Proceedings Paper:
Schiffer, J, Seel, T, Raisch, J et al. (2014) A consensus-based distributed voltage control for reactive power sharing in microgrids. In: Proceedings of the 2014 European Control Conference (ECC). European Control Conference (ECC14), 24-27 Jun 2014, Strasbourg, France. IEEE, pp. 1299-1305. ISBN: 978-3-9524269-1-3.

https://doi.org/10.1109/ECC.2014.6862217

A consensus-based distributed voltage control for reactive power sharing in microgrids

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***Abstract*— We propose a consensus-based distributed voltage control (DVC), which solves the problem of reactive power sharing in autonomous meshed inverter-based microgrids with inductive power lines. Opposed to other control strategies available thus far, the DVC does guarantee reactive power sharing in steady-state while only requiring distributed communication among inverters, i.e. no central computing nor communication unit is needed. Moreover, we provide a necessary and sufficient condition for local exponential stability. In addition, the performance of the proposed control is compared to the usual voltage droop control [1] in a simulation example based on the CIGRE benchmark medium voltage distribution network.**

I. Introduction

Microgrids represent a promising concept to facilitate the integration of distributed renewable sources into the electrical grid [2], [3], [4]. Two main motivating facts for the need of such concepts are: *(i)* the increasing installation of renewable energy sources world-wide – a process motivated by political, environmental and economic factors; *(ii)* a large portion of these renewable sources consists of small-scale distributed generation (DG) units connected at the low (LV) and medium voltage (MV) levels via AC inverters. Since the physical characteristics of inverters largely differ from those of conventional electrical generators, i.e. synchronous generators (SGs), different control approaches are required [5].

A microgrid addresses these issues by gathering a combination of generation units, loads and energy storage elements at distribution level into a locally controllable system. This system can be operated either connected to or completely isolated from the main transmission grid. An autonomous or islanded microgrid is operated in the latter way.

Besides frequency and voltage stability, power sharing is an important performance criterion in the operation of microgrids [5]. Power sharing is understood as the ability of the local controls of the individual generation sources to achieve a desired steady-state distribution of the power outputs of all generation sources *relative* to each other, while satisfying the load demand in the network. The relevance of this control objective lies within the fact that it allows to prespecify the utilization of the generation units in operation.

When generation sources are connected to the network via SGs, droop control is often used to achieve the objective of active power sharing [6]. Under this approach, the rotational speed of each SG in the network is monitored locally to derive how much power each SG needs to provide.

Inspired hereby, researchers have proposed to apply a similar control to AC inverters [1], [7]. It has been shown [8], [9], [10] that this heuristic decentralized control law indeed locally stabilizes the network frequency and that the control gains and setpoints can be chosen such that a desired active power distribution is achieved in steady-state without any explicit communication among the different sources. The nonnecessity of an explicit communication system is explained by the fact that the network frequency serves as a common implicit communication signal.

Furthermore, in microgrids, droop control is typically also applied with the objective to achieve a desired *reactive power* distribution. The most common approach is to set the voltage amplitude with a proportional control, the feedback signal of which is the reactive power generation relative to a reference setpoint [1], [11]. However, this control does in general not guarantee a desired reactive power sharing [10], [12], [13]. As a consequence, several other (heuristic) decentralized voltage control laws have been proposed [12], [13], [14], [15], [16], but no general conditions or formal guarantees for reactive power sharing are given.

Therefore, we propose in this work a consensus-based distributed voltage control (DVC), which solves the open problem of reactive power sharing in autonomous meshed inverter-based microgrids with inductive power lines. Unlike most other related communication-based control concepts, e.g. [17], [18], the present approach only requires distributed communication among inverters, i.e. it does not require a central communication or computing unit nor all-to-all communication among the inverters.

Furthermore, due to the lack in performance with respect to reactive power sharing of the voltage control [1], the control presented here is meant to replace the voltage control [1] rather than complementing it in a secondary control-like manner, as e.g. in [18], [19]. Moreover, unlike e.g. [19], we characterize uniqueness properties of equilibrium points of the closed-loop system and provide a necessary and sufficient condition for local exponential stability.

We would like to emphasize that reactive power sharing is only of practical interest in networks or clusters of networks, where the generation units are in close electrical proximity. This is often the case in microgrids and we only consider such networks in this paper. Close electrical proximity usually implies close geographical distance between the different units, which facilitates the practical implementation of a distributed communication network.

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Partial support from the HYCON2 Network of excellence (FP7 ICT 257462) is acknowledged.

II. PRELIMINARIES AND NOTATION

We define the sets $\mathcal{N} := \{1, \ldots, n\}$, $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} | x \geq 0\}$, $\mathbb{R}_{>0} := \{x \in \mathbb{R} | x > 0\}$, $\mathbb{R}_{<0} := \{x \in \mathbb{R} | x < 0\}$ and $\mathbb{T} := [0, 2\pi)$. For a set $\mathcal{U}$, $i \sim \mathcal{U}$ denotes "for all $i \in \mathcal{U}$" and $|\mathcal{U}|$ its cardinality. Let $x := \operatorname{col}(x_i) \in \mathbb{R}^n$ denote a vector with entries x_i, $i \sim \mathcal{N}$; $\underline{0}_n \in \mathbb{R}^n$ the vector of all zeros; $\underline{1}_n \in \mathbb{R}^n$ the vector with all ones; I_n the $n \times n$ identity matrix; $0_{n\times n}$ the $n \times n$ matrix of all zeros and $\operatorname{diag}(a_i), i \sim \mathcal{N}$, an $n \times n$ diagonal matrix with entries a_i. For $z \in \mathbb{C}$, $\Re(z)$ denotes the real part of z and $\Im(z)$ its imaginary part. Let j denote the imaginary unit. The conjugate transpose of a vector v is denoted by v^*. For a matrix $A \in \mathbb{R}^{n\times n}$, let $\sigma(A) := \{\lambda \in \mathbb{C} : \det(\lambda I_n - A) = 0\}$ denote its spectrum. The numerical range or field of values of A is defined as $W(A) := \{x^* Ax : x \in \mathbb{C}^n, x^* x = 1\}$. It holds that $\sigma(A) \subseteq W(A)$ [20]. If A is symmetric then $W(A) \subseteq \mathbb{R}$ and $\min(\sigma(A)) \leq W(A) \leq \max(\sigma(A))$ [20]. Let $A_{\text{sy}} = \frac{1}{2}(A + A^T)$, respectively $A_{\text{sk}} = \frac{1}{2}(A - A^T)$ be the symmetric, respectively skew-symmetric part of A. Then $\Re(W(A)) = W(A_{\text{sy}})$ and $\Im(W(A)) = W(A_{\text{sk}})$ [20].

The following two results are used in the paper.

Lemma 2.1: [20] Let A and B be matrices of appropriate dimensions and let B be positive semidefinite. Then,

$$\sigma(AB) \subseteq W(A)W(B) := \{\lambda = \alpha\beta | \alpha \in W(A), \beta \in W(B)\}.$$

Lemma 2.2: [21]. Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ be a matrix with constant entries. Let $F : \mathbb{R}^n \to \mathbb{R}^n$, $F(x) := \operatorname{col}(f_1(x_1), \ldots, f_n(x_n))$, where $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, are nonlinear strictly monotonically increasing functions. Consider the nonlinear algebraic equation

$$F(x) + Ax = y. \tag{1}$$

Then, (1) possesses a unique solution in x for each y if A is positive semidefinite.

A. Network model

We consider a generic meshed microgrid and assume that loads are modeled by constant impedances. This leads to a set of nonlinear differential-algebraic equations (DAE). Then, a network reduction (called Kron-reduction [6]) is carried out to eliminate all algebraic equations and to obtain a set of differential equations. We assume this process has been conducted and work with the Kron-reduced network.

In the reduced network, each node represents a DG unit interfaced via an AC inverter. The set of nodes of this network is denoted by $\mathcal{N}$. We associate a time-dependent phase angle $\delta_i : \mathbb{R}_{\geq 0} \to \mathbb{T}$ and a voltage amplitude $V_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ to each node $i \in \mathcal{N}$ in the microgrid. Two nodes i and k of the microgrid are connected via a complex admittance $Y_{ik} = Y_{ki} \in \mathbb{C}$. For convenience, we define $Y_{ik} := 0$ whenever i and k are not directly connected via an admittance. We denote the set of neighbors of a node $i \in \mathcal{N}$ by $\mathcal{N}_i := \{k \mid k \in \mathcal{N},\ k \neq i\ , Y_{ik} \neq 0\}$.

We assume that the power lines of the microgrid are *lossless*, i.e. all lines can be represented by purely inductive admittances. This may be justified as follows [8], [10]. In medium (MV) and low voltage (LV) networks the line impedance is usually not purely inductive, but has a non-negligible resistive part. On the other hand, the inverter output impedance is typically inductive (due to the output inductor and/or the possible presence of an output transformer). Under these circumstances, the inductive parts dominate the resistive parts in the admittances for some particular microgrids, especially on the MV level. We only consider such microgrids and absorb the inverter output admittances (together with possible transformer admittances) into the line admittances Y_{ik}, while neglecting all resistive effects.

Then, an admittance connecting two nodes i and k can be represented by $Y_{ik} := jB_{ik}$ with $B_{ik} \in \mathbb{R}_{<0}$. The representation of loads as constant impedances in the original network leads to shunt-admittances at at least some of the nodes in the Kron-reduced network, i.e. $\hat{Y}_{ii} = \hat{G}_{ii} + j\hat{B}_{ii} \neq 0$ for some $i \in \mathcal{N}$, where $\hat{G}_{ii} \in \mathbb{R}_{>0}$ is the shunt conductance and $\hat{B}_{ii} \in \mathbb{R}_{<0}$ denotes the shunt susceptance.

In this work, we focus on the reactive power flows. The reactive power flow $Q_{ik} : \mathbb{T}^2 \times \mathbb{R}^2_{>0} \to \mathbb{R}$ from node $i \in \mathcal{N}$ to node $k \in \mathcal{N}$ is given by [6]

$$\begin{aligned} &Q_{ik}(\delta_i(t), \delta_k(t), V_i(t), V_k(t)) = \\ &\qquad |B_{ik}|V_i^2(t) - |B_{ik}|V_i(t)V_k(t)\cos(\delta_i(t) - \delta_k(t)). \end{aligned} \tag{2}$$

Furthermore, we make use of the standard decoupling assumption, i.e. we assume that $\delta_i(t) - \delta_k(t) \approx 0$ and hence $\cos(\delta_i(t) - \delta_k(t)) \approx 1$, for all $t \geq 0$ and for $i \sim \mathcal{N}$, $k \sim \mathcal{N}_i$, see [6], [13][1]. Then, the reactive power flow $Q_i : \mathbb{R}^n_{>0} \to \mathbb{R}$ at a node $i \in \mathcal{N}$ is obtained as[2]

$$Q_i(V_1, \ldots, V_n) = |B_{ii}|V_i^2 - \sum_{k\sim\mathcal{N}_i} |B_{ik}|V_i V_k \tag{3}$$

with $B_{ii} := \hat{B}_{ii} + \sum_{k\sim\mathcal{N}_i} B_{ik}$. Hence,

$$|B_{ii}| \geq \sum_{k\sim\mathcal{N}_i} |B_{ik}|. \tag{4}$$

It follows from (3) that the reactive power Q_i can be controlled by controlling the voltage amplitudes V_i and V_k, $i \in \mathcal{N}$, $k \in \mathcal{N}$. This fact is used when designing a distributed voltage control for reactive power sharing in Section III-B.

The focus of this work is on generation units. Hence, we express all power flows in "Generator Reference-Arrow System".

B. Graph theory

Since the proposed voltage control is distributed, it requires communication among the generation units in the network. We employ a graph theoretic notation [22] to describe the high-level properties of the communication network.

An undirected graph of order n is a tuple $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{1, \ldots, n\}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, $\mathcal{E} := \{e_1, \ldots, e_m\}$ is the set of undirected edges. The l-th edge connecting nodes i and k is denoted as $e_l = \{i, k\} = \{k, i\}$. A node represents an individual agent, i.e. a generation source in the present case. The set of neighbors of a node i is denoted by $\mathcal{C}_i$ and contains all k for which $e_l = \{i, k\} \in \mathcal{E}$. If there is an edge between two nodes i and k, then i and k can exchange their local measurements with

[1] All our results also hold for arbitrary, but constant angle differences, i.e. $\delta_i(t) - \delta_k(t) = \delta_{ik}$, $\delta_{ik} \in \mathbb{T}$, at the cost of a more complex notation.

[2] To simplify notation the time argument of all signals is omitted from now on.

each other. We assume that the graph contains no self-loops, i.e. there is no edge $e_l = \{i, i\}$.

The $|\mathcal{V}| \times |\mathcal{V}|$ adjacency matrix $\mathcal{A}$ has entries $a_{ik} = a_{ki} = 1$ if an edge between i and k exists and $a_{ik} = 0$ otherwise. The degree of a node i is given by $d_i = \sum_{k=1}^{n} a_{ik}$. With $\mathcal{D} := \mathrm{diag}(d_i) \in \mathbb{R}^{n\times n}$, the Laplacian matrix of an undirected graph is given by $\mathcal{L} := \mathcal{D} - \mathcal{A}$ and is symmetric positive semidefinite [22].

A path in a graph is an ordered sequence of nodes such that any pair of consecutive nodes in the sequence is connected by an edge. $\mathcal{G}$ is called connected if for all pairs $(i, k) \in \mathcal{V} \times \mathcal{V}$, $i \neq k$, there exists a path from i to k. Given an undirected graph, zero is a simple eigenvalue of its Laplacian matrix $\mathcal{L}$ if and only if the graph is connected. Moreover, a corresponding right eigenvector to this simple zero eigenvalue is then $\underline{1}_n$, i.e. $\mathcal{L}\underline{1}_n = \underline{0}_n$ [22].

The nodes in the communication and in the electrical network are identical, i.e. $\mathcal{N} \equiv \mathcal{V}$. Note that the communication topology may, but does not necessarily have to, coincide with the topology of the electrical network, i.e. we may allow $\mathcal{C}_i \neq \mathcal{N}_i$ for any $i \in \mathcal{V}$.

III. Inverter modeling and distributed voltage control for reactive power sharing

A. Inverter model

We model the inverters as controllable AC voltage sources the amplitude and frequency of which can be defined by the designer [23].[3] Then, the inverter at the i-th node can be represented by [10], [24]

$$\begin{aligned} \dot{\delta}_i &= u_i^{\delta}, \\ \tau_{V_i}\dot{V}_i &= -V_i + u_i^{\mathrm{V}}, \end{aligned} \tag{5}$$

where $u_i^{\delta} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $u_i^{V} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ are controls and $\tau_{V_i} \in \mathbb{R}_{>0}$ is the time constant of a low-pass filter representing an input delay in the voltage. It is also assumed that the reactive power output is measured and processed through a low pass filter [7]

$$\tau_{P_i}\dot{Q}_i^m = -Q_i^m + Q_i, \tag{6}$$

where Q_i is given in (3) and $\tau_{P_i} \in \mathbb{R}_{>0}$ is the time constant of the filter. We furthermore associate to each inverter its power rating $S_i^N \in \mathbb{R}_{>0}$, $i \sim \mathcal{N}$.

Due to the decoupling assumption in II-A, we neglect the dynamics of δ_i in the following. Furthermore, in practice $\tau_{P_i} \gg \tau_{V_i}$, hence we assume $\tau_{V_i} = 0$. The model (5), (6) can then be reduced to

$$\begin{aligned} V_i &= u_i^{\mathrm{V}}, \\ \tau_{P_i}\dot{Q}_i^m &= -Q_i^m + Q_i, \end{aligned} \tag{7}$$

on which our further analysis is focused.

B. Distributed voltage control for reactive power sharing

We employ the following definition of proportional reactive power sharing.

Definition 3.1: Let $\chi_i \in \mathbb{R}_{>0}$ denote weighting factors and Q_i^s the steady-state reactive power flow, $i \sim \mathcal{N}$. Then, two inverters at nodes i and k are said to share their reactive powers proportionally according to χ_i and χ_k if

$$\frac{Q_i^s}{\chi_i} = \frac{Q_k^s}{\chi_k}.$$

Remark 3.2: From (7) it follows that in steady-state $\dot{Q}_i^m = 0$ and hence $Q_i^{m,s} = Q_i^s$, where the superscript s denotes signals in steady-state.

Remark 3.3: A practical choice for χ_i would, e.g. be $\chi_i = S_i^N$, where $S_i^N \in \mathbb{R}_{>0}$ is the nominal power rating of the i-th inverter.

Inspired by consensus-algorithms [25], we propose the following distributed voltage control (DVC) u_i^V for an inverter at node $i \in \mathcal{N}$

$$\begin{aligned} u_i^V &= V_i^d - k_i \int_0^t e_i(\tau)d\tau, \\ e_i &:= \sum_{k\sim\mathcal{C}_i} \left(\frac{Q_i^m}{\chi_i} - \frac{Q_k^m}{\chi_k}\right) = \sum_{k\sim\mathcal{C}_i} (\bar{Q}_i - \bar{Q}_k), \end{aligned} \tag{8}$$

where $V_i^d \in \mathbb{R}_{>0}$ is the desired (nominal) voltage amplitude and $k_i \in \mathbb{R}_{>0}$ a feedback gain. Furthermore, for convenience we have defined the weighted reactive power flows $\bar{Q}_i := Q_i^m/\chi_i$, $i \sim \mathcal{N}$. Recall that $\mathcal{C}_i$, cf. II-B, is the set of neighbor nodes of node i in the graph induced by the communication network, i.e. the set of nodes the i-th node can exchange information with. The control scheme is illustrated for the inverter at the i-th node in Fig. 1.

Remark 3.4: The proposed DVC (8) is a *distributed* control that requires communication infrastructure. Unlike [17], [18], the DVC does not require a central control and/or communication unit nor all-to-all communication. The only requirement on the communication topology is that the graph induced by the communication network is connected.

Remark 3.5: The usual voltage droop control for microgrids with inductive power lines is given by [1], [11]

$$u_i^V = V_i^d - k_{Q_i}(Q_i^m - Q_i^d), \tag{9}$$

where $k_{Q_i} \in \mathbb{R}_{>0}$ is the feedback (droop) gain and $Q_i^d \in \mathbb{R}$ the setpoint for the reactive power output of the i-th inverter. Opposed to the DVC (8), the control law (9) is decentralized, i.e. the feedback signal is the locally injected reactive power Q_i. It does therefore not require communication. However, it does not guarantee reactive power sharing [10], [12], [13].

Remark 3.6: In addition to reactive power sharing, it may be desired that the voltage amplitudes V_i remain within certain boundaries. With the above control law (8), where the voltage amplitudes are actuator signals, this can, e.g. be ensured by saturating the control signal u_i^V. For mathematical simplicity, we do not consider this in the present analysis.

Differentiating $V_i = u_i^V$ with respect to time and combining (7) and (8), yields the following closed-loop dynamics for the i-th node, $i \in \mathcal{N}$,

$$\begin{aligned} \dot{V}_i &= -k_i \sum_{k\sim\mathcal{C}_i} \left(\frac{Q_i^m}{\chi_i} - \frac{Q_k^m}{\chi_k}\right), \quad V_i(0) = V_i^d, \\ \tau_{P_i}\dot{Q}_i^m &= -Q_i^m + Q_i, \quad Q_i^m(0) =: Q_{i0}^m, \end{aligned} \tag{10}$$

[3] An underlying assumption to this model is that whenever the inverter connects an intermittent renewable generation source, e.g. a photovoltaic plant, to the network, it is equipped with some sort of storage (e.g., battery). Thus, it can increase and decrease its power output in a certain range.

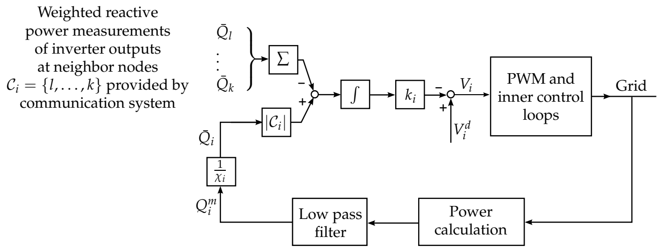


Fig. 1: Block diagram of the DVC (8) for an inverter at node $i \in \mathcal{N}$. V_i is the voltage amplitude, V_i^d its desired value, Q_i^m the measured reactive power and $\bar{Q}_i$ the weighted reactive power, where χ_i is the weighting coefficient to ensure proportional reactive power sharing and k_i is a feedback gain.

and the interaction between nodes is modeled by (3). Note that $V_i(0)$ is determined by the control law (8). By recalling from II-B that $\mathcal{L} \in \mathbb{R}^{n\times n}$ is the Laplacian matrix of the communication network and defining the matrices

$$T := \mathrm{diag}(\tau_{P_i}) \in \mathbb{R}^{n\times n},\ D := \mathrm{diag}(1/\chi_i) \in \mathbb{R}^{n\times n},$$
$$K := \mathrm{diag}(k_i) \in \mathbb{R}^{n\times n},$$

as well as the column vectors $V \in \mathbb{R}^n_{>0}, Q \in \mathbb{R}^n$ and $Q^m \in \mathbb{R}^n$

$$V := \mathrm{col}(V_i), \quad Q := \mathrm{col}(Q_i), \quad Q^m := \mathrm{col}(Q_i^m), \quad (11)$$

the complete closed-loop system dynamics can be written compactly in matrix form as

$$\begin{aligned} \dot{V} &= -K\mathcal{L}DQ^m, \\ T\dot{Q}^m &= -Q^m + Q, \end{aligned} \qquad (12)$$

where $Q_i = Q_i(V)$ is given by (3).

IV. Stability and reactive power sharing

We start by proving that the proposed DVC (8) does indeed guarantee proportional reactive power sharing in steady-state.

Claim 4.1: The DVC (8) achieves proportional reactive power sharing in steady-state in the sense of Definition 3.1.

Proof: Set $\dot{V} = 0$ in (12). Note that, since $\mathcal{L}$ is the Laplacian matrix of an undirected connected graph, it has a simple zero eigenvalue with a corresponding right eigenvector $\beta \underline{1}_n$, $\beta \in \mathbb{R}\setminus\{0\}$. All its other eigenvalues are positive real. Moreover, K is a diagonal matrix with positive diagonal entries and from (12) in steady-state $Q^s = Q^{m,s}$. Hence, for $\beta \in \mathbb{R}\setminus\{0\}$ and $i \sim \mathcal{N}$, $k \sim \mathcal{N}$

$$\underline{0}_n = -K\mathcal{L}DQ^s \Leftrightarrow DQ^s = \beta \underline{1}_n \Leftrightarrow \frac{Q_i^s}{\chi_i} = \frac{Q_k^s}{\chi_k}. \quad (13)$$

■

To analyze properties of equilibria of the system (12), (3), we make the following assumption.

Assumption 4.2: For every $Q^{m,s} = Q^s(V^s) \in \mathbb{R}^n_{>0}$ satisfying the steady-state condition (13), there exists a $V^s \in \mathbb{R}^n_{>0}$ satisfying (3), $i \sim \mathcal{N}$.

Remark 4.3: Because of (13) all entries of $Q^{m,s} = Q^s(V^s)$ must have the same sign. Since we consider networks with inductive power lines and loads, only $Q^{m,s} = Q^s(V^s) \in \mathbb{R}^n_{>0}$ is practically relevant.

The next result characterizes uniqueness properties of equilibria of the system (12), (3).

Proposition 4.4: Consider the system (12), (3) satisfying Assumption 4.2. Then to each positive vector of reactive power flows Q^s there exists a unique positive vector of voltage amplitudes V^s.

Proof: Assume $Q^s \in \mathbb{R}^n_{>0}$ given. Because of

$$Q_i^s = |B_{ii}|{V_i^s}^2 - \sum_{k\sim\mathcal{N}_i} |B_{ik}|V_i^s V_k^s, \quad i \sim \mathcal{N}, \qquad (14)$$

no element V_i^s can be zero. Hence, (14) can be rewritten as

$$-\frac{Q_i^s}{V_i^s} + |B_{ii}|V_i^s - \sum_{k\sim\mathcal{N}_i} |B_{ik}|V_k^s = 0, \quad i \sim \mathcal{N},$$

or, more compactly, $F(V^s) + RV^s = \underline{0}_n$, where $F(V^s) := \mathrm{col}(-Q_i^s/V_i^s)$ and $R \in \mathbb{R}^{n\times n}$ with entries $r_{ii} := |B_{ii}|$, $r_{ik} := -|B_{ik}|$, $i \neq k$. Clearly, for $V_i^s > 0$, the expression $(-Q_i^s/V_i^s)$ is strictly monotonically increasing in V_i^s and because of (4) R is positive semidefinite. Uniqueness then follows from Lemma 2.2. ■

We proceed by establishing a condition for local stability of equilibria of the system (12), (3). To this end, we make the following important observation. Recall that $\underline{1}_n^T\mathcal{L} = \underline{0}_n^T$. Hence, multiplying the first equation in (12) from the left with $\underline{1}_n^T K^{-1}$ yields

$$\underline{1}_n^T K^{-1}\dot{V} = \underline{0}_n^T DQ^m \quad \Rightarrow \quad \sum_{i=1}^n \frac{\dot{V}_i}{k_i} = 0. \qquad (15)$$

Consequently, by integrating (15), the motion of an arbitrary voltage V_i, $i \in \mathcal{N}$, can be expressed in terms of all other voltages V_k, $k \sim \mathcal{N}\setminus\{i\}$ for all $t \geq 0$. This implies that studying the stability properties of equilibra of the system (12), (3) with dimension $2n$, is equivalent to studying the stability properties of corresponding equilibria of a reduced system of dimension $2n-1$. For ease of notation and without loss of generality, we define the reduced voltage vector as

$$V_R := \mathrm{col}(V_i) \in \mathbb{R}^{n-1}_{>0} \qquad (16)$$

and choose to express V_n by integrating (15) as

$$\begin{aligned} V_n = V_n(V(0), V_R) &= \sum_{i=1}^n \frac{V_i(0)}{k_i} - \sum_{i=1}^{n-1} \frac{k_n}{k_i} V_i \\ &= \sum_{i=1}^n \frac{V_i^d}{k_i} - \sum_{i=1}^{n-1} \frac{k_n}{k_i} V_i, \end{aligned} \qquad (17)$$

since $V_i(0) = V_i^d$, $i \sim \mathcal{N}$. We denote the reactive power flows in the reduced coordinates by

$$\mathcal{Q}_i(V_1, \ldots, V_{n-1}) = |B_{ii}|V_i^2 - \sum_{k\sim\mathcal{N}_i} |B_{ik}|V_i V_k, \qquad (18)$$

$i \sim \mathcal{N}$, with V_n given by (17). By defining the matrix $\mathcal{L}_R$

$$\mathcal{L}_R := \begin{bmatrix} I_{n-1} & \underline{0}_{n-1} \end{bmatrix} K\mathcal{L} \in \mathbb{R}^{(n-1)\times n}, \qquad (19)$$

the system (12), (3) can be written in the reduced coordinates $\mathrm{col}(V_R, Q^m) \in \mathbb{R}^{n-1}_{>0} \times \mathbb{R}^n$ as

$$\begin{aligned} \dot{V}_R &= -\mathcal{L}_R DQ^m, \\ T\dot{Q}^m &= -Q^m + \mathcal{Q}, \end{aligned} \qquad (20)$$

with $\mathcal{Q} := \mathrm{col}(\mathcal{Q}_i) \in \mathbb{R}^n$ and $\mathcal{Q}_i$, $i \sim \mathcal{N}$, given in (18). It follows from (17) that

$$\frac{\partial V_n(V_1, \ldots, V_{n-1})}{\partial V_i} = -\frac{k_n}{k_i}, \quad i \sim \mathcal{N}\setminus\{n\}.$$

Consequently,

$$\frac{\partial \mathcal{Q}_k}{\partial V_i} = \frac{\partial Q_k}{\partial V_i} - \frac{k_n}{k_i}\frac{\partial Q_k}{\partial V_n}, \quad i \sim \mathcal{N}\setminus\{n\}. \qquad (21)$$

Let $x^s := \mathrm{col}(V_R^s, Q^{m,s})$ be the corresponding equilibrium point to $\mathrm{col}(V^s, Q^{m,s})$ defined in Assumption 4.2. By introducing the matrix

$$N := \frac{\partial Q}{\partial V}\Big|_{V^s} \in \mathbb{R}^{n\times n}$$

with entries (use (3))

$$n_{ii} := 2|B_{ii}|V_i^s - \sum_{k\sim\mathcal{N}_i} |B_{ik}|V_k^s,\ n_{ik} := -|B_{ik}|V_i^s, i \neq k, \quad (22)$$

as well as the matrix $\mathcal{R} \in \mathbb{R}^{n\times(n-1)}$

$$\mathcal{R} := \begin{bmatrix} I_{(n-1)} \\ -b^T \end{bmatrix}, \quad b := \mathrm{col}\left(\frac{k_n}{k_1}, \dots, \frac{k_n}{k_{n-1}}\right), \quad (23)$$

and making use of (21), it follows that

$$\frac{\partial \mathcal{Q}}{\partial V_R}\Big|_{V_R^s} = N\mathcal{R}. \quad (24)$$

To derive an analytic stability condition, it is convenient to assume identical low pass filter time constants.

Assumption 4.5: The time constants of the low pass filters in (12) are chosen such that $\tau = \tau_{P_1} = \ldots = \tau_{P_n}$.

Furthermore, we define the deviations of the system variables with respect to x^s as

$$\tilde{V}_R := V_R - V_R^s \in \mathbb{R}^{n-1}, \quad \tilde{Q}^m := Q^m - Q^{m,s} \in \mathbb{R}^n.$$

By making use of (24) together with Assumption 4.5, linearizing the system (20), (18) at x^s yields

$$\begin{bmatrix} \dot{\tilde{V}}_R \\ \dot{\tilde{Q}}^m \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{(n-1)\times(n-1)} & -\mathcal{L}_R D \\ \frac{1}{\tau} N\mathcal{R} & -\frac{1}{\tau} I_n \end{bmatrix}}_{:=A} \begin{bmatrix} \tilde{V}_R \\ \tilde{Q}^m \end{bmatrix}. \quad (25)$$

The following two relations are helpful to establish our claim

$$\begin{aligned} \mathcal{R}\mathcal{L}_R &= \mathcal{R}\begin{bmatrix} I_{n-1} & \underline{0}_{n-1} \end{bmatrix} K\mathcal{L} \\ &= K \begin{bmatrix} I_{n-1} & \underline{0}_{n-1} \\ -\underline{1}_{n-1}^T & 0 \end{bmatrix} \mathcal{L} = K\mathcal{L}, \end{aligned} \quad (26)$$

and

$$\mathcal{R}^T K^{-1} \underline{1}_n = \underline{0}_{n-1}. \quad (27)$$

Lemma 4.6: For $Q^s, V^s \in \mathbb{R}^n_{>0}$, all eigenvalues of N have positive real part.

Proof: Dividing (14) by $V_i^s > 0$ yields

$$\frac{Q_i^s}{V_i^s} = |B_{ii}|V_i^s - \sum_{k\sim\mathcal{N}_i} |B_{ik}|V_k^s > 0. \quad (28)$$

Furthermore, from (4) it follows that

$$|B_{ii}|V_i^s \geq \sum_{k\sim\mathcal{N}_i} |B_{ik}|V_i^s. \quad (29)$$

Hence, with n_{ii} and n_{ik} defined in (22), we have that

$$n_{ii} = 2|B_{ii}|V_i^s - \sum_{k\sim\mathcal{N}_i} |B_{ik}|V_k^s > |B_{ii}|V_i^s \geq \sum_{k\sim\mathcal{N}\setminus\{i\}} |n_{ik}|.$$

Therefore, N is a diagonally dominant matrix with positive diagonal elements and the claim follows from Gershgorin's disc theorem [20]. ■

Lemma 4.7: For $Q^s, V^s \in \mathbb{R}^n_{>0}$, the matrix product $ND\mathcal{L}D$ has a zero eigenvalue with geometric multiplicity one and a corresponding right eigenvector $\beta D^{-1}\underline{1}_n$, $\beta \in \mathbb{R}\setminus\{0\}$; all other eigenvalues have positive real part.

Proof: The matrix D is diagonal with positive diagonal entries and hence positive definite. Furthermore, $\mathcal{L}$ is the Laplacian matrix of an undirected connected graph and therefore positive semidefinite. In addition, $\mathcal{L}$ has a simple zero eigenvalue with a corresponding right eigenvector $\beta\underline{1}_n$, $\beta \in \mathbb{R}\setminus\{0\}$ and Lemma 4.6 implies that N is nonsingular. Hence, $ND\mathcal{L}D$ has a zero eigenvalue with geometric multiplicity one and a corresponding right eigenvector $\beta D^{-1}\underline{1}_n$, $\beta \in \mathbb{R}\setminus\{0\}$. In addition, $D\mathcal{L}D$ is positive semidefinite and by Lemma 2.1 it follows that

$$\sigma(ND\mathcal{L}D) \subseteq W(N)W(D\mathcal{L}D).$$

The aforementioned properties of D and $\mathcal{L}$ imply that $W(D\mathcal{L}D) \subseteq \mathbb{R}_{\geq 0}$. To prove that all eigenvalues apart from the zero eigenvalue have positive real part, we show that $\Re(W(N)) \subseteq \mathbb{R}_{>0}$. This also implies that the only element of the imaginary axis in $W(N)W(D\mathcal{L}D)$ is the origin. To see this, we recall that the real part of the numerical range of N is given by the range of its symmetric part, i.e.

$$\Re(W(N)) = W\left(\frac{1}{2}\left(N + N^T\right)\right).$$

The symmetric part of N has entries

$$\bar{n}_{ii} := n_{ii}, \quad \bar{n}_{ik} := -\frac{1}{2}|B_{ik}|(V_i^s + V_k^s),$$

where n_{ii} is defined in (22). From (28) it follows that

$$|B_{ii}|V_i^s > \sum_{k\sim\mathcal{N}_i} |B_{ik}|V_k^s.$$

Hence, together with (29), it follows that

$$|B_{ii}|V_i^s > \frac{1}{2}\sum_{k\sim\mathcal{N}_i} |B_{ik}|(V_i^s + V_k^s) = \sum_{k\sim\mathcal{N}\setminus\{i\}} |\bar{n}_{ik}|$$

and

$$\bar{n}_{ii} = 2|B_{ii}|V_i^s - \sum_{k\sim\mathcal{N}_i} |B_{ik}|V_k^s > |B_{ii}|V_i^s > \sum_{k\sim\mathcal{N}\setminus\{i\}} |\bar{n}_{ik}|.$$

Consequently, the symmetric part of N is diagonally dominant with positive diagonal entries and by Gershgorin's disc theorem its eigenvalues are all positive real. ■

We are now ready to state our main result.

Proposition 4.8: Consider the system (12), (3) satisfying Assumption 4.2. Fix D and $\tau > 0$. Select $\tau_{P_i} = \tau$, $i \sim \mathcal{N}$ and $K = D$. Denote by $x^s = \mathrm{col}(V_R^s, Q^{m,s}) \in \mathbb{R}^{2n-1}_{>0}$ the corresponding equilibrium point of the reduced system (20), (18). Let $\mu_i = a_i + jb_i$ be the i-th nonzero eigenvalue of the matrix product $ND\mathcal{L}D$ with $a_i \in \mathbb{R}$ and $b_i \in \mathbb{R}$. Then, x^s is an exponentially stable equilibrium point of the system (20), (18) if and only if

$$\tau b_i^2 < a_i. \quad (30)$$

for all μ_i. Moreover, x^s is exponentially stable for any $\tau > 0$ if and only if $ND\mathcal{L}D$ has only real eigenvalues.

Proof: We have just shown that with $\tau_{P_i} = \tau$, $i \sim \mathcal{N}$, the linear system (25) locally represents the microgrid dynamics (12), (18). The proof is thus given by deriving the spectrum of A defined in (25). Let λ be an eigenvalue of A with a corresponding right eigenvector $v = \mathrm{col}(v_1, v_2)$, $v_1 \in \mathbb{C}^{n-1}$, $v_2 \in \mathbb{C}^n$. Then,

$$-\mathcal{L}_R D v_2 = \lambda v_1, \qquad N\mathcal{R}v_1 - v_2 = \tau\lambda v_2. \quad (31)$$

We first prove by contradiction that zero is not an eigenvalue of A. Therefore, assume $\lambda = 0$. Then,

$$\mathcal{L}_R D v_2 = \underline{0}_{n-1}. \quad (32)$$

From the definition of $\mathcal{L}_R$ given in (19) it follows that (32) can only be satisfied if

$$K\mathcal{L}Dv_2 = \mathrm{col}(\underline{0}_{n-1}, a), \quad a \in \mathbb{C}.$$

The fact that $\mathcal{L} = \mathcal{L}^T$ together with $\mathcal{L}\underline{1}_n = \underline{0}_n$ implies that $\underline{1}_n^T K^{-1} K\mathcal{L}Dv = 0$ for any $v \in \mathbb{C}^n$. Therefore,

$$\underline{1}_n^T K^{-1} K\mathcal{L}Dv_2 = \underline{1}_n^T K^{-1}\mathrm{col}(\underline{0}_{n-1}, a) = 0.$$

Hence, a must be zero. Consequently, $v_2 = \beta D^{-1}\underline{1}_n$, $\beta \in \mathbb{R}$. Inserting $\lambda = 0$ and $v_2 = \beta D^{-1}\underline{1}_n$ in (31) and recalling $K = D$ yields

$$N\mathcal{R}v_1 = \beta D^{-1}\underline{1}_n = \beta K^{-1}\underline{1}_n. \tag{33}$$

Premultiplying with $v_1^*\mathcal{R}^T$ gives, because of (27),

$$v_1^*\mathcal{R}^T N\mathcal{R}v_1 = 0.$$

The proof of Lemma 4.7 implies $\Re(W(N)) \subseteq \mathbb{R}_{>0}$. Hence,

$$\mathcal{R}v_1 = \underline{0}_n. \tag{34}$$

Consequently, because of (33), $\beta = 0$ and $v_2 = \underline{0}_n$. Finally, because of (23), (34) implies $v_1 = \underline{0}_{n-1}$. Hence, (31) can only hold for $\lambda = 0$ if $v_1 = \underline{0}_{n-1}$ and $v_2 = \underline{0}_n$. Therefore, zero is not an eigenvalue of A.

To establish conditions under which all eigenvalues of A have negative real part, notice that, for $\lambda \neq 0$, (31) can be rewritten as

$$\tau\lambda^2 v_2 + \lambda v_2 + N\mathcal{R}\mathcal{L}_R Dv_2 = \underline{0}_n.$$

Recalling (26) and $K = D$ yields

$$\tau\lambda^2 v_2 + \lambda v_2 + ND\mathcal{L}Dv_2 = \underline{0}_n. \tag{35}$$

This implies that v_2 must be an eigenvector of $ND\mathcal{L}D$. Recall that Lemma 4.7 implies that $ND\mathcal{L}D$ has a zero eigenvalue with geometric multiplicity one and all its other eigenvalues have positive real part. For $ND\mathcal{L}Dv_2 = \underline{0}_n$, (35) has solutions $\lambda = 0$ and $\lambda = -1/\tau$. Recall that zero is not an eigenvalue of A. Hence, we have $\lambda_1 = -1/\tau$ as first eigenvalue (with unknown algebraic multiplicity) of A.

Denote the remaining[4] eigenvalues of $ND\mathcal{L}D$ by $\mu_i \in \mathbb{C}$. Let a corresponding right eigenvector be given by $w_i \in \mathbb{C}^n$, i.e. $ND\mathcal{L}Dw_i = \mu_i w_i$. Without loss of generality, choose w_i such that $w_i^* w_i = 1$. By multiplying (35) from the right with w_i^*, the remaining $0 \leq m \leq 2n-2$ eigenvalues of A are the solutions $\lambda_{i_{1,2}}$ of

$$\tau\lambda_{i_{1,2}}^2 + \lambda_{i_{1,2}} + \mu_i = 0. \tag{36}$$

First, consider real nonzero eigenvalues, i.e. $\mu_i = a_i$ with $a_i > 0$. Then, clearly, both solutions of (36) have negative real parts, e.g. by the Hurwitz condition. Next, consider complex eigenvalues of $ND\mathcal{L}D$, i.e. $\mu_i = a_i + jb_i$, $a_i > 0, b_i \in \mathbb{R}\setminus\{0\}$. Then, from (36) we have

$$\lambda_{i_{1,2}} = \frac{1}{2\tau}\left(-1 \pm \sqrt{1 - 4\tau(a_i + jb_i)}\right). \tag{37}$$

We define $\alpha_i := 1 - 4a_i\tau$, $\beta_i := -4b_i\tau$ and recall that the roots of a complex number $\sqrt{\alpha_i + j\beta_i}, \beta_i \neq 0$, are given by $\pm(\psi_i + j\nu_i)$, $\psi_i \in \mathbb{R}$, $\nu_i \in \mathbb{R}$, [26] with

$$\psi_i = \left(0.5\left(\alpha_i + \sqrt{\alpha_i^2 + \beta_i^2}\right)\right)^{-0.5}.$$

[4]Neither the algebraic multiplicities of the eigenvalues of $NDLD$ nor the geometric multiplicities of its nonzero eigenvalues are known in the present case. However, this information is not required, since it suffices to know that $\Re(\sigma(NDLD)) \subseteq \mathbb{R}_{\geq 0}$. This fact has been proven in Lemma 4.7.

Thus, both solutions $\lambda_{i_{1,2}}$ in (37) have negative real parts if and only if

$$\left(0.5\left(\alpha_i + \sqrt{\alpha_i^2 + \beta_i^2}\right)\right)^{-0.5} < 1 \Leftrightarrow \sqrt{\alpha_i^2 + \beta_i^2} < 2 - \alpha_i.$$

Inserting α_i and β_i gives

$$\sqrt{(1 - 4a_i\tau)^2 + 16b_i^2\tau^2} < 1 + 4a_i\tau,$$

where the right hand side is positive. The condition is therefore equivalent to

$$b_i^2\tau < a_i,$$

which is condition (30). Hence, A is Hurwitz if and only if (30) holds for all μ_i. Finally, the equilibrium point x^s is locally exponentially stable if and only if A is Hurwitz [27]. ∎

V. Simulation Example

The performance of the proposed DVC (8) is now demonstrated and compared with the usual voltage droop control (9) via a simulation example based on the inner ring of the islanded Subnetwork 1 of the CIGRE benchmark MV distribution network [28]. In particular, we show the ability of the DVC (8) to quickly achieve a desired reactive power distribution after changes in the load.

The network consists of eight main buses and is shown in Fig. 2. We assume that the generation sources at buses 9b, 9c, 10b and 10c are operated with the DVC (8), respectively the droop control (9). The remaining sources are operated in PQ-mode. The distributed communication network is also depicted in Fig. 2. Note that the communication is not all-to-all and that there is no central unit. The simulations are carried out in Plecs [29].

We associate to each inverter a power rating $S^N = [0.517, 0.353, 0.333, 0.023]$ pu, where pu denotes per unit values with respect to the base power $S_{\text{base}} = 3$ MVA. For the DVC (8), we select a multiple of the nominal power rate of each source as weighting coefficient, i.e. $\chi_i = 5S_i^N$, $i \sim \mathcal{N}$ (cf. Remark 3.3) and, following Proposition 4.8, we set $K = D$. On the basis of selection criteria for frequency-active power droop [10], the parameters of the control (9) are set to $Q_i^d = 0.2S_i^N$ pu and $k_{Q_i} = 0.1/S_i^N$ pu/pu. For both controls, we set $V_i^d = 1$ pu. To satisfy Assumption 4.5, the low pass filter time constants are set to $\tau_{P_i} = 0.2$ s, $i \sim \mathcal{N}$. The frequencies of the inverters are controlled with the usual frequency droop control, see e.g. [10], [11].

We consider the following scenario: at first, the system is operated under nominal loading conditions. Then, at $t = 0.5$ s, there is an increase in load at buses 3 and 9. We compare the reactive power outputs and voltage trajectories of the inverters under the controls (8), respectively (9).

The simulation results are shown for the system (7) operated with the DVC (8) in Fig. 3a and with the droop control (9) in Fig. 3b. The system quickly reaches a steady-state under both controls after the load change at $t = 0.5$ s. However, under the droop control (9), the reactive power is not shared by all inverters in proportion of their ratings. On the contrary and as predicted, the DVC (8) does achieve a desired reactive power distribution in steady-state.

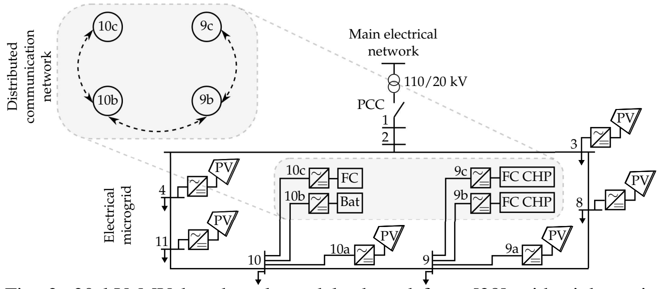


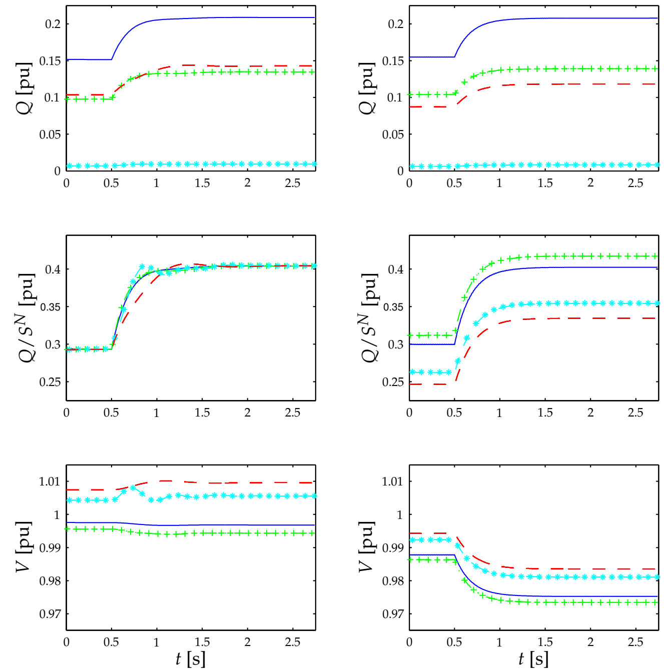
Fig. 2: 20 kV MV benchmark model adapted from [28] with eight main buses and generation sources of type: PV-Photovoltaic, FC-fuel cell, Bat-battery, CHP-CHP fuel cell. The symbol ↓ denotes a load and PCC denotes the point of common coupling to the main grid.

Moreover, compared to the droop control (9), the voltage levels remain close to the nominal value $V^d = 1$. This becomes especially obvious from the voltage trajectories after the load step at $t = 0.5$ s, where all voltages are decreased under the droop control (9). Here, the DVC (8) merely causes small variations in the voltage amplitudes in order to satisfy the increased reactive power demand by the loads. This is an indication that no secondary voltage control may be necessary when operating the inverters with the DVC (8) – a clear advantage over the droop control (9).

Local stability under the control (8) is confirmed for both operating points via Proposition 4.8.

VI. Conclusion

We have proposed a consensus-based distributed voltage control (DVC), which solves the problem of reactive power sharing in inverter-based microgrids with inductive power lines. Unlike the widely used voltage droop control [1], [11], the DVC does guarantee a desired reactive power distribution in steady-state. Moreover, we have provided a necessary and sufficient condition for local exponential stability. The performance gain in terms of power sharing compared to the usual voltage droop control has been demonstrated in a simulation example. Future research will address the analysis of microgrids, in which the generation units are equipped with frequency droop control together with the DVC.



(a) Trajectories of system (7) operated with DVC (8)

(b) Trajectories of system (7) operated with voltage droop control (9)

Fig. 3: Comparison of DVC and droop control with absolute power outputs Q_i in pu, power outputs relative to source ratings Q_i/S_i^N and voltage amplitudes V_i in pu corresponding to: FC CHP 9b $i = 1$ '–', FC CHP 9c $i = 2$ '- -', battery 10 b $i = 3$ '- +', FC 10c $i = 4$ '-*'.

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