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Partial mirror symmetry, lattice presentations and algebraic monoids

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ABSTRACT

This is the second in a series of papers that develops the theory of reflection monoids, motivated by the theory of reflection groups. Reflection monoids were first introduced in [**7**]. In this paper we study their presentations as abstract monoids. Along the way we also find general presentations for certain join-semilattices (as monoids under join) which we interpret for two special classes of examples: the face lattices of convex polytopes and the geometric lattices, particularly the intersection lattices of hyperplane arrangements. Another spin-off is a general presentation for the Renner monoid of an algebraic monoid, which we illustrate in the special case of the "classical" algebraic monoids.

Introduction

"Numbers measure size, groups measure symmetry", and inverse monoids measure partial symmetry. In [**7**] we initiated the formal study of partial *mirror* symmetry via the theory of what we call reflection monoids. The aim is three-fold: (i). to wrap up a reflection group and a naturally associated combinatorial object into a single algebraic entity having nice properties, (ii). to unify various unrelated (until now) parts of the theory of inverse monoids under one umbrella, and (iii). to provide workers interested in partial symmetry with the appropriate tools to study the phenomenon systematically.

This paper continues the programme by studying presentations for reflection monoids. As one of the distinguishing features of real reflection (or Coxeter) groups are their presentations this is a natural thing to do. Broadly, our approach is to adapt the presentation found in [**4**] to our purposes.

Roughly speaking, an inverse monoid (of the type considered in this paper) is made up out of a group W (the *units*), a poset E with joins $\vee$ (the *idempotents*) and an action of W on E. A presentation for an inverse monoid thus has relations pertaining to each of these three components. In particular, we need presentations for W as a group and E as a monoid under $\vee$.

For a reflection monoid, W is a reflection group. If it is a real reflection group, as all in this paper turn out to be, then it has a Coxeter presentation; so that part is already nicely taken care of.

The poset E is a commutative monoid of idempotents, and we invest a certain amount of effort in finding presentations for these (§1). We imagine that much of this material is of independent interest. Here we are motivated by the notion of independence in a geometric lattice (see for instance [**21**]), which we first generalize to the setting of graded atomic $\vee$-semilattices. The idea is that relations arise when we have dependent sets of atoms. Our first examples are the face monoids of convex polytopes, and it turns out that simple polytopes have particularly simple presentations. The pay-off comes in §5, where these face monoids are

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the idempotents in the Renner monoid of a linear algebraic monoid (Renner monoids being to algebraic monoids as Weyl groups are to algebraic groups). We then specialize to geometric lattices–their presentations turn out to be nicer (Theorem 1.6). We finally come full circle with the intersection lattices of the reflecting hyperplanes of a finite Coxeter group (§1.4), where we work though the details for the classical Weyl groups. These reappear in §4 as the idempotents of the Coxeter arrangement monoids.

Historically, presentations for reflection monoids start with Popova's presentation for the symmetric inverse monoid $\mathscr{I}_n$ [**22**]. Just as the symmetric group $\mathfrak{S}_n$ is one of the simplest examples of a reflection group (being the Weyl group of type A_{n-1}) so $\mathscr{I}_n$ is one of the simplest examples of a reflection monoid (the Boolean monoid of type A_{n-1}; see [**7**, §5]). Our general presentation for a reflection monoid (Theorem 2.1 of §2) specializes to Popova's in this special case, unlike those found in [**11**, **25**]. In the resulting presentation there is one relation that seems less obvious than the others. This turns out to always be true. The units in a reflection monoid form a reflection group W and each relation in this non-obvious family arises from an orbit of the W-action on the reflecting hyperplanes of W. So, the interaction between a reflection group and a naturally associated combinatorial object – in this case the intersection lattice of the reflecting hyperplanes of the group – manifests itself in the presentation for the resulting reflection monoid.

Sections 3 and 4 work out explicit presentations for the two main families of reflection monoids that were introduced in [**7**]: the Boolean monoids and the Coxeter arrangement monoids.

Finally, we turn our attention to algebraic monoids and Renner monoids in §5. We are not aware of a length function for general reflection monoids or even for Coxeter arrangement monoids. Moreover, a Coxeter arrangement monoid does not seem to be associated with a particular reductive monoid. Thus the techniques used in [**10**] and [**11**] to find presentations for Renner monoids are not available to us in getting presentations for reflection monoids. We therefore use only elementary combinatorics. Although not all Renner monoids are reflection monoids (see [**7**, Example 8.3]), our methods are easily adapted to get presentations for Renner monoids and our general result involves fewer generators and relations than those found in [**10**] and [**11**]. The basic principle here is to build an abstract monoid of partial isomorphisms from a reflection group acting on a combinatorial description of a rational polytope. This abstract monoid is then isomorphic to the Renner monoid of an algebraic monoid–the reflection group corresponds to the Weyl group of the underlying algebraic group and the polytope arises from the weights of a representation of the Weyl group (a reflection group and naturally associated combinatorial structure being wrapped up!). We work the details for the "classical" algebraic monoids (special linear, orthogonal, symplectic) as well as another nice family of examples introduced by Solomon in [**26**].

As a final remark, we recall that although every Renner monoid is a homomorphic image of a geometric lattice reflection monoid (see [**7**, Theorem 8.1]), it is not clear (to us) what connection, if any, there is between the presentations of the two monoids. In particular, we do not know if it is possible to obtain a presentation for a Renner monoid from one for the corresponding reflection monoid.

We end this section by setting notation for reflection groups (see [**3**, **14**, **16**] for more details) and recalling the monoids of partial symmetries introduced in [**7**].

Let V be a finite dimensional vector space over $\mathbb{R}$ and $W = W(\Phi) \subset GL(V)$ finite, be the reflection group generated by the reflections $\{s_v\}_{v\in\Phi}$ for $\Phi \subset V$ a finite root system. Then there is a distinguished set S of generating reflections giving W the structure of a Coxeter group with generators S and relations $(st)^{m_{st}} = 1$ for $m_{st} \in \mathbb{Z}^{\geq 1}$. For future reference we list in Table 1 the Φ, the S and the Coxeter symbol for the Weyl groups of types A_n, B_n and D_n.

The full set T of reflections in W is the set of W-conjugates of S. Write $\mathscr{A} = \{H_t \subset V \mid t \in T\}$ for the set of reflecting hyperplanes of W. Then W naturally acts on $\mathscr{A}$ and every orbit contains an H_s with $s \in S$. Moreover, if $s, s' \in S$ then H_s and $H_{s'}$ lie in the same orbit if and only if s and s' are joined in the Coxeter symbol by a path of edges labeled entirely by *odd* m_{st}. Thus the number of orbits of W on $\mathscr{A}$ is the number of connected components of the Coxeter symbol once the even labeled edges have been dropped. This number will appear later on as the number of relations in a certain family in the presentation for a reflection monoid.

As in [**7**], when $G \subseteq GL(V)$ is any group and $X \subseteq V$, a key role is played by the isotropy groups $G_X = \{g \in G \,|\, vg = v \text{ for all } v \in X\}$. A theorem of Steinberg [**29**, Theorem 1.5] asserts that for $G = W(\Phi)$, the isotropy group $W(\Phi)_X$ is itself a reflection group; indeed, generated by the reflections s_v for $v \in \Phi \cap X^\perp$.

Now to partial mirror symmetry. Recall from [**7**, §2] that if $G \subset GL(V)$ and $\mathcal{S}$ is a system of subspaces for G then the monoid of partial isomorphisms is defined by $M(G, \mathcal{S}) := \{g_X \,|\, g \in G, X \in \mathcal{S}\}$ where g_X is the partial isomorphism with domain X. It is a factorizable inverse monoid. If $G = W$ is a reflection group then $M(W, \mathcal{S})$ is called a *reflection monoid*.

Similarly we have monoids of partial permutations: replace V by a finite set E; the group G is now $G \subseteq \mathfrak{S}_E$ and $\mathcal{S}$ is a system of subsets of E for G [**7**, §9.2]. In all the examples in this paper E will turn out to have more structure: it will be a $\vee$-semilattice with a unique minimal element $\mathbf{0}$ and with the G-action by poset isomorphisms. The system of subsets $\mathcal{S}$ consists of intervals in E, namely, for any $a \in E$ the sets $E_{\geq a} := \{b \in E \,|\, b \geq a\}$. Then $E = E_{\geq \mathbf{0}}$, $E_{\geq a} \cdot g = E_{\geq a \cdot g}$ for $g \in G$, and $E_{\geq a} \cap E_{\geq b} = E_{\geq a \vee b}$. Ordering $\mathcal{S}$ by *reverse* inclusion, the map $E \to \mathcal{S}$ given by $a \mapsto E_{\geq a}$ is a poset isomorphism that is equivariant with respect to the G-actions on E and $\mathcal{S}$.

1. *Idempotents*

1.1. *Generalities*

Let E be a finite commutative monoid of idempotents. It is a fundamental result that E acquires, via the ordering $x \leq y$ if and only if $xy = y$, the structure of a join semi-lattice with a unique minimal element. Conversely, any join semi-lattice with unique minimal element is a commutative monoid of idempotents via $xy := x \vee y$. Moreover, in either case we also have a unique maximal element–the join of all the elements of (finite) E. From now on we will apply monoid and poset terminology (see [**28**, Chapter 3]) interchangeably to E and write $\mathbf{0}$ for the unique minimal element and $\mathbf{1}$ for the unique maximal one. The reader should beware: the $\mathbf{0}$ of the poset E is the multiplicative 1 of the monoid E and the $\mathbf{1}$ is the multiplicative 0. Recall that a poset map $f : E \to E'$ is a map with $fx \leq' fy$ when $x \leq y$.

All of our examples will turn out to have slightly more structure: E is *graded* if for every $x \in E$, any two saturated chains $\mathbf{0} = x_0 < x_1 \cdots < x_k = x$ have the same length. In this case E has a rank function $\text{rk} : E \to \mathbb{Z}^{\geq 0}$ with $\text{rk}(x) = k$. In particular $\text{rk}(\mathbf{0}) = 0$, and if x and y are such that $x \leq z \leq y$ implies $z = x$ or $z = y$, then $\text{rk}(y) = \text{rk}(x) + 1$. Write $\text{rk}E := \text{rk}(\mathbf{1})$. The elements of rank 1 are called the *atoms*, and E is said to be *atomic* if every element is a join of atoms. In particular, an atomic E is generated as a monoid by its atoms.

For example, the *Boolean lattice* $\mathscr{B}_X$ *of rank* n is the lattice of subsets of $X = \{1, \ldots, n\}$ ordered by *reverse* inclusion. It is graded with $\text{rk}(Y) = |X \setminus Y|$ and atomic with atoms the $a_i := \{1, \ldots, \widehat{i}, \ldots, n\}$. The monoid operation is just intersection.

Writing $\bigvee S$ for the join of the elements in a subset $S \subseteq E$, call a set S of atoms *independent* if $\bigvee S \setminus \{s\} < \bigvee S$ for all $s \in S$, and *dependent* otherwise; S is *minimally dependent* if it is dependent and every proper subset is independent. These notions satisfy the following properties, most of which are clear, although some hints are given:

(I1). If $|S| \leq 2$ then S is independent; in particular, any three element set of dependent atoms is minimally dependent.

(I2). If S is dependent then there exists $T \subset S$ with T independent and $\bigvee T = \bigvee S$ (successively remove those s for which $\bigvee S \setminus \{s\} = \bigvee S$).
(I3). Any subset of an independent set is independent.
(I4). If T is independent and $S = T \cup \{b\}$ is dependent then there is a $T' \subseteq T$ with $T' \cup \{b\}$ minimally dependent (this is clear if $|S| = 3$; if S arbitrary is not minimally dependent already then there is an $s \in S$ with $S \setminus \{s\}$ dependent, and in particular $s \neq b$. The result then follows by induction applied to $S \setminus \{s\}$.)
(I5). If S is independent then there is an injective map of posets $\mathscr{B}_S \hookrightarrow E$, not necessarily grading preserving (send $T \subseteq S$ to $\bigvee T$ in E); consequently, if S is independent then $|S| \leq \mathrm{rk}E$.

There is an obvious analogy here with linear algebra, which becomes stronger in §1.3 when E is a geometric lattice.

Here is our first presentation. Throughout this paper we adopt the standard abuse whereby the same symbol is used to denote an element of an abstract monoid given by a presentation and the corresponding element of the concrete monoid that is being presented. Apart from the proof of the following (where we temporarily introduce new notation to separate these out) the context ought to make clear what is being denoted.

Proposition 1.1. *Let E be a finite graded atomic commutative monoid of idempotents with atoms A. Then E has a presentation with:*

generators: $a \in A$.

relations: $ab = ba \, (a, b \in A)$, *(Idem2)*

$a_1 \dots a_k = a_1 \dots a_k b \, (a_i, b \in A)$, *(Idem3)*

for $a_1, \dots, a_k$, $(1 \leq k \leq \mathrm{rk}E)$ *independent and* $b \leq \bigvee a_i$.

Notice that when $k = 1$ the *(Idem3)* relations are $a = a^2$ for $a \in A$. To emphasise the point we separate these from the rest of the *(Idem3)* relations and call them family *(Idem1)*. Note also that the $\{a_1, \dots, a_k, b\}$ appearing in *(Idem3)* are dependent.

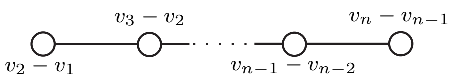
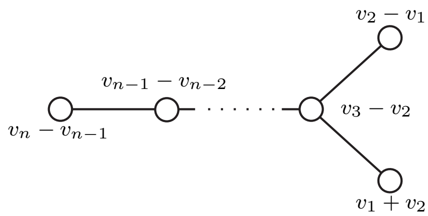
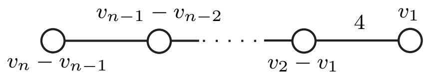
Type	Root system Φ	Coxeter symbol and simple system
$A_{n-1}\,(n \geq 2)$	$\{v_i - v_j \ (1 \leq i \neq j \leq n)\}$	
$D_n\,(n \geq 4)$	$\{\pm v_i \pm v_j \ (1 \leq i < j \leq n)\}$	
$B_n\,(n \geq 2)$	$\{\pm v_i \ (1 \leq i \leq n)$, $\pm v_i \pm v_j \ (1 \leq i < j \leq n)\}$	

Table 1. *Root systems, simple systems and Coxeter symbols for the classical Weyl groups.*

Proof. We temporarily introduce alternative notation for the atoms and then remove it at the end of the proof: we use Roman letters $a, b, \ldots$ for the atoms A of E and their Greek equivalents $\alpha, \beta, \ldots$ for a set in 1-1 correspondence with A. Let M be the quotient of the free monoid on the $\alpha \in A$ by the congruence generated by the relations *(Idem2)*-*(Idem3)*, with Greek letters rather than Roman. We have already observed that E is generated by the $a \in A$, and the relations *(Idem2)*-*(Idem3)* clearly hold in E, so the map $\alpha \mapsto a$ induces an epimorphism $M \to E$. To see that this map is injective, we choose representative words: for any $e \in E \setminus \{\mathbf{0}\}$, let $A_e := \{a \in A \,|\, a \leq e\}$ and

$$\underline{e} = \prod_{a \in A_e} \alpha.$$

It remains to show that any word in the α's mapping to e can be transformed into the representative word $\underline{e}$ using the relations *(Idem1)*-*(Idem3)*. Let $\alpha_1 \ldots \alpha_k$ be such a word and let $b \in A_e$ be such that $b \neq a_i$ for any i. If no such b exists then the word is $\underline{e}$ already and we are done. Otherwise, there is an independent subset $\{a_{i_1} \ldots, a_{i_\ell}\}$ with $e = a_{i_1} \vee \cdots \vee a_{i_\ell}$, and so we have an *(Idem3)* relation $\alpha_{i_1} \cdots \alpha_{i_\ell} = \alpha_{i_1} \cdots \alpha_{i_\ell}\beta$. Multiplying both sides by $\alpha_1 \ldots \alpha_k$, reordering using *(Idem1)* and removing redundancies using *(Idem2)*, we obtain $\alpha_1 \ldots \alpha_k = \alpha_1 \ldots \alpha_k \beta$. Repeat this until the word is $\underline{e}$. $\square$

For a simple example, the Boolean lattice $\mathscr{B}_X$ of rank n has atoms the $a_i = \{1, \ldots, \widehat{i}, \ldots, n\}$ with $\bigvee a_{i_j}$ the set X with the indices i_j omitted. Removing an atom from this join has the effect of re-admitting the corresponding index. The resulting join is thus strictly smaller than $\bigvee a_{i_j}$, and we conclude that any set of atoms is independent. As an *(Idem3)* relation in Proposition 2.3 arises as a result of a set $\{a_1, \ldots, a_k, b\}$ of dependent atoms, the *(Idem3)* relations are vacuous when $k > 1$ and we have a presentation with generators $a_1, \ldots, a_n$ and relations $a_i^2 = a_i$ and $a_i a_j = a_j a_i$ for all i, j.

1.2. *Face monoids of polytopes*

In §5 we will encounter a class of commutative monoids of idempotents that are isomorphic to the face lattices of convex polytopes. It is to these that we now turn.

A *(convex) polytope* P in a real vector space V is the convex hull of a finite set of points. The standard references for convex polytopes are [**12**, **31**]. An r-dimensional *face* is defined in [**12**, §2.4]. We consider P itself to be a face, and say P is a d-polytope when $\dim P = d$, and $\varnothing$ to be the unique face of dimension -1. A $(d-1)$-face of a d-polytope is called a *facet*.

Let $\mathscr{F}(P)$ be the faces of P ordered by *reverse* inclusion (this is the opposite order to that normally used in the polytope literature). In any case, it is well known that $\mathscr{F}(P)$ is a graded (with $\mathrm{rk} f = \mathrm{codim}_P f := \dim P - \dim f$) atomic lattice with atoms the facets, join $f_1 \vee f_2 = f_1 \cap f_2$, meet $f_1 \wedge f_2$ the smallest face containing f_1 and f_2, unique minimal element $\mathbf{0} = P$ and maximal element $\mathbf{1} = \varnothing$ (hence $\mathrm{rk}\mathscr{F}(P) = \dim P$). We call the associated monoid the *face monoid of the polytope P*.

Two polytopes are combinatorially equivalent if their face lattices are isomorphic as lattices. The *combinatorial type* of a polytope is the isomorphism class of its face lattice, and when one talks of a combinatorial description of a polytope, one means a description of $\mathscr{F}(P)$. In this paper, all statements about polytopes are true *up to combinatorial type*.

Example 1 (the d-simplex Δ^d). Let V be a $(d+1)$-dimensional Euclidean space with basis $\{v_1, \ldots, v_{d+1}\}$ and Δ^d the convex hull of these basis vectors. Any subset of the v_i of size $k+1$ spans a k-simplex. If $X = \{1, \ldots, d+1\}$ then $\mathscr{F}(\Delta^d)$ is isomorphic to the Boolean lattice $\mathscr{B}_X$ by the map sending $Y \subseteq X$ to the convex hull of the points $\{v_i \,|\, i \in Y\}$. Thus, any set of facets is independent.

In particular $\mathscr{F}(\Delta^d)$ has a presentation with generators $a_1, \ldots, a_{d+1}$ and relations $a_i^2 = a_i$ and $a_i a_j = a_j a_i$. We will meet this commutative monoid of idempotents twice more in this paper: as the idempotents of the Boolean reflection monoids in §3, and as the idempotents of the Renner monoid of the "classical" linear monoid $\overline{k^\times \mathbf{SL}_d}$ in §5.2.

A d-polytope is *simplicial* when each facet has the combinatorial type of a $(d-1)$-simplex.

EXAMPLE 2 (the d-octahedron or cross-polytope $\diamond^d$). Let V be d-dimensional Euclidean and let $\diamond^d$ be the convex hull of the vectors $\{\pm v_1, \ldots, \pm v_d\}$. To describe $\diamond^d$ combinatorially, let $\pm X = \{\pm 1, \ldots, \pm d\}$ and call a subset $J \subset \pm X$ admissible whenever $J \cap (-J) = \varnothing$. Alternatively, if $J^+ = J \cap X$ and $J^- = J \cap (-X)$ then $-J^+ \cap J^- = \varnothing$. Note that the admissible sets are closed under passing to subsets (hence under intersection) but not under unions. Let E_0 be the admissible subsets of $\pm X$ ordered by reverse inclusion. This poset has a number of minimal elements, namely, any set of the form $J^+ \cup J^-$ with $J^+ \subseteq X$ and $J^- = -X \setminus -J^+$. In particular these sets are completely determined by J^+. Let E be E_0, together with $\pm X$, and ordered by reverse inclusion. Then the map sending $J \in E$ to the convex hull of the points

$$\{v_i \,|\, i \in J^+\} \cup \{-v_{-i} \,|\, i \in J^-\},$$

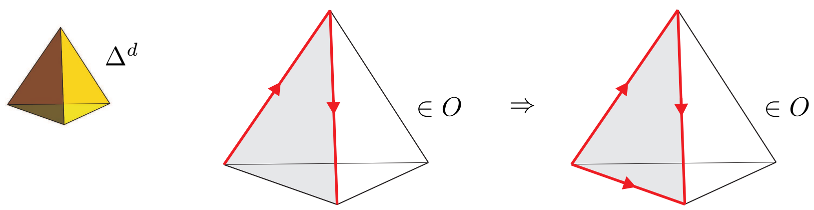
is a lattice isomorphism $E \to \mathscr{F}(\diamond^d)$. In particular, if f_J, f_K are faces corresponding to $J, K \in E$ then $f_J \vee f_K = f_{J \cap K}$. We will meet the monoid E again in §5.2 as the idempotents of the Renner monoids of the classical monoids $\overline{k^\times \mathbf{SO}_{2d}}$, $\overline{k^\times \mathbf{SO}_{2d+1}}$ and $\overline{k^\times \mathbf{Sp}_{2d}}$.

For a convex polytope P there is a *dual polytope* P^*, unique up to combinatorial type, with the property that $\mathscr{F}(P^*) = \mathscr{F}(P)^{\mathrm{opp}}$, the opposite lattice to $\mathscr{F}(P)$, i.e. $\mathscr{F}(P)^{\mathrm{opp}}$ has the same elements as $\mathscr{F}(P)$ and order $f_1 \leq f_2$ in $\mathscr{F}(P)^{\mathrm{opp}}$ if and only if $f_2 \leq f_1$ in $\mathscr{F}(P)$ (see [**12**, 3.4]). Call P *simple* if and only if its dual P^* is simplicial. Equivalently, each vertex (0-face) of P is contained in exactly $\dim P$ facets. The d-simplex is self dual, corresponding to the fact that a Boolean lattice is isomorphic as a lattice to its opposite. Another simple polytope is:

EXAMPLE 3 (the d-permutohedron). Let V be $(d+1)$-dimensional Euclidean and let the symmetric group $\mathfrak{S}_{d+1}$ act on V via $v_i\pi = v_{i\pi}$ for $\pi \in \mathfrak{S}_{d+1}$. Writing $v \cdot \mathfrak{S}_{d+1}$ for the orbit of $v \in V$, let $0 \leq m_1 < \cdots < m_{d+1}$ be integers and define a d-permutohedron P to be the convex hull of the orbit $(\sum m_i\, v_i) \cdot \mathfrak{S}_{d+1}$. The combinatorial type of P does not depend on the m_i, so we will just say *the* d-permutohedron. Figure 1(c) shows the 3-permutohedron. Our interest in permutohedra comes about as the lattice $\mathscr{F}(P)$ is isomorphic to the idempotents of the Renner monoid of §5.3.

To get a presentation for $\mathscr{F}(P)$ it is useful to have a combinatorial version of $\mathscr{F}(P)$, and we now describe such a version in some detail. To this end, an orientation of a 1-face (i.e. edge) $\{v_i, v_j\}$ $v_i \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet v_j$ of the d-simplex Δ^d has the form $v_i \bullet\!\!\longrightarrow\!\!\bullet v_j$, written (v_i, v_j), or $v_i \bullet\!\!\longleftarrow\!\!\bullet v_j$ written (v_j, v_i). If Δ^d is a d-simplex with *some* edges oriented, then we say that the oriented edges form a *partial orientation* O of Δ^d.

A partial orientation O is *admissible* when (i). for any 2-face $\{v_i, v_j, v_k\}$ in Δ^d with (v_i, v_j) and $(v_j, v_k) \in O$ we have $(v_i, v_k) \in O$ also, i.e.



and (ii). every 2-face in Δ^d has either 0 or ≥ 2 of its incident edges in O. We call these two properties *transitivity* and *incomparability*.

Let E_0 be the set of admissible partial orientations of Δ^d and define $O_1 \leq O_2$ iff every edge in O_1 is also in O_2 and with the same orientation, i.e. the order is just inclusion. This is a partial order on E_0 with a unique minimal element $\varnothing$ (i.e. no edges oriented) and maximal elements with every edge oriented. Formally adjoin a unique maximal element $\mathbf{1}$ to get the poset E. Define $O_1 \vee O_2$ to be the union of the oriented edges in O_1 and O_2 if this gives an admissible partial orientation, or $\mathbf{1}$ if it doesn't. Then E has the structure of a join semi-lattice. If there is an edge oriented one way in O_1 and the other way in O_2 then $O_1 \vee O_2$ is not even a partial orientation. It turns out that this is the only obstacle to $O_1 \vee O_2$ being admissible:

– If O_1, O_2 are admissible with $O_1 \vee O_2$ a partial orientation, then $O_1 \vee O_2$ is transitive.
– If O_1, O_2 are partial orientations satisfying incomparability and with $O_1 \vee O_2$ a partial orientation, then $O_1 \vee O_2$ satisfies incomparability.

Thus for $O_i \in E$ we have $\bigvee O_i < \mathbf{1}$ exactly when $\bigvee O_i$ is a partial orientation, i.e. each edge is oriented consistently, if at all, among the O_i.

For J a non-empty proper subset of $X = \{1, \dots, d+1\}$, let Δ_J be the sub-simplex of Δ^d spanned by the vertices $\{v_j \mid j \in J\}$ and $\Delta_{X \setminus J}$ similarly. Let O_J be the partial orientation where the only edges oriented are those not contained in either Δ_J or $\Delta_{X \setminus J}$; necessarily such edges have one vertex v_j ($j \in J$) and the other v_i ($i \in X \setminus J$). Orient the edge with the orientation running from the latter vertex to the former. We leave it to the reader to show that the O_J are admissible partial orientations and moreover, are minimal non-empty elements in the poset E, i.e. $O_J \in E$, and if $O \in E$ with $O < O_J$ then $O = \varnothing$.

For any $O \in E$ define a relation $\sim$ on the vertices of Δ^d by $u \sim v$ exactly when there is no path of (consistently) oriented edges from u to v or from v to u. This is easily seen to be reflexive and symmetric, and also transitive, the last using the incomparability and transitivity of the partial orientation O. Let $\{\Lambda_1, \dots, \Lambda_p\}$ be the resulting equivalence classes. It is easy to show that given Λ_i, Λ_j and vertices $u \in \Lambda_i, v \in \Lambda_j$ that the edge connecting them lies in O, oriented say from u to v. Moreover, given any other such pair u', v', the edge connecting them is also oriented from u' to v'. Define an order on the Λ's by $\Lambda_i \preceq \Lambda_j$ whenever the pairs are oriented from Λ_i to Λ_j in this way. In particular, $\preceq$ is a total order and so we write the equivalence classes (after relabelling) as a tuple $(\Lambda_1, \dots, \Lambda_p)$, i.e. we have an *ordered partition*.

For the O_J above we just get $(X \setminus J, J)$ via this process. If $O \in E$ and $(\Lambda_1, \dots, \Lambda_p)$ is the corresponding ordered partition then let $J_k = \Lambda_k \cup \dots \cup \Lambda_p$. We leave the reader to see that we can then write

$$O = \bigvee_{k=2}^{p} O_{J_k}, \tag{1.1}$$

an expression for O as a join of atomic O_J. In particular the O_J comprise *all* the atoms in E.

PROPOSITION 1.2. *Let P be the d-permutohedron and E the poset of admissible partial orientations of the d-simplex with a formal $\mathbf{1}$ adjoined. If $O \in E$ is given by (1.1), let f_O be the convex hull of those vertices $\sum m_{i\pi} v_i$ such that*

$$\textstyle\sum_{j \in J_k} m_{j\pi} = m_1 + \dots + m_{|J_k|}$$

for all k. Then $O \mapsto f_O$ is an isomorphism $E \cong \mathscr{F}(P)$ of lattices. Moreover, the two facets $f_J := f_{O_J}, f_K := f_{O_K}$ are disjoint if and only if neither of J, K is contained in the other, i.e. $J \neq J \cap K \neq K$.

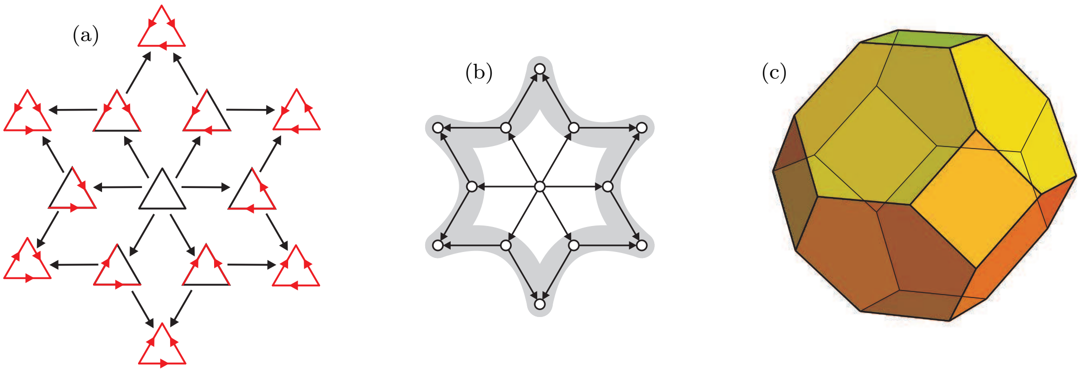


FIGURE 1. *(a). the poset E_0 of admissible partial orientations of Δ^2 with $O_1 \to O_2$ indicating $O_1 < O_2$ (b). the poset E_0 superimposed on a distorted 2-permutohedron (or hexagon) (c). the 3-permutohedron.*

Proof. That $O \mapsto f_O$ is a well defined map and a bijection is well known (see, e.g.: [**31**, Lecture 0]). If $O_1 \leq O_2$ in E then each J_{2k} coincides with some $J_{1k'}$. Thus, if the f_{O_i} are the convex hulls of sets of vertices S_i as in the Proposition, we have $S_2 \subseteq S_1$ and so $f_{O_1} \leq f_{O_2}$. This argument can be run backwards, so that we have a poset isomorphism. For the final part, $f_J \cap f_K = \varnothing$ iff $O_J \vee O_K = \mathbf{1}$, and it is easy to check that this happens exactly when $J \neq J \cap K \neq K$. $\square$

Returning to generalities, it turns out that the face lattices of simple polytopes have particularly simple presentations as commutative monoids of idempotents. Recalling the definition of independent atoms from §1.1, we lay the groundwork for this with the following result:

PROPOSITION 1.3. *Let P be a simple d-polytope.*

(i) *If v is a vertex of P then the interval $[P, v] := \{f \in \mathscr{F}(P) \,|\, P \leq f \leq v\}$ is a Boolean lattice of rank d. In particular, facets $f_1, \ldots, f_k$ with $\bigvee f_i < \varnothing$ are independent.*
(ii) *Let P be the d-permutohedron and $f_1, \ldots, f_k \in \mathscr{F}(P)$ independent facets such that $\bigvee f_i = \varnothing$. Then $k \leq 2$.*

The first part is standard; indeed it is often stated as an equivalent definition of a simple polytope as in [**31**, Proposition 2.16]. The second part is not true for an arbitrary simple polytope: consider the triangular prism $\Delta^2 \times [0, 1]$. To see it, we show that if $f_1, \ldots, f_k$ are facets with $\bigvee f_i = \varnothing$ then there are $1 \leq j < m \leq k$ with $f_j \vee f_m = \varnothing$; in particular $k \geq 3$ facets with join $\varnothing$ are dependent. This uses the combinatorial description of the d-permutohedron. Let the facets f_i correspond to admissible partial orientations O_i of Δ^d. We have $\bigvee f_i = \varnothing$ exactly when $\bigvee O_i$ is not a partial orientation. Thus there is an edge of Δ^d and O_j, O_m with the edge oriented in different directions in these two. But then $f_j \vee f_m = \varnothing$.

Part 1 of Proposition 1.3 means that for a simple polytope the *(Idem3)* relations in Proposition 2.3 are vacuous when $\bigvee a_i < \varnothing$; part 2 means that for the d-permutohedron the *(Idem3)* relations further reduce to $a_1 a_2 = a_1 a_2 b$ for each pair a_1, a_2 of disjoint facets.

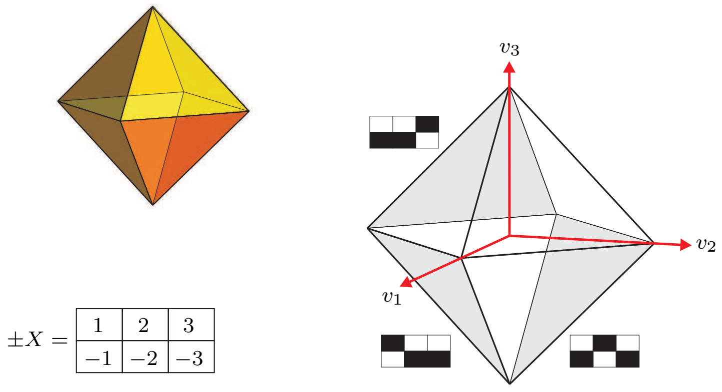


FIGURE 2. *Independent triples of facets in the 3-octahedron* $\diamond^3$*: we have* $\{x_1, x_2, x_3\} = X = \{1, 2, 3\}$ *and option (1) of Proposition 1.5 is chosen for each* j*. The atoms in* E *are depicted by blackened boxes and the corresponding facets of the octahedron shaded. Every other triple is equivalent to this one via a symmetry of* $\diamond^3$.

PROPOSITION 1.4. *Let* E *be the face monoid of a simple polytope* P *with facets* A*. Then* E *has a presentation with:*

generators: $a \in A$.

relations:
$$a^2 = a \,(a \in A), \tag{Idem1}$$
$$ab = ba \,(a, b \in A), \tag{Idem2}$$
$$a_1 \dots a_k = a_1 \dots a_k b \,(a_i, b \in A), \tag{Idem3}$$
for $a_1, \dots, a_k\,, (2 \le k \le \dim P)$ *independent with* $\bigvee a_i = \varnothing$.

Combining this presentation with part 2 of Proposition 1.3 and the combinatorial description of the d-permutohedron gives:

The d-permutohedron: has presentation with generators a_J for $\varnothing \ne J \subsetneq X = \{1, \dots, d+1\}$ and relations $a_J^2 = a_J$ for all J; $a_J a_K = a_K a_J$ for all J, K, and
$$a_J a_K = a_J a_K a_L$$
for all $J \ne J \cap K \ne K$ and all L.

The facets of the d-permutohedron are parametrized by the admissible partial orientations O_J, for J a non-empty proper subset of X, and two facets have join $\varnothing$ exactly when they correspond to O_J, O_K with $J \ne J \cap K \ne K$. The presentation follows.

Finally, we return to the d-octahedron $\diamond^d$, where things are not so simple (pun intended). Recalling the poset E of Example 2, let $J \subseteq X = \{1, \dots, d\}$ and write $a(J) := J \cup (-X \setminus -J)$ for the atoms in E (note that J is now a subset of X rather than $\pm X$). The independent sets can be described by the following, the proof of which is [**6**, Proposition 5]:

PROPOSITION 1.5. *Let* $\{x_1, \dots, x_k\} \subseteq X$ *with* $k \ge 3$ *and* $d \ge 3$*, and* $J_{10}, \dots, J_{k0}$ *subsets of* $X \setminus \{x_1, \dots, x_k\}$*. For* $j = 1, \dots, k$ *we recursively define sets* $J_{1j}, \dots, J_{kj}$ *as follows: either,*
(0). *do not add* x_j *to* $J_{j,j-1}$ *but do add* x_j *to all other* $J_{i,j-1}$ *for* $i \ne j$*; or*

(1). *do add x_j to $J_{j,j-1}$ but do not add x_j to all other $J_{i,j-1}$ for $i \neq j$.*
Then, if $J_j := J_{jk}$, the $a(J_1), \dots, a(J_k)$ are independent atoms in E, and every set of k independent atoms arises in this way.

Thus, at the 0-th step we have the sets $J_{10}, \dots, J_{k0}$; at the 1-st step either add x_1 to J_{10} and not to the others, or vice-versa; iterate. Figure 2 illustrates the independent triples of facets in the 3-octahedron: we have $X = \{1, 2, 3\}$ and 2^3 independent triples corresponding to a choice of the (0)-(1) options in Proposition 1.5. Letting $x_j = j$ (hence $J_{j0} = \varnothing$) and choosing option (1) for each j gives the atoms $a(1) = \{1, -2, -3\}, a(2) = \{-1, 2, -3\}$ and $a(3) = \{-1, -2, 3\}$ corresponding to the shaded triple of faces. Any other triple of independent facets is equivalent to this one via a symmetry of the octahedron.

Let Ind_k be the set of independent tuples $(a(J_1), \dots, a(J_k))$ arising via Proposition 1.5.

The d-octahedron $\diamond^d$: has a presentation with generators a_J for $J \subseteq X = \{1, \dots, d\}$ and relations $a_J^2 = a_J$ for all J; $a_J a_K = a_K a_J$ for all J, K and

$$a_{J_1} \dots a_{J_k} = a_{J_1} \dots a_{J_k} a_K$$

for all $(a(J_1), \dots, a(J_k)) \in Ind_k$ with $2 \leq k \leq d$ and all $a(K) \supseteq \bigcap a(J_i)$.

1.3. *Geometric monoids*

Suppose now that E is a lattice, hence with both joins $\vee$ and meets $\wedge$. A graded atomic lattice E is *geometric* when

$$\mathrm{rk}(a \vee b) + \mathrm{rk}(a \wedge b) \leq \mathrm{rk}(a) + \mathrm{rk}(b), \tag{1.2}$$

for any $a, b \in E$. We will call the corresponding commutative monoid of idempotents *geometric*.

Beginning with a non-example, the face lattices of polytopes are not in general geometric: if f_1, f_2 are non-intersecting facets of a d-polytope then the left hand side of (1.2) is d and the right hand side is 2.

The canonical example of a geometric lattice is the collection of all subspaces of a vector space under either inclusion/reverse inclusion, where (1.2) is a well known equality. The example that will preoccupy us is the following: a *hyperplane arrangement* is a finite set $\mathscr{A}$ of linear hyperplanes in a vector space V, and the intersection lattice $\mathcal{H}$ is the set of all intersections of elements of $\mathscr{A}$ ordered by reverse inclusion, with the null intersection taken to be V. The result is a geometric lattice [**20**, §2.1] with $\mathrm{rk}(A) = \mathrm{codim}\, A$, atoms the hyperplanes $\mathscr{A}$; $\mathbf{0} = V$ and $\mathbf{1} = \bigcap_{H \in \mathscr{A}} H$. If $\mathscr{A}$ are the reflecting hyperplanes of a reflection group $W \subset GL(V)$ then $\mathscr{A}$ is called a *reflection* or *Coxeter* arrangement. If $W = W(\Phi)$ for Φ some finite root system, we will write $\mathcal{H}(\Phi)$ for the intersection lattice of the Coxeter arrangement.

The linear algebraic analogy of §1.1 can be pushed a little further in a geometric lattice:
(I6). $\mathrm{rk}(\bigvee S) \leq |S|$ for atoms S, with S independent if and only if $\mathrm{rk}(\bigvee S) = |S|$.
(I7). If S is minimally dependent then $\bigvee S \setminus \{s\} = \bigvee S$ for all $s \in S$.

That $\mathrm{rk}(\bigvee S) \leq |S|$ is a well known property of geometric lattices that follows from (1.2)–see for example [**21**]. Indeed, (I6) is the normal definition of independence in a geometric lattice.

To see it, we show first by induction on the size of $|S|$ that if $\mathrm{rk}(\bigvee S) < |S|$ then S is dependent: a three element set with $\mathrm{rk}(\bigvee S) < 3$ is the join of any two of its atoms, hence dependent, as the join of two atoms always has rank two (the result is vacuous if $|S| = 2$ as the join of two distinct atoms has rank 2). If S is arbitrary and $\bigvee S \setminus \{s\} = \bigvee S$ for all s then S is clearly dependent. Otherwise, if $\bigvee S \setminus \{s\} < \bigvee S$ for some $s \in S$ with $\mathrm{rk}(\bigvee S \setminus \{s\}) < \mathrm{rk}(\bigvee S) < |S|$, then $\mathrm{rk}(\bigvee S \setminus \{s\}) < |S \setminus \{s\}|$. By induction, $S \setminus \{s\}$ is dependent, hence so is S.

On the other hand, if $\mathrm{rk}(\bigvee S) = |S|$ but $\bigvee S \setminus \{s\} = \bigvee S$ for some s, then $\mathrm{rk}(\bigvee S \setminus \{s\}) = \mathrm{rk}(\bigvee S) = |S| > |\bigvee S \setminus \{s\}|$, a contradiction. Thus $\mathrm{rk}(\bigvee S) = |S|$ implies that S is independent, and we have established (I6).

Condition (I7) is a straightforward comparison of ranks. Taking three facets of the 2-octahedron (diamond) gives a minimally dependent set S in the face lattice where $\bigvee S \setminus \{s\} = \bigvee S$ is true for only one of the three s, so this property is not enjoyed by arbitrary graded atomic lattices.

Minimal dependence comes into its own when we have a geometric lattice. In particular we can replace the *(Idem3)* relations of Proposition 2.3 with a smaller set:

THEOREM 1.6. *Let E be a finite geometric commutative monoid of idempotents with atoms A. Then E has a presentation with:*

generators: $a \in A$.

relations: $a^2 = a\, (a \in A)$, *(Idem1)*

$ab = ba\, (a, b \in A)$, *(Idem2)*

$\widehat{a}_1 \dots a_k = \cdots = a_1 \dots \widehat{a}_k\, (a_i \in A)$, *(Idem3a)*

for all $\{a_1, \dots, a_k\}$*minimally dependent.*

Proof. The Theorem is proved if we can deduce the *(Idem3)* relations of Proposition 2.3 from the relations above. Suppose then that $a_1 \dots a_k = a_1 \dots a_k b$ is an *(Idem3)* relation with $\{a_1, \dots,\ a_k\}$ independent in E and $b \leq \bigvee a_i$. Thus $\{a_1, \dots,\ a_k, b\}$ is dependent, so by (I4) of §1.1 there are $a_{i_1}, \dots, a_{i_k}$ with $\{a_{i_1}, \dots, a_{i_k}, b\}$ minimally dependent. In particular, we have $a_{i_1}, \dots a_{i_k} = a_{i_1} \dots a_{i_k} b$ by *(Idem3a)*, and multiplying both sides by $a_1 \dots a_k$ and using *(Idem1)*-*(Idem2)* gives the result. ☐

It is sometimes convenient to use the *(Idem3a)* relations in the form:

$$a_1 \dots a_k = a_1 \dots \widehat{a}_i \dots a_k \qquad \textit{(Idem3b)}$$

for all $\{a_1, \dots, a_k\}$ minimally dependent and all $1 \leq i \leq k$.

1.4. *Coxeter arrangements*

In §4 we will encounter a class of commutative monoids of idempotents isomorphic to the Coxeter arrangements $\mathcal{H}(\Phi)$ for Φ the root systems of types A_{n-1}, B_n and D_n. In this section we interpret Theorem 1.6 for these monoids. We follow a similar pattern to §1.2: first we give the arrangement, then a combinatorial description (which as in §1.2 means a description of the lattice $\mathcal{H}$) and then use this to identify the independent and minimally dependent sets of atoms. It turns out to be convenient to expand on an idea of Fitzgerald [**8**].

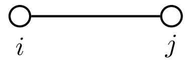
EXAMPLE 4 ($\mathcal{H}(A_{n-1})$ and the partition lattice $\Pi(n)$). Let V be Euclidean with orthonormal basis $\{v_1, \dots, v_n\}$ and $\mathscr{A}$ the hyperplanes with equations $x_i - x_j = 0$ for all $i \neq j$. Equivalently, if Φ is the type A_{n-1} root system from Table 1 then $\mathscr{A}$ consists of the hyperplanes $\{v^\perp \mid v \in \Phi\}$ and $W(\Phi)$ is the symmetric group acting on V by permuting the v_i.

We remind the reader of a well known description of $\mathcal{H}(A_{n-1})$. Let $X = \{1, \dots, n\}$ and consider the partitions $\Lambda = \{\Lambda_1, \dots, \Lambda_p\}$ of X ordered by refinement: $\Lambda \leq \Lambda'$ iff every block Λ_i of Λ is contained in some block Λ'_j of Λ'. This is a graded atomic lattice with

$$\mathrm{rk}\Lambda = \sum(|\Lambda_i| - 1) \qquad (1.3)$$

and atoms the partitions having a single non-trivial block of the form $\{i, j\}$. The map sending $(v_i - v_j)^\perp$ to the atomic partition $\{i, j\}$ extends to a lattice isomorphism $\mathcal{H} \to \Pi(n)$ given by $X(\Lambda) \mapsto \Lambda$ where $\sum t_i v_i \in X(\Lambda)$ whenever $t_i = t_j$ for i, j in the same block of Λ.

To proceed further we borrow an idea from [**8**]: for a set S of atoms in either $\mathcal{H}(A_{n-1})$ or $\Pi(n)$, form the graph Γ_S with vertex set X and $|S|$ edges of the form:



for each atom $(v_i - v_j)^\perp$ or $\{i, j\} \in S$. Recall that a connected graph (possibly with multiple edges and loops) having fewer edges than vertices cannot contain a circuit. If $\Lambda = \bigvee S$ is the join in $\Pi(n)$, then the blocks of the partition Λ are the vertices in the connected components of Γ_S. Thus, by (1.3), S is independent when the component corresponding to the block Λ_i has $|\Lambda_i| - 1$ edges, i.e. has a number of edges that is one less than the number of its vertices. Such a connected graph is a tree, so S is independent exactly when Γ_S is a forest.

The atoms S are thus dependent when Γ_S contains a circuit, and minimally dependent when Γ_S *is* just a circuit.

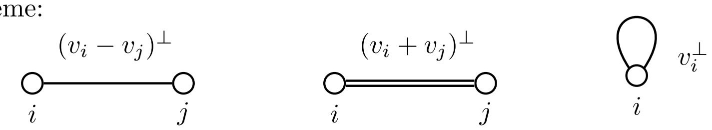
EXAMPLE 5 ($\mathcal{H}(B_n)$). Let V be as in the previous example and $\mathscr{A}$ the hyperplanes with equations $x_i = x_i \pm x_j = 0$ for all $i \neq j$; equivalently, if Φ is the type B_n root system from Table 1 then $\mathscr{A}$ consists of the hyperplanes $v_i^\perp$ and $(v_i \pm v_j)^\perp$, with $W(\Phi)$ acting on V by signed permutations of the v_i (see also the end of §5.2).

A combinatorial description of $\mathcal{H}(B_n)$ appears in [**7**, §6.2] (see also [**20**, §6.4]): a coupled partition is a partition of the form $\Lambda = \{\Lambda_{11} + \Lambda_{12}, \ldots, \Lambda_{q1} + \Lambda_{q2}, \Lambda_1, \ldots, \Lambda_p\}$, where the Λ_{ij} and Λ_i are blocks and $\Lambda_{i1} + \Lambda_{i2}$ is a "coupled" block. The + sign is purely formal. Let $\mathcal{T}$ be the set of pairs (Δ, Λ) where $\Delta \subseteq X = \{1, \ldots, n\}$ and Λ is a coupled partition of $X \setminus \Delta$. An order is defined in [**7**, §5.2] making $\mathcal{T}$ a graded atomic lattice with

$$\mathrm{rk}(\Delta, \Lambda) = |\Delta| + \sum(|\Lambda_{i1}| + |\Lambda_{i2}| - 1) + \sum(|\Lambda_i| - 1). \tag{1.4}$$

Let $X(\Delta, \Lambda) \subseteq V$ be the subspace with $v = \sum t_i v_i \in X(\Delta, \Lambda)$ exactly when $t_i = 0$ for $i \in \Delta$; $t_i = t_j$ if i, j lie in the same block of Λ (either uncoupled or in a couple); and $t_i = -t_j$ if i, j lie in different blocks of the same coupled block. Then the map $X(\Delta, \Lambda) \mapsto (\Delta, \Lambda)$ is a lattice isomorphism $\mathcal{H}(B_n) \to \mathcal{T}$.

If S is a set of atoms in $\mathcal{H}(B_n)$, let Γ_S be the graph with vertex set $\{1, \ldots, n\}$ and edges given by the scheme:



(a) (b) (c)

A *circuit* is a closed path of type (a) and (b) edges, and a circuit is *odd* if it contains an odd number of (b) type edges, and *even* otherwise.

If $\bigvee S = X(\Delta, \Lambda) \in \mathcal{H}(B_n)$, then a vertex i of Γ_S is contained in Δ if and only if for all $v = \sum t_i v_i \in X(\Delta, \Lambda)$ we have $t_i = 0$. In particular, $i \in \Delta$ if and only if every vertex in the connected component of i is in Δ. Otherwise, the vertices in this component form a block or coupled block of Λ.

If a component contains a vertex i incident with an edge of type (c) above, then $t_i = 0$, and so $t_j = 0$, for all $v \in X(\Delta, \Lambda)$ and all the vertices j in the component. We thus have all the vertices of the component in Δ. Similarly if the connected component contains an odd circuit, for then $t_i = -t_i$ for each vertex i in the circuit, and all the vertices are in Δ too. On the other hand, suppose the component has no (c) edges and all circuits even. Label a vertex by 1, and propagate the labelling through the component by giving vertices joined by (a) edges the same label and vertices joined by (b) edges labels that are negatives of each other. The absence of odd circuits means this labelling can be carried out consistently. Label the remaining vertices of Γ_S by 0, to give an $v \in X(\Delta, \Lambda)$ with $t_i \neq 0$ for i some vertex of our component, and so the component gives a block or coupled block.

We conclude that the vertices of a component of Γ_S lie in Δ exactly when the component has a (c) edge or contains an odd circuit. We claim that S is independent exactly when each component of Γ_S has one of the forms

– a tree of (a) and (b) type edges together with at most one (c) type edge;
– contains a unique odd circuit, no (c) type edges, and removing one (hence any) edge of the circuit gives a tree.

For, if the component contains no (c) edges and no odd circuits, then its vertices contribute a block or coupled block to Λ, and by (1.4), its edges are independent exactly when there are $\sum(|\Lambda_{i1}| + |\Lambda_{i2}| - 1) + \sum(|\Lambda_i| - 1)$ of them; in other words, when the number of edges is one less than the number of vertices. Thus we have a tree of (a) and (b) edges.

If the component contains a (c) edge then its vertices are in Δ, and by (1.4) its edges are independent when there are the same number of them as there are vertices. Removing the (c) edge gives a connected graph with number of edges one less than the number of vertices, hence a tree. The original component was thus a tree of (a) and (b) edges with a single (c) edge.

Finally, if the component contains an odd circuit, then for the edges to be independent it cannot have any (c) edges by the previous paragraph. Again the vertices are in Δ and so for independence the numbers of edges and vertices must be the same. Removing an edge from the circuit must give a tree as in the previous paragraph. In particular, the circuit is unique.

We finish with the minimally dependent sets. A *branch vertex* of a tree of (a) and (b) edges is a vertex incident with at least three edges. A *line* is a tree of (a) and (b) edges containing at least one edge and no branch vertices. It contains exactly two vertices (its *ends*) incident with a single edge.

Proposition 1.7. *A set S of atoms in $\mathcal{H}(B_n)$ is minimally dependent precisely when Γ_S has one of the forms:*

(i) *an even circuit;*
(ii) *an odd circuit with a single (c) edge, or two odd circuits intersecting only in a single vertex;*
(iii) *a line, each end of which is incident with either a (c) edge or an odd circuit intersecting the line only in this end vertex.*

The proof splits into various cases depending on the number of type (c) edges in Γ_S. For the details we refer the reader to [**6**, Proposition 6].

Example 6 ($\mathcal{H}(D_n)$). This is very similar to the previous example, so we will be briefer. Let V be as before and $\mathscr{A}$ the hyperplanes with equations $x_i \pm x_j = 0$ for all $i \neq j$; equivalently, if Φ is the type D_n root system of Table 1 then $\mathscr{A}$ consists of the hyperplanes $(v_i \pm v_j)^\perp$, with $W(\Phi)$ acting on V by even signed permutations of the v_i (see also the end of §5.2). In particular we have a sub-arrangement of $\mathcal{H}(B_n)$. If $\mathcal{T}^\circ \subset \mathcal{T}$ consists of those (Δ, Λ) with $|\Delta| \neq 1$ then the isomorphism $\mathcal{H}(B_n) \to \mathcal{T}$ restricts to an isomorphism $\mathcal{H}(D_n) \to \mathcal{T}^\circ$. We have the same expression (1.4) for $\mathrm{rk}(\Delta, \Lambda)$ and the same conditions for a v to lie in $X(\Delta, \Lambda)$ as in the previous example.

If S is set of atoms in $\mathcal{H}(D_n)$, let Γ_S be the graph with vertex set $\{1, \ldots, n\}$ and edges of types (a) and (b) above. The arguments from here on are what you get if you drop the (c) type edges from all the arguments in the previous example. Thus, the vertices of a component of Γ_S

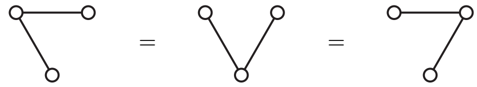
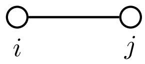
(A1) 

FIGURE 3. *Relations for the intersection lattice $\mathcal{H}(A_n)$.*

lie in Δ exactly when the component contains an odd circuit. It follows that S is independent when each component of Γ_S is either a tree of (a) and (b) type edges or contains a unique odd circuit, removing one (hence any) edge of which gives a tree. The equivalent version of Proposition 1.7 gives S minimally dependent when Γ_S is one of the forms: (i). an even circuit; (ii). two odd circuits intersecting only in a single vertex; or (iii). a line, each end of which is incident with an odd circuit intersecting Γ_S only in this end vertex.

We are now ready to give our presentations for the three classical reflection arrangements. In each case we have replaced the *(Idem3a)* family of relations given in Theorem 1.6 by a smaller set.

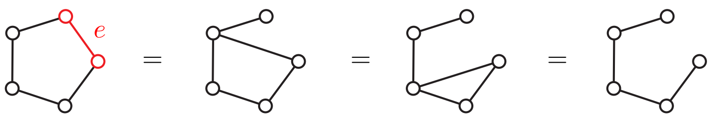
The intersection lattice $\mathcal{H}(A_{n-1})$ or partition lattice $\Pi(n)$: has generators



$(1 \leq i \neq j \leq n)$

and relations $(A0)$: the generators are commuting idempotents, and the relation $(A1)$ of Figure 3, which holds for all triples $\{i,j,k\}$. See also [**8**, Theorem 2]. As the generators commute the relations in Figure 3 can be interpreted unambiguously and can be applied to a graph fragment while leaving the rest untouched. Observe that multiplying together any two of the graphs in Figure 3 gives an *(Idem3b)* relation of the form: "a triangle equals a triangle minus an edge".

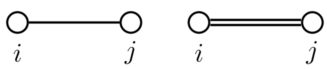
To see the presentation, the *(Idem3a)* relations for $\mathcal{H}(A_{n-1})$ are of the form $\Gamma_S = \Gamma_S \setminus \{e\}$ where Γ_S is a circuit and e some edge of it. Given such a circuit, repeated applications of the relations $(A1)$, as in say,

$\Gamma_S =$ 

$= \Gamma_S \setminus \{e\}$

move e anticlockwise around the circuit until we have a triangle, from which the edge can then be removed. Thus the *(Idem3a)* relations follow from the relations $(A0)$-$(A1)$.

The intersection lattice $\mathcal{H}(B_n)$: has generators



$(1 \leq i \neq j \leq n)$

$(1 \leq i \leq n)$

and relations $(B0)$: the generators are commuting idempotents and the $(B1)$-$(B4)$ of Figure 4. The relations $(B1)$, $(B2)$ and $(B4)$ hold for all triples $\{i,j,k\}$ and $(B3)$ holds for all pairs $\{i,j\}$.

That these relations hold in $\mathcal{H}(B_n)$ follows by checking that the corresponding subspaces are the same, i.e. if $\Gamma_S, \Gamma_{S'}$ are two graphs differing only by applying one of these relations to

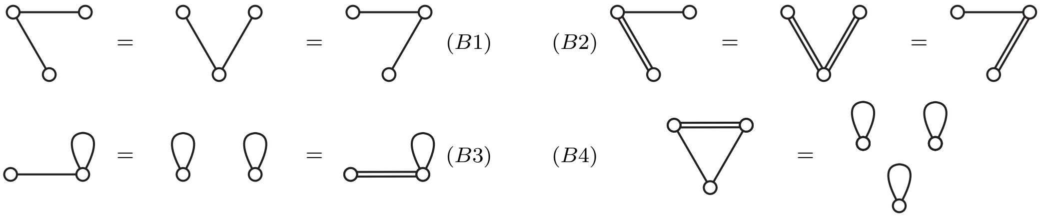


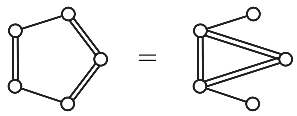
FIGURE 4. *Relations for the intersection lattice $\mathcal{H}(B_n)$.*

some fragment, then $\bigvee S = \bigvee S'$ in the intersection lattice. For example, let the vertices in the relations $(B4)$ be labelled anti-clockwise as i, j and k. If $v = \sum t_i v_i \in \bigvee S$ (the left hand side) then we have $t_i = t_j = t_k$ and $t_i = -t_k$, hence $t_i = t_j = t_k = 0$; similarly for $v \in \bigvee S'$. As all the other t's are the same we get our equality.

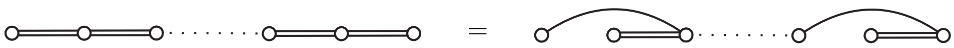
The presentation follows by showing that if Γ_S is one of the graphs in Proposition 1.7, and e is some edge of it, then the *(Idem3b)* relation $\Gamma_S = \Gamma_S \setminus \{e\}$ follows from $(B0)$-$(B4)$. We start with a series of relations that can be deduced from (B0)-(B4):

(i). If Γ is a circuit of type (a) edges, then $\Gamma = \Gamma \setminus \{e\}$ for any edge e.

(ii). If Γ is a circuit of type (a) and (b) edges then it can be replaced by a circuit composed entirely of (b) edges, as for example in:

$\Gamma =$  $= \Gamma'$

(iii). A fragment of $2m$ consecutive (b) edges can, by the relations $(B2)$, be replaced by a connected fragment containing m consecutive (a) edges:

$\Gamma =$  $= \Gamma'$

(iv). A fragment of consecutive (a) edges can be augmented:

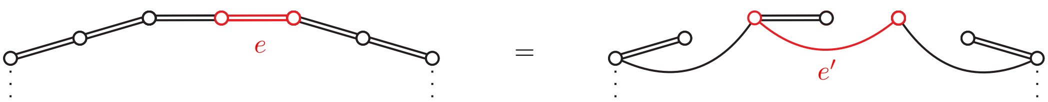
$\Gamma =$  $= \Gamma'$

(v). Let Γ be connected containing a (c) edge, and Γ' on the same vertices, *each* incident with a (c) edge, and having no other edges. Then, using the relations $(B3)$, $\Gamma = \Gamma'$.

(vi). Let Γ contain an odd circuit and Γ' a graph on the same vertex set as Γ, each of which is incident with a (c) edge, and having no other edges. Then $\Gamma = \Gamma'$.

Now let Γ_S be a graph of the form given in part 3 of Proposition 1.7 and $e \in \Gamma_S$ some edge. Then every component of both Γ and $\Gamma_S \setminus \{e\}$ contains an odd circuit and/or a (c) edge, hence by (v)-(vi) there is a Γ' such that $\Gamma = \Gamma' = \Gamma \setminus \{e\}$ can be deduced from $(B1)$-$(B4)$. Similarly for Γ_S of the form given in part 2 of Proposition 1.7.

Finally, let Γ_S be an even circuit as in part 1 of Proposition 1.7 and e a type (b) edge in this circuit (if no such e exists then we are done by (i)). Applying (ii) gives the fragment below left, and this equals the fragment at the right by (iii):



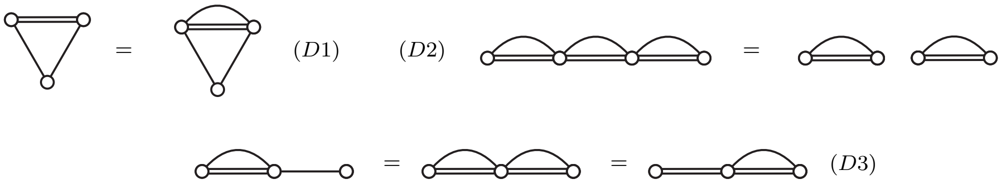
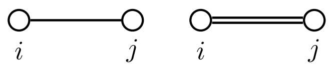


FIGURE 5. *Relations for the intersection lattice $\mathcal{H}(D_n)$: relations $(B1)$ and $(B2)$ from Figure 4 together with relations $(D1)$-$(D3)$ above.*

Applying (i), we can remove the edge e' and then run the process backwards to get $\Gamma_S \setminus \{e\}$. If instead e is a type (a) edge then the argument is similar.

The intersection lattice $\mathcal{H}(D_n)$: has generators



$(1 \leq i \neq j \leq n)$

and relations $(D0)$: the generators are commuting idempotents, $(B1)$ and $(B2)$ of Figure 4, and $(D1)$-$(D3)$ of Figure 5. The relations $(D1)$ and $(D3)$ in Figure 5 hold for all triples $\{i, j, k\}$ and the relations $(D2)$ for all 4-tuples $\{i, j, k, \ell\}$.

The proof of the presentation is very similar to the $\mathcal{H}(B_n)$ case. See [**6**] for more details.

2. *A presentation for reflection monoids*

We now return to the specifics of reflection monoids and give a presentation (Theorem 2.1 below) for those reflection monoids $M(W, \mathcal{S})$ where $W \subset GL(V)$ is a finite reflection group and $\mathcal{S}$ a graded atomic system of subspaces of V for W. We also give the analogous presentation when $\mathcal{S}$ is a system of subsets of some set E. The main technical tool is the presentation for factorizable inverse monoids found in [**4**].

Let V be a finite dimensional real vector space and $W \subset GL(V)$ a finite real reflection group with generating reflections S and full set of reflections $T = W^{-1}SW$. Let $\mathscr{A} = \{H_t \subset V \mid t \in T\}$ be the reflecting hyperplanes of W. Suppose also that:

(P1). $\mathcal{S}$ is a finite system of subspaces in V for W, and that via $X \leq Y$ if and only if $X \supseteq Y$, the system is a graded (by $\text{rk}X = \text{codim}\, X := \dim V - \dim X$) atomic $\vee$-semilattice with atoms A. We have $\text{rk}\mathcal{S} = \dim V - \dim \bigvee_{\mathcal{S}} X$. The W-action preserves the grading, and in particular we have $AW = A$. If $a_1, \ldots, a_k$ are distinct atoms let O_k be a set of orbit representatives for the W-action $\{a_1, \ldots, a_k\} \overset{w}{\mapsto} \{a_1 w, \ldots, a_k w\}$.

(P2). We use the Greek equivalents of Roman letters to indicate a fixed word for an element in terms of generators. In particular, the reflection group W has presentation with generators the $s \in S$ and relations $(st)^{m_{st}} = 1$ for $s, t \in S$, where $m_{st} = m_{ts} \in \mathbb{Z}^{\geq 1} \cup \{\infty\}$ with $m_{st} = 1$ if and only if $s = t$. For each $w \in W$ we fix a word ω for w in the reflections $s \in S$ (subject to $\sigma = s$).

(P3). Now for the action of W on $\mathcal{S}$: For each $a \in A$ fix a representative atom $a' \in O_1$ and a $w \in W$ with $a = a'w$ subject to $w = 1$ if $a \in O_1$. Now define the word α to be $\omega^{-1} a' \omega$. If w is an arbitrary element of W and $a \in A$ then by α^ω we mean the word obtained in this way for $aw \in A$. Note that this is *not* necessarily $\omega^{-1} a \omega$. For $e \in \mathcal{S}$, fix a join $e = \bigvee a_i \, (a_i \in A)$ and define $\varepsilon := \prod \alpha_i$.

(P4). Let $\{H_{s_1}, \ldots, H_{s_\ell}\}$ be representatives for the W-action on $\mathscr{A}$ with the $s_i \in S$. For example, drop the even labeled edges in the Coxeter symbol for W and choose one s from each component of the resulting graph. For each $i = 1 \ldots, \ell$ consider the set of $X \in \mathcal{S}$ with the property that $H_{s_i} \supseteq X$. If this set is non-empty them form the pairs (e, s_i) for each $e \in \mathcal{S}$ minimal in this set. Let *Iso* be the set of all such pairs.

With the notation established we have:

THEOREM 2.1. *Let $W \subset GL(V)$ be a finite real reflection group and $\mathcal{S}$ a graded atomic system of subspaces for W. Then the reflection monoid $M(W, \mathcal{S})$ has a presentation with*

$$\begin{array}{rll}
\textit{generators:} & s \in S, a \in O_1. & \\
\textit{relations:} & (st)^{m_{st}} = 1,\ (s,t \in S), & \textit{(Units)}\\
& a^2 = a,\ (a \in O_1), & \textit{(Idem1)}\\
& \alpha_1\alpha_2 = \alpha_2\alpha_1,\ (\{a_1, a_2\} \in O_2), & \textit{(Idem2)}\\
& \alpha_1 \ldots \alpha_{k-1} = \alpha_1 \ldots \alpha_{k-1}\alpha,\ (\{a_1, \ldots, a_{k-1}, a\} \in O_k) & \\
& \quad \textit{with } a_1, \ldots, a_{k-1}, (3 \leq k \leq rk\mathcal{S}) \textit{ independent and } a \leq \bigvee a_i, & \textit{(Idem3)}\\
& s\alpha = \alpha^s s,\ (s \in S, a \in A), & \textit{(RefIdem)}\\
& \varepsilon s = \varepsilon,\ (e, s) \in \textit{Iso}. & \textit{(Iso)}
\end{array}$$

To prove Theorem 2.1 we start with a presentation for an arbitrary factorizable inverse monoid [**4**, Theorem 6], interpret the various ingredients in the setting of a reflection monoid, and then remove relations and generators.

Suppose then that M is a factorizable inverse monoid with units $W = W(M)$ and idempotents $E = E(M)$. Let $\langle S \,|\, R_W \rangle$ and $\langle A \,|\, R_E \rangle$ be monoid presentations for W and E. For $w \in W$, fix a word ω for w in the $s \in S$ and similarly for $e \in E$ fix a word ε in the $a \in A$, with the usual conventions applying when $w \in S$ and $e \in A$. For $w \in W$ and $e \in E$ we have $w^{-1}ew \in E$, and by ε^ω we mean the chosen word for $w^{-1}ew$ in the $a \in A$ (it turns out that we will only have need for the notation ε^ω in the case that $w \in S$ and $e \in A$). For each $e \in E$ let $W_e = \{w \in W \,|\, ew = e\}$ be the idempotent stabilizer, and $S_e \subseteq W_e$ a set of monoid generators for W_e.

THEOREM 2.2 ([**4**]). *The factorizable inverse monoid M has a presentation with,*

$$\begin{array}{rl}
\textit{generators:} & s \in S, a \in A,\\
\textit{relations:} & R_W, R_E,\\
& sa = a^s s,\ (s \in S, a \in A),\\
& \varepsilon\omega = \varepsilon,\ (e \in E, w \in S_e).
\end{array}$$

This theorem will give presentations for *arbitrary* reflection monoids. Here, W is a reflection group, and in the real case we take the standard Coxeter presentation for W. If moreover W is finite, then by Steinberg's theorem the W_e are parabolic subgroups of W, so S_e consists of reflections. Although the presentation thus obtained looks much the same as that in Theorem 2.2, it is, in fact, more precise and economical. However, under the assumptions (P1)-(P4) we can be much more explicit and as all the natural examples satisfy these conditions, we concentrate on this case.

We now interpret the various ingredients in the presentation. The s of Theorem 2.2 are the generating reflections s of the reflection group W. Identifying $X \in \mathcal{S}$ with the partial identity on X, the A of Theorem 2.2 are the atomic subspaces A of the system $\mathcal{S}$. If $s \in S$ and $a \in A$ then $as \in A$, so that α^s is just another one of the symbols in A, and we write a^s for this symbol. If $X \in \mathcal{S}$ then W_X is the isotropy group $W_X = \{w \in W \mid yw = y \text{ for all } y \in X\}$, a group generated by reflections, and so we can take S_X to consist of those $t \in T$ with $H_t \supseteq X$.

As an intermediate step we thus have the presentation for $M(W, \mathcal{S})$ with

$$
\begin{array}{rlr}
\text{generators:} & s \in S, a \in A, & \\
\text{relations:} & (st)^{m_{st}} = 1\, (s, t \in S), & (a) \\
& a^2 = a\, (a \in A), & (b) \\
& a_1 a_2 = a_2 a_1\, (a_1, a_2 \in A), & (c) \qquad (*) \\
& a_1 \dots a_{k-1} = a_1 \dots a_{k-1} a\, (a_i, a \in A) & (d) \\
& \quad \text{for } a_1, \dots, a_{k-1}, (3 \leq k \leq \text{rk}\mathcal{S}) \text{ independent and } a \leq \bigvee a_i. & \\
& sa = a^s s\, (s \in S, a \in A), & (e) \\
& \varepsilon\tau = \varepsilon,\, (e \in \mathcal{S}, t \in S_e). & (f)
\end{array}
$$

Deducing Theorem 2.1 is now a matter of thinning out relations and generators from $(*)$, using the W-action on $\mathcal{S}$.

LEMMA 2.3. *Let $w \in W$ and for $j = 1, \dots, k$ let $a_j \in A$ and $a'_j = a_j \cdot w$. Let $W_i(x_1, \dots, x_k)$ for $i = 1, 2$ be words in the free monoid on the x_i. Then the relation $W_1(a_1, \dots, a_k) = W_2(a_1, \dots, a_k)$ and the relations (e) imply the relation $W_1(a'_1, \dots, a'_k) = W_2(a'_1, \dots, a'_k)$.*

Proof. If $s \in S, a \in A$ and $as = a' \in A$, then relations (e) of $(*)$ give a relation $sa = a's$, hence $a' = sas$. By induction, if $a' = aw$ for some $w \in W$ we have the relation $a' = \omega^{-1} a \omega$. Thus, for all j we have $a'_j = \omega^{-1} a_j \omega$ so that $W_i(a'_1, \dots, a'_k) = W_i(\omega^{-1} a_1 \omega, \dots, \omega^{-1} a_k \omega) = \omega^{-1} W_i(a_1, \dots, a_k) \omega$, and the result follows. □

LEMMA 2.4. *Let $e \in \mathcal{S}$ and $t \in T$ be such that $H_t \supseteq e$. Then there is an $(e', s) \in$ Iso and $w \in W$ with $t = w^{-1} s w$ and $e' w \supseteq e$ and the relation $\varepsilon\tau = \varepsilon$ of $(*)$ implied by the (Iso) relation $\varepsilon' \sigma = \varepsilon'$ and the relations (a)-(e).*

Proof. There is an element s with $H_s w = H_t$. Thus $H_s \supseteq e w^{-1}$ and there is an element e' with $H_s \supseteq e' \supseteq e w^{-1}$ and $(e', s) \in$ *Iso*. This pair satisfies the requirements of the Lemma and moreover, $e = e \cdot e' w$ in (the monoid) $\mathcal{S}$, so that the relations (a)-(e) of $(*)$ give $\varepsilon = \varepsilon \omega^{-1} \varepsilon' \omega$ and $\tau = \omega^{-1} s \omega$. Thus

$$\varepsilon\tau = \varepsilon \omega^{-1} \varepsilon' \omega \omega^{-1} s \omega = \varepsilon \omega^{-1} \varepsilon' s \omega = \varepsilon \omega^{-1} \varepsilon' \omega = \varepsilon.$$

□

Proof of Theorem 2.1. Lemma 2.4 allows us to thin out family (f) in $(*)$ to give the *(Iso)* relations of Theorem 2.1. Lemma 2.3 allows us to thin out the families (b)-(d) in $(*)$ to involve just the orbit representatives as in Theorem 2.1. The *(Units)* relations we leave untouched.

Finally a generator $a \in A$ can be expressed as $\alpha = \omega^{-1} a_0 \omega$ for some $a_0 \in O_1$, and this allows us to thin these generators, replacing each occurrence of a in the *(RefIdem)* relations by α. $\square$

REMARKS. There are a number of variations on Theorem 2.1:

(i) There is a completely analogous presentation when $M(W, \mathcal{S})$ is a monoid of partial permutations. Let E be a set, $W = (W, S)$ a (not necessarily finite) Coxeter system acting faithfully on E and $\mathcal{S}$ a graded atomic system of subsets of E for W. For $t \in T = W^{-1}SW$ let $H_t \subseteq E$ be the set of fixed points of t and $\mathscr{A} = \{H_t \,|\, t \in T\}$. Notice that $H_t \cdot w = H_{w^{-1}tw}$ so there is an induced W-action on $\mathscr{A}$. There is one condition that we must impose: for any $e \in \mathcal{S}$ the isotropy group W_e is generated by reflections; indeed, by the $t \in T$ with $H_t \supseteq e$. Adapting (P1)-(P4), the presentation of Theorem 2.1 now goes straight through for $M(W, \mathcal{S})$.

(ii) If $\mathcal{S}$ is a geometric lattice, then the *(Idem3)* relations of Theorem 2.1 can be replaced by

$$\widehat{\alpha}_1 \ldots \alpha_k = \cdots = \alpha_1 \ldots \widehat{\alpha}_k,\ (a_1, \ldots, a_k \in O_k)$$

with $a_1, \ldots, a_k$ minimally dependent and $3 \leq k \leq \mathrm{rk}\mathcal{S}$.

(iii) The sets O_k of (P1) can sometimes be hard to describe; their definitions can be varied in two ways–either by changing the group or the set on which it acts (or both, as in §5.2). This has the effect of introducing more relations. If W' is a subgroup of W we can replace O_k by O'_k, a set of orbit representatives for the W-action restricted to W'. On the other hand, it may be more convenient to describe orbit representatives for the W-action $(a_1, \ldots, a_k) \overset{w}{\mapsto} (a_1 \cdot w, \ldots, a_k \cdot w)$ on ordered *k-tuples* of distinct atoms. The commuting of the idempotents then allows us to return to sets $\{a_1, \ldots, a_k\}$.

3. *Boolean reflection monoids*

In [**7**, §5] we introduced the Boolean reflection monoids, formed from a Weyl group $W(\Phi)$ for $\Phi = A_{n-1}, B_n$ or D_n, and the Boolean system $\mathscr{B}$. In this section we find the presentations given by Theorem 2.1. In particular, we recover Popova's presentation [**22**] for the symmetric inverse monoid by interpreting it as the Boolean reflection monoid of type A.

Recall from [**7**, §5] that V is a Euclidean space with basis $\{v_1, \ldots, v_n\}$ and inner product $(v_i, v_j) = \delta_{ij}$, with $\Phi \subset V$ a root system from Table 1 and $W(\Phi) \subset GL(V)$ the associated reflection group. The Coxeter generators for $W(\Phi)$ are given in the third column of Table 1: let s_i $(1 \leq i \leq n-1)$ be the reflection in the hyperplane orthogonal to $v_{i+1} - v_i$, with s_0 the reflection in v_1 (type B) or in $v_1 + v_2$ (type D).

For $J \subseteq X = \{1, \ldots, n\}$ let

$$X(J) = \bigoplus_{j \in J} \mathbb{R} v_j \subseteq V,$$

and $\mathscr{B} = \{X(J) \,|\, J \subseteq X\}$ with $X(\varnothing) = \mathbf{0}$. Then by [**7**, §5], $\mathscr{B}$ is a system in V for $W(\Phi)$–the *Boolean* system–and $M(\Phi, \mathscr{B}) := M(W(\Phi), \mathscr{B})$ is the *Boolean reflection monoid of type* Φ.

We've obviously seen the poset $(\mathscr{B}, \supseteq)$ before: it is isomorphic to the Boolean lattice $\mathscr{B}_X$ of §1.1, with atoms A the $a_i := X(1, \ldots, \widehat{i}, \ldots, n) = v_i^{\perp}$. For $k \leq n$ the $W(\Phi)$-action on A is k-fold transitive, so the O_k each contain a single element. We choose $O_1 = \{a_1\}$ and O_2 the pair $\{a_1, a_2\}$. Rather than a_1 we will write $a \in O_1$ for our single idempotent generator. If $i > 1$ then let w be the reflection $s_{v_1 - v_i}$ so that $a_i = aw$; let $\omega := s_1 \ldots s_{i-1}$ so that

$$\alpha_i := (s_{i-1} \ldots s_1) a (s_1 \ldots s_{i-1}) \tag{3.1}$$

Any $e \in \mathscr{B}$ can be written uniquely as $e = a_{i_1} \vee \cdots \vee a_{i_k}$ for $i_1 < \cdots < i_k$; write $\varepsilon := \alpha_{i_1} \ldots \alpha_{i_k}$.

The result is that the Boolean reflection monoids have generators the s_i and a single idempotent a, with the *(Idem1)* relations $a^2 = a$, and the *(Idem2)* relations $a\alpha_2 = \alpha_2 a$, or $as_1as_1 = s_1as_1a$.

We saw at the end of §1.1 that any set of atoms in $\mathscr{B}$ is independent, so the *(Idem3)* relations are vacuous. Note also the "thinning" effect of the $W(\Phi)$-action: the n generators and $n + \frac{1}{2}n(n-1)$ relations of §1.1 have been reduced to just one generator and two relations.

Now to the *(Iso)* relations. Dropping the even labeled edges from the symbols in Table 1 and choosing an $s \in S$ from each resulting component gives representatives H_{s_1} in types A and D and H_{s_0}, H_{s_1} in type B. If $X(J) \in \mathscr{B}$ is to be minimal with $H_{s_1} \supseteq X(J)$ then J is minimal with $1, 2 \notin J$, i.e. $J = \{3, \dots, n\}$ and $X(J) = a_1 \vee a_2$ (compare this with the calculation at the end of Example 7 in §5.2). Similarly with H_{s_0} we have $X(J) = a_1$, and so

Φ	*Iso*
A_{n-1}	$(a_1 \vee a_2, s_1)$
B_n	$(a_1 \vee a_2, s_1), (a_1, s_0)$
D_n	$(a_1 \vee a_2, s_1)$

The *(Iso)* relations are thus $as_1a = as_1as_1$ in all cases, together with $as_0 = a$ in type B.

This completes the presentation given by Theorem 2.1 for the Boolean reflection monoids. But it turns out that the *(RefIdem)* relations can be significantly reduced in number. For all three Φ we have *(Units)* relations $(s_is_{i+1})^3 = 1$ for $1 \leq i \leq n-2$, which we use in their "braid" form, $s_{i+1}s_is_{i+1} = s_is_{i+1}s_i$. Then:

LEMMA 3.1. *The relations $s_i\alpha_j = \alpha_j^{s_i}s_i$ for $1 \leq i \leq n-1$ and $1 \leq j \leq n$ are implied by the (Units) relations and the relations $s_ia = \alpha^{s_i}s_i$ for $1 \leq i \leq n-1$, ie.: the relations $s_ia = as_i$, $(i \neq 1)$.*

Proof. We have

$$a_js_i = \begin{cases} a_{j-1}, & i = j-1, \\ a_{j+1}, & i = j, \\ a_j, & i \neq j-1, j, \end{cases}$$

hence $\alpha_j^{s_i}$ is one of the words α_{j-1} $(i = j-1)$ or α_{j+1} $(i = j)$ or α_j (otherwise) chosen in (3.1). There are then four cases to consider: (i). $1 \leq i < j-1$:

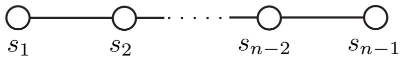
$$\begin{aligned} s_i\alpha_j &= s_i(s_{j-1}\dots s_1)a(s_1\dots s_{j-1}) = (s_{j-1}\dots s_is_{i+1}s_i\dots s_1)a(s_1\dots s_{j-1}) \\ &= (s_{j-1}\dots s_{i+1}s_is_{i+1}\dots s_1)a(s_1\dots s_{j-1}) = (s_{j-1}\dots s_1)s_{i+1}a(s_1\dots s_{j-1}) \\ &= (s_{j-1}\dots s_1)as_{i+1}(s_1\dots s_{j-1}) = (s_{j-1}\dots s_1)a(s_1\dots s_{i+1}s_is_{i+1}\dots s_{j-1}) \\ &= (s_{j-1}\dots s_1)a(s_1\dots s_is_{i+1}s_i\dots s_{j-1}) = (s_{j-1}\dots s_1)a(s_1\dots s_{j-1})s_i \\ &= \alpha_js_i, \end{aligned}$$

where we have used the braid relations and the commuting of s_{i+1} and a. (ii). Suppose that $j < i \leq n-1$: s_i commutes with $s_1, \dots, s_{j-1}$ and a, giving the result immediately. (iii). $i = j-1$: $s_{j-1}\alpha_j = s_{j-1}(s_{j-1}\dots s_1)a(s_1\dots s_{j-1}) = (s_{j-2}\dots s_1)a(s_1\dots s_{j-1})s_{j-1}s_{j-1} = \alpha_{j-1}s_{j-1}$ (iv). $i = j$: $s_j\alpha_j = s_j(s_{j-1}\dots s_1)a(s_1\dots s_{j-1}) = (s_j\dots s_1)a(s_1\dots s_{j-1})s_js_j = \alpha_{j+1}s_j$. □

Putting it all together in the type A case we get the following presentation:

The Boolean reflection monoid of type A:

$$M(A_{n-1},\mathscr{B})=\langle s_1,\dots,s_{n-1},a\,|\quad \begin{aligned}&(s_is_j)^{m_{ij}}=1,\, a^2=a,\\ &s_ia=as_i\ (i\neq 1),\\ &as_1a=as_1as_1=s_1as_1a\rangle\end{aligned}$$



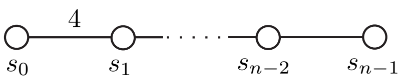
Recall that the m_{ij} can be read off the Coxeter symbol, with the nodes joined by an edge labelled m_{ij} if $m_{ij}\geq 4$, an unlabelled edge if $m_{ij}=3$, no edge if $m_{ij}=2$ (and $m_{ij}=1$ when $i=j$). The relation $s_ia=\alpha^{s_i}s_i$ is vacuous when $i=1$.

Remark. We saw in [**7**, §3.1] that $M(A_{n-1},\mathscr{B})$ is isomorphic to the symmetric inverse monoid $\mathscr{I}_n$ – we thus recover Popova's presentation [**22**] for the symmetric inverse monoid.

Now to the type B Boolean reflection monoids, where the relations $s_0\alpha_j=\alpha_j^{s_0}s_0$, $(1\leq j\leq n)$ are implied by the (Units) relations and the relation $s_0\alpha_2=\alpha_2s_0$, i.e. $s_0s_1as_1=s_1as_1s_0$.

The Boolean reflection monoid of type B:

$$M(B_n,\mathscr{B})=\langle s_0,\dots,s_{n-1},a\,|\quad \begin{aligned}&(s_is_j)^{m_{ij}}=1,\, a^2=a,\\ &s_ia=as_i\ (i\neq 1),\ as_0=a,\\ &s_0s_1as_1=s_1as_1s_0,\\ &as_1as_1=s_1as_1a=as_1a\rangle.\end{aligned}$$

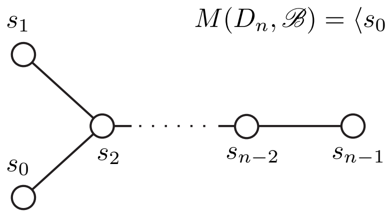


Remark. We saw in [**7**, §5] that just as the Weyl group $W(B_n)$ is isomorphic to the group $\mathfrak{S}_{\pm n}$ of signed permutations of $X=\{1,2,\dots,n\}$ (see also §5.2), so the Boolean reflection monoid $M(B_n,\mathscr{B})$ turns out to be isomorphic to the monoid of *partial* signed permutations $\mathscr{I}_{\pm n}:=\{\pi\in\mathscr{I}_{X\cup -X}\,|\,(-x)\pi=-(x\pi)\text{ and }x\in\operatorname{dom}\pi\Leftrightarrow -x\in\operatorname{dom}\pi\}$. See also [**30**].

And so finally to the type D Boolean reflection monoids, where the relations $s_0\alpha_j=\alpha_j^{s_0}s_0$ for $1\leq j\leq n$ are implied by $s_0a=\alpha_2s_0$, $s_0\alpha_3=\alpha_3s_0$ and the relations for W.

The Boolean reflection monoid of type D:

$$M(D_n,\mathscr{B})=\langle s_0,\dots,s_{n-1},a\,|\quad \begin{aligned}&(s_is_j)^{m_{ij}}=1,\, a^2=a,\\ &s_ia=as_i\ (i>1),\ s_0a=s_1as_1s_0,\\ &as_1a=as_1as_1=s_1as_1a,\\ &s_0s_2s_1as_1s_2=s_2s_1as_1s_2s_0\rangle.\end{aligned}$$



4. *Coxeter arrangement monoids*

We repeat §3 for the Coxeter arrangement monoids [**7**, §6]. Let $W=W(\Phi)\subset GL(V)$ be a reflection group with reflecting hyperplanes $\mathscr{A}=\{v^\perp\,|\,v\in\Phi\}$ and $\mathcal{H}=\mathcal{H}(\Phi)$ the intersection lattice of §§1.3-1.4: this is a system for W in V–the *Coxeter arrangement* system. We write

$M(\Phi, \mathcal{H})$ for the resulting *Coxeter arrangement monoid* of type Φ. We use the notation for the Coxeter generators from §3.

The atoms A for the system and the hyperplanes $\mathscr{A}$ coincide now. Drop even labeled edges from the symbols in Table 1 to get $O_1 = \{a\}$ in types A and D, or $\{a_1, a_2\}$ in type B, where

$$a = a_1 := (v_2 - v_1)^\perp \text{ and } a_2 := v_1^\perp;$$

giving generators the s_i of §3 and a for types A and D, or the s_i and a_1, a_2 for type B.

The *(Iso)* relations are particularly simple when the system and the intersection lattice are the same: the $(e, s) \in \mathit{Iso}$ consist of the representative $a = H_s$ and the s above. Thus, the relations are $as_1 = a$ for types A and D, or $a_1 s_1 = a_1$ and $a_2 s_0 = a_2$ in type B.

We deal with the remaining relations on a case by case basis.

4.1. *The Coxeter arrangement monoids of type A*

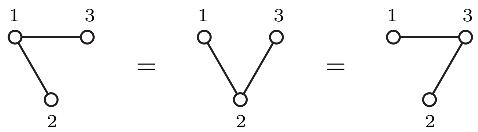
We have $\mathscr{A} = \{a_{ij} := (v_i - v_j)^\perp \,|\, 1 \leq i < j \leq n\}$, and write

$$\alpha_{ij} := \begin{cases} (s_{i-1} \dots s_1)(s_{j-1} \dots s_2)a(s_2 \dots s_{j-1})(s_1 \dots s_{i-1}), & \text{for } 2 \leq i < j \leq n, \\ (s_{j-1} \dots s_2)a(s_2 \dots s_{j-1}) & \text{for } i = 1 \text{ and } 2 \leq j \leq n, \end{cases} \tag{4.1}$$

with $\alpha_{12} := a_1$.

The isomorphism $\mathcal{H}(A_{n-1}) \to \Pi(n)$ of §1.4 and the isomorphism $W(A_{n-1}) \to \mathfrak{S}_n$ (written as $g(\pi) \mapsto \pi$) gives the $W(A_{n-1})$-action on $\mathcal{H}(A_{n-1})$ as $X(\Lambda)g(\pi) = X(\Lambda\pi)$, where $\Lambda\pi = \{\Lambda_1\pi, \dots, \Lambda_p\pi\}$. Thus, as $\mathfrak{S}_n$ acts 4-fold transitively on $\{1, \dots, n\}$, we take O_2 to be $\{a_{12}, a_{34}\}$ and $\{a_{12}, a_{23}\}$ when $n \geq 4$, giving *(Idem2)* relations $a\alpha_{34} = \alpha_{34}a$ and $a\alpha_{23} = \alpha_{23}a$. When $n = 2$ there is only one idempotent (hence no *(Idem2)* relations) and when $n = 3$ we take O_2 to be $\{a_{12}, a_{23}\}$.

We have the presentation for $\mathcal{H}(A_{n-1})$ of §1.4. Lemma 2.3 and the triple transitivity of $\mathfrak{S}_n$ on $\{1, \dots, n\}$ reduce the (A1) relations to:

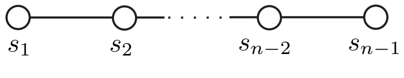


that is, $a\alpha_{13} = a\alpha_{23} = \alpha_{13}\alpha_{23}$.

As with the Boolean monoids, the *(RefIdem)* relations can be reduced in number. The relations $s_i\alpha_{jk} = \alpha_{jk}^{s_i}s_i$ for $1 \leq j < k \leq n$ and $1 \leq i \leq n-1$ are implied by the the *(Units)* relations and the relations $s_i a = \alpha_{12}^{s_i} s_i$ for $1 \leq i \leq n-1$, i.e. the relations $s_i a = as_i$, $(i \neq 2)$. The proof, which is a similar but more elaborate version of that for Lemma 3.1, is left to the reader. Finally, the relation $a\alpha_{23} = \alpha_{23}a$ simplifies to $a\alpha_{13} = \alpha_{13}a$ and this allows us to show that the final $(A1)$ relation $a\alpha_{13} = a\alpha_{23} = \alpha_{13}\alpha_{23}$ is redundant. Putting it all together we get:

The Coxeter arrangement monoid of type A_{n-1}:

$$M(A_{n-1}, \mathcal{H}) = \langle s_1, \dots, s_{n-1}, a \,|\, \begin{array}{l} (s_i s_j)^{m_{ij}} = 1,\, a^2 = a,\, as_1 = a, \\ s_i a = as_i \,(i \neq 2), a\alpha_{13} = \alpha_{13}a, \\ a\alpha_{34} = \alpha_{34}a\rangle. \end{array}$$



for $n \geq 4$ and where α_{ij} is given by (4.1). We leave the reader to make the necessary adjustments in the $n = 2, 3$ cases. We saw in [**7**, §2.2] that the type A Coxeter arrangement monoid is isomorphic to the monoid of uniform block permutations. See also [**8**].

4.2. *The Coxeter arrangement monoids of type B*

As in §4.1 we work with $n \geq 4$ and leave the simpler cases $n = 2, 3$ to the reader. We have $\mathscr{A}$ the a_{ij} from §4.1 together with

$$\{d_{ij} := (v_i + v_j)^\perp \,|\, 1 \leq i < j \leq n\} \text{ and } \{e_i := v_i^\perp \,|\, 1 \leq i \leq n\}.$$

Let α_{ij} be the expression defined in (4.1), except with a_1 instead of a, and

$$\delta_{ij} = \begin{cases} (s_{i-1} \dots s_1)(s_{j-1} \dots s_2) s_0 a_1 s_0 (s_2 \dots s_{j-1})(s_1 \dots s_{i-1}), & 2 \leq i < j \leq n, \\ (s_{j-1} \dots s_2) s_0 a_1 s_0 (s_2 \dots s_{j-1}), & i = 1 \text{ and } 2 < j \leq n, \end{cases} \tag{4.2}$$

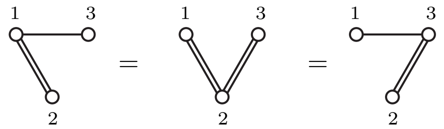
with $\delta_{12} := s_0 a_1 s_0$; and

$$\varepsilon_i := (s_{i-1} \dots s_1) a_2 (s_1 \dots s_{i-1}). \tag{4.3}$$

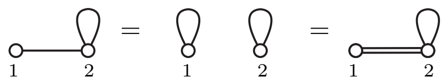
for $i > 1$ with $\varepsilon_1 := a_2$. One can build a combinatorial model for the action of $W(B_n)$ on $\mathcal{H}(B_n)$ much as in §4.1: the isomorphism $\mathcal{H}(B_n) \to \mathfrak{T}$ of §1.4 and the well known isomorphism $W(B_n) \to \mathfrak{S}_n \ltimes \mathbf{2}^n$ (see [**7**, §6.2] for notation) give the $W(B_n)$-action on $\mathcal{H}(B_n)$ as $X(\Delta, \Lambda) g(\pi, T) = X(\Delta\pi, \Lambda^T \pi)$. One deduces from this that $O_2 = \{\{a_1, a_{34}\}, \{a_1, a_{23}\}, \{a_1, d_{12}\}, \{a_2, a_{23}\}, \{a_1, a_2\} \; \{a_2, e_2\}\}$, and hence that *(Idem2)* relations are

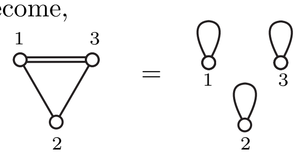
$$a_1\alpha_{34} = \alpha_{34}a_1, a_1\alpha_{23} = \alpha_{23}a_1, a_1\delta_{12} = \delta_{12}a_1, a_2\alpha_{23} = \alpha_{23}a_2, a_1a_2 = a_2a_1 \text{ and } a_2\varepsilon_2 = \varepsilon_2a_2.$$

We have the presentation for $\mathcal{H}(B_n)$ of §1.4, hence the relations $a_1\alpha_{13} = a_1\alpha_{23} = \alpha_{13}\alpha_{23}$ of §4.1; the family (B2) reduces to:



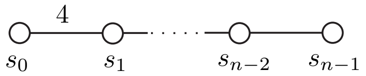
that is, $a_1\delta_{13} = \delta_{13}\delta_{23} = a_1\delta_{23}$; families (B3) and (B4) become,





or $a_1a_2 = \varepsilon_2a_2 = \delta_{12}a_2$ and $a_1\alpha_{23}\delta_{13} = a_2\varepsilon_2\varepsilon_3$. The (RefIdem) relations can be deduced from the (Units) relations and the relations $s_ia_1 = a_1s_i \, (i \neq 0, 2)$, $s_ia_2 = a_2s_i \, (i \neq 1)$, $s_0\alpha_{2j} = \alpha_{2j}s_0 \, (j > 2)$, $s_0\delta_{2j} = \delta_{2j}s_0 \, (j > 2)$, $s_1\delta_{12} = \delta_{12}s_1$ and $s_0\varepsilon_2 = \varepsilon_2s_0$. Finally, as in the type A case, the single $(A1)$ relation is redundant; moreover, the $a_1a_2 = \varepsilon_2a_2 = \delta_{12}a_2$ can be reduced to $a_1a_2 = \varepsilon_2a_2$.

The Coxeter arrangement monoid of type B_n*:*



$$\begin{aligned} M(B_n, \mathcal{H}) = \langle s_0, \dots, s_{n-1}, a_1, a_2 \,|\; & (s_is_j)^{m_{ij}} = 1, \, a_j^2 = a_j, a_1s_1 = a_1, a_2s_0 = a_2, \\ & s_ia_1 = a_1s_i \, (i \neq 0, 2), s_ia_2 = a_2s_i \, (i \neq 1), \\ & s_0\alpha_{2j} = \alpha_{2j}s_0, \, (j > 2), s_0\delta_{2j} = \delta_{2j}s_0, \, (j > 2), \\ & s_1\delta_{12} = \delta_{12}s_1, s_0\varepsilon_2 = \varepsilon_2s_0, a_j\alpha_{23} = \alpha_{23}a_j, \\ & a_1a_2 = a_2a_1 = a_2\varepsilon_2 = \varepsilon_2a_2, a_1\delta_{12} = \delta_{12}a_1, \\ & a_1\alpha_{34} = \alpha_{34}a_1, a_1\delta_{13} = \delta_{13}\delta_{23} = a_1\delta_{23}, \\ & a_1\alpha_{23}\delta_{13} = a_2\varepsilon_2\varepsilon_3\rangle. \end{aligned}$$

where α_{ij}, δ_{ij} and ε_i are given by (4.1)-(4.3).

REMARK. Just as the type A Coxeter arrangement monoid is isomorphic to the monoid of uniform block permutations, so the type B reflection monoid is isomorphic to the monoid of "uniform block signed permutations". See [**7**, §6.2] for details.

4.3. *The Coxeter arrangement monoids of type D*

The $\mathscr{A}$ are the a_{ij} and d_{ij} of §4.2; let α_{ij} be defined as in (4.1) and

$$\delta_{ij} = \begin{cases} (s_{i-1}\dots s_1)(s_{j-1}\dots s_2)g^{-1}ag(s_2\dots s_{j-1})(s_1\dots s_{i-1}), & 2 \le i < j \le n, \\ (s_{j-1}\dots s_2)g^{-1}ag(s_2\dots s_{j-2}), & i = 1 \text{ and } 2 < j \le n, \end{cases} \tag{4.4}$$

with $g = s_2 s_1 s_0 s_2$ and $\delta_{12} := g^{-1}ag$.

There is also a combinatorial model for the action of $W(D_n)$ on $\mathcal{H}(D_n)$. We refer the reader to [**7**, §6.2] or [**20**, §6.4] for details, noting that $O_2 = \{\{a, a_{34}\}, \{a, a_{23}\}, \{a, d_{12}\}\}$ when $n > 4$, while for $n = 4$ we have $\{a, d_{34}\}$ as well. The *(Idem2)* relations are thus

$$a\alpha_{34} = \alpha_{34}a, a\alpha_{23} = \alpha_{23}a, a\delta_{12} = \delta_{12}a,$$

together with $a\delta_{34} = \delta_{34}$ when $n = 4$.

The presentation for $\mathcal{H}(D_n)$ of §1.4 together with Lemma 2.3 give the relations $a\alpha_{13} = a\alpha_{23} = \alpha_{13}\alpha_{23}$ and $a\delta_{13} = \delta_{13}\delta_{23} = a\delta_{23}$ of §4.2, as well as

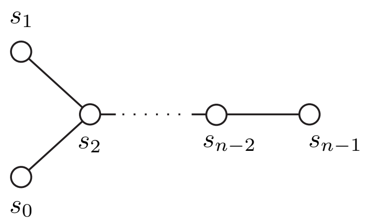
$$a\alpha_{23}\delta_{23} = a\delta_{12}\delta_{23} = a\alpha_{23}\delta_{12}\delta_{23},$$

$a\alpha_{23}\delta_{13} = a\alpha_{13}\alpha_{23}\delta_{13}$ and

$$a\alpha_{23}\alpha_{34}\delta_{12}\delta_{23}\delta_{34} = a\alpha_{34}\delta_{12}\delta_{34}.$$

The *(RefIdem)* relations can be deduced from the relations $s_i a = a s_i\, (i \ne 2)$, $s_0\alpha_{3k} = \alpha_{3k}s_0\, (k > 3)$, $s_0\delta_{3k} = \delta_{3k}s_0\, (k > 3)$ and $s_3\delta_{12} = \delta_{12}s_3$. Finally, we have the running redundancies of the previous two cases together with $a\alpha_{23}\delta_{13} = a\alpha_{13}\alpha_{23}\delta_{13}$ also redundant. All of which leads us to:

The arrangement monoid of type D_n ($n > 4$):



$$\begin{aligned} M(D_n, \mathcal{H}) = \langle s_0, \dots, s_{n-1}, a \mid\ & (s_i s_j)^{m_{ij}} = 1,\ a^2 = a, a s_1 = a,\ s_i a = a s_i\ (i \ne 2), \\ & s_0\alpha_{3j} = \alpha_{3j}s_0, s_0\delta_{3j} = \delta_{3j}s_0,\ (\text{both } j > 3), \\ & s_3\delta_{12} = \delta_{12}s_3,\ a\alpha_{34} = \alpha_{34}a, a\delta_{12} = \delta_{12}a, \\ & a\alpha_{13} = \alpha_{13}a, a\delta_{13} = \delta_{13}\delta_{23} = a\delta_{23}, \\ & a\alpha_{23}\delta_{23} = a\delta_{12}\delta_{23} = a\alpha_{23}\delta_{12}\delta_{23}, \\ & a\alpha_{23}\alpha_{34}\delta_{12}\delta_{23}\delta_{34} = a\alpha_{34}\delta_{12}\delta_{34}\rangle. \end{aligned}$$

together with $a\delta_{34} = \delta_{34}a$ when $n = 4$, and where α_{ij} and δ_{ij} are given by (4.1) and (4.4).

5. *Renner monoids*

5.1. *Generalities*

The principal objects of study in this section are algebraic monoids: affine algebraic varieties that carry the structure of a monoid. The theory builds on that of linear algebraic groups, and there are many parallels between the two. Standard references for both the groups and the monoids are [**2**, **14**, **15**, **23**, **24**]. The beginner should start with the survey [**26**].

Much of the structure of an algebraic group is encoded by the Weyl group W. The analogous role is played for algebraic monoids by the Renner monoid R. It turns out that the Renner monoid can be realized as a monoid $M(W,\mathcal{S})$ of partial permutations. Moreover, the system $\mathcal{S}$ is isomorphic (as a $\vee$-semilattice with $\mathbf{0}$) to the face lattice $\mathscr{F}(P)$ of a convex polytope P. We use these facts to obtain presentations for Renner monoids. Very different presentations have been found by Godelle [**11**, **10**] using a completely different approach.

We start by establishing notation from algebraic groups and monoids. Let $k=\overline{k}$ be an algebraically closed field and M an irreducible algebraic monoid over k. We will assume throughout that M has a 0. Let G be the group of units, and assume that G is a reductive algebraic group. In particular, M is reductive.

All the examples in this paper will arise via the following construction. Let G_0 be a connected semisimple algebraic group and $\rho : G_0 \to GL(V)$ a rational representation with finite kernel. Let $M = M(G_0,\rho) := \overline{k^\times\rho(G_0)}$, where $k^\times = k\setminus\{0\}$. Then M is a reductive irreducible algebraic monoid with 0 and units $G := k^\times\rho(G_0)$–see [**26**, §2]. If $G_0\subset GL_n$ is a classical algebraic group and $\rho : G_0 \hookrightarrow GL_n$ is the natural representation then we call the resulting M a *classical algebraic monoid*. Thus, if $G_0=\mathbf{SL}_n,\mathbf{SO}_n$ and $\mathbf{Sp}_n$, we have the general linear monoid $\mathbf{M}_n=\overline{k^\times\mathbf{SL}_n}$ (all $n\times n$ matrices over k), the orthogonal monoids $\mathbf{MSO}_n=\overline{k^\times\mathbf{SO}_n}$ and the symplectic monoids $\mathbf{MSp}_n=\overline{k^\times\mathbf{Sp}_n}$.

Returning to generalities, let $T\subset G$ be a maximal torus and $\overline{T}\subset M$ its (Zariski) closure. Let $\mathfrak{X}(T)=\mathrm{Hom}(T,k^\times)$ be the character group and $\mathfrak{X}:=\mathfrak{X}(T)\otimes\mathbb{R}$. Then $\mathfrak{X}(T)$ is a free $\mathbb{Z}$-module with rank equal to $\dim T$. In the construction above, if $T_0\subset G_0$ is a maximal torus then $T=k^\times\rho(T_0)\subset G$ is a maximal torus with $\dim T=\dim T_0+1$. If $v\in\mathfrak{X}(T_0)$ then the map given by $t\rho(t')\mapsto v(t')$, $(t\in k^\times, t'\in T)$ is a character in $\mathfrak{X}(T)$, and so we can identify $\mathfrak{X}(T_0)$ with a submodule of $\mathfrak{X}(T)$ with $\mathrm{rk}_\mathbb{Z}\mathfrak{X}(T)=\mathrm{rk}_\mathbb{Z}\mathfrak{X}(T_0)+1$. If $\mathfrak{X}_0=\mathfrak{X}(T_0)\otimes\mathbb{R}\subset\mathfrak{X}$ then $\dim\mathfrak{X}=\dim\mathfrak{X}_0+1$.

Let $\Phi=\Phi(G,T)\subset\mathfrak{X}(T)$ be the root system determined by T. If $\Phi(G_0,T_0)$ is the system for (G_0,T_0) above, then by [**26**, §2] or [**27**, Chapter 7] the character $t\rho(t')\mapsto v(t')$, $(t\in k^\times, t'\in T)$ is a root in $\Phi(G,T)$. Thus we can identify $\Phi(G_0,T_0)$ with a subset of $\Phi(G,T)$ where $|\Phi(G,T)|=\dim G-\dim T=(\dim G_0+1)-(\dim T_0+1)=|\Phi(G_0,T_0)|$. In particular, we can identify the root systems of G_0 and G. The root systems for the examples considered in this section are given in Table 2.

If $v\in\Phi$ let s_v be the reflection of $\mathfrak{X}$ in v and $W(\Phi)=\langle s_v\,|\,v\in\Phi\rangle$ the resulting reflection group. Let $\Delta\subset\Phi$ be a simple system (determined by the choice of a Borel subgroup $T\subset B$) so that $W(\Phi)$ is a Coxeter system (W,S) with S the set of reflections s_v in the simple roots $v\in\Delta$. Let $W(G,T)=N(T)/T$ be the Weyl group. If $w\in W$ and $\overline{w}\in G$ with $w=\overline{w}T$ then we will abuse notation throughout and write $w^{-1}tw$ rather than $\overline{w}^{-1}t\overline{w}$. In particular, W acts faithfully on $\mathfrak{X}$ via $v^w(t)=v(w^{-1}tw)$, realizing an injection $W\hookrightarrow GL(\mathfrak{X})$ and an isomorphism $W(G,T)\cong W(\Phi)$. We will identify these two groups in what follows and just write W for both. If $G=k^\times\rho(G_0)$ we identify the Weyl groups $W(G_0,T_0)$ and $W(G,T)$ via the identifications of their root systems.

We will also need the duals of these notions: let $\mathfrak{X}^\vee(T)=\mathrm{Hom}(k^\times,T)$ be the cocharacter group of T (i.e. 1-parameter subgroups of T). The Weyl group acts on $\mathfrak{X}^\vee(T)$ via $\lambda\mapsto\lambda w$ where $(\lambda w)(t)=w^{-1}\lambda(t)w$ for $t\in k^\times$. If $\langle\cdot,\cdot\rangle:\mathfrak{X}(T)\times\mathfrak{X}^\vee(T)\to\mathbb{Z}$ is the natural pairing, let $\Phi^\vee=\{v^\vee\in\mathfrak{X}^\vee(T)\,|\,v\in\Phi \text{ and } \langle v,v^\vee\rangle=2\}$ be the coroots and $\Delta^\vee=\{v^\vee\}_{v\in\Delta}$ the simple coroots.

The idempotents $E(\overline{T})$ in $\overline{T}$ are a finite commutative monoid, and we adopt the partial order of §1.1. The resulting poset is a graded atomic lattice with $\mathrm{rk}(e)=\dim T-\dim Te$ and atoms $A=\{e\in E(\overline{T})\,|\,\dim Te=\dim T-1\}$. The Weyl group W acts faithfully on $E(\overline{T})$ via $e\mapsto w^{-1}ew$, giving an injection $W\hookrightarrow\mathfrak{S}_{E(\overline{T})}$. This action preserves the partial order and the grading, so in particular restricts to the atoms A.

It turns out that there is a convex polytope P (see §1.2) with face lattice $\mathscr{F}(P)$ isomorphic to the lattice E. We describe, following [**26**, §5], how this polytope comes about in the situation that $M = \overline{k^\times \rho(G_0)}$ for $\rho : G_0 \to GL(V)$. Let $m = \dim V$, $\ell = \dim T_0$ and $\Phi_0 = \Phi(G_0, T_0)$ with simple roots $\Delta_0 = \{v_1, \dots, v_\ell\}$ and simple coroots $\Delta_0^\vee$. For each $i = 1, \dots, \ell$ and simple coroot $v_i^\vee$, let $\chi_i^\vee := \rho v_i^\vee \in \mathfrak{X}^\vee(T)$. We can write

$$\chi_i^\vee(t) := \chi(\mathbf{a}_i)^\vee(t) = \mathrm{diag}(t^{a_{i1}}, \dots, t^{a_{im}}) \tag{5.1}$$

with the $\mathbf{a}_i = (a_{i1}, \dots, a_{im}) \in \mathbb{Z}^m$. Let $\mathbb{R}^\ell$ be the space of column vectors and P the convex hull in $\mathbb{R}^\ell$ of the m vectors $(a_{1j}, \dots, a_{\ell j})^T$. Thus, if A is the $\ell \times m$ matrix with rows the $\mathbf{a}_i$ then P is the convex hull in $\mathbb{R}^\ell$ of the columns.

If $f \in \mathscr{F}(P)$ is a face of P then define $e_f := \sum_j E_{jj}$, the sum over those $1 \leq j \leq m$ such that $(a_{1j}, \dots, a_{\ell j})^T \in f$, and with E_{ij} the matrix with 1 in row i, column j, and 0's elsewhere. Then the map $\zeta : \mathscr{F}(P) \to E(\overline{T})$ given by $\zeta(f) = e_f$ is an isomorphism of posets. The P for the examples of this section are given in Table 2 (these will be justified later). Actually, even more is true. The Weyl group acts on $\mathfrak{X}^\vee(T)$ via $(\lambda w)(t) = \rho(w)^{-1}\lambda(t)\rho(w)$ so that $\chi(\mathbf{a}_i)^\vee w = \chi(\mathbf{b}_i)^\vee$, with $\mathbf{b}_i = (b_{i1}, \dots, b_{im})$ a permutation of $\mathbf{a}_i$. In particular, W permutes the vertices of P inducing an action of W on $\mathscr{F}(P)$. Then the poset isomorphism $\zeta : \mathscr{F}(P) \to E(\overline{T})$ is equivariant with respect to the Weyl group actions on $\mathscr{F}(P)$ and $E(\overline{T})$.

Let $R = \overline{N_G(T)}/T$ be the Renner monoid of M – a finite factorizable inverse monoid with units W and idempotents $E(\overline{T})$. It turns out that R is *not* in general a reflection monoid, although it is the image of a reflection monoid with units W and system of subspaces in $\mathfrak{X}$ (see [**7**, Theorem 8.1]). For us the Renner monoid will be a monoid of partial permutations using the construction described at the end of the Introduction. To see why we will need a result [**7**, Proposition 2.1] which we restate here in abbreviated form:

Proposition 5.1. *Let $M = EG$ and $N = FH$ be factorizable inverse monoids and let $\theta : G \to H$ and $\zeta : E \to F$ isomorphisms such that*
– *ζ is equivariant: $(geg^{-1})\zeta = (g\theta)(e\zeta)(g\theta)^{-1}$ for all $g \in G$ and $e \in E$, and*
– *θ respects stablizers: $G_e\theta = H_{e\zeta}$ for all $e \in E$.*
Then the map $\varphi : M \to N$ given by $(eg)\varphi = (e\zeta)(g\theta)$ is an isomorphism.

Now let $E = \mathscr{F}(P)$ above and $\mathcal{S}_P$ be the system of intervals for W given, as at the end of the Introduction, by $E_{\geq f} = \{f' \in \mathscr{F}(P) \,|\, f' \subseteq f\}$. Let $M(W, \mathcal{S}_P)$ be the resulting monoid of partial permutations, in which every element can be written in the form $\mathrm{id}_{E_{\geq f}} w$ for f a face of P and $w \in W$. The following is then an immediate application of Proposition 5.1 (with θ the identity):

Proposition 5.2. *If W is the Weyl group of $G = G(M)$, $\mathcal{S}_P$ the system arising from the polytope P and R is the Renner monoid of M, then the map $\mathrm{id}_{E_{\geq f}} w \mapsto e_f w$ is an isomorphism $M(W, \mathcal{S}_P) \to R$.*

For $e \in E(\overline{T})$ let $\Phi_e = \{v \in \Phi \,|\, s_v a = a s_v \text{ for all } a \in E(\overline{T})_{\geq e}\}$. The proof of [**7**, Theorem 9.2] shows that if $X = E(\overline{T})_{\geq e}$, the isotropy group W_X is equal to $W(\Phi_e)$, the subgroup of W

M	G_0	Φ	polytope P
general linear $\mathbf{M}_n$	$\mathbf{SL}_n$	A_{n-1}	Δ^n
special orthogonal $\mathbf{MSO}_{2\ell+1}$	$\mathbf{SO}_{2\ell+1}$	B_ℓ	$\diamond^\ell$
symplectic $\mathbf{MSp}_{2\ell}$	$\mathbf{Sp}_{2\ell}$	C_ℓ	$\diamond^\ell$
special orthgonal $\mathbf{MSO}_{2\ell}$	$\mathbf{SO}_{2\ell}$	D_ℓ	$\diamond^\ell$
Solomon's example §5.3	$\mathbf{SL}_n$	A_{n-1}	$(n-1)$-permutohedron

Table 2. *Basic data for the algebraic monoids considered in §5.*

generated by the $\{s_v\}_{v\in\Phi_e}$. Moreover, for $v \in \Phi$ and $t = s_v$, we have $H_t \supseteq E(\overline{T})_{\geq e}$ if and only if $v \in \Phi_e$.

The conditions of Remark 1 at the end of §2 are thus satisfied and we are ready to set up our presentation for the Renner monoid:

(R1). Let $A = \{e \in E \mid \dim Te = \dim T - 1\}$ be the atoms of $E(\overline{T})$. Let O_k be sets defined as in (P1) of §2.

(R2). As before W has a presentation with generators the $s \in S$ for $S = \{s_v \mid v \in \Delta\}$ and relations $(st)^{m_{st}} = 1$. For each $w \in W$ we fix an expression ω for w in the simple reflections $s \in S$ (subject to $\sigma = s$).

(R3). The action of W on $\mathcal{S}$ is represented notationally as before: for $a \in A$ fix an $a' \in O_1$ and a $w \in W$ with $a = w^{-1}a'w$ (subject to $w = 1$ when $a \in O_1$) and define $\alpha := \omega^{-1}a'\omega$. If w is an arbitrary element of W and $a \in A$ then by α^{ω} we mean the word obtained in this way for $w^{-1}aw \in A$. As before this is not necessarily $\omega^{-1}\alpha\omega$. For $e \in \mathcal{S}$, fix a join $e = \bigvee a_i\, (a_i \in A)$ and define $\varepsilon := \prod \alpha_i$.

(R4). For $e \in E(\overline{T})$ let $\Phi_e = \{v \in \Phi \mid s_v a = a s_v$ for all $a \in E(\overline{T})_{\geq e}\}$ and let $\{v_1, \dots, v_\ell\}$ be representatives, with $v_i \in \Delta$, for the W-action on Φ. For $i = 1, \dots, \ell$, enumerate the pairs $(e, s_i := s_{v_i})$ where $e \in E(\overline{T})$ is minimal in the partial order on $E(\overline{T})$ with the property that $v_i \in \Phi_e$. Let *Iso* be the set of all such pairs.

THEOREM 5.3. *Let M be an reductive irreducible algebraic monoid with* 0. *Then the Renner monoid of M has a presentation with*

generators: $s \in S, a \in O_1$.

relations:
$$(st)^{m_{st}} = 1,\ (s,t \in S), \qquad \textit{(Units)}$$
$$a^2 = a,\ (a \in O_1), \qquad \textit{(Idem1)}$$
$$\alpha_1\alpha_2 = \alpha_2\alpha_1,\ (\{a_1,a_2\} \in O_2), \qquad \textit{(Idem2)}$$
$$\alpha_1 \dots \alpha_{k-1} = \alpha_1 \dots \alpha_{k-1}\alpha,\ (\{a_1, \dots, a_{k-1}, a\} \in O_k)$$
$$\text{with } a_1, \dots, a_{k-1}, (3 \leq k \leq \dim T) \text{ independent and } a \leq \bigvee a_i, \qquad \textit{(Idem3)}$$
$$s\alpha = \alpha^s s,\ (s \in S, a \in A), \qquad \textit{(RefIdem)}$$
$$\varepsilon s = \varepsilon,\ (e,s) \in \textit{Iso}. \qquad \textit{(Iso)}$$

All the presentations in this section can be obtained in an algorithmic way, and so can be implemented in a computer algebra package for specific calculations.

5.2. *The classical monoids*

We illustrate the results of the previous section by giving presentations for the Renner monoids of the $\overline{k^\times G_0} \subseteq \mathbf{M}_n$ where G_0 is one of the classical groups $\mathbf{SL}_n, \mathbf{Sp}_n, \mathbf{SO}_n$ (see also [**11**]). We see from Table 2 that while the root systems for $\mathbf{SO}_{2\ell+1}$ and $\mathbf{Sp}_{2\ell}$ are different, the resulting Weyl groups $W(B_\ell)$ and $W(C_\ell)$ turn out to be isomorphic. Indeed, the Weyl groups $W(A_{n-1}), W(B_n) \cong W(C_n)$ and $W(D_n)$ all have alternative descriptions as permutation groups: namely, the symmetric group $\mathfrak{S}_n$ and the groups of signed and even signed permutations $\mathfrak{S}_{\pm n}$ and $\mathfrak{S}^e_{\pm n}$ (see below for the definitions of these).

The same is true for the Renner monoids: $\mathbf{MSO}_{2\ell+1}$ and $\mathbf{MSp}_{2\ell}$ have isomorphic Weyl groups and isomorphic idempotents, both isomorphic to $\mathscr{F}(\diamond^\ell)$, so it is not surprising that their Renner monoids are isomorphic. Indeed, the four Renner monoids can be realized as monoids of partial permutations, with units one of $\mathfrak{S}_\ell, \mathfrak{S}_{\pm\ell}$ or $\mathfrak{S}^e_{\pm\ell}$, and E one of the combinatorial descriptions of $\mathscr{F}(P)$ given in §1.2.

Consequently there are two ways to get their presentations, and for variety we illustrate both. For $\mathbf{M}_n = \overline{k^\times \mathbf{SL}_n}$ we just apply (R1)-(R4) and Theorem 5.3 directly. For the other three we work instead with their realizations as monoids of partial permutations, applying the adapted versions of (P1)-(P4), as in Remark 1 at the end of §2, and then Theorem 2.1. We then give an isomorphism from these to the Renner monoids.

Throughout $\mathbf{T}_n \subset \mathbf{GL}_n$ is the group of invertible diagonal matrices.

Example 7 (the general linear monoid $\mathbf{M}_n$). Let $G_0 = \mathbf{SL}_n$ with $T_0 = \mathbf{SL}_n \cap \mathbf{T}_n$ a maximal torus; $G = k^\times G_0 = \mathbf{GL}_n$ with maximal torus $T = k^\times T_0 = \mathbf{T}_n$. The general linear monoid is then $\mathbf{M}_n = \overline{k^\times \mathbf{SL}_n}$ with $\overline{T}$ the diagonal matrices.

For $\mathrm{diag}(t_1, \ldots, t_n) \in T$ let $v_i \in \mathfrak{X}(T)$ be given by $v_i \,\mathrm{diag}(t_1, \ldots, t_n) = t_i$. Then $\mathfrak{X}(T)$ is the free $\mathbb{Z}$-module with basis $\{v_1, \ldots, v_n\}$ and $\mathfrak{X}(T_0)$ the submodule consisting of those $\sum t_i v_i$ with $\sum t_i = 0$. The root system $\Phi(G_0, T_0) = \Phi(G, T)$ has type A_{n-1}:

$$\{v_i - v_j \,(1 \leq i \neq j \leq n)\},$$

with simple system $\Delta = \{v_{i+1} - v_i \,(1 \leq i \leq n-1)\}$ arising from the Borel subgroup of upper triangular matrices.

In this case the Weyl group $W(G, T)$ can be identified with a subgroup of G, namely the set of permutation matrices $A(\pi) := \sum_i E_{i,i\pi}$ as π varies over the symmetric group $\mathfrak{S}_n$. Indeed, the Weyl group is easily seen to be isomorphic to $\mathfrak{S}_n$, but we will stay inside the world of algebraic groups in this example. The isomorphism $W(G, T) \to W(A_{n-1})$ is induced by $A(i, j) \mapsto s_{v_i - v_j}$.

The idempotents $E = E(\overline{T})$ are the diagonal matrices $\mathrm{diag}(t_1, \ldots, t_n)$ with $t_i \in \{0, 1\}$ for all i. Alternatively, for $J \subseteq X = \{1, \ldots, n\}$, let $e_J := \sum_{j \in J} E_{jj}$, so that $E(\overline{T})$ consists of the e_J for $J \in \mathscr{B}_X$ (and indeed, $E(\overline{T})$ is easily seen to be isomorphic to $\mathscr{B}_X$, but again we stay inside algebraic groups). The Weyl group action on $E(\overline{T})$ is given by

$$e_J \mapsto A(\pi)^{-1} e_J A(\pi) = e_{J\pi}.$$

Let $e_i := e_J$ for $J = \{1, \ldots, \widehat{i}, \ldots, n\}$. Running through (R1)-(R4), the atoms in $E(\overline{T})$ are $A = \{e_i \,|\, 1 \leq i \leq n\}$. There is a single W-orbit on A and we choose $e := e_1$ for O_1. There is a single W-orbit on pairs of atoms and we choose the pair $\{e, e_2\}$ for O_2. We will see below that there is no need for O_k for $k > 2$. If $e_i \in A$, $(i > 1)$, we have $e_i = s_{i-1} \ldots s_1 e s_1 \ldots s_{i-1}$, so let

$$\varepsilon_i = s_{i-1} \ldots s_1 e s_1 \ldots s_{i-1},$$

with $\varepsilon_1 = e$. Let $e_J \in E$ with $X \setminus J = \{i_1, \ldots, i_k\}$, giving $e_J = e_{i_1} \vee \cdots \vee e_{i_k}$, and let

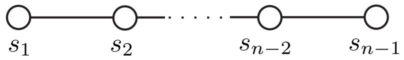
$$\varepsilon_J = \varepsilon_{i_1} \ldots \varepsilon_{i_k}.$$

We have $A(\pi)^{-1} e_J A(\pi) = e_J$ exactly when $J\pi = J$; moreover, $E_{\geq e_J} = \{e_I \,|\, J \supseteq I\}$. The result is that

$$\Phi_{e_J} = \{v_i - v_j \,|\, i, j \notin J\}.$$

There is a single W-orbit on the roots Φ and we choose $v_2 - v_1 \in \Delta$ as representative. If e_J is to be minimal in E with the property that $v_2 - v_1 \in \Phi_{e_J}$ then J is minimal (under reverse inclusion!) with $1, 2 \notin J$. Thus $J = \{3, \ldots, n\}$, and the set *Iso* consists of the single pair $(e_{\{3,\ldots,n\}}, s_1)$ with $\varepsilon_{\{3,\ldots,n\}} = \varepsilon_1 \varepsilon_2 = e s_1 e s_1$.

A presentation for the Renner monoid of $\mathbf{M}_n$*:* By Theorem 5.3 we have generators $s_1, \ldots, s_{n-1}$, e with *(Units)* relations $(s_i s_j)^{m_{ij}} = 1$, where the m_{ij} are given by the Coxeter symbol



The *(Idem1)* relation is $e^2 = e$ the *(Idem2)* relations are

$$\varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1, \text{ or, } es_1es_1 = s_1es_1e.$$

We saw in §1.1 that in $\mathscr{B}_X$ – or in §1.2 that in $\mathscr{F}(\Delta^n)$ – all subsets of atoms are independent and so the *(Idem3)* relations are vacuous. The *(RefIdem)* relations are $s_i\varepsilon_j = \varepsilon_j^{s_i}s_i$ for $1 \leq i \leq n-1$ and $1 \leq j \leq n$; as in Lemma 3.1 of §3 we can prune these to $s_ie = \varepsilon^{s_i}s_i\,(1 \leq i \leq n-1)$. We have $s_1es_1 = s_2$ and $s_ies_i = e\,(i > 1)$ so that $\varepsilon^{s_1} = \varepsilon_2 = s_1es_1$, $\varepsilon^{s_i} = e\,(i > 1)$ and the relations are

$$s_ie = es_i\,(i > 1).$$

Finally, the *(Iso)* are $\varepsilon_Js_1 = \varepsilon_J$ for $J = \{3, \dots, n\}$, or

$$es_1e = es_1es_1.$$

Remark. It is well known that R is isomorphic to the symmetric inverse monoid $\mathscr{I}_n$ where the s_i correspond to the (full) permutation $(i, i+1)$ and e to the partial identity on the set $\{2, \dots, n\}$, and so yet again we have the Popova presentation.

As promised we now introduce two families of monoids of partial permutations. Let $\pm X = \{\pm 1, \dots, \pm\ell\}$ and define the group $\mathfrak{S}_{\pm X}$ of signed permutations of X to be

$$\mathfrak{S}_{\pm X} = \{\pi \in \mathfrak{S}_{X \cup -X} \,|\, (-x)\pi = -x\pi \text{ for all } x \in \pm X\}.$$

(the reason for the change in notation from n to ℓ will become apparent in Example 8 below). A signed permutation π is *even* if the number of $x \in X$ with $x\pi \in -X$ is even, and the even signed permutations $\mathfrak{S}_{\pm X}^e$ form a subgroup of index two in $\mathfrak{S}_{\pm X}$.

The symmetric group is a subgroup in an obvious way: let $\pi \in \mathfrak{S}_{\pm X}$ be such that x and $x\pi$ have the same sign for all $x \in \pm X$. In particular π is even. Any such π has a unique expression $\pi = \pi_+\pi_-$ with $\pi_+ \in \mathfrak{S}_X$, $\pi_- \in \mathfrak{S}_{-X}$ and $\pi_+(x) = \pi_-(-x)$. The map $\pi \mapsto \pi_+$ is then an isomorphism from the set of such π to $\mathfrak{S}_X$. We will just write $\mathfrak{S}_X \subset \mathfrak{S}_{\pm X}$ (or $\subset \mathfrak{S}_{\pm X}^e$) from now on to mean this subgroup.

We require Coxeter system structures for $\mathfrak{S}_{\pm X}$ and $\mathfrak{S}_{\pm X}^e$. Indeed, we have $\mathfrak{S}_{\pm X} \cong W(B_\ell) \cong W(C_\ell)$ via s_{v_1} or $s_{2v_1} \mapsto (1, -1)$ and $s_{v_{i+1}-v_i} \mapsto (i, i+1)(-i, -i-1)$ and $\mathfrak{S}_{\pm X} \cong W(D_\ell)$ via $s_{v_1+v_2} \mapsto (1, -2)(-1, 2)$ and $s_{v_{i+1}-v_i} \mapsto (i, i+1)(-i, -i-1)$.

Now to a system of subsets for $\mathfrak{S}_{\pm X}$ and $\mathfrak{S}_{\pm X}^e$. In [**7**, §5] we used the elements of $\mathscr{B}_X$ to give a system for $\mathfrak{S}_{\pm X}$ and this led to the monoid $\mathscr{I}_{\pm n}$ of partial signed permutations. Here we want something different. Recall from Example 2 the poset E of admissible subsets of $\pm X$, with $\pm X$ adjoined. If $\pi \in \mathfrak{S}_{\pm X}$ and J is admissible, then it is easy to see that $J\pi$ is also admissible, and so the action of $\mathfrak{S}_{\pm X}$ on $\pm X$ restricts to E. Our system consists of the intervals $E_{\geq J} = \{I \in E \,|\, J \supseteq I\}$ as in the Introduction.

Write $M(\mathfrak{S}_{\pm X}, \mathcal{S})$ and $M(\mathfrak{S}_{\pm X}^e, \mathcal{S})$ for the resulting monoids of partial permutations.

A brief interlude: We detour to parametrize the orbits of the action of the symmetric group $\mathfrak{S}_X$ on k-tuples of distinct subsets of X given by $(J_1, \dots, J_k) \stackrel{\pi}{\mapsto} (J_1\pi, \dots, J_k\pi)$. The results here may well be part of the folklore of the combinatorics of the symmetric group; full details are in [**6**, §6.3]. Let $X = \{1, \dots, \ell\}, Y = \{1, \dots, k\}$ with $\mathscr{B}_Y$ the Boolean lattice on Y, ordered as usual by reverse inclusion, and $[0, \ell] \subset \mathbb{Z}$, with this interval inheriting the usual order from $\mathbb{Z}$.

Let $f : \mathscr{B}_Y \to [0, \ell]$ be a poset map and define $f^* : \mathscr{B}_Y \to \mathbb{Z}$ (not necessarily a poset map) by

$$f^*(I) = \sum_{J \supseteq I} (-1)^{|J \setminus I|} f(J). \tag{5.2}$$

Then f is a *characteristic map* if $f^*(I) \geq 0$ for all I, and $f^*(\varnothing) = 0$.

In [**6**, §6.3] (see also below) it is shown that a tuple $(J_1, \ldots, J_k)$ gives rise to a characteristic map and every characteristic map arises from some tuple. Moreover, two tuples lie in the same $\mathfrak{S}_X$-orbit exactly when they give rise to identical characteristic maps. Thus,

Lemma 5.4. *With X, Y as above, the orbits of the diagonal action of $\mathfrak{S}_X$ on k-tuples of distinct subsets of X are parametrized by the characteristic maps.*

We write $Char_k$ for the set of characteristic maps $f : \mathscr{B}_Y \to [0, \ell]$ when $|Y| = k$. Although $Char_k$ depends on both k and ℓ, in the examples below ℓ will be fixed. Write $(J_1 \ldots, J_k)_f$ for the tuple arising from $f \in Char_k$ in the following way: let disjoint sets K_I, $(\varnothing \neq I \in \mathscr{B}_Y)$ be defined by first setting $K_Y := \{1, \ldots, f^*(Y) = f(Y)\}$ if $f(Y) > 0$, or $K_Y := \varnothing$ if $f(Y) = 0$. Now choose some total ordering $\preceq$ on $\mathscr{B}_Y$ having minimal element Y, and for general I let K_I be the next $f^*(I)$ points of $[0, \ell] \setminus \bigcup_{J \prec I} K_J$. Although the choice of $\preceq$ is not important, for definiteness we take $J \prec I$ when $|I| < |J|$ and order sets of the same size lexicographically. Then for $i = 1, \ldots, k$, let $J_i = \bigcup K_I$, the (disjoint) union over those I with $i \in I$. See [**6**, §6.3] for a proof using Möbius inversion that this construction works.

For fixed k the possible characteristic maps in $Char_k$ can be enumerated by letting $f(Y) = f^*(Y) = n_0 \geq 0$. If $I = Y \setminus \{i\}$ then $f^*(I) = f(I) - f(Y) \geq 0$ gives $f(I) = n_i \geq n_0$. In general, if $I = Y \setminus J$, $(J \subset Y)$ then $f(I)$ can equal any $n_J \in [0, \ell]$ satisfying $n_J \geq \sum_{K \subseteq J} (-1)^{|J \setminus K|} n_K$ (and $f(\varnothing) = \sum_{J \neq \varnothing} (-1)^{|J|+1} n_J$).

For example, if $k = 1$ then $\mathscr{B}_Y$ is the two element poset $Y < \varnothing$. We have $f^*(Y) = f(Y) \geq 0$ and $f(\varnothing) = f(Y)$. Thus $Char_1$ consists of the $f(\varnothing) = f(Y) = n_0$, for each $n_0 \in [0, \ell]$, of which there are $\ell + 1$. This coincides with the fact that $\mathfrak{S}_X$ acts t-fold transitively on X for each $0 \leq t \leq \ell$, hence there are $\ell + 1$ orbits. As another example, explicit orbit representatives $(J_1, J_2, J_3)_f$ when $k = 3$ can be obtained as follows: let $n_0, \ldots, n_3, n_{12}, n_{13}, n_{23}$ be integers such that $0 \leq n_0 \leq n_i \leq \ell$, $n_i + n_j - n_0 \leq n_{ij} \leq \ell$ and $\sum n_{ij} - \sum n_i + n_0 \leq \ell$. The following picture depicts $X = \{1, \ldots, \ell\}$, with 1 at the left:

n_0	$n_1 - n_0$	$n_2 - n_0$	$n_3 - n_0$	$n_{12} - n_1 - n_2 + n_0$	$n_{13} - n_1 - n_3 + n_0$	$n_{23} - n_2 - n_3 + n_0$	(†)
{1,2,3}	{2,3}	{1,3}	{1,2}	{3}	{2}	{1}	

and the number in each box gives the number of points in the box (so the left most box represents the points $\{1, \ldots, n_0\}$, the second the points $\{n_0 + 1, \ldots, n_1\}$, and so on). Each box is also labeled below by a subset of Y. Then J_i is the union of those boxes for which i appears in the subset below it; e.g.: J_1 is the union of the grey boxes.

We return to the monoids $M(\mathfrak{S}_{\pm X}, \mathcal{S})$ and $M(\mathfrak{S}^e_{\pm X}, \mathcal{S})$. For the rest of the paper, all mention of (P1)-(P4) refers to the adapted versions of these as in Remark 1 at the end of §2. As observed in the Introduction, the map $J \mapsto E_{\geq J}$ is a poset isomorphism $E \cong \mathcal{S}$ which is equivariant with respect to the $\mathfrak{S}_{\pm X}$ and $\mathfrak{S}^e_{\pm X}$ actions. Thus, in running through (P1)-(P4) we can work with the admissible $J \in E$ rather than the corresponding intervals $E_{\geq J} \in \mathcal{S}$. This makes the notation a little less cumbersome.

(1). The monoid $M(\mathfrak{S}_{\pm X}, \mathcal{S})$. The atoms in E are the $a(I) := I \cup (-X \setminus -I)$, $I \subseteq X = \{1, \ldots, \ell\}$ of §1.2. There are thus 2^ℓ atoms here versus the ℓ in the $\mathbf{SL}_n$ case. Now to the sets

O_k for $k \geq 1$. Let $a(I)$ be an atom of E with $I = \{i_1, \dots, i_k\}$. Then

$$a(I) \cdot (i_1, -i_1) \cdots (i_k, -i_k) = -X = a(\varnothing), \tag{5.3}$$

so there is a single $\mathfrak{S}_{\pm X}$-orbit on the atoms, and we take $O_1 = \{a\}$ with $a := a(\varnothing)$.

For O_2 we can use the set *Char*$_2$, although it turns out that with the $\mathfrak{S}_{\pm X}$ action we do can do a little more. Let $a(I), a(K)$ be a pair of atoms with $|I| \leq |K|$ and $I \cap K = \{i_1, \dots, i_k\}$. Then the pair $(a(I), a(K)) \cdot (i_1, -i_1) \cdots (i_k, -i_k) = (a(I_1), a(K_1))$ with $I_1 = I \setminus (I \cap K)$ and $K_1 = K \setminus (I \cap K)$ disjoint. The pair I_1, K_1 can then be moved by the $\mathfrak{S}_X$-action as far as possible to the left of $\{1, \dots, \ell\}$ while remaining disjoint. Thus, for O_2 we take the pairs $\{a(I), a(K)\}$ with $I = \{1, \dots, j_1\}$, $K = \{j_1 + 1, \dots, j_2\}$ for all $0 \leq j_1 < j_2 \leq \ell$.

For $k = 3$ we can play a similar game, but this doesn't work for $k > 3$. Instead, for $k > 2$ we restrict the $\mathfrak{S}_{\pm X}$-action on E to the subgroup $\mathfrak{S}_X \subset \mathfrak{S}_{\pm X}$ and consider orbit representatives on the k-tuples as in remark 3 at the end of §2. Thus the O_k , $(k > 2)$ will be sets of representatives with possible redundancies. If $f \in$ *Char*$_k$ is a characteristic map, then by the construction following Lemma 5.4 we have a unique tuple $(I_1, \dots, I_k)_f$ with characteristic map f. For O_k we take the set of $\{a(I_1), \dots, a(I_k)\}$ where $(I_1, \dots, I_k)_f$ arises via $f \in$ *Char*$_k$.

Write $s_i := (i, i+1)(-i, -i-1)$, $(1 \leq i \leq \ell - 1)$; $s_0 := (1, -1)$ and $\omega_i := s_{i-1} \cdots s_1 s_0 s_1 \cdots s_{i-1}$ for $i > 1$ and $\omega_1 = s_0$. If $a(I)$ is an atom with $I = \{i_1, \dots, i_k\}$, let

$$\alpha(I) := \omega_{i_1} \dots \omega_{i_k} a \, \omega_{i_k} \dots \omega_{i_1}. \tag{5.4}$$

Let $J \in E$ be admissible with $\pm X \setminus \pm J = \{\pm i_1, \dots, \pm i_k\}$. Then, recalling that $J^+ = J \cap X$, we take as fixed word for J

$$\alpha(\widehat{i_1}, \dots, i_k, J^+) \cdots \alpha(i_1, \dots, \widehat{i_k}, J^+) \tag{5.5}$$

when $k > 1$ (and where $\alpha(\widehat{i_1}, \dots, i_k, J^+)$ means $\alpha(\{\widehat{i_1}, \dots, i_k\} \cup J^+)$), or $\alpha(J^+)\alpha(i_1, J^+)$ when $k = 1$.

Finally then to (P4) and $\mathscr{A} = \{H_t\}$ where $H_t = \{J \in E \,|\, Jt = J\}$. Every $t \in T$ in $\mathfrak{S}_{\pm X}$ is conjugate to s_0 or s_1 (using the Coxeter group structure) so there are two $\mathfrak{S}_{\pm X}$ orbits on $\mathscr{A}$ with representatives $H_0 := H_{s_0}$ and $H_1 := H_{s_1}$, where H_0 consists of those $J \in E$ with $\pm 1 \notin J$ and H_1 those J with either $\pm 1, \pm 2 \notin J$ or $1, 2 \in J$ or $-1, -2 \in J$. If J is to be minimal with $H_0 \supseteq E_{\geq J}$ then J has the form

(5.6)

which is $\alpha(J^+)\alpha(1, J^+)$. Similarly, if J is to be minimal with $H_1 \supseteq E_{\geq J}$ then J has the form

(5.7)

which is $\alpha(1, J^+)\alpha(2, J^+)$.

The set *Iso* thus consists of the pairs $(\alpha(1, I)\alpha(2, I), s_1)$ for all $I \subseteq X \setminus \{1, 2\}$ and the pairs $(\alpha(I)\alpha(1, I), s_0)$ for all $I \subseteq X \setminus \{1\}$. Rather than write out the resulting presentation for this monoid here, we save it for Example 8 below.

(2). The monoid $M(\mathfrak{S}^e_{\pm X}, \mathcal{S})$. The difference here is that we pass to the subgroup $\mathfrak{S}^e_{\pm X}$ of $\mathfrak{S}_{\pm X}$ and its action on E. The atoms are the $a(I) := I \cup (-X \setminus -I)$, $I \subseteq X$ as before. If $a(I)$ is one such with $I = \{i_1, \dots, i_k\}$, then for k even

$$a(I) \cdot (i_1, -i_2)(-i_1, i_2) \cdots (i_{k-1}, -i_k)(-i_{k-1}, i_k) = -X = a(\varnothing), \tag{5.8}$$

and for k odd

$$a(I)\cdot(i_1,-i_2)(-i_1,i_2)\cdots(i_{k-2},-i_{k-1})(-i_{k-2},i_{k-1})(i_{k-1},i_k)(-i_{k-1},-i_k)\cdots(1,2)(-1,-2) \tag{5.9}$$

gives $a(1)$. Although (5.3) still holds in the even case, we change here to the version (5.8) because of our choice of generators for $\mathfrak{S}^e_{\pm X}$ below. Thus, $O_1=\{a_1:=a(\varnothing),a_2=a(1)\}$.

Let $a(I),a(K)$ be a pair of atoms with $I\cap K=\{i_1,\dots,i_k\}$. Then a similar argument as in the $\mathfrak{S}_{\pm X}$ case gives O_2 the pairs $\{a(I),a(K)\}$ with $I=\{1,\dots,j_1\}$, $K=\{j_1+1,\dots,j_2\}$ (when k is even) or $I=\{1,\dots,j_1\}$, $K=\{j_1,\dots,j_2\}$ (when k is odd), with $0\leq j_1<j_2\leq\ell$ in both cases.

The $O_k\,,(k>2)$ are exactly as in the $\mathfrak{S}_{\pm X}$ case, since $\mathfrak{S}_X\subset\mathfrak{S}^e_{\pm X}$. Thus O_k is the set of $\{a(I_1),\dots,a(I_k)\}$ where $(I_1,\dots,I_k)_f$ arises via $f\in \mathit{Char}_k$.

Write $s_i:=(i,i+1)(-i,-i-1)$, $(1\leq i\leq\ell-1)$ and $s_0:=(1,-2)(-1,2)$ and let $\omega_{ij}:=s_{i-1}\cdots s_1s_{j-1}\cdots s_2s_0s_2\cdots s_{j-1}s_1\cdots s_{i-1}$ for $1<i<j\leq\ell-1$, or $\omega_{1j}:=s_{j-1}\cdots s_2s_0s_2\cdots s_{j-1}$ for $(j>2)$ or $\omega_{12}:=s_0$. If $a(I)$ is an atom with $I=\{i_1,\dots,i_k\}$ and k even, let

$$\alpha(I):=\omega_{i_1i_2}\dots\omega_{i_{k-1}i_k}a_1\,\omega_{i_{k-1}i_k}\dots\omega_{i_1i_2}, \tag{5.10}$$

or if k is odd

$$\alpha(I):=\omega_{i_1i_2}\dots\omega_{i_{k-2}i_{k-1}}s_{i_{k-1}}\cdots s_1a_2\,s_1\cdots s_{i_{k-1}}\omega_{i_{k-2}i_{k-1}}\dots\omega_{i_1i_2}. \tag{5.11}$$

If $J\in E$ is admissible then it is represented by the word (5.5) and the comments following it. The treatment of (P4) is also virtually identical to the previous case: every $t\in T$ is conjugate in $\mathfrak{S}^e_{\pm X}$ to s_1, so there is a single $\mathfrak{S}^e_{\pm X}$-orbit on $\mathscr{A}$ with representative $H_1:=H_{s_1}$ consisting of the $J\in E$ with either $\pm1,\pm2\notin J$ or $1,2\in J$ or $-1,-2\in J$. The set *Iso* thus consists of the pairs $(\alpha(1,I)\alpha(2,I),s_1)$ for all $I\subseteq X\setminus\{1,2\}$. Again, we save the presentation of this monoid for Example 10 below.

Example 8 (the symplectic monoids $\mathbf{MSp}_n$). Let $n=2\ell$ and

$$G_0=\mathbf{Sp}_n=\{g\in\mathbf{GL}_n\,|\,g^TJg=J\}\text{ for }J=\begin{bmatrix}0&J_0\\-J_0&0\end{bmatrix},$$

where $J_0=\sum_{i=1}^{\ell}E_{i,\ell-i+1}$ is $\ell\times\ell$. Note that as in [**19**], this is the version of the symplectic group given by Humphreys [**15**] rather than the version used by Solomon in [**26**]. Let $T_0=\mathbf{Sp}_n\cap\mathbf{T}_n$, the matrices of the form $\mathrm{diag}(t_1,\dots,t_\ell,t_\ell^{-1},\dots,t_1^{-1})$ with the $t_i\in k^\times$. Let $G=k^\times\mathbf{Sp}_n$ with maximal torus $T=k^\times T_0$, and let the symplectic monoid $\mathbf{MSp}_n=\overline{k^\times\mathbf{Sp}_n}\subset\mathbf{M}_n$.

For $i=0,\dots,\ell$ let $v_i\in\mathfrak{X}(T)$ be given by $v_i\,t_0\cdot\mathrm{diag}(t_1,\dots,t_\ell,t_\ell^{-1},\dots,t_1^{-1})=t_i$ so that $\mathfrak{X}(T)$ is the free $\mathbb{Z}$-module on $\{v_0,\dots,v_\ell\}$. The roots $\Phi(G_0,T_0)=\Phi(G,T)$ have type C_ℓ:

$$\{\pm v_i\pm v_j\,(1\leq i<j\leq\ell)\}\cup\{\pm2v_i\,(1\leq i\leq\ell)\}$$

lying in an ℓ-dimensional subspace of $\mathfrak{X}=\mathfrak{X}(T)\otimes\mathbb{R}$. The group G has rank $\ell+1$ and semisimple rank ℓ. We use the simple system $\Delta=\{2v_1,v_{i+1}-v_i\,(1\leq i\leq\ell-1)\}$ as described in the Introduction.

We now describe an isomorphism between the Renner monoid R of $\mathbf{MSp}_{2\ell}$ and the monoid $M(\mathfrak{S}_{\pm\ell},\mathcal{S})$ of partial isomorphisms described in (1) above. The units in $M(\mathfrak{S}_{\pm\ell},\mathcal{S})$ are $\mathfrak{S}_{\pm\ell}$ and the units in the Renner monoid R are the Weyl group $W(C_\ell)$ so we have the isomorphism $\mathfrak{S}_{\pm\ell}\cong W(C_\ell)$ given before the interlude. Let $s_0,\dots,s_{\ell-1}$ denote either the signed permutations of $\mathfrak{S}_{\pm\ell}$ introduced in (1) above or the simple reflections in $W(C_\ell)$.

The idempotents in $M(\mathfrak{S}_{\pm\ell},\mathcal{S})$ are the partial identities $\mathrm{id}_{E_{\geq J}}$ on the $E_{\geq J}\in\mathcal{S}$, and the idempotents in R are $E(\overline{T})$, the matrices $\mathrm{diag}(t_1,\dots,t_\ell,t_\ell^{-1},\dots,t_1^{-1})$ with $t_i\in\{0,1\}$. We write E for the idempotents in $M(\mathfrak{S}_{\pm\ell},\mathcal{S})$ as well as for the poset of admissible subsets. Let the map

$\eta : \pm X \to \{1, \ldots, n = 2\ell\}$ be given by

$$\eta(i) := \begin{cases} i, & i > 0 \\ 2\ell + 1 + i, & i < 0. \end{cases}$$

Define $\zeta : E \to E(\overline{T})$ by $\mathrm{id}_{E_{\geq J}} \mapsto e(J) := \sum_{j \in \eta J} E_{jj}$ for $J \subset \pm X$ admissible, and $\mathrm{id}_{\pm X} \mapsto I_n$. Then $\zeta : E \to E(\overline{T})$ is an isomorphism that is equivariant with respect to the $\mathfrak{S}_{\pm\ell}$-action on E and the $W(C_\ell)$-action on $E(\overline{T})$ (see [**26**, Example 5.5]).

Finally, if $e = \mathrm{id}_{E_{\geq J}}$ is an idempotent in $M(\mathfrak{S}_{\pm\ell}, \mathcal{S})$ and $G = \mathfrak{S}_{\pm\ell}$, then the idempotent stabilizer G_e consists of those $\pi \in \mathfrak{S}_{\pm\ell}$ that fix the admissible set J pointwise. Similarly, we have $W(C_\ell)_{e\zeta}$ consisting of those $\pi\theta \in W(C_\ell)$ with $e(J)\pi\theta = e(J)$. This is also equivalent to π fixing J pointwise. We thus have our isomorphism $M(\mathfrak{S}_{\pm\ell}, \mathcal{S}) \cong R$ by Proposition 5.1 (we could also have used Proposition 5.2 but the above is more direct).

A presentation for the Renner monoid of $\mathbf{MSp}_{2\ell}$*:* It remains to take the (P1)-(P4) data for $M(\mathfrak{S}_{\pm\ell}, \mathcal{S})$ listed in (1) above and apply Theorem 2.1. We have generators $s_0, \ldots, s_{\ell-1}, a$ with *(Units)* relations $(s_i s_j)^{m_{ij}} = 1$ where the m_{ij} are given by

The *(Idem1)* relation is $a^2 = a$, and the *(Idem2)* relations are

$$\alpha(1, \ldots, j_1)\alpha(j_1 + 1, \ldots, j_2) = \alpha(j_1 + 1, \ldots, j_2)\alpha(1, \ldots, j_1),$$

for all $0 \leq j_1 < j_2 \leq \ell$, with $\alpha(I)$ given by (5.4). The *(Idem3)* relations are

$$\alpha(I_1) \ldots \alpha(I_{k-1}) = \alpha(I_1) \ldots \alpha(I_{k-1})\alpha(I)$$

for $(I_1, \ldots, I_{k-1}, I)_f$ arising from $f \in \mathit{Char}_k$, and with $(a(I_1), \ldots, a(I_{k-1})) \in \mathit{Ind}_{k-1}$, $(k \geq 2)$ and all $a(K) \supseteq \bigcap a(I_i)$, where Ind_{k-1} is given by Proposition 1.5. The *(RefIdem)* relations consist of three families:

$$s_0\, \alpha(I) = \alpha(I)s_0, \text{ and } s_i\, \alpha(I) = \alpha(I)s_i, \text{ and } s_i\, \alpha(I) = \alpha(I)^{s_i} s_i.$$

The first is for all $I \subseteq X$ with $1 \notin I$ (if $1 \in I$ then the relations $s_0\, \alpha(I) = \alpha(I)^{s_0} s_0$ are vacuous); the second for $1 \leq i \leq \ell - 1$ and $i, i+1 \in I$ or $i, i+1 \notin I$; the third when exactly one of $i, i+1$ lies in I; finally, $\alpha(I)^{s_i} = s_i\omega_{i+1}\omega_{i_2} \cdots \omega_{i_k} a \omega_{i_k} \cdots \omega_{i_2}\omega_{i+1}$ when $I = \{i, i_2, \ldots, i_k\}$, and when $i + 1 \in I$ is similar. Finally, the *(Iso)* relations are

$$\alpha(1, I)\alpha(2, I)s_1 = \alpha(1, I)\alpha(2, I), \text{ and } \alpha(I)\alpha(1, I)s_0 = \alpha(I)\alpha(1, I),$$

the first for all $I \subseteq X \setminus \{1, 2\}$ and the second for all $I \subseteq X \setminus \{1\}$.

Example 9 (the odd dimensional special orthogonal monoids $\mathbf{MSO}_n$). This is very similar to the previous case. Let $n = 2\ell + 1$ and

$$G_0 = \mathbf{SO}_n = \{g \in \mathbf{GL}_n \,|\, g^T J g = J\} \text{ for } J = \begin{bmatrix} 0 & 0 & J_0 \\ 0 & 1 & 0 \\ -J_0 & 0 & 0 \end{bmatrix},$$

with J_0 as in Example 8. We have taken the definition of $\mathbf{SO}_n$ given in [**19**] rather than [**15**] to make the similarity with $\mathbf{Sp}_n$ more apparent. We have $T_0 = \mathbf{SO}_n \cap \mathbf{T}_n$, the matrices of the form $\mathrm{diag}(t_1, \ldots, t_\ell, \pm 1, t_\ell^{-1}, \ldots, t_1^{-1})$ with the $t_i \in k^\times$; $G = k^\times \mathbf{SO}_n$ with T as before and the orthogonal monoid $\mathbf{MSO}_n = \overline{k^\times \mathbf{SO}_n} \subset \mathbf{M}_n$.

The roots have type B_ℓ, so are the same as C_ℓ except with $\pm v_i$ instead of $\pm 2v_i$. Nevertheless, as is well known, the Weyl group $W(B_\ell)$ is isomorphic to $W(C_\ell)$ and we take the simple system $\Delta = \{v_1, v_{i+1} - v_i\, (1 \leq i \leq \ell - 1)\}$.

If R is the Renner monoid of $\mathbf{MSO}_n$ then the isomorphism $M(\mathfrak{S}_{\pm\ell}, \mathcal{S}) \cong R$ is analogous to Example 8, except in the isomorphism $\theta : \mathfrak{S}_{\pm\ell} \to W(B_\ell)$ we have $(1, -1) \mapsto s_0 := s_{v_1}$ and in the isomorphism $\zeta : E \to E(\overline{T})$ we have $\eta : \pm X \to \{1, \ldots, n = 2\ell + 1\}$ given by

$$\eta(i) := \begin{cases} i, & i > 0 \\ 2\ell + 2 + i, & i < 0 \end{cases}$$

and $e(J) := E_{\ell+1,\ell+1} + \sum_{j\in\eta J} E_{jj}$ for $J \subset \pm X$ admissible.

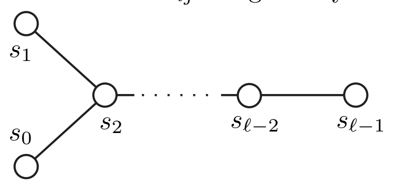
The presentation for the Renner monoid of $\mathbf{MSO}_{2\ell+1}$ is identical to the presentation in the $\mathbf{MSp}_{2\ell}$ case of Example 8.

Example 10 (the even dimensional special orthogonal monoids $\mathbf{MSO}_n$). Let $n = 2\ell$ and

$$G_0 = \mathbf{SO}_n = \{g \in \mathbf{GL}_n \,|\, g^T Jg = J\} \text{ for } J = \begin{bmatrix} 0 & J_0 \\ J_0 & 0 \end{bmatrix},$$

with J_0 and $\mathbf{MSO}_n$ as above. The roots have type D_ℓ: $\{\pm v_i \pm v_j \ (1 \leq i < j \leq \ell)\}$ with simple roots $\Delta = \{v_1 + v_2, v_{i+1} - v_i \ (1 \leq i \leq \ell - 1)\}$. If R is the Renner monoid of $\mathbf{MSO}_{2\ell}$, the isomorphism $M(\mathfrak{S}^e_{\pm\ell}, \mathcal{S}) \cong R$ is built from the isomorphism $\theta : \mathfrak{S}^e_{\pm\ell} \to W(D_\ell)$ given before the interlude together with $\zeta : E \to E(\overline{T})$ exactly as for $\mathbf{MSp}_n$.

A presentation for the Renner monoid of $\mathbf{MSO}_{2\ell}$*:* We take the (P1)-(P4) data for $M(\mathfrak{S}^e_{\pm\ell}, \mathcal{S})$ listed in (2) above and apply Theorem 2.1. We have generators $s_0, \ldots, s_{\ell-1}, a_1, a_2$ with *(Units)* relations $(s_i s_j)^{m_{ij}} = 1$ where the m_{ij} are given by



The *(Idem1)* relations are $a_1^2 = a_1, a_2^2 = a_2$, and the *(Idem2)* relations are

$$\alpha(1, \ldots, j_1)\alpha(j_1 + \varepsilon, \ldots, j_2) = \alpha(j_1 + \varepsilon, \ldots, j_2)\alpha(1, \ldots, j_1),$$

for all $0 \leq j_1 < j_2 \leq \ell$, $\varepsilon = 0, 1$ and with $\alpha(I)$ given by (5.10)-(5.11). The *(Idem3)* relations are exactly as in the $\mathbf{MSp}_n$ case. The *(RefIdem)* relations are the same as for $\mathbf{MSp}_n$ for $s_i \ (1 \leq i \leq \ell - 1)$; the relations involving s_0 are slightly different. We get:

$$s_0\alpha(I) = \alpha(I)s_0, \text{ and } s_0\alpha(I) = \alpha(I)^{s_0} s_0.$$

with the first for all I with $1, 2 \notin I$ and the second where at least one (or both) of $1, 2$ are in I It is straightforward to give an expression for $\alpha(I)^{s_0}$. Finally, the *(Iso)* relations are

$$\alpha(1, I)\alpha(2, I)s_1 = \alpha(1, I)\alpha(2, I),$$

for all $I \subseteq X \setminus \{1, 2\}$.

5.3. *An example of Solomon*

For the beautiful interplay between group theory and combinatorics that results, we look at a family of examples considered by Solomon in [**26**, Example 5.7]. We follow the pattern of the last section, defining first an algebraic monoid M, followed by an abstract monoid of partial isomorphisms which turns out to be isomorphic to the Renner monoid of M.

Let $G_0 = \mathbf{SL}_n$ and V_0 the natural module for G_0. Let $\bigwedge^p V_0$ be the p-th exterior power and let

$$V = \bigotimes_{p=1}^{n-1} \bigwedge^p V_0, \text{ with } \dim V := m = \prod_{p=1}^{n-1} \binom{n}{p}.$$

If $\rho : G_0 \to GL(V)$ is the corresponding representation then let $M = \overline{k^\times \rho(G_0)} \subset \mathbf{M}_m$. Let R be the Renner monoid of M.

Now to a monoid of partial isomorphisms. Take an n-dimensional Euclidean space with basis $\{u_1, \ldots, u_n\}$ and $\mathfrak{S}_n$ acting by $u_i\pi = u_{i\pi}$ for $\pi \in \mathfrak{S}_n$. The $(n-1)$-simplex Δ^{n-1} is the convex hull of the u_i, and as the $\mathfrak{S}_n$-action is linear, it restricts to an action on Δ^{n-1}. This is just the action of the group of reflections and rotations of Δ^{n-1}. In particular, if O is an admissible partial orientation of Δ^{n-1} as in Example 3 of §1.2, then it is clear that the image $O\pi$ is also admissible. Consider the induced $\mathfrak{S}_n$-action on the set E_0 of admissible partial orientations and extend it to the poset E of Example 3 by defining $\mathbf{1}\pi = \mathbf{1}$ for all $\pi \in \mathfrak{S}_n$. This action is clearly by poset isomorphisms.

Thus the collection of intervals $E_{\geq O} = \{O' \in E \,|\, O \leq O'\}$ forms a system $\mathcal{S}$ of subsets of E for $\mathfrak{S}_n$ with $M(\mathfrak{S}_n, \mathcal{S})$ the corresponding monoid of partial isomorphisms.

The isomorphism $M(\mathfrak{S}_n, \mathcal{S}) \cong R$. By Proposition 5.2 it suffices to establish an isomorphism from $M(\mathfrak{S}_n, \mathcal{S})$ to $M(W, \mathcal{S}_P)$ where W is the Weyl group of G (or G_0) and P is the polytope described in §5.1. The Weyl group is $W(A_{n-1})$ and we take $\theta : \mathfrak{S}_n \to W(A_{n-1})$ the standard isomorphism given by $(i, i+1) \mapsto s_i := s_{v_{i+1}-v_i}$.

We now describe P, following [**26**, Example 5.7]. It turns out to be convenient to describe another abstract polytope P' first, and then relate this back to the P we are interested in. Let $X = \{1, \ldots, n\}$ and $\tau = \{J_1, \ldots, J_{n-1}\}$ be a collection of subsets of X with $|J_i| = i$. Thus, τ contains exactly one non-empty proper set of each possible cardinality. Let Σ be the set of all such τ. Given $\tau \in \Sigma$, let a_j be the number of J_i in which j occurs, and let v_τ be the vector $(a_1, \ldots, a_n)^T \in \mathbb{R}^n$.

PROPOSITION 5.5. *The convex hull P' of the v_τ, for $\tau \in \Sigma$, is the $(n-1)$-permutohedron having the parameters $m_1, \ldots, m_n = 0, \ldots, n-1$.*

The proof is in [**6**, §6.5]. The polytope P' is not quite the P described in §5.1. To get it back, we need to compute the columns of the matrix A whose rows $\mathbf{a}_i = (a_{i1}, \ldots, a_{im})$ are given by (5.1). Recall the simple roots $v_{p+1} - v_p$ from Example 7 and let $(v_{p+1} - v_p)^\vee(t) = \mathrm{diag}(1, \ldots, t, t^{-1}, \ldots, 1)$ for $1 \leq p \leq n-1$ be the corresponding coroots with the t in the p-th position. If $v_\tau = (a_1, \ldots, a_n)^T$ arises from $\tau = \{J_1, \ldots, J_{n-1}\}$ with $J_1 = \{i\}, J_2 = \{j, k\}, \ldots, J_{n-1} = \{1, \ldots, \widehat{q}, \ldots, n\}$, then V has basis the v of the form

$$v = v_i \otimes (v_j \wedge v_k) \otimes \cdots \otimes (v_1 \wedge \cdots \wedge \widehat{v}_q \wedge \cdots \wedge v_n),$$

as τ ranges over Σ and where $\{v_1, \ldots, v_n\}$ is a basis for V_0. Then $\rho(v_{p+1} - v_p)^\vee(t)v = t^{a_p - a_{p+1}}v$, and so the columns of A are the $(a_1 - a_2, \ldots, a_{n-1} - a_n)^T$. In particular the map $(x_1, \ldots, x_n)^T \mapsto (x_1 - x_2, \ldots, x_{n-1} - x_n)^T$ sends the permutohedron P' of Proposition 5.5 to the polytope P described in §5.1.

Let O be an admissible partial orientation of Δ^{n-1} and $O \mapsto f'_O$ be the isomorphism $E \to \mathscr{F}(P')$ of Proposition 2.4, with $m_1, \ldots, m_n = 0, \ldots, n-1$. The map $\mathbb{R}^n \to \mathbb{R}^{n-1}$ given by $(x_1, \ldots, x_n) \mapsto (x_1 - x_2, \ldots, x_{n-1} - x_n)$ induces an isomorphism $\mathscr{F}(P') \to \mathscr{F}(P)$ which we write as $f'_O \mapsto f_O$. Finally, let ζ send the partial identity on the interval $E_{\geq O}$ of E to the partial identity on the interval $E_{\geq f_O}$ of $\mathscr{F}(P)$.

That ζ is equivariant and θ preserves idempotent stabilizers (which actually turn out to be trivial) we leave to the reader, although we supply the following hint: the vertices of P can be labeled (in a one to one fashion) by the $g \in W(A_{n-1})$ and the edges can be labeled by the s_i so that there is an s_i-labeled edge connecting g to g' if and only if $g' = gs_i$ (in the language of [**5**, Chapter 3], the 1-skeleton of P is the universal cover, or Cayley graph, of the presentation 2-complex of W with respect to its presentation as Coxeter group). The action of $W(A_{n-1})$

on the vertices of P can then be described as follows: if $g = s_{i_1} \dots s_{i_k} \in W(A_{n-1})$ and v is the vertex of P labeled by the identity, then let v' be the terminal vertex of a path starting at v and with edges labeled $s_{i_1}, \dots, s_{i_k}$. For any vertex u, let $s_{j_1} \dots s_{j_\ell}$ be the label of a path from v to u, and let u' be the terminal vertex of a path starting at v' and with label $s_{j_1} \dots s_{j_\ell}$. Then g maps u to u' (and in particular v to v'). In the language of [**5**, Chapter 4], the $W(A_{n-1})$-action is as the Galois group of the covering of 2-complexes.

Define $\varphi : M(\mathfrak{S}_n, \mathcal{S}) \to M(W(A_{n-1}), \mathcal{S}_P)$ as in Proposition 5.1.

Presentation data for the monoid $M(\mathfrak{S}_n, \mathcal{S})$. The atoms are the partial orientations a_J from §1.2 for J a non-empty proper subset of $X = \{1, \dots, n\}$. The $\mathfrak{S}_n$-action on the partial orientations induces an action on the atoms given by $a_J \cdot \pi = a_{J\pi}$ for $\pi \in \mathfrak{S}_n$. Thus, we just have the action of $\mathfrak{S}_n$ on the subsets of X, so for the representatives O_k we can appeal to the interlude of the previous section.

The set $Char_1$ corresponds to the n_0 with $0 \leq n_0 \leq n$, and we take $O_1 = \{a_1, \dots, a_{n-1}\}$ with $a_i := a_{\{1\dots,i\}}$. The absence of an a_0 and a_n is because we have restricted to the action on the non-empty proper subsets of X. The set $Char_2$ corresponds to the n_0, n_1, n_2 such that $0 \leq n_0 \leq n_1, n_2$ with $n_1 + n_2 - n_0 \leq n$ and $0 < n_i < n$. From the interlude we get a tuple (J, K) where

$$J = \{1, \dots, n_0\} \cup \{n_0 + 1, \dots, n_1\} \text{ and } K = \{1, \dots, n_0\} \cup \{n_1 + 1, \dots, n_1 + n_2 - n_0\}$$

are representatives for the corresponding orbit. Thus we take O_2 to be the pairs $\{a_J, a_K\}$. The set $Char_3$ corresponds to the $n_0, \dots, n_3, n_{ij}$ satisfying the conditions given in the example at the end of the interlude, together with $0 < n_{ij} < n$. We get a corresponding tuple (J_1, J_2, J_3) using the scheme (†) above, and we take O_3 to be the the set of $\{a_{J_1}, a_{J_2}, a_{J_3}\}$.

For J a non-empty proper subset of X, fix an element $w_J \in \mathfrak{S}_n$ with $Jw_J = \{1, \dots, |J|\}$ and let

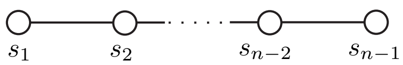
$$\alpha_J := \omega_J a_k \omega_J^{-1}. \tag{5.12}$$

It turns out that for an arbitrary $O \in E$ we do not require an expression in the atoms for O, except in the case $O = \mathbf{1}$, the formally adjoined unique maximal element. We take $\mathbf{1} := \bigvee a_J$, the join over all the atoms, i.e. over all non-empty proper subsets J of X.

Finally we have the set *Iso*. The set T consists of the transpositions $(i, j) \in \mathfrak{S}_n$ and for $t \in T$, $H_t = \{O \in E_0 \,|\, Ot = O\} \cup \{\mathbf{1}\}$ with $\mathscr{A} = \{H_t \,|\, t \in T\}$. There is a single $\mathfrak{S}_n$-orbit on $\mathscr{A}$ with representative $H_1 := H_{s_1}$ where $s_i := (i, i + 1)$. The O are the partial admissible orientations of Δ^{n-1}, and one such is fixed by s_1 exactly when the edge joining v_1 and v_2 is not in O, and for all $i > 2$, the edge joining v_1 and v_i lies in O if and only if the edge joining v_2 and v_i lies in O. We want O minimal with the property that $H_1 \supseteq E_{\geq O}$. But if $O < \mathbf{1}$ then the interval $E_{\geq O}$ contains an admissible partial orientation in which all the edges of Δ^{n-1} are oriented (i.e. a total order). Thus, $E_{\geq O}$ contains an O' in which the edge joining v_1 and v_2 is oriented, and so $O's_1 \neq O'$. The result is that $H_1 \not\supseteq E_{\geq O}$.

The only element of E then that is minimal with $H_1 \supseteq E_{\geq O}$ is $\mathbf{1}$, and *Iso* consists of the single pair $(\mathbf{1}, s_1)$.

EXAMPLE 11 (the presentation for the Renner monoid of M). We have generators $s_1, \dots, s_{n-1}$ and $a_1, \dots, a_{n-1}$ with *(Units)* relations $(s_i s_j)^{m_{ij}} = 1$ where the m_{ij} are given by the symbol



The *(Idem1)* relations are $a_i^2 = a_i$ $(1 \leq i \leq n-1)$ and the *(Idem2)* relations are $\alpha_J\alpha_K = \alpha_K\alpha_L$ where the α's are given by (5.12),

$$J = \{1, \dots, n_0\} \cup \{n_0+1, \dots, n_1\}, K = \{1, \dots, n_0\} \cup \{n_1+1, \dots, n_1+n_2-n_0\}$$

and $0 \leq n_0 \leq n_1, n_2$ are such that $n_1+n_2-n_0 \leq n$ and $0 < n_i < n$.

The presentation for the permutohedron from §1.2 gives *(Idem3)* relations $\alpha_{J_1}\alpha_{J_2} = \alpha_{J_1}\alpha_{J_2}\alpha_{J_3}$ for all $\{a_{J_1}, a_{J_2}, a_{J_3}\} \in O_3$ where J_1, J_2 satisfy $J_1 \neq J_1 \cap J_2 \neq J_2$; that is, $n_1 - n_0$, $n_{13} - n_1 - n_3 + n_0$ are not both zero, and $n_2 - n_0$, $n_{23} - n_2 - n_3 + n_0$ are not both zero.

The *(RefIdem)* are $s_i\alpha_J = \alpha_{Js_i}s_i$ for $1 \leq i \leq n-1$ and J a non-empty proper subset of X. Finally the *(Iso)* are the single relation

$$\prod \alpha_J \cdot s_1 = \prod \alpha_J,$$

where the product is over all proper non-empty subsets J of X.

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