Time-to-build and the Capital Structure*

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Abstract

Choosing the numerical value for the time-to-build parameter in models with one capital good is not trivial because capital includes for example stock of plant, equipment and consumer durables, each of them characterized by different gestation lags. In this paper we shed some light on this issue by proving under which conditions, the long-run dynamics of a Ramsey model with one capital good and of the same model with N capital goods with heterogeneous gestation lags are equivalent.

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1 Introduction

Since the seminal article of Kydland and Prescott [5], several contributions have investigated models where capital takes time to be built. These contributions have provided a sound understanding of the qualitative dynamics of exogenous growth (e.g. Asea and Zak [1]) and endogenous growth (e.g. Bambi [2], Bambi et al. [3], Collard et al. [4]) models with time-to-build.

On the other hand, a quantitative analysis of these models is indeed far from obvious. To explain this point, it is illustrative to mention the following argument in Kydland and Prescott ([5], page 1361) "Capital for our model reflects all tangible capital, including stocks of plant and equipment, consumer durables and housing. (...) Different types of capital have different construction periods and patterns of resource requirements. (...) Having but one type of capital, we assume, as a compromise, that four quarters are required." In other words, Kydland and Prescott choose, in their calibration, a value of the gestation lag by "averaging" across the different gestation lags of the different types of capital included in tangible capital.

While such a "compromise" seems sound and reasonable, it is, nevertheless, based on the implicit assumption that a model with one capital good and time-to-build can be analogous (at least in its long-run dynamics) to a model with multiple capital goods and heterogeneous lags. However, this analogy is not proved, but rather assumed, and it is not clear under which conditions, if any, it may indeed hold. Furthermore, it is probably even less clear that such a connection may still exist in an AK endogenous growth model since the presence of constant

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return to scale in the accumulating factor of production could imply all the resources to be invested in the capital good with the lowest time-to-build.

In this contribution, we address these issues and, in particular, we identify conditions under which the long-run dynamics of an economy with one capital good and time-to-build is equal to that one of another economy with N capital goods and heterogeneous gestation lags. Most importantly, we provide an analytical ground to the Kydland and Prescott's suggestion by showing that one of these conditions requires the gestation lag parameter in the model with one capital good to be a weighted average of the gestation lags in the model with multiple capital goods.

2 A Ramsey model with one capital good and time-to-build.

Consider the social planner version of a Ramsey model with time-to-build:

$$\max_{\tilde{c}(t)} \int_0^\infty e^{-\rho t} \log \tilde{c}(t) dt \qquad (P1)$$

subject to

$$Z\tilde{k}(t)^\alpha = \tilde{c}(t) + \tilde{i}(t) \tag{1}$$

$$\tilde{i}(t) = \dot{\tilde{k}}(t+d) \tag{2}$$

All the variables appear with a $\tilde{}$ to differentiate them from those in the economy described in the next section. Furthermore $Z > 0$ and $d > 0$ indicate the level of technology and the (finite) time-to-build parameter respectively. Using a modified version of the Pontryagin's Maximum Principle, we may write the Hamiltonian $\mathcal{H} = e^{-\rho t} \log \tilde{c}(t) + \varrho(t) \left[Z\tilde{k}(t-d)^\alpha - \tilde{c}(t-d) \right]$ where $\varrho(t)$ is the costate variable, and derive the first order conditions:

$$\frac{1}{\tilde{c}(t)} e^{-\rho t} = \varrho(t+d) \tag{3}$$

$$\varrho(t+d)\alpha Z\tilde{k}(t)^{\alpha-1} = -\dot{\varrho}(t) \tag{4}$$

An optimal plan is defined as it follows:

Definition 1 *An optimal plan is any path $(\tilde{c}(t), \tilde{k}(t))_{t\geq 0}$ which solves the capital accumulation equation*

$$\dot{\tilde{k}}(t+d) = Z\tilde{k}(t)^\alpha - \tilde{c}(t) \tag{5}$$

the Euler equation

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \alpha Z \frac{\tilde{c}(t)}{\tilde{c}(t+d)} e^{-\rho d} \tilde{k}(t+d)^{\alpha-1} - \rho \tag{6}$$

a standard transversality condition, given the initial history of capital.

3 A N capital goods with heterogeneous gestation lags model

Consider now an economy with N capital goods used as inputs to produce a final good through a constant return to scale Cobb-Douglas production function. In intensive form, the production function, the resource constraint and the net investments write respectively

$$y(t) = A\prod_{j=1}^{N} x_j(t)^{\alpha_j}, \qquad y(t) = c(t) + \sum_{j=1}^{N} i_j(t) \qquad and \qquad i_j(t) = \dot{x}_j(t+d_j)$$

with $\sum_{j=1}^{N} \alpha_j = \alpha \leq 1$, and $d_j > 0$ indicating the time-to-build of capital good j. Consistently with what previously done, we assume $\delta_j = 0$ for any j.[1] The initial conditions of the capital goods are also given. The social planner problem of this economy is:

$$\max_{c(t)} \int_0^{\infty} e^{-\rho t} \log c(t) dt \qquad (P2)$$

subject to

$$A\prod_{j=1}^{N} x_j(t)^{\alpha_j} = c(t) + \sum_{j=1}^{N} i_j(t) \tag{7}$$

$$i_j(t) = \dot{x}_j(t+d_j) \qquad \forall j = 1, ..., N \tag{8}$$

whose Hamiltonian is

$$\mathcal{H} = e^{-\rho t} \log c(t) + \sum_{j=1}^{N} v_j(t) i_j(t-d_j) + w(t)\left[A\prod_{j=1}^{N} x_j(t)^{\alpha_j} - c(t) - \sum_{j=1}^{N} i_j(t)\right]$$

where $v_j(t)$, with $j = 1, ..., N$ and $w(t)$ are the multipliers. The first order conditions with respect to $c(t)$, $i_j(t)$ and $x_j(t)$ are respectively:

$$\frac{1}{c(t)} e^{-\rho t} = w(t) \tag{9}$$

$$v_j(t+d_j) = w(t) \qquad \forall j = 1, ..., N \tag{10}$$

$$w(t)\alpha_j x_j(t)^{-1} A \prod_{j=1}^{N} x_j(t)^{\alpha_j} = -\dot{v}_j(t) \qquad \forall j = 1, ..., N \tag{11}$$

The following transversality conditions are also required for the Maximum Principle to hold:

$$\lim_{t\to\infty} \frac{x_j(t)}{c(t)} e^{-\rho(t-d_j)} = 0 \qquad \forall j = 1, ..., N$$

Considering two generic types of capital, $x_m(t)$ and $x_\kappa(t)$, it follows immediately by combining the last two equations that

$$\frac{\alpha_m x_\kappa(t)}{\alpha_\kappa x_m(t)} = \frac{\dot{w}(t-d_m)}{\dot{w}(t-d_\kappa)} \tag{12}$$

and, therefore, the production function can be rewritten as follows

$$y(t) = A\prod_{j=1}^{N} x_j(t)^{\alpha_j} = \left[A\prod_{j=1}^{N}\left(\frac{\alpha_j \dot{w}(t-d_m)}{\alpha_m \dot{w}(t-d_j)}\right)^{\alpha_j}\right] x_m(t)^{\alpha} \tag{13}$$

while the dynamics of the costate variables is described by the following equation

$$\alpha_m \left[A\prod_{j=1}^{N}\left(\frac{\alpha_j \dot{w}(t-d_m)}{\alpha_m \dot{w}(t-d_j)}\right)^{\alpha_j}\right] x_m(t)^{\alpha-1} = -\frac{\dot{w}(t-d_m)}{w(t)} \tag{14}$$

Therefore the Euler equation can be obtained combining (9) with (14).

[1] All the results still hold as long as $\delta_j = \delta$ for any j.

Definition 2 *An optimal plan is any path $(c(t), x_m(t))_{t\geq 0}$ which solves the Euler equation*

$$g_c(t) = \alpha_m \frac{c(t)}{c(t+d_m)} e^{-\rho d_m} \left[A \prod_{j=1}^{N} \left(\frac{\alpha_j \cdot (g_c(t)+\rho)\cdot c(t-d_j+d_m)}{\alpha_m \cdot (g_c(t-d_j+d_m)+\rho)\cdot c(t)} e^{\rho(d_m-d_j)} \right)^{\alpha_j} \right] x_m(t+d_m)^{\alpha-1} - \rho \qquad (15)$$

the capital accumulation equation

$$A x_m^{\alpha}(t) \prod_{j=1}^{N} \left[\frac{\alpha_j \cdot (g_c(t-d_m)+\rho)\cdot c(t-d_j)}{\alpha_m \cdot (g_c(t-d_j)+\rho)\cdot c(t-d_m)} e^{\rho(d_m-d_j)} \right]^{\alpha_j} = \sum_{j=1}^{N} \frac{\alpha_j \cdot (g_c(t-d_m+d_j)+\rho)\cdot c(t)}{\alpha_m \cdot (g_c(t)+\rho)\cdot c(t-d_m+d_j)} e^{\rho(d_m-d_j)} \cdot$$

$$x_m(t+d_j) \left[g_{x_m}(t+d_j) + \left(\frac{\dot{g}_c(t-d_m+d_j)}{g_c(t-d_m+d_j)+\rho} - g_c(t-d_m+d_j) - \frac{\dot{g}_c(t)}{g_c(t)+\rho} + g_c(t) \right) \right] + c(t) \qquad (16)$$

and the transversality conditions, given the initial history of capitals. In the equations above we have used the notation $g_z(t) = \frac{\dot{z}(t)}{z(t)}$, with $z = c, x_m$. Furthermore, the optimal path of all the other capital goods $x_\kappa(t)$ with $\kappa \neq m$ can be derived by the optimal path $(c(t), x_m(t))_{t\geq 0}$ using equations (9) and (12).

In the next section we investigate in which extent the long-run dynamics of (P1) and (P2) are analogous.

4 Long-run dynamics: comparing economy (P1) with (P2)

We distinguish two cases: no economic growth ($0 < \alpha < 1$) and endogenous growth ($\alpha = 1$).

Steady state and stationary optimal plan

In the case of $0 < \alpha < 1$ and capital goods with different time-to-build, the unique, strictly positive steady state (x_m^*, c^*) of problem (P2) is the unique solution of the capital accumulation equation (16) and the Euler equation (15) evaluated at the steady state:

$$c^* = \left(A \prod_{j=1}^{N} \left(\frac{\alpha_j}{\alpha_m} \right)^{\alpha_j} \right) \cdot (x_m^*)^{\alpha} \cdot e^{\rho\left(\alpha d_m - \sum_{j=1}^{N} \alpha_j d_j\right)} \qquad (17)$$

$$\rho = e^{-\rho[(1-\alpha)d_m + \sum_{j=1}^{N} \alpha_j d_j]} \cdot \alpha \left(\frac{\alpha_m A}{\alpha} \prod_{j=1}^{N} \left(\frac{\alpha_j}{\alpha_m} \right)^{\alpha_j} \right) \cdot (x_m^*)^{\alpha-1} \qquad (18)$$

The steady state is a stationary optimal plan of problem (P2) when $x_m(0) = x_m^*$ and the exogenously given initial conditions of the other capital goods are set to satisfy $\alpha_m x_\kappa(0) = \alpha_\kappa x_m(0)$ for any $\kappa = 1, ..., N$.

Furthermore, equations (17) and (18) coincide with the two corresponding equations of problem (P1) under the following conditions:

$$d = d_m = \frac{\sum_{j=1}^{N} \alpha_j d_j}{\alpha}, \qquad Z = \frac{\alpha_m A}{\alpha} \prod_{j=1}^{N} \left(\frac{\alpha_j}{\alpha_m} \right)^{\alpha_j},$$

$$c(t) = \frac{\alpha}{\alpha_m} \tilde{c}(t), \qquad \text{and} \qquad \tilde{k}(t) = x_m(t) \qquad (19)$$

In fact, under these conditions, equations (17) and (18) rewrite:

$$\tilde{c}^* = Z(\tilde{k}^*)^{\alpha} \qquad (20)$$

$$\rho = e^{-\rho d} \alpha Z (\tilde{k}^*)^{\alpha-1} \qquad (21)$$

which are respectively the capital accumulation equation (5) and Euler equation (6) evaluated at the steady state of problem (P1). Crucially, the first of the conditions (19) requires that one of the time-to-build parameter in the economy with N capital goods, i.e. d_m, is a weighted average of all the gestation lags.[2] If conditions (19) hold then, starting from (P1) we may derive the long-run dynamics of (P2) and viceversa.

Endogenous growth and balanced growth paths

In the case of $\alpha = 1$, endogenous growth is possible. A balanced growth path (BGP) of the economy is defined as a solution of the capital accumulation equation and Euler equation such that, for a suitable $g > 0$, all the aggregate variables are purely exponential function, i.e. $c(t) = c_0 e^{gt}$, $x_m(t) = x_m(0)e^{gt}$, etc. and the initial conditions are respected.

The growth rate of the economy (P2) is $g = -g_w - \rho$ with g_w real root of the characteristic equation associated to (14):

$$\alpha Z e^{g_w \sum_{j=1}^{N} \alpha_j d_j} = -g_w$$

On the other hand, the growth rate of the economy (P1) is $g = -g_\varrho - \rho$ with g_ϱ the real root of the characteristic equation associated to (10):

$$\alpha Z e^{-g_\varrho d} = -g_\varrho$$

Observe that the two characteristic equations coincide and lead to the same result, i.e. $g_w = g_\varrho$, and, therefore, the economic growth is the same in the two models, if

$$d = \sum_{j=1}^{N} \alpha_j d_j$$

which was one of the relations identified before in (19). Based on this result, it follows that the condition on Z to have positive economic growth in problem (P1), which has been already proved by the existing literature (e.g. Lemma 2 in Bambi [2]), can be used to find the corresponding condition on parameters for problem (P2). From conditions (19), we can also find that $\tilde{y}(t) = \alpha_m y(t)$.

Main result and numerical examples

We can now summarize these findings in the following theorem

Theorem 1 *Consider two economies (P1) and (P2) characterized respectively by one capital good with a gestation lag, d, and N capital goods with different gestation lags, namely $d_1, ..., d_N$. Then the long-run dynamics of economy (P2) can be derived by the long-run dynamics of economy (P1) and viceversa when the relations found in (19) hold.*

Two numerical examples are now proposed to illustrate this result.

[2]It is worth noting that a d_m such that $d_m = \frac{\sum_{j=1}^{N} \alpha_j d_j}{\alpha}$ *always* exists when i) $d \in [0, \bar{d}]$, ii) $d_j \in [0, \bar{d}]$ with $d_j \neq d_i$ for all $i \neq j$, and iii) $N \to \infty$; in fact, under these assumptions, the sum converges to a real number in $[0, \bar{d}]$. Such a result depends on the fact that, by construction, $\lim_{N\to\infty} \sum_{j=1}^{N} \alpha_j$ is still equal to 1.

Example 1 Consider as in Kydland and Prescot [5] that the capital stock in the economy (P1) includes plants, housing and consumer durables characterized respectively by the time-to-build parameters $d_1 = 8$, $d_2 = 4$ and $d_3 = 1$ quarters. Suppose also that the capital share $\alpha = \frac{1}{3}$. Then the "compromise" of $d_m = 4$ quarters suggested by the authors can be achieved, for example, when $\alpha_1 = \frac{1}{12}$, $\alpha_2 = \frac{5}{36}$ and $\alpha_3 = \frac{5}{9}$.

Example 2 (Endogenous growth) Consider an economy with $N = 3$ capital goods with heterogeneous delays $d_1 = 1$, $d_2 = \frac{3}{2}$, $d_3 = 2$, capital shares $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$, and $A = 0.42$. Taking into account conditions (19), if we set $d = d_m = \alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_2 = \frac{3}{2}$ and $Z = \frac{1}{3}A = 0.14$ then the growth rate the two economies is the same. Suppose, instead, that the capital shares are $\alpha_1 = \alpha_3 = \frac{1}{4}$ and $\alpha_2 = \frac{1}{2}$. Then again we should set $d = \frac{3}{2}$ but now $Z = \frac{1}{\sqrt{2}}$.

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