^{-1} \tag{1}$$

More recent upper bounds include characteristics of demand. For example, for traffic networks with separable, polynomial link costs, Correa et al. (2008) showed that tighter upper bounds than those presented by Roughgarden (2003) can be derived provided the free-flow travel cost on each network link is at least a non-zero, fixed proportion of its travel cost under a UE assignment of travel demand. Englert et al. (2010) also showed that the maximum increase in the Price of Anarchy, due to an increase in demand, can be bounded for traffic networks with separable, polynomial link costs and a single origin-destination (OD) pair.

This focus on *worst-case* values of the Price of Anarchy is a pattern that is also evident in recent transport literature; for example, in the establishment of upper bounds for the maximum efficiency gains of road pricing (Han and Yang, 2008, Yang et al., 2010) and car number plate based traffic rationing schemes (Han et al., 2010). The Price of Anarchy has also been used to establish upper bounds on the maximum efficiency loss; for example, in competitions between providers of private road infrastructure (Liu et al., 2011, Xiao et al., 2007), in traffic networks where only a minority of travellers have access to advanced traveller information (Liu et al., 2007), and in traffic networks where some travellers choose to follow shortest paths, oblivious to the effects of congestion (Karakostas et al., 2011).

However, in focussing on the worst-case value of the Price of Anarchy *across* broad families of traffic networks, the above studies neglect the variation that occurs *within* families of traffic networks, between traffic networks that may have very different demand and supply structures. Evidence of this variation is revealed by numerical studies, such as those of Wu et al. (2008) and Youn et al. (2008), which illustrate how the Price of Anarchy varies with travel demand in a range of real and synthetic traffic networks. An example of this variation is shown in Figure 1, which displays how the Price of Anarchy changes as travel demand is increased in three, single Origin-Destination (OD) sub-networks of the Boston, London and New York road networks (Youn et al., 2008).

In each city, it can be seen that there are broadly three identifiably distinct regions of behaviour: an initial region in which the Price of Anarchy is one; an intermediate region of fluctuations; and a final region of decay, which has a similar characteristic shape across all three networks. The similarities in this general behaviour across the three cities suggests that there may be common mechanisms that drive this variation. Yet, focussing on the detail of the individual graphs, the patterns are obviously different. For example, the graphs for Boston and New York have single dominant peaks, which are both higher than the peak reached in London. Whereas, the graph for London remains closer to its maximum value for a longer interval of demand than in either Boston or New York. It is also evident that the Price of Anarchy is not a smooth function of demand; the peak in Boston is a prominent example of this feature.

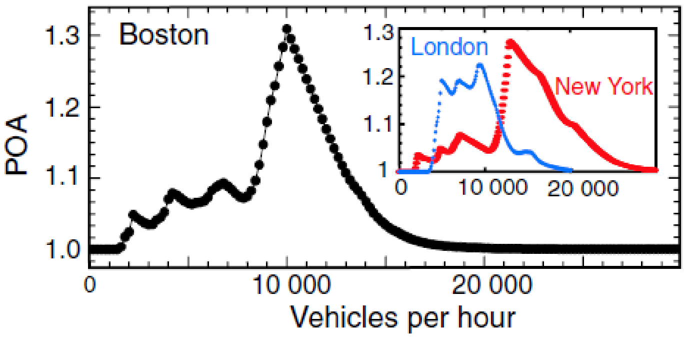

Figure 1 – Price of Anarchy against Demand for three Single OD Networks (Youn et al., 2008)

Figure 1 is only an illustration of how the Price of Anarchy has different values, at different levels of demand, in different traffic networks. However, the study from which it is taken does not provide a comprehensive explanation for the variation shown. The findings of theoretical studies, such as those of Roughgarden (2003) and Correa et al. (2008), are of little explanatory use here. This is because these studies reveal only the maximum value that the Price of Anarchy could reach, and this value is often significantly higher than the maximum value that is achieved across the demand range. For example, each network in Figure 1 has polynomial link cost functions, for which the the highest power $p = 10$. Equation (1) therefore provides an upper bound for the Price of Anarchy in these networks of approximately 3.5, which is significantly larger than the highest value of the Price of Anarchy of 1.3 in Figure 1. An explanation for this variation in the Price of Anarchy is required if we are to understand how different combinations of demand and supply, such as those that exist in real traffic networks, yield different magnitudes of the inefficiency of selfish routing.

Addressing this gap in the literature, this paper identifies and characterises four mechanisms that govern how the Price of Anarchy varies as travel demand is increased in a traffic network. Focussing on the general setting of traffic networks with multiple OD pairs and continuous, differentiable, separable and strictly increasing link cost functions, this paper reveals the source of the variations seen in Figure 1: namely that as demand increases there are expansions and contractions in the set of routes (for each OD) that are of minimum cost under UE (shorthand: 'UE route set'), or of minimum marginal total cost under SO (shorthand: 'SO route set'). The demands at which such expansions and contractions occur, under UE or SO, are defined as *route transition points*. As demand increases through these four different kinds of transition points, there are specific impacts on the Price of Anarchy. These are characterised through a combination of theoretical results and computational experiments.

Finally, in the special case of traffic networks with BPR-like cost functions (with common power), this paper proves that there is a systematic relationship between link flows under UE and SO, and that, consequently, there is also a systematic relationship between levels of demand at which route transition points occur under UE and SO. We also conjecture that, in this special case, the Price of Anarchy has power law decay for large demand; this claim is supported by numerical evidence.

The remainder of the paper is structured as follows. Section 2 defines notation and describes the UE and SO models. Section 3 then characterises the existence of expansions and contractions in minimum (marginal total) cost route sets under UE and SO, and proves that these are equivalent, under the condition of proportionality, to expansions and contractions in the sets of OD specific links that have non-zero flow. Section 3 also describes the systematic relationship between link flows under UE and SO for the special case described above. Section 4 presents theoretical results and conjectures, which characterise the effects of expansions and contractions on Total Network Travel Cost under SO, Total Network Travel Cost under UE and the Price of Anarchy. Section 5 then presents four numerical examples, which illustrate the theory of the preceding sections and also provide numerical evidence to support the proposed conjectures. Finally, section 6 presents conclusions and our outlook for future work.

2 Mathematical Preliminaries and Notation

In this paper, the topology of a traffic network is represented by a directed graph $G(V, A)$, comprising a set of nodes V and a set of directed links A. The costs of travel on each link $i \in A$ are represented by cost functions c_i. Travel demand is represented by an Origin-Destination (OD) vector Q with entries q_r denoting the volume of travel on OD movements $r = 1, \dots, R$. Each OD movement r is served by a finite number $k = 1, \dots, \kappa_r$ of acyclic routes K^r.

Using this notation, the UE principle is characterised in equation (2) (Patriksson, 1994).

$$\begin{gathered} f_k^r > 0 \Rightarrow C_k^r = \pi_r \\ f_k^r = 0 \Rightarrow C_k^r \geq \pi_r \\ \forall k \in K^r, \forall r = 1, \dots, R \end{gathered} \tag{2}$$

Here, f_k^r denotes the flow and $C_k^r = C(f_k^r)$ denotes the cost of travel on a route $k \in K^r$. The cost of travel on each route $k \in K^r$ is assumed to be the sum of link costs: $C_k^r = \sum_i c_i(x_i)\, \delta_{i,k}^r$, where $\delta_{i,k}^r = 1$, if link i is part of route $k \in K^r$, and zero otherwise. The set of links that comprise a route $k \in K^r$ is denoted $I_k^r = \{i \in A | \delta_{i,k}^r = 1\} \subset A$. The $\delta_{i,k}^r$ terms form a link-path incidence matrix, which is denoted by Δ. The minimum OD travel cost under UE, for the r^{th} OD movement is represented in this paper by $\pi_r = \min_{k \in K^r} C_k^r$.

Subject to the above conditions and the assumption that link costs c_i are continuous, positive, separable and strictly increasing functions of link flows x_i, it can be shown that there exist unique link flows x_i^{UE} satisfying the UE conditions (2) and which solve (Sheffi, 1985):

$$\min_x z(x) = \sum_{i \in A} \int_0^{x_i} c_i(\omega) d\omega$$

subject to the constraints:

$$\sum_k f_k^r = q_r \ \forall r$$
$$x_i = \sum_r \sum_{k \in K^r} f_k^r \delta_{i,k}^r \ \forall i \qquad (3)$$
$$f_k^r \geq 0 \ \forall k \in K^r, \forall r = 1, \dots, R$$

Under the same conditions, unique link flows x_i^{SO} satisfying the SO principle also exist, and solve a minimisation program with the same constraints (3) but with objective function $\tilde{z}(x) = \sum_{i \in A} x_i c_i$. Under the assumption that link costs c_i are also differentiable, the SO objective function is equivalent to the UE objective function under a transformation of link costs $\tilde{c}_i = c_i + x_i \times dc_i/dx_i$. In comparison with c_i, the cost functions $\tilde{c}_i$ include the additional cost burden that each unit of flow imparts on all other units of flow on each link (Sheffi, 1985). Sheffi (1985, p71-74) refers to $\tilde{c}_i$ as *"marginal travel costs"* and defines the *"marginal total travel cost"* on a route $k \in K^r$ as $\tilde{C}_k^r = \sum_i \tilde{c}_i(x_i)\, \delta_{i,k}^r$. The *minimum marginal total travel cost* for the r^{th} OD movement is represented in this paper by $\tilde{\pi}_r = \min_{k \in K^r} \tilde{C}_k^r$.

In order to guarantee the existence and uniqueness of link flows under UE and SO, the following assumption is presumed to hold throughout the paper:

Assumption A1: For each link $i \in A$ in a traffic network G, the cost function c_i is a continuous, differentiable, positive, separable and strictly increasing function of link flow x_i. In order to guarantee the uniqueness of the SO link flow solution, it is also assumed that $d^2c/dx^2 \geq 0$.

In addition to link flows, the above programs also guarantee the uniqueness of route costs under the UE and SO principles. However, route flows f_k^r are, in general, not unique. In fact, there are typically an infinite number of possible route flow solutions $F = \{f_k^r\}$ that satisfy the above constraints.

For a given network, having computed the UE and SO solutions, the difference between them can be measured by the Price of Anarchy, which is formally defined as follows (Roughgarden, 2005):

Definition 2.1: For a traffic network G, with cost functions c and demand vector Q, the *Price of Anarchy* ρ is defined as:

$$\rho = \frac{TTC^{UE}}{TTC^{SO}} = \frac{\sum_{i \in A} x_i^{UE} c_i\left(x_i^{UE}\right)}{\sum_{i \in A} x_i^{SO} c_i\left(x_i^{SO}\right)}$$

The Price of Anarchy is therefore unique (under Assumption A1) and does not depend on the particular equilibrium route flow solution considered.

3 The Existence of Expansions and Contractions in Minimum Cost Route Sets

This section characterises how the set of routes for an OD movement, which are of minimum cost under UE, or minimum marginal total cost under SO, can *expand* or *contract* in response to a perturbation in travel demand. This section begins with two network examples to illustrate this behaviour, and then provides definitions and notation to characterise the different types of expansions and contractions that can occur in general traffic networks. It is then shown that, under the condition of proportionality, an expansion (contraction) in the minimum cost route set (under UE or SO), for an OD movement, is equivalent to an expansion (contraction) in the set of links that have non-zero flow for that OD movement.

In the special case of traffic networks with cost functions $c_i = a_i + b_i x_i^\beta$ for which all links share a common power β, it is shown that there is a systematic relationship between link flows under UE and SO, and that, consequently, there is also a systematic relationship between the levels of demand at which expansions and contractions occur in minimum cost route sets under UE and SO.

3.1 Illustrative Examples

3.1.1 Example 1: Expansions in the Minimum Cost Route Sets under UE and SO

Consider a traffic network of N parallel links, serving a single OD pair with increasing demand $q > 0$, and with affine link cost functions of the form $c_i = a_i + b_i x_i$, where $a_i, b_i > 0$ and $a_i < a_{i+1}$ $\forall i = 1, \dots, N$. In such a network, under UE and at sufficiently low levels of demand q, all flow uses only the cheapest route, which is provided by link 1. This holds for all values of $q > 0$ for which:

$$c_1(x_1^{UE} = q) \leq c_2(x_2^{UE} = 0) \iff a_1 + b_1 q \leq a_2 \iff q \leq \frac{(a_2 - a_1)}{b_1}$$

For values of $q > (a_2 - a_1)/b_1$, link 2 activates and $c_1(x_1^{UE}) = c_2(x_2^{UE})$. Both links therefore carry flow at UE and the *set of minimum cost routes* comprises links 1 and 2. As q increases from this threshold the set of minimum cost routes remains unchanged provided:

$$\begin{aligned} c_1(x_1^{UE}) = c_2(x_2^{UE}) \leq c_3(x_3^{UE} = 0) \iff a_1 + b_1 x_1^{UE} \leq a_3 \iff \cdots \\ \iff a_1 + b_1\left(\frac{(a_2 - a_1) + b_2 q}{b_1 + b_2}\right) \leq a_3 \iff q \leq \frac{a_3 - a_2}{b_2} + \frac{a_3 - a_1}{b_1} \end{aligned} \quad (4)$$

For values of q above the threshold shown in equation (4), link 3 activates and $c_1(x_1^{UE}) = c_2(x_2^{UE}) = c_3(x_3^{UE})$; i.e. the set of minimum cost routes comprises links 1, 2 and 3.

As demand continues to increase, the minimum OD cost of travel continues to increase and further links become members of the minimum cost route set. This process continues until, at a sufficiently large level of demand, all links in the network belong to this set. It can be shown that under UE, for a given $M < N$, the set of minimum cost routes comprises M links for all values of q satisfying equation (5).

$$\sum_{i=1}^{M-1} \frac{a_M - a_i}{b_i} < q \leq \sum_{j=1}^{M} \frac{a_{M+1} - a_j}{b_j} \quad (5)$$

A similar pattern emerges under SO: increasing demand causes a sequence of links to be added to the set of minimum cost routes. Although, as travellers consider the marginal link travel costs $\tilde{c}_i$, rather than c_i, when choosing routes; it is the set of routes of minimum *marginal total* cost that changes. For the above parallel link network the cost transformation $\tilde{c}_i$ yields equation (6).

$$\tilde{c}_i = c_i + \frac{dc_i}{dx_i} x_i = (a_i + b_i x_i) + (b_i) x_i = a_i + 2 b_i x_i \quad (6)$$

The pattern of changes in the minimum marginal total cost route set under SO can therefore be obtained by redefining $b_i := 2b_i$ in the above UE derivation. It follows that under SO, for a given

$M < N$, the set of routes that are of minimum marginal total cost comprises M routes for all values of q satisfying equation (7).

$$\sum_{l=1}^{M-1} \frac{a_M - a_l}{2b_l} < q \leq \sum_{m=1}^{M} \frac{a_{M+1} - a_m}{2b_m} \tag{7}$$

It follows from the above that as demand increases, the order in which routes become minimum cost under UE is exactly the same as the order in which routes become minimum marginal total cost under SO. This follows for general multiple OD networks from the cost function transformation $\tilde{c}_i$.

This example illustrates how the set of minimum cost routes under UE, and the set of minimum marginal total cost routes under SO, can expand in response to an increase in demand. This example could also be used to demonstrate that the sets of minimum cost routes under UE and SO can also contract. This could be achieved by starting with high demand q, such that all N links belong to the minimum cost route set, and by gradually decreasing q towards zero. The example that follows in section 3.1.2 demonstrates, perhaps counter-intuitively, that the set of minimum cost routes, under UE and SO, can also contract in response to an increase in demand.

3.1.2 Example 2: Contractions in the Minimum Cost Route Sets under UE and SO

Consider the five link traffic network shown in Figure 2, which serves two OD pairs $O \to D1$ and $O \to D2$ as shown. Further suppose that the five links have the following affine link cost functions: $c_1 = 2 + x_1$, $c_2 = 3 + x_2$, $c_3 = 9 + x_3$, $c_4 = 1 + x_4$ and $c_5 = 1 + x_5$; and that demand on the $O \to D2$ movement is fixed at $q_{O\to D2} = 1$. There are two routes for each OD pair: for $O \to D1$, the routes are link $\{1\}$ and links $\{2,4\}$; for $O \to D2$, the routes are links $\{2,5\}$ and link $\{3\}$.

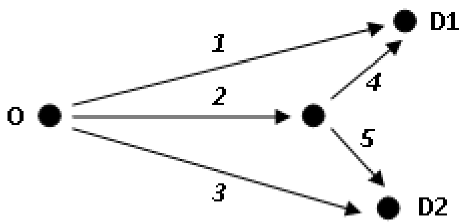

Figure 2 - Five Link Network with Two OD Pairs

Consider demand $q_{O\to D1}$ increasing from zero under SO. The variation of marginal total route costs under SO, for each of the four routes, with respect to $q_{O\to D1}$, is shown in Figure 3. In addition to providing further examples of expansions in the minimum cost route set; it can also be seen that, for $q_{O\to D1} < 11.5$, route $\{2,5\}$ is part of the minimum marginal total cost route set for OD movement $O \to D2$, but that, for demand $q_{O\to D1} > 11.5$, this route ceases to be a member of this set. This example therefore demonstrates that the set of minimum marginal total cost routes under SO can contract due to an increase in travel demand. Furthermore, this example also demonstrates that the set of minimum marginal total cost routes for one OD pair; in this case $O \to D2$, can change due to an increase in demand on a different OD movement; in this case $O \to D1$. This latter observation demonstrates the potential complexity of possible dependencies that may exist between expansions and contractions on different OD movements.

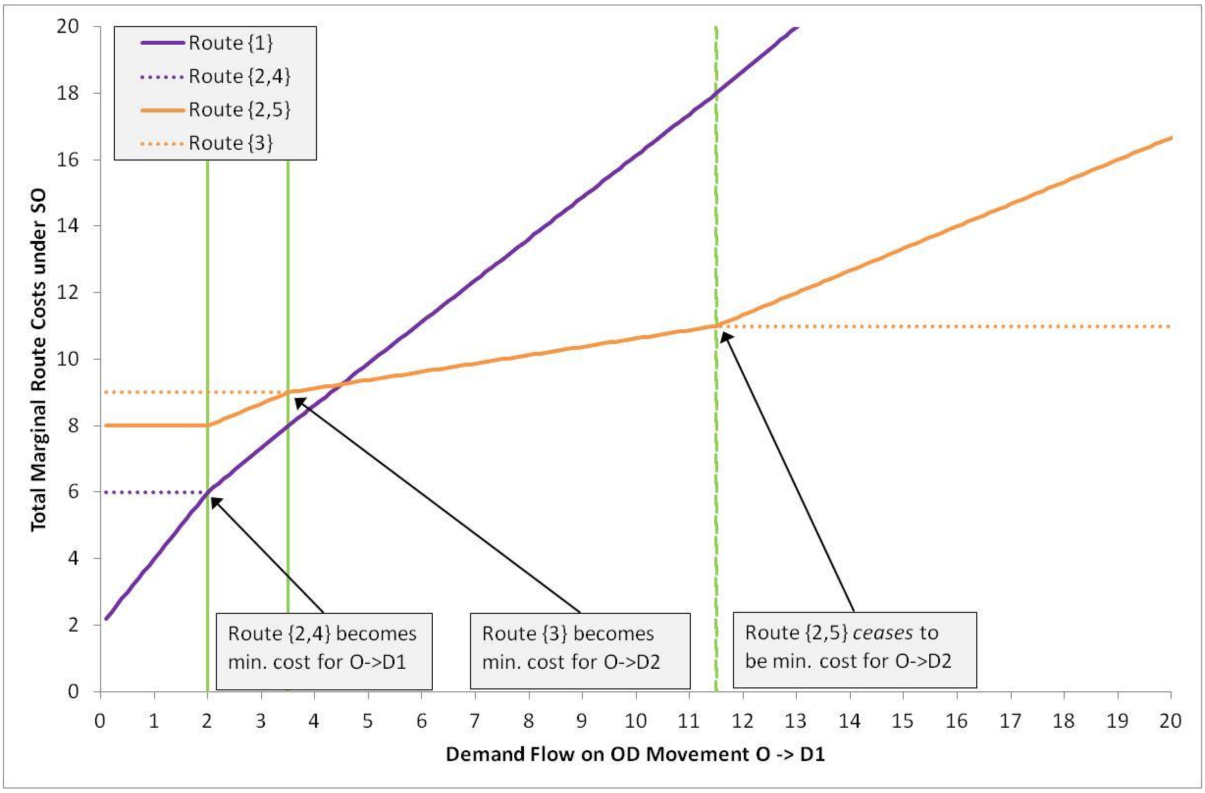

Figure 3 - Route Costs under SO against increasing demand on O->D1 for the network in Figure 2

It can be shown that exactly the same pattern of expansions and contractions also occurs under UE for this network example; although at different levels of demand $q_{O \to D1}$.

3.2 Definitions, Notation and Limiting Conditions

The examples presented in sections 3.1.1 and 3.1.2 illustrate that the set of minimum cost routes under UE, and the set of minimum marginal total cost routes under SO, for an OD movement, can expand or contract due to a perturbation in travel demand. The examples also demonstrated that an increase (or decrease) in demand on one OD movement has the potential to cause an expansion or a contraction in the route sets of another OD movement. In section 4 it is shown that expansions and contractions in these sets, under UE and SO, have a significant influence on how the Price of Anarchy varies with travel demand. As such, the following definitions and notation are proposed in order to characterise these phenomena.

Definition 3.1: The set of *minimum cost routes under UE*, for an OD movement r, at a demand Q is defined as $K_{min}^r = \{k \in K^r | C_k^r = \pi_r(Q)\}$. To track changes in K_{min}^r with respect to perturbations in demand, a vector function $\Upsilon_r^{UE}(Q)$ is defined for each OD movement r, which has entries u_k for which $u_k = 1$, if $C_k^r = \pi_r(Q)$, and $u_k = 0$, if $C_k^r > \pi_r(Q)$.

Definition 3.2: A demand vector Q is defined as a *route transition point under UE* if there exist vectors $g, h \in \mathbb{R}^R \backslash \{0\}$ for which, for at least one OD movement r:

$$\lim_{\lambda_1 \to 0} \Upsilon_r^{\mathrm{UE}}(Q - \lambda_1 g) \neq \lim_{\lambda_2 \to 0} \Upsilon_r^{\mathrm{UE}}(Q + \lambda_2 h) \tag{8}$$

where $\lambda_1, \lambda_2 > 0$.

Individual route transition points are denoted by η_{UE}, and the set of all such demand vectors for a given network G is denoted H_{UE}. As shorthand, in the remainder of the paper, we refer to the limit on the left-hand side of equation (8) as $Q \to \eta_{UE}^{-}$ and the limit on the right-hand side of equation (8) as $Q \to \eta_{UE}^{+}$. The SO equivalent definitions 3.1 and 3.2 can be obtained with appropriate changes to superscripts and notation. For example, the set of minimum marginal total cost routes under SO, for an OD movement r, is defined as $\widetilde{K}_{min}^{r} = \{k \in K^r | \tilde{C}_k^r = \tilde{\pi}_r(Q)\}$. Both K_{min}^{r} and $\widetilde{K}_{min}^{r}$ are uniquely defined under Assumption A1. Individual SO route transition points are denoted by η_{SO}, and the set of all transition points is denoted H_{SO}.

In the case of a network with only one OD movement, the limits in definition 3.2 (equivalently, the directional derivatives of Υ_r^{UE}) simply correspond to increasing/decreasing the scalar OD demand. In the multiple OD case, at a particular R-dimensional demand vector Q, there are many directional derivatives. Overall, as demand increases new routes will activate. However, complex scenarios are conceivable; for example a route transition point could occur at some given Q, with an expansion in the minimum cost route set for one OD movement, a contraction in the minimum cost route set for a different OD movement, and no change in the minimum cost route sets for a third OD movement. Analysis of route transition points and the existence/non-existence of directional derivatives at such points is a significant focus of attention in the UE Sensitivity Analysis literature (Josefsson and Patriksson, 2007, Patriksson, 2004)[1]. In this paper we are subject to the same analytical difficulties identified in such works. For example, Josefsson and Patriksson (2007) remark that directional derivatives cannot be guaranteed for the BPR cost functional form because it has zero cost derivative at zero flow.

We will show the importance of these route transition points in determining the PoA. However, we will only consider the cases where one or more routes activate for one or more OD movements, or one or more routes deactivate for one or more OD movements. We wish to exclude the complex scenarios noted above where routes simultaneously activate on one OD movement and deactivate on a different OD movement (at the same OD demand). We restrict our attention to changes that occur at route transition points η (satisfying conditions C1-C3 below) when travel demand Q *increases* (as in definition 3.3).

Definition 3.3: Consider two demand vectors $Q^1, Q^2 \in \mathbb{R}^R$ with $Q^i = [q_1^i, \ldots, q_R^i]$. Demand is said to have *increased* from Q^1 to Q^2 if and only if $q_j^1 \leq q_j^2 \ \forall j = 1, \ldots, R$, and $\exists j'$ for which $q_{j'}^1 < q_{j'}^2$.

Route Transition Point Conditions:

Conditions C1-C3 hold when they are satisfied for both UE and SO transition points. We therefore use generic notation e.g. H to indicate $\mathrm{H_{UE}}$ or $\mathrm{H_{SO}}$ as appropriate.

C1. Demand vectors g and h in (8) satisfy $g_r \geq 0$ and $h_r \geq 0$, $\forall r = 1, \ldots, R$

C2. For the vectors g, h in C1, $\exists \lambda_1, \lambda_2 > 0$ such that $\forall \theta_1 \in [0, \lambda_1]$, $Q - \theta_1 g \notin \mathrm{H}$ and $\forall \theta_2 \in [0, \lambda_2]$ $Q + \theta_2 h \notin \mathrm{H}$

C3. At each route transition point η, either:

(i) $\forall r = 1, \ldots, R$; for each $k \in K^r$, $\lim_{Q \to \eta^-} u_k \leq \lim_{Q \to \eta^+} u_k$ for entries u_k in Υ_r

[1] This literature is covered in more detail in section 4.3.2.

(ii) $\forall r = 1, \dots, R$; for each $k \in K^r$, $\lim_{Q \to \eta^-} u_k \geq \lim_{Q \to \eta^+} u_k$ for entries u_k in Υ_r

Condition C1 is the most restrictive of the three conditions, as it excludes all cases in which a route transition occurs as demand decreases on one or more OD movements. Condition C2 excludes cases in which two route transition points are adjacent to each other. Condition C3 excludes cases at which there is an expansion in the minimum cost route set for at least one OD movement that occurs simultaneously with a contraction in the minimum cost route set for at least one different OD movement.

3.3 An Alternative Characterisation of Minimum Cost Route Sets under UE and SO

When conducting numerical experiments it would be convenient if route transition points could be identified by examining the unique UE (and SO) link flows, rather than requiring explicit computation of route flows. To this end, before moving on to explain how route transition points affect the Price of Anarchy, the next sections describe a link-flow characterisation of expansions and contractions in the minimum cost route set. In addition, for a specific family of traffic networks, a systematic relationship between route transition points under UE and SO is shown to exist that allows quick calculation of one set of transition points from the other.

As already noted, the UE and SO route flows are not typically unique. A uniquely identifiable route flow solution F^* can be defined using the *condition of proportionality*, which was first proposed by Bar-Gera and Boyce (1999).

Definition 3.4: *"The condition of proportionality states that the same proportions apply to all travellers facing a choice between a pair of alternative segments (PASs), regardless of their origins and destinations, where a segment is defined as a sequence of one or more links"* (Bar-Gera et al., 2012).

This route flow solution has the useful property that *"any route that can be used under the UE conditions will be used"* (Bar-Gera et al., 2012). Lu and Nie (2010) have shown that route flows under the condition of proportionality vary continuously with respect to travel demand Q. As the SO problem can be transformed into an equivalent UE problem, it follows that there also exists a unique SO route flow solution, which we denote by $\tilde{F}^*$, that satisfies the condition of proportionality. In networks with only a single origin, the route flow solutions F^* and $\tilde{F}^*$ can be derived from the approach proportions produced by the Origin Based Assignment (OBA) algorithm (Bar-Gera, 2002, Bar-Gera et al., 2012). In networks with multiple origins, these route flow solutions cannot be derived using OBA; the Traffic Assignment by Paired Alternative Segments (TAPAS) algorithm can be used instead (Bar-Gera, 2010, Bar-Gera et al., 2012).

The results that follow prove that, for each OD movement r, an expansion (contraction) in the set K^r_{min} or $\tilde{K}^r_{min}$, is equivalent, under the condition of proportionality, to an expansion (contraction) in the set of links, under UE or SO, that have non-zero flow for that OD movement. These sets are referred to as the *Origin Specific Active Network* for an OD movement r and are formally defined in definition 3.5 as follows.

Definition 3.5: The *OD Specific Active Network under UE and Proportionality*, for an OD movement r, at a demand Q is the set $X^{UE}_r = \{ i \in A \,|\, \exists k \in K^r, s.t. \, i \in I^r_k \;\&\; f^r_k > 0 \} \subseteq A$, where $f^r_k \in F^*$, the route flow solution that satisfies the condition of proportionality. To track changes in X^{UE}_r with respect to perturbations in demand, a vector function $\Phi^{UE}_r(Q)$ is defined for each OD movement r, which has entries v_i for which $v_i = 1$, if $\exists k \in K^r$ for which $i \in I_k^r$ and $f_k^r > 0$, and $v_i = 0$, if $\forall k \in K^r$ for which $i \in I_k^r, f_k^r = 0$.

An equivalent version of definition 3.5 is also defined for SO, with appropriate changes to superscripts and notation. The sets X_r^{UE} and X_r^{SO} are both uniquely defined under the condition of proportionality. Levels of demand at which these sets change are referred to as *link transition points*, which are formally defined in definition 3.6.

Definition 3.6: A demand vector Q is defined as a *link transition point under UE* if there exist vectors $g, h \in \mathbb{R}^R \backslash \{0\}$ for which, for at least one OD movement r:

$$\lim_{\mu_1 \to 0} \Phi_r^{\mathrm{UE}}(Q - \mu_1 g) \neq \lim_{\mu_2 \to 0} \Phi_r^{\mathrm{UE}}(Q + \mu_2 h) \tag{9}$$

where $\mu_1, \mu_2 > 0$. Individual link transition points are denoted by ω_{UE}, and the set of all such demand vectors for a given network G is denoted Ω_{UE}. Again, an equivalent version of definition 3.6 is also defined for SO. Note that, since the link flows are unique, the link transition points do not depend on the particular route flow solution considered.

This alternative characterisation of the changing nature with which demand is assigned to a traffic network is useful because it is often significantly easier to identify the set of active links in the route flow solution produced under the condition of proportionality, for each OD movement, than it is to identify the set of routes that are of minimum cost. This is because there are often many more routes than there are links, especially in large traffic networks, and the enumeration of routes is a computationally expensive procedure. Accordingly, these results are used in the examples in section 5 to track expansions and contractions in K_{min}^r and $\widetilde{K}_{min}^r$.

Proposition 3.1 and corollary 3.2 characterise the relationship between the sets K_{min}^r and X_r^{UE}, and the sets $\widetilde{K}_{min}^r$ and X_r^{SO} for an OD movement r.

Proposition 3.1: Consider a traffic network G for which Assumption A1 holds, and let $F^* = \{f_k^r\}$ represent the route flow solution under UE that satisfies the condition of proportionality. Suppose that Q represents a demand vector that is not a route transition point, i.e. $Q \notin \mathrm{H}_{UE}$. For a given OD movement r, further suppose that Q does not correspond to a level of demand at which X_r^{UE} changes. Then, for that OD movement r:

(i) A link $i \in X_r^{UE}$ if and only if $\exists k' \in K_{min}^r$ for which $i \in I_{k'}^r$.
(ii) A route $k' \in K_{min}^r$ if and only if $I_{k'}^r \subset X_r^{UE}$.

Part (i) describes how, for an OD movement r, the OD Specific Active Network under UE and Proportionality can be constructed from the set of minimum cost routes for the OD movement r. Part (ii) describes how, for an OD movement r, the set of minimum cost routes under UE can be constructed from the OD Specific Active Network for the OD movement r.

Proof: For parts (i) and (ii), the only if and if statements are addressed in turn.

(i) Only if statement: For a given link $i \in A$ suppose that $i \in X_r^{UE}$ for an OD movement r. Then by equation (3) $\exists k' \in K^r$ for which $f_{k'}^r > 0$. For this k', the UE conditions (2) imply that $C_{k'}^r = \pi_r$ and that therefore $k' \in K_{min}^r$.

(i) If statement: For a given link $i \in A$, suppose that $\exists k' \in K_{min}^r$ for which $i \in I_{k'}^r$. By definition 3.1, for this route k', it follows that $C_{k'}^r = \pi_r$. Under the condition of proportionality, a route flow solution F^* can be constructed for which $f_k^r > 0\ \forall k$ for which $C_k^r = \pi_r$. By equation (3) this route flow solution provides that link i has positive flow for the OD movement r. It therefore follows that $i \in X_r^{UE}$.

(ii) Only if statement: For a given route $k' \in K^r$ suppose that $k' \in K_{min}^r$. Then, by definition 3.1, $C_{k'}^r = \pi_r$. Under the condition of proportionality, a route flow solution F can be constructed for which $f_{k'}^r > 0$. As $f_{k'}^r > 0$ and, by equation (3), all links $i \in I_{k'}^r$ contain the flow $f_{k'}^r$ as part of their summation, it follows that each such link i has positive flow for the OD movement r. In other words, $i \in X_r^{UE}, \forall i \in I_{k'}^r$ and therefore $I_{k'}^r \subset X_r^{UE}$.

(ii) If statement: For a given route $k' \in K^r$, suppose that $I_{k'}^r \subset X_r^{UE}$. Suppose, for a contradiction, that $k' \notin K_{min}^r$. Then by equation (2), $C_{k'}^r > \pi_r$. By starting assumption, all links $i \in I_{k'}^r$ carry flow for this OD, i.e. $i \in X_r^{UE}$. Hence each such link must lie on at least one route $k^* \in K^r \backslash \{k'\}$ for which $f_{k^*}^r > 0$ and hence, from equation (2), $C_{k^*}^r = \pi_r$. Therefore, each $k^* \in K_{min}^r$ and it follows, from the only if statement of part(ii), which has just been proven, that $I_{k*}^r \subset X_r^{UE}$.

Therefore both $I_{k'}^r \subset X_r^{UE}$ and $I_{k*}^r \subset X_r^{UE}$. Consider the pair(s) of alternative segments defined by the set of links $(I_{k'}^r \cup I_{k*}^r) \backslash (I_{k'}^r \cap I_{k*}^r) \subset X_r^{UE}$ i.e. both alternative segments (in each pair) are used. Under the condition of proportionality, it follows from Bar-Gera (2006) that *"for every used pair of alternative segments and every used route that contains one of the segments, there will be a similar used route containing the alternative segment"* (Bar-Gera et al., 2012). In this statement, the *"similar used route"* refers to a route that only differs from the *"used route"* in the pair of alternative segments; i.e. the *"used route"* and the *"similar used route"* overlap each other in the rest of their composition. This proportionality implies that $f_{k'}^r > 0$, which implies that $C_{k'}^r = \pi_r$. This contradicts the assumption that $k' \notin K_{min}^r$. ■

The equivalent statement of proposition 3.1 for SO is stated as follows.

Corollary 3.2: Consider a traffic network G for which Assumption A1 holds, and let $\tilde{F}^* = \{f_k^r\}$ represent the route flow solution under SO that satisfies the condition of proportionality. Suppose that Q represents a demand vector that is not a route transition point, i.e. $Q \notin \mathrm{H}_{SO}$. For a given OD movement r, further suppose that Q does not correspond to a level of demand at which X_r^{SO} changes. Then, for that OD movement r:

(i) A link $i \in X_r^{SO}$ if and only if $\exists k' \in \tilde{K}_{min}^r$ for which $i \in I_{k'}^r$.

(ii) A route $k' \in \tilde{K}_{min}^r$ if and only if $I_{k'}^r \subset X_r^{SO}$.

Proof: Traces that of proposition 3.1 with appropriate changes in notation from UE to SO. ■

The following results prove that the sets K_{min}^r ($\tilde{K}_{min}^r$) and X_r^{UE} (X_r^{SO}) expand and contract at identical levels of demand.

Proposition 3.3: Consider a traffic network G for which Assumption A1 holds, and let $F^* = \{f_k^r\}$ represent the route flow solution under UE that satisfies the condition of proportionality. There is a one-to-one correspondence between route transition points η_{UE} and link transition points ω_{UE}.

Proof: This statement is proved by contradiction.

There are four cases to consider: i) $\exists\eta_{UE}$ corresponding to an expansion in K^r_{min} for which $\nexists\omega_{UE}$ corresponding to an expansion in X^{UE}_r, ii) $\exists\omega_{UE}$ corresponding to an expansion in X^{UE}_r for which $\nexists\eta_{UE}$ corresponding to an expansion in K^r_{min}, iii) $\exists\eta_{UE}$ corresponding to a contraction in K^r_{min} for which $\nexists\omega_{UE}$ corresponding to a contraction in X^{UE}_r, iv) $\exists\omega_{UE}$ corresponding to a contraction in X^{UE}_r for which $\nexists\eta_{UE}$ corresponding to an contraction in K^r_{min}. Proofs are provided for cases i) and ii); the proofs of iii) and iv) are similar.

Case i) Suppose, for a contradiction, that there exists an instance of demand η_{UE}, at which K^r_{min} expands for some OD movement r, but for which there does not exist a corresponding point ω_{UE}, at which X^{UE}_r expands for the same OD movement. Therefore, there is a perturbation of demand for which $\exists k \in K^r$, such that as $Q \to \eta^-_{UE}$, $k \notin K^r_{min}$, but that as $Q \to \eta^+_{UE}$, $k \in K^r_{min}$. It follows, from proposition 3.1(ii), that as $k \notin K^r_{min}$ as $Q \to \eta^-_{UE}$, $\exists i \in I^r_k$ for which $i \notin X^{UE}_r$. It also follows, from proposition 3.1(ii), that as $k \in K^r_{min}$ as $Q \to \eta^+_{UE}$, $i \in X^{UE}_r$, $\forall i \in I^r_k$. Hence $\exists i \in A$ that is added to X^{UE}_r at η_{UE}. This contradicts our starting assumption.

Case ii) Now suppose, for a contradiction, that there exists an instance of demand ω_{UE}, at which X^{UE}_r expands for some OD movement r, for which there does not exist a corresponding point η_{UE}, at which K^r_{min} expands for the same OD movement. Therefore, there is a perturbation of demand for which $\exists i \in A$, such that as $Q \to \omega^-_{UE}$, $i \notin X^{UE}_r$, but that as $Q \to \omega^+_{UE}$, $i \in X^{UE}_r$. It follows, from proposition 3.1(i), that as $i \notin X^{UE}_r$ as $Q \to \omega^-_{UE}$, $k \notin K^r_{min}$, $\forall k \in K^r$ for which $i \in I^r_k$. It also follows, from proposition 3.1(i), that as $i \in X^{UE}_r$ as $Q \to \omega^+_{UE}$, $\exists k \in K^r_{min}$ for which $i \in I^r_k$. Hence $\exists k \in K^r$ that is added to K^r_{min} at ω_{UE}. This contradicts our starting assumption. ■

Corollary 3.4: Consider a traffic network G for which Assumption A1 holds, and let $\tilde{F}^* = \{f^r_k\}$ represent the route flow solution under SO that satisfies the condition of proportionality. There is a one-to-one correspondence between route transition points η_{SO} and link transition points ω_{SO}.

Proof: Traces that of proposition 3.3 with appropriate changes in notation from UE to SO. ■

3.4 A Systematic Relationship between UE and SO Link Flows and Route Transition Points

This section establishes two results for the special case[2] of traffic networks with link cost functions of the form $c_i = a_i + b_i x_i^\beta$ for which the coefficients $a_i, b_i > 0\ \forall i$ and $\beta > 0$ is common to all links. This set of cost functions includes, but is not limited to, the well-known BPR cost function $t_i = t_{i0}[1 + 0.15(x_i/cap_i)^4]$, for which t_{i0} represents the free-flow travel time and cap_i represents link capacity.

In this narrower context, it is proven, in theorem 3.5, that there is a systematic relationship between link flows under UE and SO. As a consequence of this, it is proven, in corollary 3.6, that there is also a systematic relationship between the levels of demand under UE and SO at which expansions and contractions occur in the minimum cost route sets K^r_{min} and $\tilde{K}^r_{min}$. This systematic relationship can be observed in the parallel link example of section 3.1.1, for which the cost functions c_i belong to the cost function set considered here. A comparison of equations (5) and (7) reveals that the level of

[2] Elsewhere in the paper link cost functions are assumed separable and monotonic as stated in Section 2.

demand at which each route $k \in K^r$ is added to K^r_{min} is exactly half the level of demand at which the same route k is added to $\tilde{K}^r_{min}$. Corollary 3.6 proves that this indicative of a result that applies more generally to networks with multiple OD pairs.

Theorem 3.5: Consider a traffic network G that serves a demand matrix Q with entries $q_r > 0$, and that has cost functions of the form $c_i = a_i + b_i x_i^\beta$ ($a_i, b_i, \beta > 0$), which satisfy Assumption A1. Let $x_i^{UE}(Q)$ and $x_i^{SO}(Q)$ denote UE and SO link flows respectively, which are defined as functions of the demand vector Q. Then, under these conditions, $\forall i \in A$:

$$x_i^{SO}\left(\frac{Q}{\sqrt[\beta]{\beta+1}}\right) = \frac{1}{\sqrt[\beta]{\beta+1}} x_i^{UE}(Q) \tag{10}$$

Proof: Consider a traffic network G, with demand matrix Q and link cost functions $c_i = a_i + b_i x_i^\beta$ has a UE link flow solution $x_i^{UE}(Q)$. The proof begins by noting that we can define a different traffic assignment problem on G, with demand matrix $\hat{Q} = \lambda Q$ and link cost functions $\hat{c}_i = a_i + b_i(x_i/\lambda)^\beta$, which has a UE link flow solution $\hat{x}_i^{UE}(\hat{Q}) = \lambda x_i^{UE}(Q)$. In other words, the traffic assignment problem has been rescaled by λ.

Now consider the problem of finding an SO link flow solution $x_i^{SO}\left(Q/\sqrt[\beta]{\beta+1}\right)$ for a given road network G serving a demand matrix $Q/\sqrt[\beta]{\beta+1}$ with link costs $c_i = a_i + b_i x_i^\beta$ as defined in the left hand side of equation (10).

As noted in section 2, this problem is equivalent to finding a UE link flow solution $x_i^{UE}\left(Q/\sqrt[\beta]{\beta+1}\right)$ on G for a demand matrix $Q/\sqrt[\beta]{\beta+1}$ with transformed cost functions (Sheffi, 1985, p73):

$$\tilde{c}_i = c_i + \frac{dc_i}{dx_i} x_i = \left(a_i + b_i x_i^\beta\right) + \left(b_i \beta x_i^{\beta-1}\right) x_i = a_i + b_i(\beta+1) x_i^\beta = a_i + b_i\left(\sqrt[\beta]{\beta+1}\, x_i\right)^\beta$$

Setting $\lambda = 1/\sqrt[\beta]{\beta+1}$ to simplify notation, this problem can be restated as one of finding a UE link flow solution $x_i^{UE}(\lambda Q)$ on G for a demand matrix λQ with cost functions $c_i = a_i + b_i(x_i/\lambda)^\beta$.

Applying the earlier scaling note, the UE link flow solution $x_i^{UE}(\lambda Q)$ in this restated problem is equivalent to a rescaled UE problem on G, which has link flow solution $\lambda \hat{x}_i^{UE}(Q)$ with demand matrix Q and link cost functions $c_i = a_i + b_i x_i^\beta$. However, this scaled problem is exactly the problem on the right hand side of equation (10), and it therefore follows that:

$$x_i^{SO}\left(Q/\sqrt[\beta]{\beta+1}\right) = \frac{1}{\sqrt[\beta]{\beta+1}} x_i^{UE}(Q) \quad \blacksquare$$

The following corollary describes the relationship between route transition points under UE and SO.

Corollary 3.6: Consider a traffic network G that serves a demand matrix Q with entries $q_r > 0$, and that has cost functions of the form $c_i = a_i + b_i x_i^\beta$ ($a_i, b_i, \beta > 0$), which satisfy Assumption A1. Suppose that the condition of proportionality holds and that $f_k^{r,UE} \in F^*$ and $f_k^{r,SO} \in \tilde{F}^*$ represent the uniquely defined route flow for each route $k \in K^r$, under UE and SO respectively. Let η_{UE} and

η_{SO} represent instances of demand Q at which, for some OD movements r, the same routes $k \in K^r$ are added to or removed from the sets K^r_{min} and $\tilde{K}^r_{min}$, respectively[3]. Then:

$$\eta_{SO} = \frac{1}{\sqrt[\beta]{\beta+1}} \eta_{UE} \tag{11}$$

Proof: Consider a given level of demand η_{UE} at which the set K^r_{min} expands for some OD movement r. Therefore, $\exists k \in K^r$, for some OD movement r, for which $k \notin K^r_{min}$ as $Q \to \eta^-_{UE}$, but for which $k \in K^r_{min}$ as $Q \to \eta^+_{UE}$. As route flows, under the condition of proportionality, are uniquely defined and vary continuously with respect to Q, it follows, from theorem 3.5, that $f^{r,UE}_k(Q) = \sqrt[\beta]{\beta+1} * f^{r,SO}_k\left(Q/\sqrt[\beta]{\beta+1}\right)$.

Now, if $k \notin K^r_{min}$ as $Q \to \eta^-_{UE}$, such that $f^{r,UE}_k = 0$ under the condition of proportionality, then it follows that $f^{r,SO}_k = 0$ and that $k \notin \tilde{K}^r_{min}$ as $Q \to \left(1/\sqrt[\beta]{\beta+1}\right)\eta^-_{UE}$. In addition, if $\in K^r_{min}$ as $Q \to \eta^+_{UE}$, such that $f^{r,UE}_k > 0$ under the condition of proportionality, then it follows that $f^{r,SO}_k > 0$ and that $k \in \tilde{K}^r_{min}$ as $Q \to \left(1/\sqrt[\beta]{\beta+1}\right)\eta^+_{UE}$. This implies that $\tilde{K}^r_{min}$ expands under SO at $Q = \left(1/\sqrt[\beta]{\beta+1}\right)\eta_{UE}$, i.e. that $\exists \eta_{SO} = \left(1/\sqrt[\beta]{\beta+1}\right)\eta_{UE}$.

A similar argument works for when η_{UE} corresponds to a contraction of K^r_{min}, for some OD movement r. ■

It is important to note that corollary 3.6 does not predict the levels of demand at which the sets K^r_{min} or $\tilde{K}^r_{min}$ will change; rather, it provides a method to identify the levels of demand at which, for example, K^r_{min} changes, given the levels of demand at which $\tilde{K}^r_{min}$ changes.

4 The Variation of the Price of Anarchy with Travel Demand

This section presents theory that describes how the Price of Anarchy varies with travel demand. In order to provide motivation and context for this theory, this section begins, in section 4.1, by illustrating how the Price of Anarchy varies with travel demand in the network examples presented in section 3.1. Sections 4.2, 4.3 and 4.4 then present theory to describe the mechanisms that govern how the Price of Anarchy varies in general networks for low, intermediate and high levels of travel demand respectively.

For intermediate levels of demand, it was established in section 3 that as travel demand Q changes, the sets K^r_{min} (X^{UE}_r) and $\tilde{K}^r_{min}$ (X^{SO}_r) can expand and contract, for one or more OD movements r. The points at which these expansions and contractions occur were defined as route transition points and several types were identified. The theory presented in this section applies to route transition points that occur under increasing demand and which satisfy conditions C1-C3, which were set out in section 3.2.

The behaviour of the Price of Anarchy is dependent, by construction, on Total Network Travel Cost under SO (TTC^{SO}) and Total Network Travel Cost under UE (TTC^{UE}). This is important for the analysis that follows.

[3] For such traffic networks, the existence of instances of demand, under UE and SO, at which the same routes $k \in K^r$ are added to or removed from K^r_{min} and $\tilde{K}^r_{min}$ follows from the relationship described in theorem 3.5. In general traffic networks, this statement can be shown to follow from the SO cost transformation $\tilde{c}_i$.

4.1 Illustrative Examples

4.1.1 Example 1: Parallel Link Network - Single OD Example

Recalling the example of section 3.1.1; consider increasing demand q in nine versions of a parallel link network with total links $N = 2,3, \dots ,10$ and coefficients $a_i = i$, $b_i = 1$ for $i = 1, \dots ,10$. Figure 4 displays the variation of the Price of Anarchy ρ_N for each of these nine networks, and also identifies the levels of demand under UE (green vertical lines) and SO (red vertical lines) at which expansions occur in the sets K_{min} and $\widetilde{K}_{min}$ respectively. These levels of demand correspond to those identified in equations (5) and (7) respectively.

For levels of demand q up to the first route transition point under SO, the Price of Anarchy is 1. Beyond this level of demand, Figure 4 illustrates, for each N, that levels of demand at which K_{min} expands coincide with all levels of demand at which the Price of Anarchy is non-differentiable. Furthermore, there is also a decrease in the gradient of the Price of Anarchy at each of these points. In contrast the Price of Anarchy appears to be differentiable at all levels of demand at which there is an expansion in $\widetilde{K}_{min}$. However, it is also evident, for each $M = 2, \dots ,10$, that the graphs of ρ_{M-1} and ρ_M depart from each other at each of these points. This demonstrates that the new routes that are available in the M parallel link case have a material effect on the trajectory of the Price of Anarchy. Overall, Figure 4 suggests that expansions under UE lead to decreases in the Price of Anarchy whereas expansions under SO lead to increases in the Price of Anarchy.

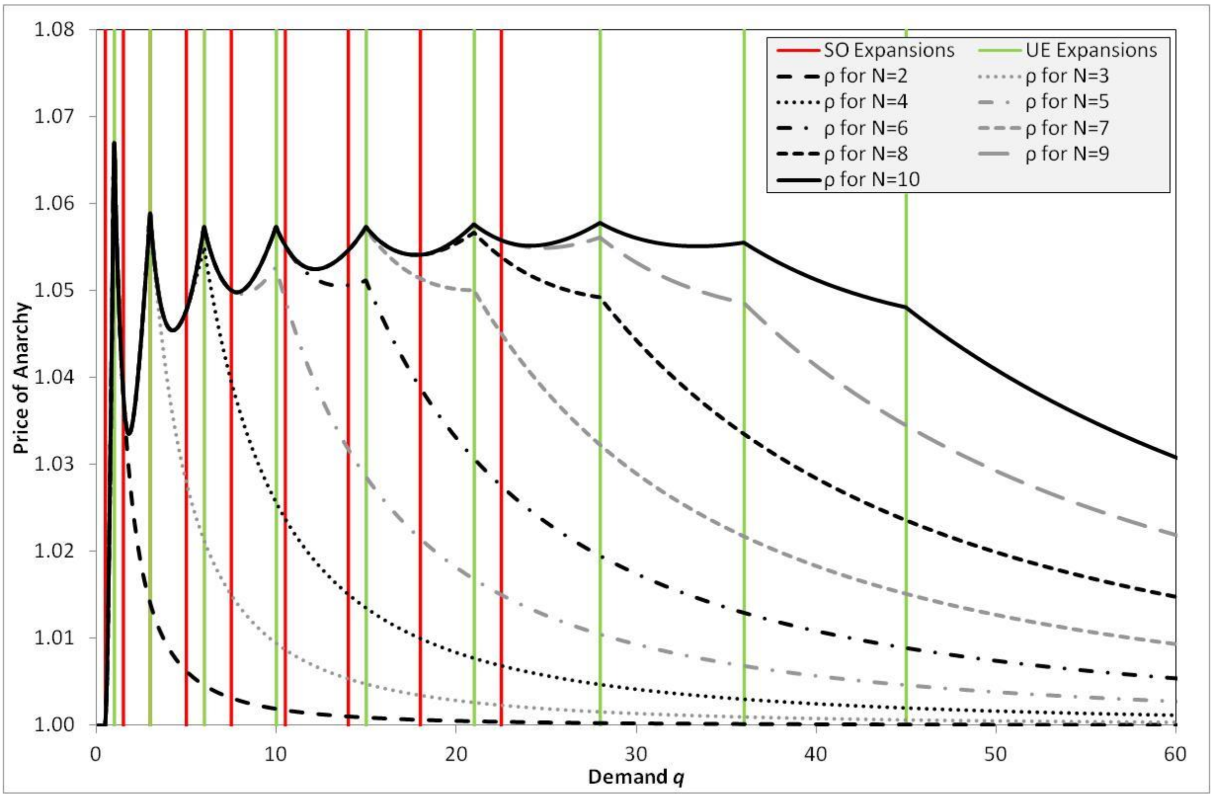

Figure 4 - The Variation of the Price of Anarchy against Demand in $N = 2, \dots, 10$ Parallel Link Network

As demand increases the Price of Anarchy eventually begins to decay back towards 1. The start of this decay coincides with the last route transition point under UE. An explicit formula for the Price of Anarchy in this region, for each network N, is shown in equation (12). This formula was derived analytically. The parameters α and γ are constants that depend on the coefficients a_i and b_i. This equation reveals that the leading order term of this decay is $O(1/q^2)$, which suggests that the similar characteristic shapes of decay, observed for the networks in Figure 1, are a systematic and more general feature of the behaviour of the Price of Anarchy for high demand.

$$\rho = 1 + \frac{1}{\alpha q^2 + \gamma q - 1} \tag{12}$$

4.1.2 Example 2: Five Link Network - Two OD Example

Now recall the five link network example of section 3.1.2, and consider increasing demand on OD movement $q_{O\rightarrow D1}$. The variation of the Price of Anarchy with demand $q_{O\rightarrow D1}$ is shown in Figure 5. The vertical lines signify levels of demand under UE (green) and SO (red) at which K_{min} and $\widetilde{K}_{min}$ expand (solid lines) and contract (dashed lines).

As was observed in Figure 4, this figures shows that the Price of Anarchy is 1 for all levels of demand $q_{O\rightarrow D1}$ up to the first route transition point under SO. Figure 5 also illustrates that at expansions in K^r_{min}, the Price of Anarchy is non-differentiable and that there is a decrease in gradient; this is the same as the behaviour in Figure 4. Figure 5 also illustrates that the Price of Anarchy is non-differentiable at the single demand level corresponding to a contraction in K^r_{min}, and that this coincides with an increase in gradient. Under SO, the Price of Anarchy is differentiable at both points of expansion and also the point of contraction in $\widetilde{K}^r_{min}$; the former leads to an increase in the gradient of the Price of Anarchy whereas the latter leads to a decrease in the gradient of the Price of Anarchy. Therefore, this example suggests that the effects of expansions in K^r_{min} and $\widetilde{K}^r_{min}$, on the Price of Anarchy, are the opposite of the effects of contractions in K^r_{min} and $\widetilde{K}^r_{min}$.

Finally, for demand beyond the final route transition point under UE, the Price of Anarchy again decays back towards 1. Although not included here, this rate of decay also satisfies $O(1/q^2)$ behaviour.

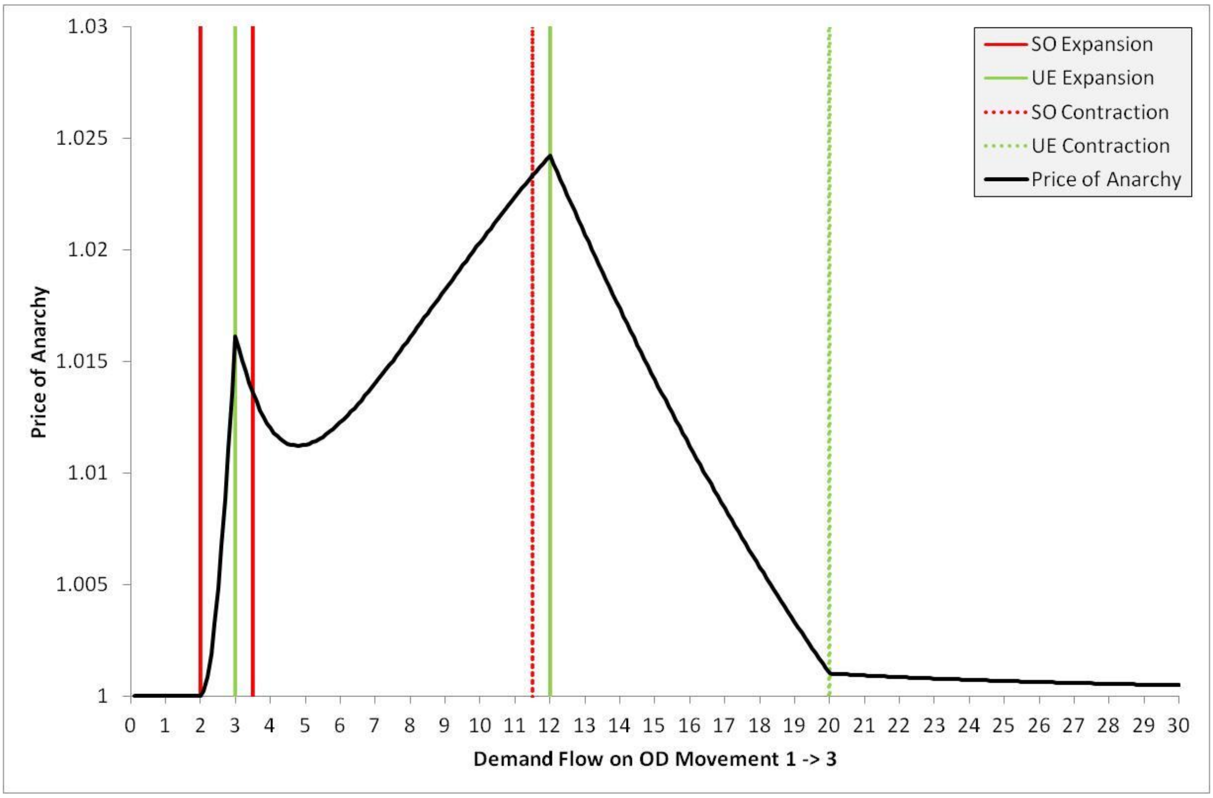

Figure 5 - The Variation of the Price of Anarchy against Demand in the Five Link Network of Figure 2

4.2 The Variation of the Price of Anarchy for Low Travel Demand

In traffic networks in which demand $q_r \to 0$ on all OD movements r, the cost of travel on each route $k \in K^r$, $\forall r$, is dictated by the free-flow travel cost component. In such cases, for such small levels of demand, the routes that are of minimum cost for each OD movement correspond to the shortest path or paths for each OD movement. This is true under both UE and SO; to see this, consider the cost function transformation $\tilde{c}_i = c_i + x_i * dc_i/dx_i$ for the SO problem. When $x_i \to 0$, the additional marginal cost term disappears and the cost of travel on each link is identical under UE and SO. In such cases, it follows that $x_i^{UE} = x_i^{SO}\ \forall i \in A$, that $TTC^{UE} = TTC^{SO}$ and that the Price of Anarchy $\rho = 1$. As demand q_r increases from zero, the shortest path(s) for each OD movement r still provide the minimum (marginal total) cost routes under UE and SO, provided that the second shortest routes have greater free-flow travel cost for each OD movement. The Price of Anarchy remains $\rho = 1$ until, for some OD movement r, the second shortest route in K^r becomes minimum cost, at which point there is a route transition point under SO. This discussion provides an explanation for the initial intervals of demand shown in Figure 1, Figure 4 and Figure 5.

4.3 The Variation of the Price of Anarchy for Intermediate Regions of Travel Demand

It is known that TTC^{UE} and TTC^{SO} are continuous and increasing functions of travel demand (Dafermos and Nagurney, 1984). As $x_i^{UE} = x_i^{SO}\ \forall i \in A$, for very low demand regions, it follows that TTC^{UE} and TTC^{SO} both increase at the same rates, at least until a route transition point occurs. This section describes the effects of route transition points, of the types described in conditions C1-C3, on the rates of change of TTC^{SO} (section 4.3.1), TTC^{UE} (section 4.3.2) and the Price of Anarchy (section 4.3.3).

4.3.1 The Sensitivity of Total Network Travel Cost under SO to Route Transition Points

The first result in this section proves that TTC^{SO} is also differentiable with respect to all demand Q, which, in particular, includes all demands $Q \in \mathrm{H}_{SO}$ that correspond to route transition points.

Proposition 4.1: Consider a traffic network G for which Assumption A1 holds. TTC^{SO} is differentiable with respect to all demand movements r for which $q_r > 0$.

Proof: Proof follows from the Envelope Theorem, which is stated as follows. For the constrained extremum problem:

$$V(z) = \max_{x_1, x_2, \dots, x_n} f(x_1, x_2, \dots, x_n, z)$$

$$\text{s.t.} g_j(x_1, x_2, \dots, x_n, z) \geq 0 \text{ for } j = 1,2, \dots, m$$

the Envelope Theorem states that, if the constraints satisfy the Slater condition and if $x_i(z)$ solve the first-order and complementary slackness conditions for the above problem, $\forall i$, then:

$$\frac{\partial V}{\partial z} = \frac{\partial f(x_1, x_2, \dots, x_n, z)}{\partial z} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial z}$$

where λ_j are Kuhn-Tucker multipliers. The Slater condition requires that there exists a point $(x_1, x_2, \dots, x_n)$ for which $g_j(x_1, x_2, \dots, x_n) > 0\ \forall j$.

The SO minimisation problem has objective function $\tilde{z}(x_1, x_2, \dots, x_n) = TTC^{SO}$ and is subject to constraints set out in equation (3). For this problem to satisfy the Slater condition, requires that there exists a link flow vector $(x_1, x_2, \dots, x_n)$ satisfying the equality constraints in (3) and which produces route flows $f_k^r > 0$, $\forall k \in K^r$, for all OD movements r. In other words a vector of link flows is required that satisfies the equality constraints and which produces positive route flows on all routes between all OD pairs. This can easily be achieved by setting $f_k^r = \gamma_k^r q_r$ where $0 < \gamma_k^r < 1$ and such that $\sum_k \gamma_k^r = 1\ \forall r$, i.e. a link flow vector produced by assigning demand flows q_r to all routes $k \in K^r$ such that all routes receive a non-zero proportion of flow.

The SO minimisation problem therefore satisfies all of the conditions of the Envelope Theorem, which guarantees that the objective function $z(x_1, x_2, \dots, x_n, q_r)$ is differentiable with respect to demand q_r. As $\tilde{z}(x_1, x_2, \dots, x_n, q_r) = TTC^{SO}$, this guarantees that TTC^{SO} is a differentiable function of $q_r > 0$. ■

The next result considers the effect on TTC^{SO} of an increase in demand through a route transition point η_{SO} of the type described in condition C3(i); at which, for each OD movement r, either:

a) $\widetilde{K}_{min}^r$ remains unchanged as demand passes through η_{SO}; or
b) $\widetilde{K}_{min}^r$ expands to include one or more additional routes k.

Proposition 4.2: Consider a traffic network G with link path incidence matrix Δ and for which Assumption A1 holds. Let η_{SO} represent a route transition point satisfying conditions C1, C2 and C3(i); as described in a) and b) above. Denote the OD movements r that satisfy b) by r'. Label routes $k \in K^{r'}$ such that: for routes $k = 1, \dots, n_{r'}$, $k \in \widetilde{K}_{min}^{r'}$ for all demand values $Q \to \eta_{SO}^-$ and $Q \to \eta_{SO}^+$; and for routes $k = n_{r'} + 1, \dots, \kappa_{r'}$, $k \notin \widetilde{K}_{min}^{r'}$ for demand $Q \to \eta_{SO}^-$, but $k \in \widetilde{K}_{min}^{r'}$ for demand $Q \to \eta_{SO}^+$.

Suppose that $\hat{G}$ denotes an adjusted version of the network G, which has an identical link path incidence matrix $\widehat{\Delta}$, except that for the OD movements r', all routes $k = n_{r'} + 1, \dots, \kappa_{r'}$ are omitted from $\widehat{\Delta}$. Then for demand $Q \to \eta_{SO}^{+}$:

$$TTC_{\hat{G}}^{SO}(Q) > TTC_{G}^{SO}(Q)$$

Proof: Let $\Psi_G^*(Q) = \left\{ \tilde{C}_k^r(Q) \middle| k \in \widetilde{K}_{min}^r,\ \forall r \right\}$ represent the unique set of route costs with minimum marginal total cost under SO, at demand Q. The route costs $\Psi_G^*(Q)$ are therefore associated with the vector of link flows $x_G^*(Q)$, which produce the minimum value of the objective function $\tilde{z}_G(Q)$ in the SO minimisation program defined in section 2. Note that $\tilde{z}_G(Q) = TTC_G^{SO}(Q)$.

By the starting assumptions; for demand levels $Q \to \eta_{SO}^{-}$, $\Psi_G^*(Q)$ uniquely minimises $\tilde{z}_G(Q)$, such that all routes $k = n_{r'} + 1, \dots, \kappa_{r'}$ for the OD movements r', satisfy $k \notin \widetilde{K}_{min}^{r'}$. Whereas, for demand levels $Q \to \eta_{SO}^{+}$, $\Psi_G^*(Q)$ uniquely minimises $\tilde{z}_G(Q)$, such that all routes $k = n_{r'} + 1, \dots, \kappa_{r'}$ for the OD movements r', satisfy $k \in \widetilde{K}_{min}^{r'}$. All other feasible route cost sets $\Psi_G(Q)$, satisfy $\tilde{z}\big(\Psi_G(Q)\big) > \tilde{z}\big(\Psi_G^*(Q)\big)$. In particular, all route cost sets $\Psi_G(Q)$, for demand levels $Q \to \eta_{SO}^{+}$, in which routes $k = n_{r'} + 1, \dots, \kappa_{r'}$ for the OD movements r', are restricted from $\widetilde{K}_{min'}^{r'}$, satisfy this condition.

Now consider the network $\hat{G}$. For demand levels $Q \to \eta_{SO}^{-}$, $\Psi_G^*(Q) = \Psi_{\hat{G}}^*(Q)$ and therefore $TTC_G^{SO}(Q) = TTC_{\hat{G}}^{SO}(Q)$. However, for demand levels $Q \to \eta_{SO}^{+}$, $\Psi_G^*(Q) \neq \Psi_{\hat{G}}^*(Q)$. This is because, in $\Psi_G^*(Q)$, the routes $k = n_{r'} + 1, \dots, \kappa_{r'}$ for the OD movements r', satisfy $k \in \widetilde{K}_{min}^{r'}$. Whereas, in $\Psi_{\hat{G}}^*(Q)$, the same routes are not in $\widetilde{K}_{min}^{r'}$ because they were omitted from the link path incidence matrix $\widehat{\Delta}$ for $\hat{G}$ by starting assumption. However, $\Psi_{\hat{G}}^*(Q)$ is still feasible for the network G. It therefore follows that $\tilde{z}\left(\Psi_{\hat{G}}^*(Q)\right) > \tilde{z}\big(\Psi_G^*(Q)\big)$ for demand levels $Q \to \eta_{SO}^{+}$. This equation is equivalent to $TTC_{\hat{G}}^{SO}(Q) > TTC_G^{SO}(Q)$. ■

A visualisation of this result is provided in Figure 6. In this figure, it is assumed that at η_{SO}, the sets $\widetilde{K}_{min}^{r}$, for one or more OD movements r in a network G, expand such that the total number of minimum marginal total cost routes over all OD movements increases from N to $> N$. Under the terms of the assumptions of proposition 4.2, the traffic network $\hat{G}$ does not contain any of these additional routes. It can be seen, in Figure 6, that as $Q \to \eta_{SO}^{-}$, $TTC_G^{SO}(Q) = TTC_{\hat{G}}^{SO}(Q)$. However, at $Q = \eta_{SO}$, these functions diverge. $TTC_{\hat{G}}^{SO}$ represents what would have happened to TTC_G^{SO} if the routes that were added to the minimum cost route sets for the OD movements r, did not exist in G. As demand $Q \to \eta_{SO}^{+}$, TTC^{SO} does not continue to follow the trajectory that it was on for $Q \to \eta_{SO}^{-}$, for which there were N minimum cost routes in total; but instead shifts onto a *lower* trajectory for which there are $> N$ minimum cost routes in total, thereby slowing the rate of increase in TTC^{SO}.

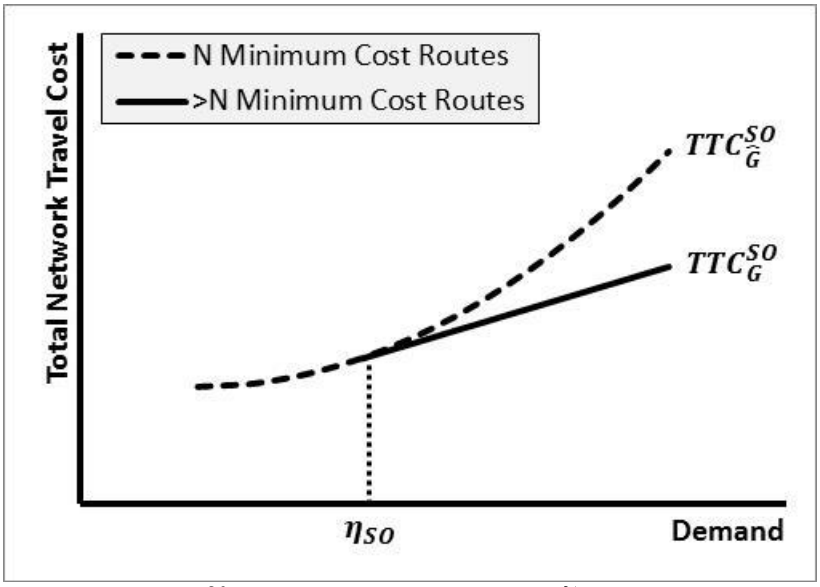

Figure 6 - The effect on TTC^{SO} of one or more expansions in $\widetilde{K}^{r}_{min}$, for some OD movements r

The final result of this subsection describes the effect on TTC^{SO} of an increase in demand through a route transition point η_{SO} of the type described in condition C3(ii); at which, for each OD movement r, either:

c) $\widetilde{K}^{r}_{min}$ remains unchanged as demand passes through η_{SO}; or
d) $\widetilde{K}^{r}_{min}$ contracts as one or more routes k are no longer of minimum marginal total cost.

Proposition 4.3: Consider a traffic network G with link path incidence matrix Δ and for which Assumption A1 holds. Let η_{SO} represent a route transition point satisfying the conditions C1, C2 and C3(ii); as described in c) and d) above. Denote the OD movements r that satisfy condition d) by r'. Label routes $k \in K^{r'}$ such that: for routes $k = 1, \dots, n_{r'}$, $k \in \widetilde{K}^{r'}_{min}$ for all demand values $Q \to \eta_{SO}^{-}$ and $Q \to \eta_{SO}^{+}$; and for routes $k = n_{r'} + 1, \dots, \kappa_{r'}$, $k \in \widetilde{K}^{r'}_{min}$ for demand $Q \to \eta_{SO}^{-}$, but $k \notin \widetilde{K}^{r'}_{min}$ for demand $Q \to \eta_{SO}^{+}$.

Suppose that $\widehat{G}$ denotes an adjusted version of the network G, which has an identical link path incidence matrix $\widehat{\Delta}$, except that for the OD movements r', all routes $k = n_{r'} + 1, \dots, \kappa_{r'}$ are omitted from $\widehat{\Delta}$. Then for demand levels $Q \to \eta_{SO}^{-}$:

$$TTC^{SO}_{\widehat{G}}(Q) > TTC^{SO}_{G}(Q)$$

Proof: This proof uses similar arguments to those used to prove Proposition 4.2. ■

A visualisation of this result is provided in Figure 7. In this figure, it is assumed that at η_{SO}, the sets $\widetilde{K}^{r}_{min}$, for one or more OD movements r in a network G, contract such that the total number of minimum marginal total cost routes over all OD movements decreases from $> N$ to N. Under the terms of the assumptions of proposition 4.3, the traffic network $\widehat{G}$ does not contain any of the routes that leave the minimum marginal total cost route set at η_{SO}. To understand the implication of this graph, it is easiest to visualise what happens as demand decreases from the right hand side. It can be seen that for demand values $Q \to \eta_{SO}^{+}$, $TTC^{SO}_{G}(Q) = TTC^{SO}_{\widehat{G}}(Q)$. At $Q = \eta_{SO}$, these functions diverge. For demand values $Q \to \eta_{SO}^{-}$, TTC^{SO} does not continue to follow the trajectory that it was on for $Q \to \eta_{SO}^{+}$, because, in the direction of decreasing demand, the set of routes of minimum marginal total cost expands from N to $> N$ routes in total, which leads to a lower value of TTC^{SO}.

The effect of this behaviour, when considering increasing demand, is that as demand moves through a route transition point of type C3(ii), TTC^{SO} transfers onto a *higher* trajectory. There is, therefore, an acceleration in the rate of increase of TTC^{SO}.

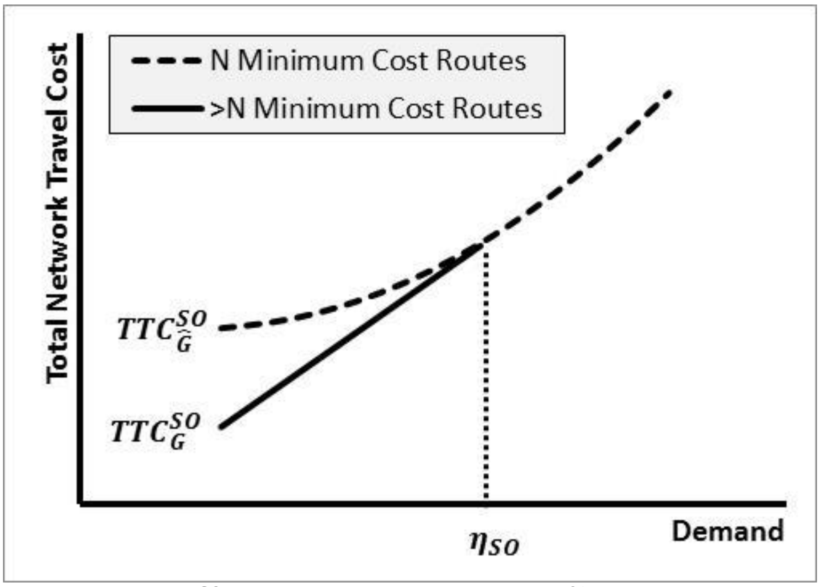

Figure 7 - The effect on $TTC^{SO}$of one or more contractions in $\widetilde{K}^r_{min}$, for some OD movements r

4.3.2 The Sensitivity of Total Network Travel Cost under UE to Route Transition Points

Similarly to section 4.3.1, this section begins by characterising the existence of derivatives of TTC^{UE} with respect to demand. These derivatives depend, by construction, upon the sensitivity of link flows x_i^{UE} with respect to increases in demand.

In the context of the UE traffic assignment problem, Patriksson (2004) provides a characterisation of the existence of directional derivatives and full derivatives of links flows. This is achieved through the derivation of a sensitivity problem, which yields directional derivatives of links flows provided that it has a unique solution. Josefsson and Patriksson (2007) built on Patriksson (2004) to show that, in traffic networks with separable link cost functions, a sufficient condition for the existence of a directional derivative of a link flow x_i^{UE}, is that the corresponding cost function c_i has a strictly positive derivative. For demands Q at which directional derivatives of link flows do exist, it follows from theorem 10 of Patriksson (2004) that full derivatives of those link flows also exist; provided it can be shown that $\partial f_k^r/\partial q = 0$, $\forall k \in K^r$ for which $f_k^r = 0$ in every possible route flow solution F, and for any perturbation of demand. Within this statement, the derivatives $\partial f_k/\partial q$ must be consistent with the set of derivatives $\partial x_i/\partial q$, which uniquely solves the sensitivity problem.

The following result proves that TTC^{UE} is differentiable for $\forall Q \neq \eta_{UE}$. In this proof $K^r_{\sim min}$ denotes the set of routes $k \in K^r$ for which $C_k^r > \pi_r$.

Proposition 4.4: Consider a traffic network G for which Assumption A1 holds. $TTC^{UE}(Q)$ is differentiable $\forall Q \notin \mathrm{H}_{UE}$.

Proof: Suppose, for a traffic network G, that a demand $Q \notin \mathrm{H}_{UE}$ is given. By definition 3.2 it follows that $\lim_{Q\to\eta_{UE}^-} \Upsilon(Q) = \lim_{Q\to\eta_{UE}^+} \Upsilon(Q)$, for all trajectories of demand about η_{UE}. It therefore follows that there exists a neighbourhood of demand about Q for which, $\forall k \in K^r$ for each OD movement r, either (a) $k \in K^r_{min}$ for all Q in this neighbourhood, or (b) $k \in K^r_{\sim min}$ for all Q in this neighbourhood. In other words, the set K^r_{min} of routes that are of minimum cost, for each OD movement r, and the set $K_{\sim min}$ of routes that have costs strictly greater than minimum cost, for each OD movement r, do not change due to a small perturbation of demand.

By the UE conditions (2), it follows that $\forall k \in K_{\sim min}^{r}$ that $f_k^r = 0$, for each OD movement r. It also follows from the above argument that, for any small perturbation of demand, $f_k^r = 0$ will remain true. It consequently follows, from theorem 10 of Patriksson (2004), that link flows x_i^{UE} are differentiable for each link i for which $x_i^{UE} > 0$. As all link flows x_i^{UE} are differentiable functions of Q for all links i for which $x_i^{UE} > 0$, and all other links, for which there is no information about differentiability, have $x_i^{UE} = 0$, it follows that TTC^{UE} is differentiable at Q. This is by construction of TTC^{UE}, because it is a sum of products of differentiable functions. ∎

The contrapositive result of proposition 4.4 is that all instances of demand Q, at which TTC^{UE} is not differentiable, must correspond to route transition points η_{UE}.

Conjectures 4.5 and 4.6 present claims for the behaviour of TTC^{UE} at route transition points of the types described in conditions C3(i) and C3(ii) respectively.

Conjecture 4.5: Consider a traffic network G for which Assumption A1 holds, and let η_{UE} represent a route transition point of type C3(i). Then:

$$\lim_{Q \to \eta_{UE}^-} \left(\frac{\partial}{\partial q} TTC^{UE} \right) > \lim_{Q \to \eta_{UE}^+} \left(\frac{\partial}{\partial q} TTC^{UE} \right)$$

Conjecture 4.6: Consider a traffic network G for which Assumption A1 holds, and let η_{UE} represent a route transition point of type C3(ii). Then:

$$\lim_{Q \to \eta_{UE}^-} \left(\frac{\partial}{\partial q} TTC^{UE} \right) < \lim_{Q \to \eta_{UE}^+} \left(\frac{\partial}{\partial q} TTC^{UE} \right)$$

The above conjectures are stated without proof. We remark that proof of these conjectures is challenging because it is not possible to guarantee that the directional derivatives, stated in Conjectures 4.5 and 4.6, always exist (Josefsson and Patriksson, 2007). Given this difficulty, it is particularly noteworthy that TTC^{SO} is fully differentiable at all points of demand Q, including all route transition points η_{SO}, given the similarities that exist between the UE and SO models.

Numerical evidence supporting these conjectures can be found in the examples in section 5.

4.3.3 The Sensitivity of the Price of Anarchy to Route Transition Points

This section describes the implications of the results of sections 4.3.1 and 4.3.2 for the Price of Anarchy; starting with differentiability.

Corollary 4.7: Consider a traffic network G for which Assumption A1 holds. The Price of Anarchy is a differentiable function for all demand movements $q_r > 0$, for which $Q \notin \mathrm{H}_{UE}$.

Proof: Follows from propositions 4.1 and 4.4. ∎

The result that follow describes the differing effects on the Price of Anarchy of route transition points of the types described in conditions C3(i) and C3(ii), under UE and SO.

Theorem 4.8: Consider a traffic network G for which Assumption A1 holds.

(i) For a demand η_{SO}, which corresponds to a route transition point that satisfies condition C3(i):

$$\tilde{\rho}(Q) < \rho(Q), \forall Q \to \eta_{SO}^+$$

where $\tilde{\rho}$ represents a continuation of the trajectory of ρ for $Q \to \eta_{SO}^{-}$, into $Q \to \eta_{SO}^{+}$.

(ii) For a demand η_{SO}, which corresponds to a route transition point that satisfies condition C3(ii):

$$\tilde{\rho}(Q) > \rho(Q), \forall Q \to \eta_{SO}^{+}$$

where $\tilde{\rho}$ represents a continuation of the trajectory of ρ for $Q \to \eta_{SO}^{-}$, into $Q \to \eta_{SO}^{+}$.

Proof: Proof of (i) follows from proposition 4.2 and the fact that TTC^{SO} is on the denominator of ρ. Proof of (ii) follows from proposition 4.3, the associated discussion that followed and the fact that TTC^{SO} is on the denominator of ρ. ■

Conjecture 4.9: Consider a traffic network G for which Assumption A1 holds.

(i) At a demand η_{UE}, which corresponds to a route transition point that satisfies condition C3(i):

$$\lim_{Q \to \eta_{UE}^{-}} \left(\frac{\partial \rho}{\partial q} \right) > \lim_{Q \to \eta_{UE}^{+}} \left(\frac{\partial \rho}{\partial q} \right)$$

(ii) At a demand η_{UE}, which corresponds to a route transition point that satisfies condition C3(ii):

$$\lim_{Q \to \eta_{UE}^{-}} \left(\frac{\partial \rho}{\partial q} \right) < \lim_{Q \to \eta_{UE}^{+}} \left(\frac{\partial \rho}{\partial q} \right)$$

Proof of Conjecture 4.9 would follow immediately from proofs of conjectures 4.5 and 4.6.

4.4 The Variation of the Price of Anarchy for High Travel Demand

As travel demand values q_r become larger, the network becomes saturated as the delay components of travel cost begin to dominate the free-flow component. In the network example in section 3.1.1, it was shown that expansions in the sets K_{min}^{r} and $\widetilde{K}_{min}^{r}$ eventually stop once demand reaches a sufficiently high threshold. This matches our observations from numerical examples.

For the special case of traffic networks with cost functions of the form $c_i = a_i + b_i x_i^{\beta}$ $(a_i, b_i, \beta > 0)$, our conjecture is that as demand Q continues to increase, the Price of Anarchy enters a region of decay that can be characterised by a power law. This characterisation is stated as conjecture 4.10, and is supported by the numerical examples that follow in section 5.

Conjecture 4.10: Consider a traffic network G that serves a demand matrix Q with entries $q_r > 0$, and that has cost functions of the form $c_i = a_i + b_i x_i^{\beta}$ $(a_i, b_i, \beta > 0)$, which satisfy Assumption A1. Let ζ represent a global demand multiplier applied to the demand matrix Q. Then, as $\zeta \to \infty$, the leading order behaviour of the Price of Anarchy is $O\left(1/\zeta^{2\beta}\right)$.

5 Numerical Examples

This section presents four numerical examples, which provide illustrations of the theoretical results presented in sections 3 and 4; and also provide numerical evidence to support conjectures 4.5, 4.6, 4.9 and 4.10. The first example in section 5.1 addresses the simplest scenario of the variation of the Price of Anarchy with increasing demand on a single OD pair. The second example in section 5.2 then presents a more complicated scenario; in which travel demand is increased, at different rates, on several OD pairs between a single origin and several destinations. The final two examples in section 5.3 then present two scenarios in which demand is uniformly increased on several OD pairs, between multiple origins and multiple destinations.

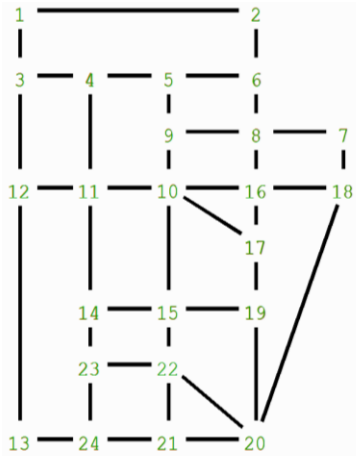

Figure 8 - Sioux Falls Network

The numerical examples in this section are based on the canonical test network of Sioux Falls[4], which is shown in Figure 8. This network comprises 24 nodes and 76 links, and the cost of travel c_i on each link i is represented by a BPR cost function with power $\beta = 4$, which is common to all links. Note that this network satisfies the conditions stated in Assumption A1, theorem 3.5 and corollary 3.6.

The results for each example are compiled from UE and SO traffic assignments undertaken at several discrete levels of travel demand. At each demand level j, travel demand q_r, on each OD movement r, is increased by a demand multiplier ζ_j^r; where $\zeta_j^r < \zeta_{j+1}^r \ \forall r, j$. This guarantees that demand is always increasing on each OD movement and therefore satisfies condition C1. As each traffic assignment is undertaken for discrete values of the demand multipliers ζ_j^r, it is not possible to identify the *exact* levels of demand at which each route transition point occurs. These levels of demand are therefore approximated in the analysis that follows by the first demand level j beyond the route transition point; this being the first level of demand at which it is possible to observe that either the minimum (marginal total) cost route set for an OD movement r or the OD specific active network for an OD movement r has changed. Each traffic assignment is calculated using the OBA algorithm, solved to an average excess cost of, at most, 10^{-9}.

5.1 Example 1: Increasing Demand in a Single OD Pair Network

In this single OD pair scenario, the variation of the Price of Anarchy is studied as travel demand $q = 10$ is increased, using demand multipliers $\zeta_j = 1,2, \ldots, 10000$, on the OD movement between node 20 and node 3 in the Sioux Falls network. Figure 9 displays the variation of the Price of Anarchy against travel demand q. The vertical lines in this figure signify levels of demand corresponding to route transition points η_{UE} and η_{SO}, at which the OD specific active networks X_1^{UE} (green lines) and X_1^{SO} (red lines), expand (solid lines) and contract (dashed lines). Recall that OD specific active networks provide an alternative characterisation for the minimum (marginal total) cost route set under UE and SO. This figure also displays graphs of the Price of Anarchy for 17 sub-networks (denoted ρ_1, ρ_2, etc), which correspond to the 17 different states of the active network, between route transition points, as demand increases.

Focussing on the graph for the full network, Figure 9 displays the same three identifiably distinct regions of behaviour of the Price of Anarchy that are evident in Figure 1: an initial region in which

[4] Network and demand matrix files for Sioux Falls were obtained from Bar-Gera (2001).

the Price of Anarchy is one, a period of fluctuations, followed by a decay back towards one. It can be seen that the Price of Anarchy varies smoothly $\forall q \in \mathrm{H}_{UE}$, which is consistent with corollary 4.7, and three of the four effects of expansions and contractions described in theorem 4.8 and conjecture 4.9 are clearly visible. For UE, at all points η_{UE} corresponding to an expansion of X^{UE}, the Price of Anarchy is non-differentiable and there is a decrease in the gradient of the Price of Anarchy, which provides numerical evidence to support conjecture 4.9(i). At the single point $\eta_{UE} \approx 38{,}000$, which corresponds to a contraction of X^{UE}, the Price of Anarchy is also non-differentiable and there is an increase in the gradient of the Price of Anarchy, which provides numerical evidence to support conjecture 4.9(ii). For SO, at all points η_{SO}, which correspond to an expansion of X^{SO}, the Price of Anarchy is smooth but transfers onto a higher trajectory than the Price of Anarchy for the sub-network that detaches, which illustrates theorem 4.8(i). The effect of a contraction in X^{SO} at a route transition point η_{SO} described in theorem 4.8(ii); for which there is a single point in this example at $\eta_{SO} \approx 25{,}000$, is less apparent.

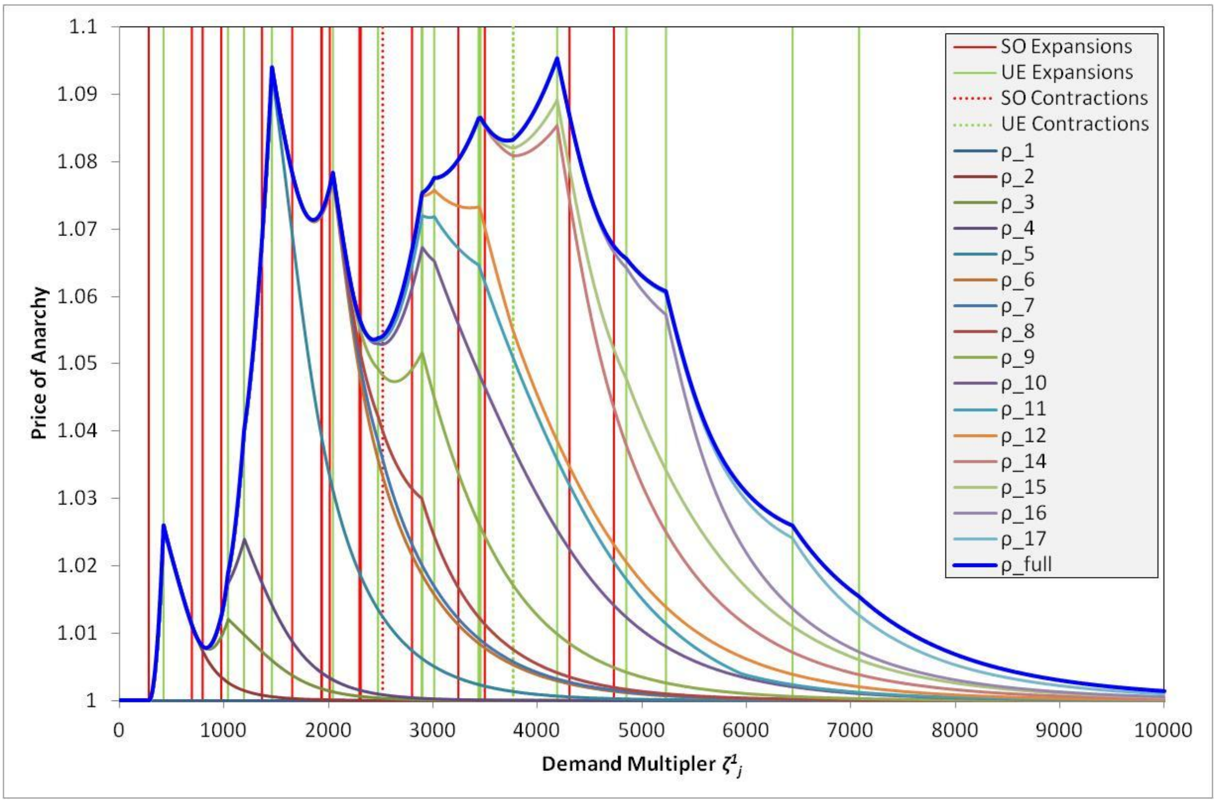

Figure 9 – The Variation of the Price of Anarchy against the Demand Multiplier ζ_j in Example 1

Turning to the systematic relationship between route transition points η_{UE} and η_{SO}, Table 1 lists the approximate levels of demand for each state of the active network as demand increases. The table also presents the value of η_{SO}/η_{UE} at each route transition point and shows the number of links that are active in each state of X_1^{UE} and X_1^{SO}. Given that, for $\beta = 4$, $1/\sqrt[\beta]{\beta + 1} \approx 0.67$, the results in this table are consistent with the conclusions of corollary 3.6.

No.	Route Transition Points		$\frac{\eta_{SO}}{\eta_{UE}}$	Number of Active Links in X_1^{UE} & X_1^{SO}
	$\eta_{SO} = \zeta_j$	$\eta_{UE} = \zeta_j$		
1	1	1	-	6
2	285	426	0.6690	12
3	697	1,042	0.6689	14

4	800	1,196	0.6689	17
5	978	1,463	0.6685	19
6	1,368	2,046	0.6686	23
7	1,657	2,478	0.6687	28
8	1,936	2,895	0.6687	29
9	1,941	2,902	0.6688	30
10	2,016	3,015	0.6687	31
11	2,300	3,439	0.6688	32
12	2,313	3,458	0.6689	33
13	2,,520	3,769	0.6686	32
14	2,803	4,191	0.6688	34
15	3,246	4,853	0.6689	35
16	3,499	5,232	0.6688	36
17	4,309	6,443	0.6688	37
18	4,734	7,079	0.6687	38

Table 1 – Route Transition Points in Example 1

Finally, Figure 10 displays the decay rate of the Price of Anarchy for demand $q > 7{,}044$, which represents the level of demand of the final route transition point η_{UE}. This figure also plots a trend-line; calculated by Ordinary Least Squares regression, which shows that the decay in the Price of Anarchy is consistent with $O\left(1/\zeta_j^{2\beta}\right)$. Figure 10 also displays decay rates of the Price of Anarchy in adjusted versions of the Sioux Falls network for values of $\beta = 1,2,3$. The decay in each of these additional scenarios, from the point of the final route transition point η_{UE}, is also $O\left(1/\zeta_j^{2\beta}\right)$. These findings are consistent with conjecture 4.10.

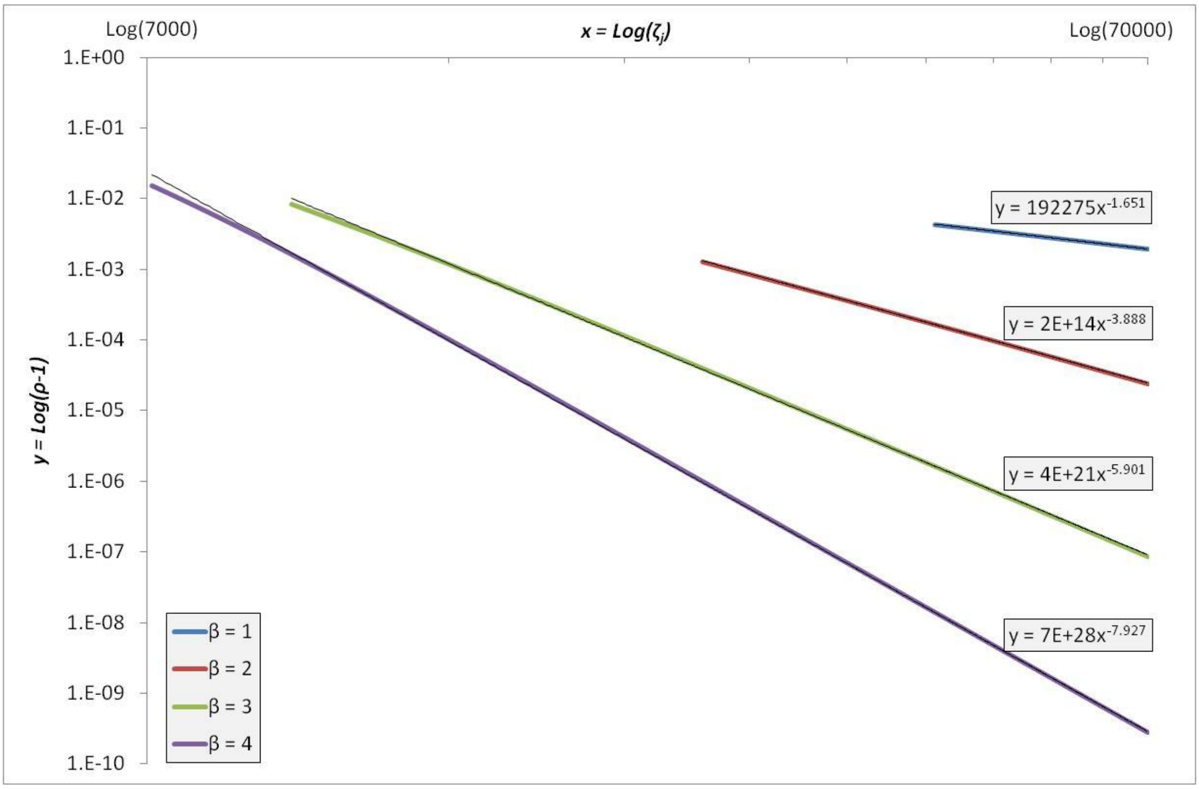

Figure 10 – Decay in the Price of Anarchy for High Demand in Example 1 for $\beta = 1, \beta = 2, \beta = 3, \beta = 4$

5.2 Example 2: Increasing Demand in a Multiple (One to Many) OD Pair Network

In this multiple OD pair scenario, the variation of the Price of Anarchy is studied as travel demand is increased on 22 OD pairs; between a single origin at node 20 and destination nodes $s_r = 1, 2, 3, \ldots, 19, 21, 22, 23$ in the Sioux Falls network. The initial amount of demand q_r and the demand multipliers ζ_j^r are different for each of the 22 OD movements. This therefore represents a more complicated scenario than the single OD example that was explored in section 5.1. The initial amount of demand on each OD movement r is set at $q_r = 24 - s_r$. The demand multipliers ζ_j^r for each OD movement r are then set at $\zeta_j^r = j \times (1 + 0.01 s_r)$, with values of $j = 1,2, \ldots ,2000$.

Figure 11 displays the variation of the Price of Anarchy against index values $j = 1,2, \ldots ,400$ for the demand multipliers ζ_j^r. Similarly to Figure 9, the vertical lines in this figure signify levels of demand corresponding to route transition points η_{UE} and η_{SO}, at which one or more OD specific active networks X_r^{UE} and X_r^{SO} expand or contract. Even with the greater complexity of this example, Figure 11 provides further numerical evidence to support conjecture 4.9 and further illustrations of theorem 4.8. In particular, the increase in gradient of the Price of Anarchy at $\eta_{UE} = 180$ is much clearer than in Figure 9.

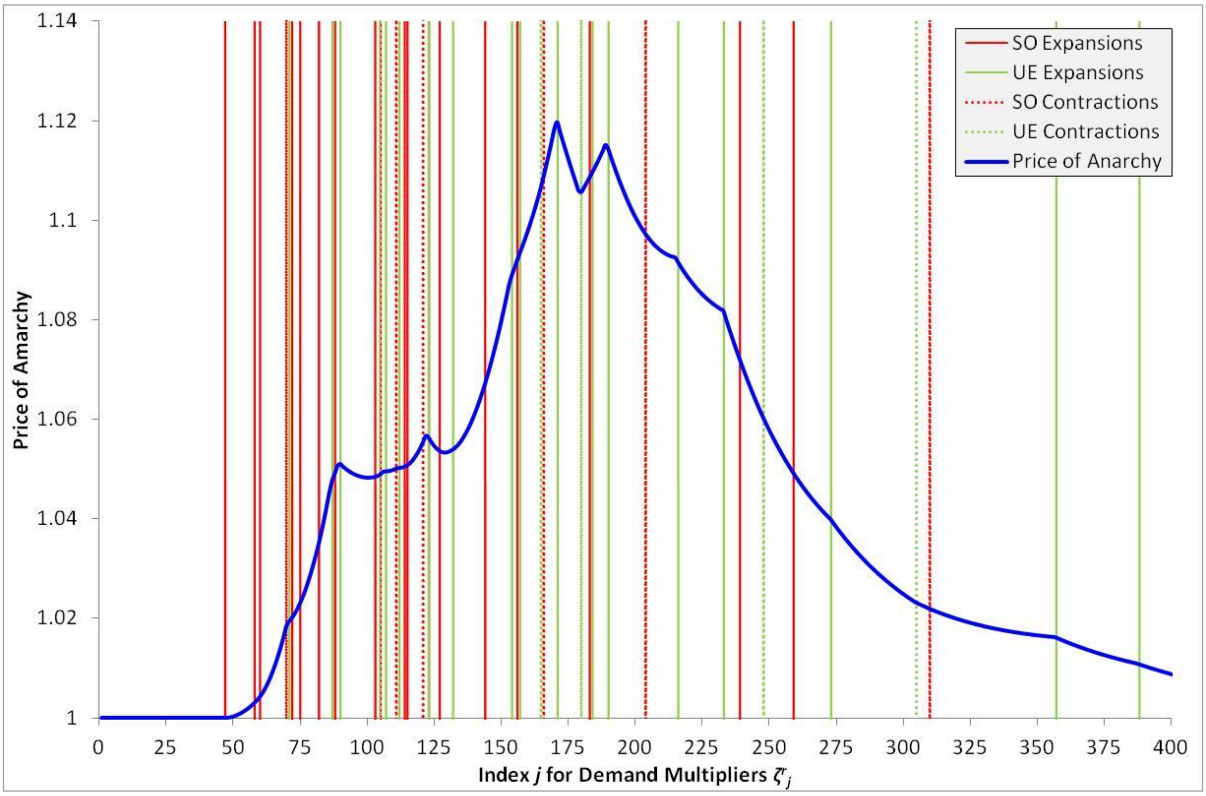

Figure 11 – The Variation of the Price of Anarchy against the Index *j* for Demand Multipliers ζ_j^r in Example 2

Turning to the systematic relationship between route transition points η_{UE} and η_{SO}, Table 2 lists the approximate levels of demand for each route transition point under UE and SO. In contrast to Table 1, the final column of this table displays the number of links with positive flow of the total of 38 links in Sioux Falls. These results are again consistent with the conclusions of corollary 3.6.

No.	Route Transition Points		η_{SO}	Number of Active Links in $\cup_r X_r^{UE}$ & $\cup_r X_r^{SO}$
	$\eta_{SO} = j$	$\eta_{UE} = j$	η_{UE}	

1	1	1	-	24
2	47	71	0.662	25
3	58	87	0.6667	26
4	60	90	0.6667	27
5	70	105	0.6667	26
6	72	107	0.6729	27
7	75	112	0.6696	28
8	82	123	0.6667	29
9	88	132	0.6667	30
10	103	154	0.6688	31
11	105	157	0.6688	32
12	111	165	0.6727	31
13	114	171	0.6667	32
14	115	171	0.6725	33
15	121	180	0.6722	32
16	123	184	0.6685	33
17	127	190	0.6684	34
18	144	216	0.6667	35
19	156	233	0.6695	36
20	166	248	0.6694	35
21	183	273	0.6703	36
22	204	305	0.6689	36
23	239	357	0.6695	37
24	259	388	0.6675	38
25	310	464	0.6681	37
26	418	625	0.6688	38

Table 2 – Route Transition Points in Example 2

Finally, Figure 12 displays the decay rate of the Price of Anarchy for demand indices $j > 625$, which represents the level of demand of the final route transition point η_{UE}. This decay is consistent with $O\left(1/j^{2\beta}\right)$ as is proposed in conjecture 4.10.

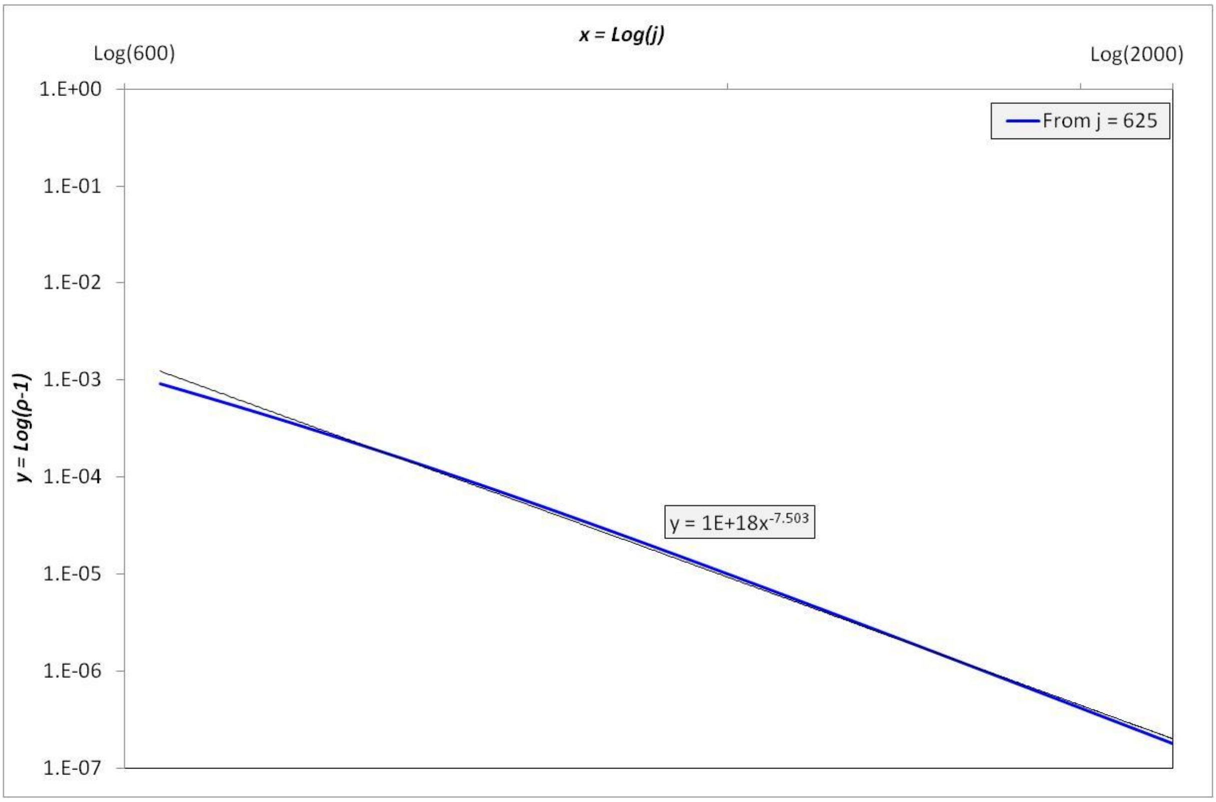

Figure 12 - Decay in the Price of Anarchy for High Demand in Example 2

5.3 Examples 3 and 4: Increasing Demand in a Multiple (Many to Many) OD Pair Network

In addition to illustrating the theoretical results and conjectures of sections 3 and 4, the two examples in this section also illustrate challenges that exist in identifying route transition points in more complicated multiple OD networks.

5.3.1 Sioux Falls Network: Five OD Example

In this first multiple OD pair scenario, the variation of the Price of Anarchy is studied as travel demand is increased on five OD pairs $r = 1, \dots, 5$in the Sioux Falls network: between node 20 and node 1; node 23 and node 2; mode 20 and node 3; node 7 and node 13; and between node 1 and node 19. The initial amounts of demand on each OD movement are set at $q_1 = 23$, $q_2 = 14$, $q_3 = 17$, $q_4 = 18$ and $q_5 = 28$. The demand multipliers for each OD movement are identical, with values $\zeta_j^1 = \cdots = \zeta_j^5 = \zeta_j = 1,2, \dots, 8000$.

Figure 13 displays the variation of the Price of Anarchy against demand multipliers up to $\zeta_j = 1000$. Similarly to previous figures the vertical lines signify levels of demand corresponding to route transition points η_{UE} and η_{SO}. As OBA is unable to identify OD specific active networks in network examples with multiple origin nodes, the vertical lines represent only those route transition points at which an expansion (contraction) in an OD specific active network coincides with an expansion (contraction) in the overall active network, which is equivalent to $\bigcup_r X_r^{UE}$ and $\bigcup_r X_r^{SO}$. The overall active network is uniquely defined by link flows. This demonstrates a limitation of using the OBA algorithm to identify changes in OD specific active networks for cases in which there are multiple origins. The consequence of this is that there may be route transition points that exist, which this method does not identify. Indeed, at $\zeta_j \approx 500$, there is a 'downward kink' in the graph of the Price of Anarchy, which suggests that there is a route transition point η_{UE} corresponding to the expansion of K_{min}^{r} for some OD movement r. The identification of route transition points, through observation of OD specific active networks, in this general case would require the TAPAS algorithm. The example in section 5.3.2 demonstrates an alternative approach to identifying route transition points, which uses route enumeration.

Despite this limitation of OBA, the behaviour of the Price of Anarchy at all other route transition points accords with the claims made in conjecture 4.9 and provides further illustrations of the statements in theorem 4.8.

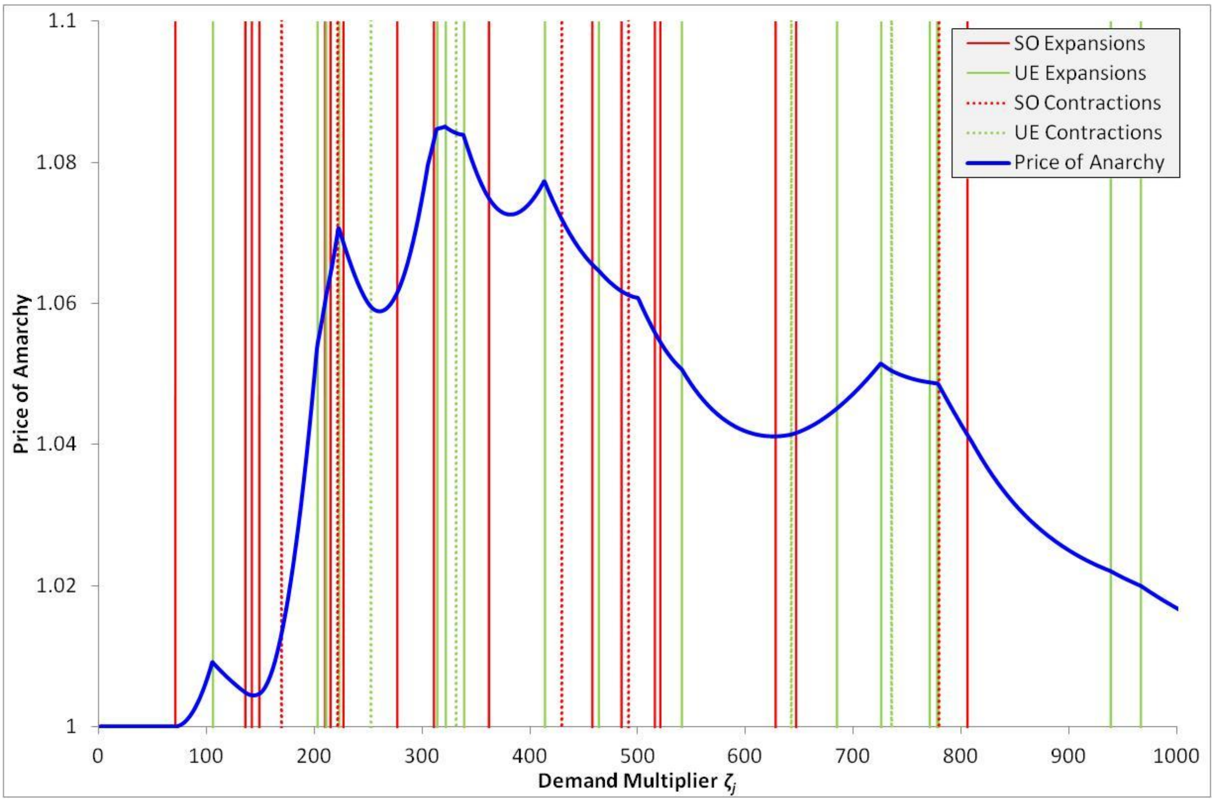

Figure 13 – The Variation of the Price of Anarchy against the Demand Multiplier ζ_j in Example 3

For each of the vertical lines in Figure 13, Table 3 lists the approximate levels of demand at which the overall active network changes as travel demand increases. Similarly to previous examples, these results are consistent with the conclusions of corollary 3.6.

No.	**Route Transition Points**		$\frac{\eta_{SO}}{\eta_{UE}}$	**Number of Active Links in** $\cup_r X_r^{UE}$ **&** $\cup_r X_r^{SO}$
	$\eta_{SO} = \zeta_j$	$\eta_{UE} = \zeta_j$		
1	1	1		21
2	71	106	0.6698	24
3	136	203	0.67	37
4	142	211	0.673	42
5	149	223	0.6682	46
6	170	253	0.6719	41
7	210	314	0.6688	46
8	215	322	0.6677	48
9	222	332	0.6687	46
10	227	339	0.6696	49

11	277	414	0.6691	50
12	311	464	0.6703	52
13	362	541	0.6691	56
14	430	643	0.6687	55
15	458	685	0.6686	56
16	485	726	0.668	57
17	492	736	0.6685	56
18	516	771	0.6693	57
19	521	778	0.6697	58
20	521	779	0.6688	59
21	628	939	0.6688	60
22	647	967	0.6691	61
23	780	1,166	0.669	60
24	806	1,205	0.6689	62
25	1,037	1,551	0.6686	63
26	1,428	2,136	0.6685	62

Table 3 – Route Transition Points in Example 3

Finally, Figure 14 displays the decay rate of the Price of Anarchy for values of the demand multiplier $\zeta_j > 2136$, which represents the level of demand of the final route transition point η_{UE}. This decay is consistent with $O\left(1/\zeta_j^{2\beta}\right)$ as is proposed in conjecture 4.10.

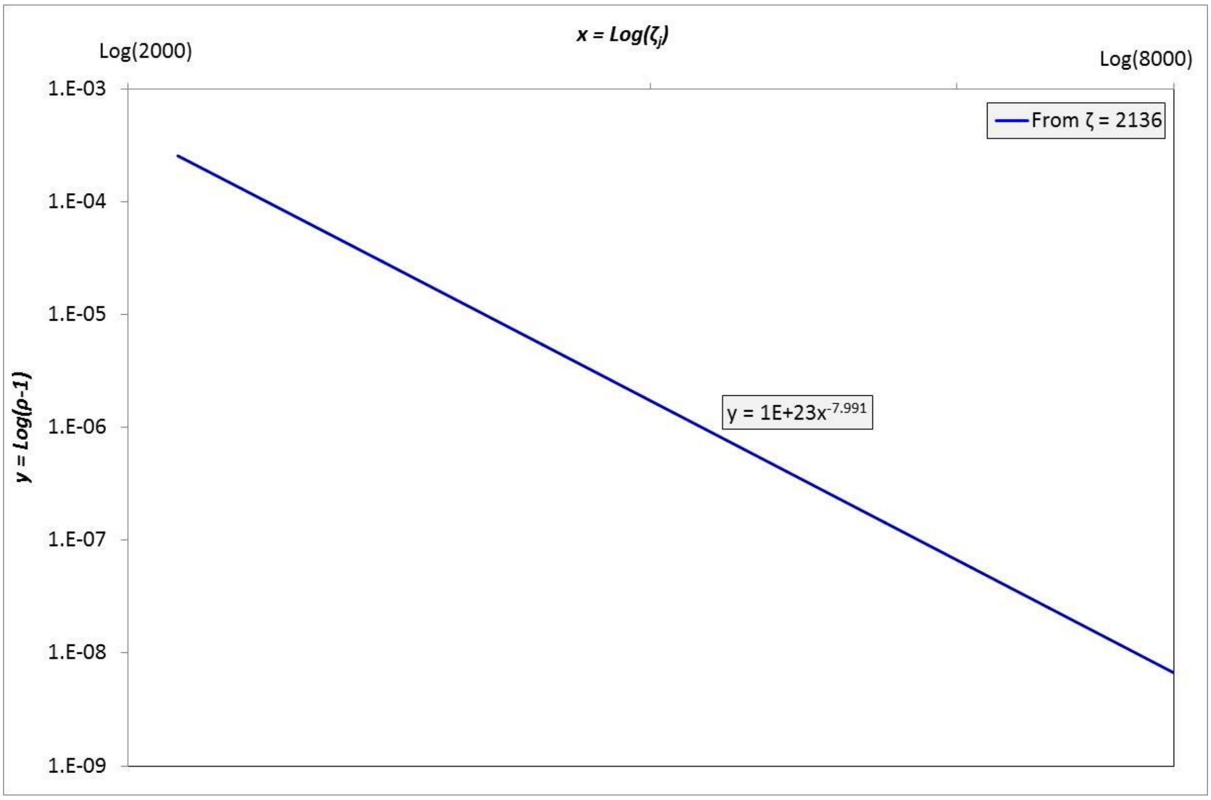

Figure 14 - Decay in the Price of Anarchy for High Demand in Example 3

5.3.2 Sioux Falls Network: 528 OD Example

In this second multiple OD pair scenario, the variation of the Price of Anarchy is studied as travel demand is increased in the Sioux Falls network, using the demand matrix file that is available at Bar-Gera (2001). This demand matrix contains 528 OD pairs. The initial amounts of demand on each OD movement are set at $q_r = 0.001q_r'$, where q_r' represents the value in the original matrix. Demand multipliers for each OD movement are then identical, with values $\zeta_j^1 = \cdots = \zeta_j^{528} = \zeta_j = 1,2,\dots,9000$.

Figure 15 displays the variation of the Price of Anarchy against demand multipliers up to $\zeta_j = 2000$. Similarly to Figure 13, this figure also uses vertical lines to signify levels of demand that correspond to route transition points η_{UE} and η_{SO} at which there is a change in the overall active networks $\bigcup_r X_r^{UE}$ and $\bigcup_r X_r^{SO}$. For this example, there are only two such route transition points, which Table 4 shows both satisfy the conditions of conclusions of corollary 3.6.

In order to better identify the full sets of route transition points H_{UE} and H_{SO}, an alternative methodology is employed in which, at each demand level, we count the number of routes, for each OD movement r, that are within a tolerance 10^{-10} of the minimum (marginal total) cost route under UE and SO. This is inspired by the approach described in Bar-Gera (2006). This method identifies a total of 364 demand levels $\zeta_j \in [0,2000]$, which correspond to route transition points η_{UE} and η_{SO} for this network. As the inclusion of a vertical line for each of these points would make Figure 15 unintelligible, we instead plot the difference between the total numbers of minimum (marginal total) cost routes under UE and SO. Referred to as the *difference measure*, this measure is calculated, for each demand level j, as $\sum_r \left|\widetilde{K}_{min}^r(\zeta_j)\right| - \sum_r \left|K_{min}^r(\zeta_j)\right|$. Although this is a particularly coarse measure, it can be seen that it has a similar overall pattern to the Price of Anarchy (though with different magnitude). This measure, therefore, provides further numerical evidence to support the claims of conjecture 4.9 and the conclusions of theorem 4.8.

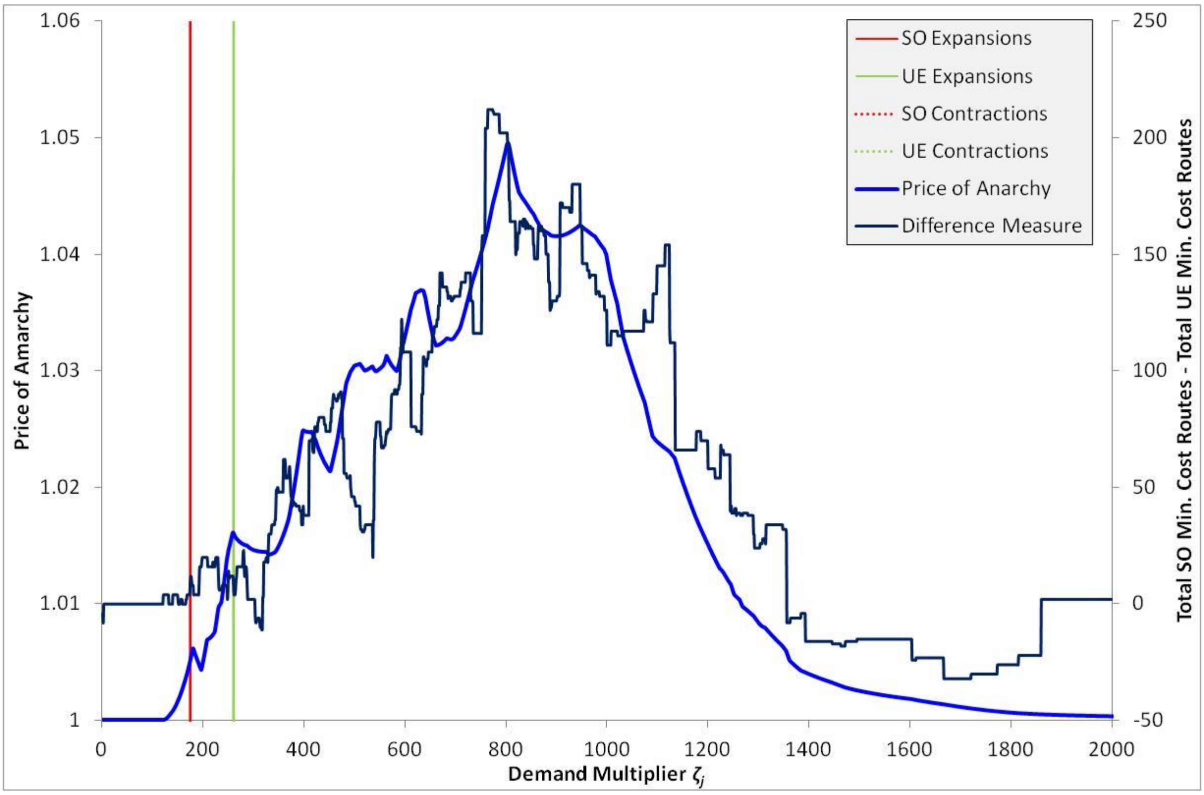

Figure 15 – The Variation of the Price of Anarchy against the Demand Multiplier ζ_j in Example 4

No.	Route Transition Points		$\frac{\eta_{SO}}{\eta_{UE}}$	Number of Active Links in $\bigcup_r X_r^{UE}$
	$\eta_{SO} = \zeta_j$	$\eta_{UE} = \zeta_j$		

				& $\cup_r X_r^{SO}$
1	1	1		74
2	71	106	0.6698	75
3	136	203	0.67	76

Table 4 – Route Transition Points in Example 4

For values of $\zeta_j > 2000$, the difference measure becomes increasingly unstable as demand increases. This is an indicator that the level of convergence of 10^{-9} eventually (and inevitably) becomes unable to clearly identify expansions and contractions because of the magnitudes of travel costs. For this example, it is therefore not possible to identify the exact level of demand at which the final region of decay in the Price of Anarchy begins. For this reason, Figure 16 displays the decay rate of the Price of Anarchy for values of the demand multiplier $\zeta_j > 934$, which signifies the first point in Figure 15 at which the Price of Anarchy begins to steadily fall. It can be seen from this figure that the decay rate of the Price of Anarchy eventually becomes consistent with $O\left(1/\zeta_j^{2\beta}\right)$, as is proposed in conjecture 4.10, for values of $\zeta_j > 4036$.

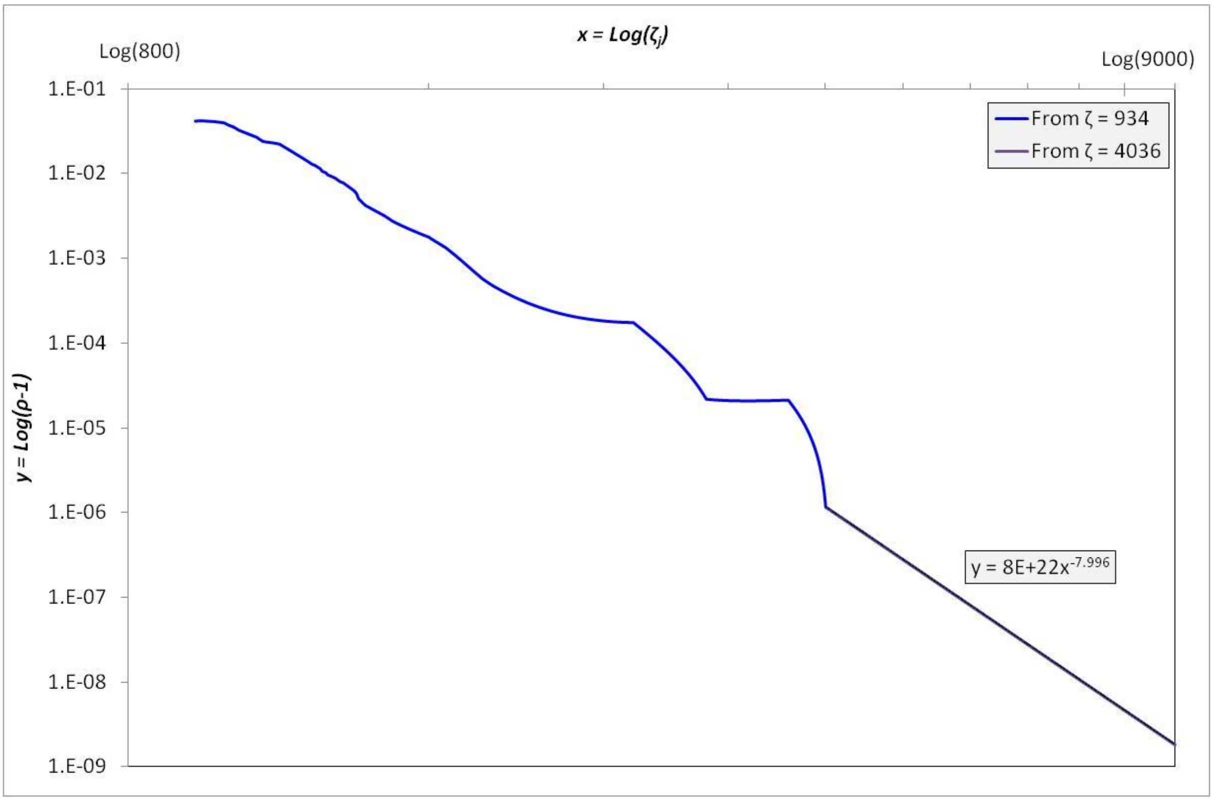

Figure 16 - Decay in the Price of Anarchy for High Demand in Example 4

6 Conclusions

Selfish routing is known to be inefficient for society as a whole but it is unknown how or why the extent of this inefficiency, which can be measured by the Price of Anarchy, varies with the underlying structures of demand and supply that have been shown to exist in real traffic networks. Focussing on how the Price of Anarchy varies as travel demand is increased in traffic networks with separable and strictly increasing cost functions, this paper has identified and described the effects of four mechanisms that govern this variation. These are, specifically, expansions and contractions in the sets of routes, for each OD movement, of minimum (marginal total) cost under UE and SO. In the special case of traffic networks with cost functions of the form $c_i = a_i + b_i x_i^{\beta}$, for which $a_i, b_i, \beta > 0$, this paper has also proven that there is a systematic relationship between link flows under UE and SO, and has conjectured that the Price of Anarchy has power law decay for large demand.

There are several opportunities for the work presented in this paper to be extended. Firstly, the numerical evidence presented in section 5 supports the claims stated in conjectures 4.9 and 4.10, with respect to the gradient of the Price of Anarchy, and also the power law decay in the Price of Anarchy for large demand. Strict proofs for each of these statements are still required. The theory could also be generalised further by easing the restrictions that were imposed on the types of demand movements and route transition points in this paper; specifically, by allowing demand movements to move freely both up and down, allowing adjacent route transition points and also allowing simultaneous expansions and contractions in minimum cost route sets. This is likely to require further numerical work to explore what exactly can be established.

More generally, by providing a description of the mechanisms that govern how the Price of Anarchy varies with travel demand, this paper provides a starting point from which the effects of patterns in demand and supply structure can begin to be explored. In particular this paper highlights the differing nature of how demand is assigned across routes under the UE and SO principles. This is an important consideration when making comparisons of the inefficiency of selfish routing across different supply topologies, which will be the general focus of our future research.

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