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Feasible parallel-update distributed MPC for uncertain linear systems sharing convex constraints☆

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Abstract

A distributed MPC approach for linear uncertain systems sharing convex constraints is presented. The systems, which are dynamically decoupled but share constraints on state and/or inputs, optimize once, in parallel, at each time step and exchange plans with neighbours thereafter. Coupled constraint satisfaction is guaranteed, despite the simultaneous decision making, by extra constraint tightening in each local problem. Necessary and sufficient conditions are given on the margins for coupled constraint satisfaction, and a simple on-line scheme for selecting margins is proposed that satisfies the conditions. Robust feasibility and stability of the overall system are guaranteed by use of the tube MPC concept in conjunction with the extra coupled constraint tightening.

Keywords: control of constrained systems; predictive control; decentralization; time-invariant

1. Introduction

Providing optimal control and decision-making to a system is very desirable. For such purposes, model predictive control (MPC) [1] has achieved more widespread adoption and greater impact in industry than any other modern control technology; for example, MPC has largely replaced traditional PID loops as the controller of choice in the process control industry [2]. The popularity of MPC is not restricted to industry, and significant advances have been made by academic researchers on theoretical properties such as stability and robustness [3].

When the system to be controlled is large in scale, or physically or organizationally disjoint, centralized MPC may be impractical or undesirable for reasons of computation, communication and the single point of failure. Completely decentralized MPC, on the other hand, in which subsystem controllers make decisions independently and without coordination, can result in poor performance and even instability [4]. Thus, attention has focused on *distributed* MPC [5], wherein controllers share information. The challenge is then how should computation and communication be used to coordinate actions and achieve system-wide feasibility, stability and optimality.

Many approaches to distributed MPC have now been proposed, and comprehensive surveys are given in [6, 7]. Algorithms are broadly divisible according to the classes of system to which they apply [5]: for instance, linear versus nonlinear dynamics; coupling via the dynamics versus coupled via constraints. The focus of this paper is on systems comprising multiple, dynamically-decoupled subsystems, each with linear time-invariant dynamics. The subsystems, which are subject to bounded, persistent disturbances, are coupled via shared constraints on states and/or inputs. The presence of such constraints has been identified as a key open research problem for DMPC [2]. One of the main difficulties is in determining the set of conditions under which coupled constraint satisfaction is ensured despite the decision-making of independent controllers. Algorithms are either hierarchical or distributed (*i.e.*, with or without a supervisory, coordinating agent), iterative or non-iterative, and sequential or parallel in the timing of updates [5].

Iterative distributed approaches include those based on primal decomposition, in which controllers share information, and bargain or coordinate with local neighbours [8–10]; dual decomposition approaches where iteration is to primal feasibility (satisfaction of coupled constraints) [11–13]; and, a cooperative scheme wherein distributed control agents augment their decision spaces to include the inputs subject to shared constraints [14].

Distributed approaches that *do not* rely on iteration and negotiation to achieve feasible solutions at each time step lead to lower levels of communication, yet the problem of guaranteeing feasibility is more challenging. Most approaches use serial or sequential, rather than parallel, updates. For example, Richards and How [15] proposed a sequential approach to robust DMPC for subsystems sharing constraints, using constraint tightening and disturbance feedback to guarantee robust feasibility. The subsystem controllers optimize in a fixed sequence within each sampling interval, transmitted new plans as they become available. An extension of the approach has been proposed for nonlinear subsystems [16]. In [17], a single-update robust DMPC approach was proposed. Based on tube-based robust MPC [18], each subsystem controller designs a tube, rather than a single trajectory, of predicted states, and employs a local feedback con-

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troller to maintain the state within the tube for any realization of the disturbance. Similar to [15], constraint tightening is used to guarantee feasibility in the presence of uncertainty; however, the sequence dependency and the need for all subsystems to optimize at each time step is removed, leading to a scheme with low and flexible levels of communication [17]. Both approaches, however, have limitations imposed by their sequential/serial nature: [15] requires sufficient time within a sampling interval for the entire sequence of optimization problems to be solved. On the other hand, [17] permits only one (or, strictly, non-coupled) subsystems to optimize at each time step, which can lead to poor performance.

The feasible parallel-update DMPC proposed in this paper avoids these limitations by permitting the *simultaneous* optimizing of subsystems' plans at each time step while maintaining robust feasibility and stability. The advantage of low and flexible communication is retained, since no inter-agent iteration or negotiation is required, and *any* number of subsystems may optimize at a time step. The approach is a significant extension of [15, 17], in that the reliance on sequential or serial updating is removed. Subsystems maintain satisfaction of convex coupled constraints on states and/or inputs, despite optimizing simultaneously, by tightening their local representations of the coupled constraints. Comparable approaches include the tube-based schemes recently proposed by Farina and Scattolini [19] and Riverso and Ferrari-Trecate [20] for dynamically-coupled, deterministic subsystems sharing constraints. These also achieve coupled constraint satisfaction despite parallel updating: in the former, predicted state and input trajectories are constrained to lie within time-invariant neighbourhoods around known-feasible references, and coupled constraints are tightened accordingly. In the latter, the tube MPC concept is applied twice, leading to a double tightening of constraints. Other approaches include those iterative methods that maintain primal feasibility across iterates [9, 14, 21] and, therefore, can be terminated after a single iteration. However, in none of these papers is an explicit mechanism given for selecting the margins by which coupled constraints are tightened. A key contribution of this paper is that a simple and explicit scheme is proposed for the on-line calculation of margins by which to tighten coupled constraints. The margins are time-varying, both with sampling time and along the prediction horizon, and are calculated from information transmitted between controllers at the previous time step. Necessary and sufficient conditions are given on the size of margins for robust coupled constraint satisfaction. Moreover, robust feasibility and stability of the closed-loop system is established for any number of subsystems optimizing simultaneously at each time step.

The paper is organized as follows. The problem is stated in Section 2. This is followed by a review of single-update tube DMPC [17] in Section 3. In Section 4, the necessary and sufficient margins for simultaneous coupled constraint satisfaction are developed, followed by the presentation of the proposed feasible parallel-update DMPC in Section 5. The approach is demonstrated by numerical examples in Section 6. Finally, Section 7 concludes the paper.

Notation and conventions:. The non-negative and positive reals (integers) are denoted, respectively, $\mathbb{R}_{0+}$ and $\mathbb{R}_{+}$ ($\mathbb{N}_{0+}$ and $\mathbb{N}_{+}$). Given $a, b \in \mathbb{N}_{0+}$, with $b > a$, $\mathbb{N}_{[a,b]} \triangleq \{a, a+1, \ldots, b-1, b\}$. $\mathbb{N}_b$ denotes $\mathbb{N}_{[0,b]}$. The cardinality of a finite set $\mathcal{A}$ is $n(\mathcal{A})$. For $x_i \in \mathbb{R}^n$, $i \in \mathbb{N}_{[a,b]}$, with $b > a$, $(x_i)_{i \in \mathbb{N}_{[a,b]}}$ means $(x_a, x_{a+1}, \ldots, x_{b-1}, x_b) \in \mathbb{R}^{(b-a)n}$. $x_{(-i)}$ means $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. For $a, b \in \mathbb{R}^n$, $a \leq b$ applies element by element. For $X, Y \subset \mathbb{R}^n$, the Minkowski sum is $X \oplus Y \triangleq \{x + y : x \in X, y \in Y\}$; for $Y \subset X$, the Pontryagin difference is $X \ominus Y \triangleq \{x \in \mathbb{R}^n : Y + x \subset X\}$. For $X \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$, $X \oplus a$ means $X \oplus \{a\}$. AX denotes the image of a set $X \subset \mathbb{R}^n$ under the linear mapping $A \colon \mathbb{R}^n \mapsto \mathbb{R}^p$, and is given by $\{Ax : x \in X\}$. A polyhedron is the intersection of a finite number of halfspaces, which is convex, and a polytope is a closed and bounded polyhedron, and is also convex. For $X \subset \mathbb{R}^n$, the support function is $h(X, y) \triangleq \sup\{y^\top x : x \in X\}$ for $y \in \mathbb{R}^n$. A set $X \subset \mathbb{R}^n$ is positively invariant (PI) for a system $x^+ = f(x)$ if and only if for all $x \in X$ it holds that $f(x) \in X$. A set $X \subset \mathbb{R}^n$ is robust positively invariant (RPI) for a system $x^+ = f(x, w)$ if and only if for all $x \in X$ and all $w \in \mathbb{W}$ it holds that $f(x, w) \in X$. The notation $x(k + j|k)$ indicates a prediction of x for j steps ahead from k.

2. Problem statement

2.1. System dynamics

Consider a set of dynamically decoupled subsystems, $\mathcal{I} = \{1, \ldots, N_l\}$. A subsystem $i \in \mathcal{I}$ has the linear time-invariant, discrete-time dynamics

$$x_i^+ = A_i x_i + B_i u_i + w_i, \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $w_i \in \mathbb{R}^{n_i}$ are, respectively, its state, control input and disturbance. x_i^+ is the successor state. The existence of a stabilizing control law K_i for each (A_i, B_i) is assured by the following assumption.

Assumption 1. For each $i \in \mathcal{I}$, (A_i, B_i) is stabilizable, and the state x_i is known exactly by the controller for i at each sampling instant.

2.2. Local constraints

The state and input of each subsystem $i \in \mathcal{I}$ are subject to local constraints

$$x_i \in \mathbb{X}_i, \qquad u_i \in \mathbb{U}_i,$$

while the disturbance w_i is unknown *a priori* but lies in a set $\mathbb{W}_i$.

Assumption 2. For each $i \in \mathcal{I}$, $\mathbb{X}_i$ is closed and convex, $\mathbb{U}_i$ is compact and convex, and each contains the origin in its interior. $\mathbb{W}_i \subset \mathbb{X}_i$ is compact and convex, and contains the origin (but not necessarily in its interior).

2.3. Shared constraints

Coupling between subsystems exists in the form of a set of shared constraints, $\mathcal{C} = \{1, \dots, N_c\}$. A shared constraint $c \in \mathcal{C}$ involves a subset of subsystems, $\mathcal{I}_c \subseteq \mathcal{I}$, and acts on the collection of *coupling outputs* of those subsystems as follows.

$$z_c \triangleq (z_{ci})_{i \in \mathcal{I}_c} \in \mathbb{Z}_c \text{ where } z_{ci} = E_{ci} x_i + F_{ci} u_i, \forall i \in \mathcal{I}_c. \quad (2)$$

Here, $z_{ci} \in \mathbb{R}^{r_{ci}}$ and $z_c \in \mathbb{R}^{r_c}$ where $r_c = \sum_{i \in \mathcal{I}_c} r_{ci}$. This is a general form of coupling constraint: a constraint c permits coupling between the states and/or inputs of any subset of subsystems.

Assumption 3. For each $c \in \mathcal{C}$, $\mathbb{Z}_c$ is a closed, convex polyhedron, containing the origin in its interior.

It follows that $\mathbb{Z}_c$ may be represented by M_c linear inequalities:

$$\mathbb{Z}_c = \mathbb{Z}_c(q_c) \triangleq \{z \in \mathbb{R}^{r_c} : p_{cm}^\top z \le q_{cm}, \forall m \in \mathbb{N}_{[1,M_c]}\} \quad (3)$$

where $p_{cm} \in \mathbb{R}^{r_c}$, $q_{cm} \in \mathbb{R}_+$, for all $m \in \mathbb{N}_{[1,M_c]}$. The matrix and vector that collect p_{cm} and q_{cm}, respectively, are $P_c \in \mathbb{R}^{M_c \times r_c}$ and $q_c = (q_{c1}, q_{c2}, \dots, q_{cM_c}) \in \mathbb{R}_+^{M_c}$, so that (3) may also be written as $\mathbb{Z}_c = \{z \in \mathbb{R}^{r_c} : P_c z \le q_c\}$.

2.4. Coupling structure

The following definitions identify structure in the coupling between subsystems, and are used to determine what information a local subsystem controller needs. By construction, $\mathcal{I}_c = \{i \in \mathcal{I} : (E_{ci}, F_{ci}) \neq 0\}$, and the subset of constraints in which subsystem $i \in \mathcal{I}$ is involved is $\mathcal{C}_i = \{c \in \mathcal{C} : (E_{ci}, F_{ci}) \neq 0\}$. Then, the set of other subsystems sharing constraints with a subsystem i is $\mathcal{Q}_i = \left(\bigcup_{c \in \mathcal{C}_i} \mathcal{I}_c\right) \setminus \{i\}$.

2.5. Control objective

The control objective is to regulate the state of each subsystem to the origin while satisfying all constraints and minimizing the infinite-horizon, system-wide cost function

$$\sum_{k=0}^{\infty} \sum_{i \in \mathcal{I}} l_i(x_i(k), u_i(k)), \quad (4)$$

where $l_i \colon \mathbb{R}^{n_i \times m_i} \mapsto \mathbb{R}_{0+}$, $l_i(x_i, u_i) \ge k \|(x_i, u_i)\|$ for some $k > 0$ and $l_i(0,0) = 0$.

3. Overview of single-update tube MPC

The single-update tube DMPC approach [17] is based on the "tube MPC" concept [18], wherein the controller designs a sequence of disturbance-invariant state sets for the system to follow. The sets are centered on the nominal trajectory; that is, the state predictions obtained by applying the optimized control sequence to the disturbance-free dynamics. In a distributed setting, each subsystem controller designs a tube for its local subsystem to follow. Use of a local feedback controller K_i alongside the implicit MPC control law then guarantees that each subsystem state remains within its tube, despite the action of the disturbance w_i, and without the need to re-optimize at every time step (as is done in conventional MPC and DMPC). Therefore, by permitting only a single subsystem to optimize at each time step, and subsequently communicating to other subsystems information about its new tube, robust coupled constraint satisfaction, feasibility and stability are guaranteed [17]. The remainder of this section more formally describes, and introduces key assumptions and definitions used later in the paper.

3.1. Distributed optimal control problem

With subsystem i at a state $x_i(k)$ at time k, the distributed optimal control problem (DOCP-i) is

$$J_i^0(x_i(k), \mathbf{z}_i^*(k)) = \min_{\mathbf{u}_i(k)} \Big\{ J_i(\mathbf{u}_i(k)) : \mathbf{u}_i(k) \in \mathcal{U}_i(x_i(k), \mathbf{z}_i^*(k)) \Big\}. \quad (5)$$

The vector $\mathbf{z}_i^*(k)$ denotes coupling output information from other subsystems needed by i to solve its problem at time k, and is described later; it is included as an index to the optimal cost J_i^0 and feasible set $\mathcal{U}_i$ to highlight the dependency of each on the coupling outputs of other subsystems, and the coupling between DOCPs. The decision variable $\mathbf{u}_i(k)$ contains the initial state prediction, $\bar{x}_i(k|k)$, and the sequence of future controls, $\{\bar{u}_i(k|k), \bar{u}_i(k+1|k), \dots, \bar{u}_i(k+N-1|k)\}$. The cost function is a finite-horizon approximation to the infinite-horizon, local cost in (4):

$$J_i(\mathbf{u}_i(k)) \triangleq F_i(\bar{x}_i(k+N|k)) + \sum_{j=0}^{N-1} l_i(\bar{x}_i(k+j|k), \bar{u}_i(k+j|k)),$$

where $F_i \colon \mathbb{R}^{n_i} \mapsto \mathbb{R}_{0+}$. The feasible set $\mathcal{U}_i(x_i(k), \mathbf{z}_i^*(k))$ is defined by the following constraints for all $j \in \mathbb{N}_{N-1}$.

$$x_i(k) - \bar{x}_i(k|k) \in \mathcal{R}_i, \quad (6a)$$

$$\bar{x}_i(k+j+1|k) = A_i \bar{x}_i(k+j|k) + B_i \bar{u}_i(k+j|k), \quad (6b)$$

$$\bar{x}_i(k+j|k) \in \mathbb{X}_i \ominus \mathcal{R}_i, \quad (6c)$$

$$\bar{u}_i(k+j|k) \in \mathbb{U}_i \ominus \mathcal{S}_i, \quad (6d)$$

$$\bar{x}_i(k+N|k) \in \mathcal{X}_i^{\mathrm{f}}, \quad (6e)$$

$$\bar{z}_{ci}(k+j|k) = E_{ci} \bar{x}_i(k+j|k) + F_{ci} \bar{u}_i(k+j|k), \forall c \in \mathcal{C}_i, \quad (6f)$$

$$(\bar{z}_{ci}(k+j|k), \bar{z}_{c(-i)}^*(k+j)) \in \mathbb{Z}_c \ominus \mathcal{T}_c, \forall c \in \mathcal{C}_i. \quad (6g)$$

The details of this feasible set are now described. $\mathcal{R}_i$ in (6a) is an RPI set for the uncertain subsystem i under the local feedback law $u_i = K_i x_i$, *i.e.*, for the closed-loop dynamics $x_i^+ = (A_i + B_i K_i) x_i + w_i$. Note the existence of $\mathcal{R}_i$ is assured by Assumptions 1 and 2. In this paper, we assume the following.

Assumption 4. For each $i \in \mathcal{I}$, $\mathcal{R}_i$ is a polytope with $0 \in \mathcal{R}_i$.

Note that this assumption is not restrictive, and tools and methods are available for computing polytopic invariant sets—or approximations to them—and corresponding control laws, e.g. [22–24]. To minimize conservativeness, it is desirable that $\mathcal{R}_i$ be chosen as small as possible [18].

Constraint (6b) is the nominal subsystem dynamics. In (6c), (6d) and (6g), the constraint sets are tightened by margins for robustness, by taking the Pontryagin difference between sets $\mathbb{X}_i$,

$\mathbb{U}_i$, $\mathbb{Z}_c$ and, respectively, sets $\mathcal{R}_i$, $\mathcal{S}_i \triangleq K_i\mathcal{R}_i$, $\mathcal{T}_c \triangleq \prod_{i\in\mathcal{I}_c} E_{ci}R_i \oplus F_{ci}\mathcal{S}_i$, for $c \in C_i$. (Note that, by Assumption 4 and linearity, $\mathcal{S}_i$ and $\mathcal{T}_c$ are polytopic and contain the origin [25]). The following assumption limits the size of these tightening sets, and is mild for most applications.

Assumption 5. For each $i \in \mathcal{I}$, $\mathcal{R}_i \subset \mathbb{X}_i$, $\mathcal{S}_i \subset \mathbb{U}_i$ and $\mathcal{T}_c \subset \mathbb{Z}_c, \forall c \in C$.

The terminal set $\mathcal{X}_i^{\mathrm{f}}$ in (6e) is a PI set for the nominal subsystem dynamics under the local terminal control law $u_i = \kappa_i^{\mathrm{f}}(x_i)$, *i.e.*, for the closed-loop dynamics $x_i^+ = A_i x_i + B_i\kappa_i^{\mathrm{f}}(x_i)$.

Assumption 6. For each $i \in \mathcal{I}$, $\mathcal{X}_i^{\mathrm{f}}$ is a polytope with $0 \in \mathcal{X}_i^{\mathrm{f}}$, and $\mathcal{X}_i^{\mathrm{f}} \subseteq \mathbb{X}_i \ominus \mathcal{R}_i$, $\kappa_i^{\mathrm{f}}(\mathcal{X}_i^{\mathrm{f}}) \subseteq \mathbb{U}_i \ominus K_i\mathcal{R}_i$, and $\prod_{i\in\mathcal{I}_c}(E_{ci}\mathcal{X}_i^{\mathrm{f}} \oplus F_{ci}\kappa_i^{\mathrm{f}}(\mathcal{X}_i^{\mathrm{f}})) \subseteq \mathbb{Z}_c \ominus \mathcal{T}_c$ for $c \in C_i$.

The terminal set is used in conjunction with the terminal cost F_i, under the following assumption. Note that Assumptions 6 and 7 are common, and correspond to A1–A4 in [3].

Assumption 7. For each $i \in \mathcal{I}$, $F_i(A_i x_i + B_i\kappa_i^{\mathrm{f}}(x_i)) - F_i(x_i) \leq -l_i(x_i, \kappa_i^{\mathrm{f}}(x_i)), \forall x_i \in \mathcal{X}_i^{\mathrm{f}}$.

Finally, as previously mentioned, the feasible set $\mathcal{U}_i$ depends not only on the sampled local state $x_i(k)$ but also on the coupling outputs of subsystems sharing constraints with i. In (6g), $\bar{z}^*_{c(-i)}(k+j)$ denotes the collection of coupling outputs at prediction step j from subsystems sharing constraint $c \in C_i$ with subsystem i, *i.e.*, the collection of $\bar{z}^*_{cq}(k+j)$ over $q \in \mathcal{P}_c$. (Alternatively viewed, the minus subscript notation means all elements of $\bar{z}^*_c(\cdot)$ excluding $\bar{z}^*_{ci}(\cdot)$.) Then $\mathbf{z}^*_i(k)$ is defined as the collection of $\bar{z}^*_{c(-i)}(k+j)$ over all $j \in \mathbb{N}_{N-1}$ and $c \in C_i$. How this information is obtained is described later. First, the tube DMPC control law and algorithm are outlined.

3.2. The tube DMPC control law and single-update algorithm

With subsystem i at state $x_i(k)$ at time k, assume that a feasible (but not necessarily optimal) solution to DOCP-i is available, *i.e.*,

$$\mathbf{u}^*_i(k) \triangleq \{\bar{x}^*_i(k|k), \bar{u}^*_i(k|k), \bar{u}^*_i(k+1|k), \ldots, \bar{u}^*_i(k+N-1|k)\}.$$

Then the control applied to a subsystem i is

$$u^*_i(k) = \bar{u}^*_i(k|k) + K_i(x_i(k) - \bar{x}^*_i(k|k)). \tag{7}$$

By construction, all constraints are satisfied at time k: $x_i(k) \in \bar{x}^*_i(k|k) \oplus \mathcal{R}_i \subset \mathbb{X}_i$, $u^*_i(k) \in \bar{u}^*_i(k|k) \oplus \mathcal{S}_i \subset \mathbb{U}_i$ and $z^*_c(k) \in \bar{z}^*_c \oplus \mathcal{T}_c \subset \mathbb{Z}_c$. Subsequently, using the control (7), the state of subsystem i evolves as $x^*_i(k+1) \in A_i\bar{x}^*_i(k|k) + B_i\bar{u}^*_i(k|k) \oplus (A_i + B_iK_i)\mathcal{R}_i \oplus \mathbb{W}_i = \bar{x}^*_i(k+1|k) \oplus (A_i + B_iK_i)R_i \oplus \mathbb{W}_i \subseteq \bar{x}^*_i(k+1|k) \oplus \mathcal{R}_i$, and, since $\bar{x}^*_i(k+1|k) \oplus \mathcal{R}_i \subset \mathbb{X}_i$ and $\bar{u}^*_i(k+1|k) \oplus \mathcal{S}_i \subset \mathbb{U}_i$, local state and input constraints remain satisfied at time $k+1$, regardless of disturbances. Moreover, $z^*_c(k+1) \in \bar{z}^*_c(k+1|k) \oplus \mathcal{T}_c \subset \mathbb{Z}_c$, so coupled constraints are also satisfied. Therefore, it is simple to show that a feasible solution to each DOCP-i can be constructed without solving *any* optimization problem at time $k+1$:

$$\begin{aligned}\tilde{\mathbf{u}}_i(k+1) \triangleq \{&\bar{x}^*_i(k+1|k), \bar{u}^*_i(k+1|k), \ldots,\\ &\bar{u}^*_i(k+N-1|k), \kappa_i^{\mathrm{f}}(\bar{x}^*_i(k+N|k))\}.\end{aligned} \tag{8}$$

Moreover, no information exchange is needed to construct these solutions at time $k+1$. This suggests the following scheme, used in [17]: a single subsystem, say i, (or, strictly, a set of subsystems not sharing any constraints) optimizes at time $k+1$, solving its DOCP-i to obtain a solution $\mathbf{u}^0_i(k+1)$ (not necessarily equal to $\tilde{\mathbf{u}}_i(k+1)$) given $x_i(k+1)$ and the coupling information $\mathbf{z}^*_i(k+1)$, which is constructed from $\mathbf{z}^*(k)$. All other subsystems renew existing feasible plans from time k via (8). The optimizing subsystem i communicates its new plan to coupled subsystems $q \in Q_i$. At time k, therefore, the coupling information $\mathbf{z}^*_i(k)$ needed by i is the collection of $\bar{z}^*_{cq}(k+j|\hat{k}_q)$ over all $j \in \mathbb{N}_{N-1}, q \in \mathcal{P}_c, c \in C_i$, where $\hat{k}_q$ is the time at which subsystem q last updated by optimization.

When the system is controlled according to this algorithm, robust coupled constraint satisfaction, feasibility and stability of the closed-loop system is guaranteed [17].

3.3. Centralized optimal control problem

For later use, we define the corresponding centralized optimal control problem (COCP). For the system at a state $x(k) = (x_i(k))_{i\in\mathcal{I}}$ at time k:

$$J^0(x(k)) = \min_{\mathbf{u}(k)}\left\{\sum_{i\in\mathcal{I}} J_i(\mathbf{u}_i(k)) : \mathbf{u}(k) \in \mathcal{U}(x(k))\right\} \tag{9}$$

where $\mathbf{u}(k) \triangleq (\mathbf{u}_i(k))_{i\in\mathcal{I}}$, and the feasible set $\mathcal{U}(x(k))$ is defined by (6a)–(6f) for all $i \in \mathcal{I}$ and the coupling constraint

$$\bar{z}_c(k+j|k) \in \mathbb{Z}_c \ominus \mathcal{T}_c, \forall c \in C, j \in \mathbb{N}_{N-1}.$$

The next result, which is adapted from Theorem 3.1 in [17], follows from construction of the constraint sets, and states that each and every subsystem i has a feasible solution to its DOCP-i if and only if the collection of these individual solutions is a feasible solution to the COCP.

Lemma 1. $(\mathbf{u}^*_i(k))_{i\in\mathcal{I}} \in \mathcal{U}(x(k)) \iff \mathbf{u}^*_i(k) \in \mathcal{U}_i(x_i(k), \mathbf{z}^*_i(k))$*, for all $i \in \mathcal{I}$, where, for each $i \in \mathcal{I}$, $\mathbf{z}^*_i(k)$ is the collection of $\bar{z}^*_{cq}(k+j)$ (obtained from $\mathbf{u}^*_q(k)$) over all $j \in \mathbb{N}_{N-1}, q \in \mathcal{I}_c, c \in C_i$.*

4. A tightening procedure for parallel coupled constraint satisfaction

The key to the robust coupled constraint satisfaction of [17] is the single-update restriction. With the system at a state $(x_i(k))_{i\in\mathcal{I}}$, and, supposing a feasible solution $\mathbf{u}^*_i(k)$ exists to each DOCP-i, it is clear that the coupled constraints are satisfied, since (6g) holds for each i, with $(\bar{z}^*_{ci}(k), \bar{z}^*_{c(-i)}(k)) \in \mathbb{Z}_c \ominus \mathcal{T}_c$. If, then, a single subsystem $i \in \mathcal{I}$ optimizes for some $\mathbf{u}^0_i(k) \neq \mathbf{u}^*(k)$, then (6g) ensures coupled constraint satisfaction is maintained. However, if two subsystems p and q that share some constraint c were to optimize simultaneously, then coupled constraint satisfaction is not guaranteed. This is because although solving DOCP-p and DOCP-q independently, obtaining $\mathbf{u}^0_p(k)$ and $\mathbf{u}^0_q(k)$ respectively, will satisfy the individual constraints

$$(\bar{z}^0_{cp}(k+j|k), \bar{z}^*_{cq}(k+j), \bar{z}^*_{c(-(p,q))}(k+j)) \in \mathbb{Z}_c \ominus \mathcal{T}_c$$

in DOCP-p, where $\bar{z}^*_{c(-(p,q))}(\cdot) \triangleq \left(\bar{z}^*_{ci}(\cdot)\right)_{i\in\mathcal{I}_c\setminus\{p,q\}}$, and

$$(\bar{z}^0_{cq}(k+j|k), \bar{z}^*_{cp}(k+j), \bar{z}^*_{c(-(p,q))}(k+j)) \in \mathbb{Z}_c \ominus \mathcal{T}_c$$

in DOCP-q, at all steps $j \in \mathbb{N}_{N-1}$, it will not necessarily lead to satisfaction of

$$(\bar{z}^0_{cp}(k+j|k), \bar{z}^0_{cq}(k+j|k), \bar{z}^*_{c(-(p,q))}(k+j)) \in \mathbb{Z}_c \ominus \mathcal{T}_c.$$

In this paper, the single-update restriction is lifted, and any number of subsystems, a subset $\mathcal{I}^{\text{opt}} \subseteq \mathcal{I}$, is permitted to optimize simultaneously at a time step. The development that permits this is the systematic tightening of (6g) in the distributed optimal control problem, restricting the feasible region for i so that two or more coupled subsystems can optimize simultaneously. The modified DOCP is defined in the next section; subsequently, a systematic procedure for determining the modified coupled constraint is developed.

4.1. Modified coupled constraint and distributed optimal control problem

The modified distributed optimal control problem (MDOCP-i) for subsystem i at state $x_i(k)$ is

$$\tilde{J}^0_i(x_i(k), \mathbf{z}^*_i(k)) = \min_{\mathbf{u}_i(k)} \Big\{ J_i(\mathbf{u}_i(k)) : \mathbf{u}_i(k) \in \tilde{\mathcal{U}}_i(x_i(k), \mathbf{z}^*_i(k)) \Big\} \quad (10)$$

where $\tilde{\mathcal{U}}_i(x_i, \mathbf{z}^*_i)$ is defined by (6a)–(6f) and the constraint

$$(\bar{z}_{ci}(k+j|k), \bar{z}^*_{c(-i)}(k+j)) \in \tilde{\mathbb{Z}}_{ci}(j). \quad (11)$$

The set $\tilde{\mathbb{Z}}_{ci}(j)$ replaces the set $\mathbb{Z}_c \ominus \mathcal{T}_c$ in the problem, and is permitted to vary over the horizon. We require the following assumption.

Assumption 8. For each $i \in \mathcal{I}$, $c \in C_i$ and $j \in \mathbb{N}_{N-1}$, the set $\tilde{\mathbb{Z}}_{ci}(j) \subseteq \mathbb{Z}_c \ominus \mathcal{T}_c$ is a closed polyhedron.

To construct $\tilde{\mathbb{Z}}_{ci}(j)$, we use the same M_c normal vectors that define, in (3), the original coupled constraint set $\mathbb{Z}_c$, but a different right-hand side:

$$\tilde{\mathbb{Z}}_{ci}(j) \triangleq \mathbb{Z}_c(\tilde{q}_{ci}(j)) = \{z \in \mathbb{R}^{r_c} : P_c z \leq \tilde{q}_{ci}(j)\}, \quad (12)$$

where $\tilde{q}_{ci}(j) \in \mathbb{R}^{M_c}$. Then specification of $\tilde{\mathbb{Z}}_{ci}(j)$ is reduced to the problem of specifying $\tilde{q}_{ci}(j)$, and this is our aim in this Section. We derive the following conditions on $\tilde{q}_{ci}(j)$: first, a lower bound to guarantee at all times the existence of feasible solution to each subsystem's MDOCP; second, an upper bound that ensures the collection of solutions, across optimizing subsystems, satisfies all coupled constraints.

In what follows, to make clear the dependence of the feasible set for problem MDOCP-i on $\tilde{q}_{ci}(j)$, we write $\tilde{\mathcal{U}}_i(x_i(k), \mathbf{z}^*_i(k), \tilde{\mathbf{q}}_i(k))$, where $\tilde{\mathbf{q}}_i(k)$ is the collection of $\tilde{q}_{ci}(j)$ over $c \in C_i$ and $j \in \mathbb{N}_{N-1}$ for subsystem i. The following lemma, which holds because the only difference between to DOCP-i and MDOCP-i is tighter coupling constraints in the latter, will be useful in later results.

Lemma 2. *Given $x_i(k)$ and $\mathbf{z}^*_i(k) = (\bar{z}^*_{c(-i)}(k+j))_{c\in C_i, j\in\mathbb{N}_{N-1}}$ such that $\mathcal{U}_i(x_i(k), \mathbf{z}^*_i(k))$ is non-empty, $\tilde{\mathcal{U}}_i(x_i(k), \mathbf{z}^*_i(k), \tilde{\mathbf{q}}_i(k)) \subseteq \mathcal{U}_i(x_i(k), \mathbf{z}^*_i(k))$.*

4.2. Lower bound on $\tilde{q}_{ci}(j)$ to ensure existence of a feasible solution to MDOCP-i

The consequence of Lemma 2 is that a solution to MDOCP-i is also a feasible solution to DOCP-i. The result in this subsection establishes conditions under which the opposite statement is true: given a solution to DOCP-i, it is also a feasible solution to MDOCP-i. In particular, a lower bound on $\tilde{q}_{ci}(j)$ is given, so that the modified coupled constraint set (12) is not tightened so much that an existing feasible solution is excluded.

Proposition 1. *Suppose that, for a subsystem $i \in \mathcal{I}$ with state $x_i(k)$ at time k, there exists a $\mathbf{u}^*_i(k) \in \mathcal{U}_i(x_i(k), \mathbf{z}^*_i(k))$, where $\mathbf{z}^*_i(k)$ is the collection of $\bar{z}^*_{cq}(k+j)$ over all $j \in \mathbb{N}_{N-1}, q \in \mathcal{I}_c, c \in C_i$. Then $\mathbf{u}^*_i(k) \in \tilde{\mathcal{U}}_i(x_i(k), \mathbf{z}^*_i(k), \tilde{\mathbf{q}}_i(k))$ if and only if*

$$\tilde{q}_{ci}(j) \geq P_c \bar{z}^*_c(k+j), \quad (13)$$

for all $j \in \mathbb{N}_{N-1}, c \in C_i$, where $\bar{z}^*_c(k+j) = (\bar{z}^*_{cr}(k+j))_{r\in\mathcal{I}_c}$.

Proof. The solution $\mathbf{u}^*_i(k) \in \mathcal{U}_i(x_i(k), \mathbf{z}^*_i(k))$ satisfies all constraints (6) by construction, and hence $\mathbf{u}^*_i(k)$ satisfies (6a)–(6f) in MDOCP-i. Therefore, to prove that $\mathbf{u}^*_i(k) \in \tilde{\mathcal{U}}_i(x_i(k), \mathbf{z}^*_i(k), \tilde{\mathbf{q}}_i(k))$ it is necessary and sufficient to show that $\mathbf{u}^*_i(k)$ satisfies the remaining constraint in MDOCP-i, (11).

The coupling constraints (6g) in DOCP-i, satisfied by construction, have

$$(\bar{z}^*_{ci}(k+j|k), \bar{z}^*_{c(-i)}(k+j)) = \bar{z}^*_c(k+j) \in \mathbb{Z}_c(q_c) \ominus \mathcal{T}_c$$

for all $j \in \mathbb{N}_{N-1}, c \in C_i$. Satisfaction of (6g) by the same $\bar{z}^*_c(\cdot)$ means $\bar{z}^*_c(k+j) \in \tilde{\mathbb{Z}}_{ci}(j)$ for $j \in \mathbb{N}_{N-1}, c \in C_i$. Rewriting this condition in terms of support functions,

$$\bar{z}^*_c(k+j) \in \tilde{\mathbb{Z}}_{ci}(j) \iff v^\top \bar{z}^*_c(k+j) \leq h\Big(\tilde{\mathbb{Z}}_{ci}(j), v\Big), \forall v \in \mathbb{R}^{r_c},$$

and $j \in \mathbb{N}_{N-1}, c \in C$. Given the polyhedral description of $\tilde{\mathbb{Z}}_{ci}(j)$ in (12) as $\mathbb{Z}_c(\tilde{q}_{ci}(j))$, it is necessary and sufficient to evaluate these support function inequalities at $v = p_{cm}, m = 1 \ldots M_c$, thus

$$h\Big(\mathbb{Z}_c(\tilde{q}_{ci}(j)), p_{cm}\Big) \geq p^\top_{cm} \bar{z}^*_c(k+j), m = 1 \ldots M_c.$$

Finally, by definition of the support function, $h\Big(\mathbb{Z}_c(\tilde{q}_{ci}(j)), p_{cm}\Big) \leq \tilde{q}_{ci}(j)$, and so $\tilde{q}_{ci}(j) \geq P_c \bar{z}^*_c(k)$. □

4.3. Upper bound on $\tilde{q}_{ci}(j)$ to ensure system-wide coupled constraint satisfaction

Now we consider the situation where a subset of subsystems, say $\mathcal{I}^{\text{opt}}(k)$, solve their MDOCPs simultaneously at time k, while all remaining subsystems continue to follow plans from a previous time step (renewed via (8)). Given that a constraint $c \in C$ involves the set $\mathcal{I}_c \subseteq \mathcal{I}$ of subsystems (a total number $n(\mathcal{I}_c)$), $\mathcal{I}^{\text{opt}}(k)$ contains some subset $\mathcal{I}^{\text{opt}}_c(k) \triangleq \mathcal{I}^{\text{opt}}(k) \cap \mathcal{I}_c$ of the subsystems sharing constraint c, a total number $n(\mathcal{I}^{\text{opt}}_c(k)) \leq n(\mathcal{I}_c)$. A necessary condition for maintaining feasibility of the overall system is

$$\Big(\underbrace{(\bar{z}_{ci}(k+j|k))_{i\in\mathcal{I}^{\text{opt}}_c(k)}}_{\text{optimizing}}, \underbrace{(\bar{z}^*_{cr}(k+j))_{r\in\mathcal{I}_c\setminus\mathcal{I}^{\text{opt}}_c(k)}}_{\text{non-optimizing}}\Big) \in \mathbb{Z}_c(q_c) \ominus \mathcal{T}_c,$$
$$\forall j \in \mathbb{N}_{N-1}, c \in C. \quad (14)$$

That is, the coupling outputs of all the optimizing subsystems, when taken together and with those of non-optimizing subsystems, must satisfy the coupling constraints.

The result in this subsection establishes conditions under which satisfaction of (14) is guaranteed for any choice of $\mathcal{I}^{\text{opt}}(k)$. In particular, an upper bound on $\tilde{q}_{ci}(j)$ in (12) is developed, which limits the maximum size of the coupled constraint set in each MDOCP-i. Alternatively viewed, the result corresponds to a minimum amount by which the original coupled constraint set must be tightened in order to guarantee coupled constraint satisfaction when the MDOCP-i problems are solved simultaneously.

Proposition 2. *Suppose that, for each subsystem $i \in \mathcal{I}$ with state $x_i(k)$ at time k, there exists a $\mathbf{u}_i^*(k) \in \mathcal{U}_i(x_i(k), \mathbf{z}_i^*(k))$, where $\mathbf{z}_i^*(k)$ is the collection of $\bar{z}_{cq}^*(k+j)$ over all $j \in \mathbb{N}_{N-1}, q \in \mathcal{I}_c, c \in C_i$. Further suppose that $\mathbf{u}_i^0(k) \in \tilde{\mathcal{U}}_i(x_i(k), \mathbf{z}_i^*(k), \tilde{\mathbf{q}}_i(k))$, for all $i \in \mathcal{I}^{\text{opt}}(k) \subseteq \mathcal{I}$. Then*

$$\Big((\bar{z}_{ci}^0(k+j|k))_{i\in\mathcal{I}_c^{\text{opt}}(k)}, (\bar{z}_{cr}^*(k+j))_{r\in\mathcal{I}_c\setminus\mathcal{I}_c^{\text{opt}}(k)}\Big) \in \mathbb{Z}_c(q_c)\ominus\mathcal{T}_c, \quad (15)$$

for all $j \in \mathbb{N}_{N-1}$, $c \in C$, if

$$\sum_{i\in\mathcal{I}_c^{\text{opt}}(k)} \tilde{q}_{ci}(j) \leq q_c - t_c + (N_c^{\text{opt}} - 1)P_c\bar{z}_c^*(k+j), \quad (16)$$

where $t_c = [h(\mathcal{T}_c, p_{c1}), h(\mathcal{T}_c, p_{c2}), \ldots, h(\mathcal{T}_c, p_{cM_c})]^\top$.

Proof. Consider $\mathbf{u}_i^0(k) \in \tilde{\mathcal{U}}_i(x_i(k), \mathbf{z}_i^*(k), \tilde{\mathbf{q}}_i)$ for $i \in \mathcal{I}^{\text{opt}}(k)$. This satisfies (6a)–(6f) and (11) by construction. In particular, constraint (11) has

$$(\bar{z}_{ci}^0(k+j|k), \bar{z}_{c(-i)}^*(k+j)) \in \mathbb{Z}_c(\tilde{q}_{ci}(j))$$

for all $j \in \mathbb{N}_{N-1}, c \in C_i$. Summing both sides of this constraint, via Minkowski addition, over all $i \in \mathcal{I}_c^{\text{opt}}(k)$,

$$\sum_{i\in\mathcal{I}_c^{\text{opt}}(k)} (\bar{z}_{ci}^0(k+j|k), \bar{z}_{c(-i)}^*(k+j)) \in \bigoplus_{i\in\mathcal{I}_c^{\text{opt}}(k)} \mathbb{Z}_c(\tilde{q}_{ci}(j)).$$

Expanding the summation and noting that $\bar{z}_c^*(k+j) = (\bar{z}_{ci}^*(k+j))_{i\in\mathcal{I}_c}$,

$$\Big((\bar{z}_{ci}^0(k+j|k))_{i\in\mathcal{I}_c^{\text{opt}}(k)}, (\bar{z}_{cr}^*(k+j))_{r\in\mathcal{I}_c\setminus\mathcal{I}_c^{\text{opt}}(k)}\Big) + (n(\mathcal{I}_c^{\text{opt}}(k)) - 1)\bar{z}_c^*(k+j) \in \bigoplus_{i\in\mathcal{I}_c^{\text{opt}}(k)} \mathbb{Z}_c(\tilde{q}_{ci}(j)).$$

Written in terms of support functions,

$$v^\top\Big((\bar{z}_{ci}^0(k+j|k))_{i\in\mathcal{I}_c^{\text{opt}}(k)}, (\bar{z}_{cr}^*(k+j))_{r\in\mathcal{I}_c\setminus\mathcal{I}_c^{\text{opt}}(k)}\Big) \leq -(n(\mathcal{I}_c^{\text{opt}}(k))-1)v^\top\bar{z}_c^*(k+j) + \sum_{i\in\mathcal{I}_c^{\text{opt}}(k)} h\Big(\mathbb{Z}_c(\tilde{q}_{ci}(j)), v\Big),$$

for all $j \in \mathbb{N}_{N-1}, c \in C$ and $v \in \mathbb{R}^{n_c}$. Likewise, writing (15) in terms of support functions,

$$v^\top\Big((\bar{z}_{ci}^0(k+j|k))_{i\in\mathcal{I}_c^{\text{opt}}(k)}, (\bar{z}_{cr}^*(k+j))_{r\in\mathcal{I}_c\setminus\mathcal{I}_c^{\text{opt}}(k)}\Big) \leq h(\mathbb{Z}_c(q_c), v) - h(\mathcal{T}_c, v),$$

for $j \in \mathbb{N}_{N-1}, c \in C$ and $v \in \mathbb{R}^{n_c}$. It is necessary and sufficient to evaluate these support function inequalities at $v = p_{cm}, m = 1\ldots M_c$. Comparing these expressions, it follows that (15) is satisfied if

$$-(n(\mathcal{I}_c^{\text{opt}}(k)) - 1)p_{cm}^\top\bar{z}_c^*(k+j) + \sum_{i\in\mathcal{I}_c^{\text{opt}}(k)} h\Big(\mathbb{Z}_c(\tilde{q}_{ci}(j)), p_{cm}\Big) \leq h(\mathbb{Z}_c(q_c), p_{cm}) - h(\mathcal{T}_c, p_{cm}).$$

for $m = 1\ldots M_c, c \in C, j \in \mathbb{N}_{N-1}$. Therefore, noting that $h(\mathbb{Z}_c(q_c), p_{cm}) = q_{cm}$ and $h\Big(\mathbb{Z}_c(\tilde{q}_{ci}(j)), p_{cm}\Big) \leq \tilde{q}_{cim}(j)$,

$$\sum_{i\in\mathcal{I}_c^{\text{opt}}(k)} \tilde{q}_{ci}(j) \leq q_c - t_c + (N_c^{\text{opt}} - 1)P_c\bar{z}_c^*(k+j),$$

where $t_c \triangleq [h(\mathcal{T}_c, p_{c1}), h(\mathcal{T}_c, p_{c2}), \ldots, h(\mathcal{T}_c, p_{cM_c})]^\top$. □

Remark 1. The bounds (13) and (16) have interpretations in terms of the slackness of the coupled constraints. The m^{th} component of $q_c - t_c - P_c\bar{z}_c^*(k+j)$ is equal to the slack remaining in constraint c, at prediction step j, in the direction p_{cm}, given the known coupling outputs $\bar{z}_{cr}^*(k+j)$ of each $r \in \mathcal{I}_c$. Rewriting (16),

$$\sum_{i\in\mathcal{I}_c^{\text{opt}}(k)} (\tilde{q}_{ci}(j) - P_c\bar{z}_c^*(k+j)) \leq q_c - t_c - P_c\bar{z}_c^*(k+j),$$

which states that the total space allowed to simultaneously optimizing subsystems sharing constraint c, in direction p_{cm}, should not exceed the slack remaining in that direction. The lower bound (13) ensures that the solution $\mathbf{u}_i^*(k)$ remains a feasible choice for each optimizing subsystem $i \in \mathcal{I}^{\text{opt}}(k)$, by not permitting the feasible region to shrink so much that this point is excluded. Note that if no slack remains in direction p_{cm} of constraint c, then $\tilde{q}_{cim}(j) = q_{cm} - t_{cm}$: no tightening is permitted in that direction.

4.4. Main result

The main result of this Section draws together the previous results, establishing conditions under which solving MDOCPs in parallel leads to guaranteed system-wide feasibility.

Theorem 1. *Suppose that, for each subsystem $i \in \mathcal{I}$ with state $x_i(k)$ at time k, there exists a $\mathbf{u}_i^*(k) \in \mathcal{U}_i(x_i(k), \mathbf{z}_i^*(k))$, where $\mathbf{z}_i^*(k)$ is the collection of $\bar{z}_{cq}^*(k+j)$ over all $j \in \mathbb{N}_{N-1}, q \in \mathcal{I}_c, c \in C_i$. Then, for all $i \in \mathcal{I}^{\text{opt}}$ and any $\mathcal{I}^{\text{opt}} \subseteq \mathcal{I}$, if $\tilde{\mathbf{q}}_i$ satisfies (13) and (16), (i) $\tilde{\mathcal{U}}_i(x_i(k), \mathbf{z}_i^*(k), \tilde{\mathbf{q}}_i)$ is non-empty and contains $\mathbf{u}_i^*(k)$; (ii) for any $\mathbf{u}_i^s(k) \in \tilde{\mathcal{U}}_i(x_i(k), \mathbf{z}_i^*(k), \tilde{\mathbf{q}}_i)$, the collection of $\{\mathbf{u}_i^s(k)\}_{i\in\mathcal{I}^{\text{opt}}(k)}$ together with $\{\mathbf{u}_r^*(k)\}_{r\notin\mathcal{I}^{\text{opt}}(k)}$ satisfy all local and coupling constraints:*

$$\Big((\mathbf{u}_i^s(k))_{i\in\mathcal{I}^{\text{opt}}(k)}, (\mathbf{u}_r^*(k))_{r\notin\mathcal{I}^{\text{opt}}(k)}\Big) \in \mathcal{U}(x(k)).$$

Proof. (i) Existence follows from Proposition 1: for all $i \in \mathcal{I}^{\text{opt}}(k)$, and any $\mathcal{I}^{\text{opt}}(k) \subseteq \mathcal{I}$, if $\tilde{q}_{ci}(j) \geq P_c\bar{z}_c^*(k+j), \forall c \in C_i, j \in \mathbb{N}_{N-1}$ in the MDOCP-i, then there exists a feasible solution, namely $\mathbf{u}_i^0(k) = \mathbf{u}_i^*(k)$, to MDOCP-$i$. Part (ii) follows directly from Lemma 2 and Proposition 2. □

The implication is that *any* subset of subsystems may optimize simultaneously, and (i) a feasible solution to each problem is guaranteed to exist, (ii) all coupled constraints remain satisfied, if the coupled constraint set in subsystems i's MDOCP is chosen as $\mathbb{Z}_c(\tilde{q}_{ci}(j))$, with $\tilde{q}_{ci}(j)$ satisfying (13) and (16). Theorem 1 assumes the existence and availability of such $\tilde{q}_{ci}(j)$, but the question remains of whether such $\tilde{q}_{ci}(j)$ can be found easily. The upper bound (16) in particular is a coupled constraint, and therefore implies some coordination is required to determine individual $\tilde{q}_{ci}(j)$ for each $i \in \mathcal{I}_c^{\mathrm{opt}}(k)$. The following result confirms that suitable $\tilde{q}_{ci}(j)$ always exist, and suggests a simple scheme for choosing them.

Proposition 3. *For $i \in \mathcal{I}^{\mathrm{opt}}(k) \subseteq \mathcal{I}$, the choice*

$$\tilde{q}_{ci}(j) = \frac{q_c - t_c + (\beta_{ci} - 1)P_c\bar{z}_c^*(k+j)}{\beta_{ci}}, \tag{17}$$

for $j \in \mathbb{N}_{N-1}$, $c \in C_i$, satisfies (13) *and* (16) *for all $\beta_{ci} \geq n(\mathcal{I}_c^{\mathrm{opt}}(k)) \geq 1$ and $\bar{z}_c^*(k+j) \in \mathbb{Z}_c(q_c) - \mathcal{T}_c$.*

Proof. Suppose that $\bar{z}_c^*(k+j) \in \mathbb{Z}_c(q_c) - \mathcal{T}_c, \forall j \in \mathbb{N}_{N-1}, c \in C$ at time k, and consider some subset of subsystems $\mathcal{I}^{\mathrm{opt}}(k) \subseteq \mathcal{I}$ so that $N^{\mathrm{opt}}(k) \geq 1$. By construction, $n(\mathcal{I}_c^{\mathrm{opt}}(k)) \geq 1$ for all $c \in \bigcup_{i \in \mathcal{I}^{\mathrm{opt}}(k)} C_i$. For $i \in \mathcal{I}^{\mathrm{opt}}(k)$, let $\tilde{q}_{ci}(j)$ be given by (17), with some $\beta_{ci} \geq n(\mathcal{I}_c^{\mathrm{opt}}(k))$, for all $j \in \mathbb{N}_{N-1}, c \in C_i$. Then, because $\bar{z}_c^*(k+j) \in \mathbb{Z}_c(q_c) - \mathcal{T}_c$ iff $P_c\bar{z}_c^*(k+j) \leq q_c - t_c$,

$$\begin{aligned}\tilde{q}_{ci}(j) &= \frac{q_c - t_c + (\beta_{ci} - 1)P_c\bar{z}_c^*(k+j)}{\beta_{ci}} \\ &\geq \frac{P_c\bar{z}_c^*(k+j) + (\beta_{ci} - 1)P_c\bar{z}_c^*(k+j)}{\beta_{ci}} = P_c\bar{z}_c^*(k+j)\end{aligned}$$

hence satisfaction of (13). To show (16), for each $c \in \bigcup_{i \in \mathcal{I}^{\mathrm{opt}}(k)} C_i$, sum (17) over $\mathcal{I}_c^{\mathrm{opt}}$:

$$\begin{aligned}\sum_{i \in \mathcal{I}_c^{\mathrm{opt}}(k)} \tilde{q}_{ci}(j) &= \sum_{i \in \mathcal{I}_c^{\mathrm{opt}}(k)} \frac{q_c - t_c + (\beta_{ci} - 1)P_c\bar{z}_c^*(k+j)}{\beta_{ci}} \\ &= \sum_{i \in \mathcal{I}_c^{\mathrm{opt}}(k)} \frac{q_c - t_c}{\beta_{ci}} + \sum_{i \in \mathcal{I}_c^{\mathrm{opt}}(k)} \frac{(\beta_{ci} - 1)P_c\bar{z}_c^*(k+j)}{\beta_{ci}}.\end{aligned}$$

Because $\beta_{ci} \geq n(\mathcal{I}_c^{\mathrm{opt}}(k)) \geq 1, \forall c, i$, then $\sum_{i \in \mathcal{I}_c^{\mathrm{opt}}(k)} \frac{q_c - t_c}{\beta_{ci}} \leq q_c - t_c$. Likewise, $\sum_{i \in \mathcal{I}_c^{\mathrm{opt}}(k)} \frac{(\beta_{ci} - 1)P_c\bar{z}_c^*(k+j)}{\beta_{ci}} \leq \left(n(\mathcal{I}_c^{\mathrm{opt}}(k)) - 1\right) P_c\bar{z}_c^*(k+j)$ for all $\beta_{ci} \geq n(\mathcal{I}_c^{\mathrm{opt}}(k)) \geq 1$. Hence (16) is satisfied. □

Here a larger β_{ci} corresponds to more tightening of the coupling constraint set in MDOCP-i: as $\beta_{ci} \to \infty$ then $\tilde{q}_{ci}(j) \to P_c\bar{z}_c^*(k+j)$, *i.e.*, $\mathbb{Z}_c(\tilde{q}_{ci}(j)) \to \{\bar{z}_c^*(k+j)\}$. In practice, it is desirable to have $\tilde{q}_{ci}(j)$ as close as possible to the original size of the constraint set, after tightening for robustness to disturbances, *i.e.*, $(q_c - t_c)$. This suggests small β_{ci}; however, β_{ci} is lower-bounded as $\beta_{ci} \geq n(\mathcal{I}_c^{\mathrm{opt}}(k))$, where the latter is the number of optimizing subsystems sharing constraint c, implying a practical lower limit on the amount of tightening required to ensure robustness to simultaneous decision making. Note that if the optimizing set, $\mathcal{I}^{\mathrm{opt}}(k)$, is selected so that *no* two subsystems within it are coupled, then $n(\mathcal{I}_c^{\mathrm{opt}}(k)) = 1$ for all c and $\tilde{q}_{ci}(k) = q_c - t_c$ if β_{ci} is chosen equal to 1: then MDOCP-i becomes identical to the DOCP-i. For any other choice of $\mathcal{I}^{\mathrm{opt}}(k)$, so that $n(\mathcal{I}_c^{\mathrm{opt}}(k)) \geq 2$ for some c, optimizing subsystems share the slack remaining in the constraint evenly.

5. Feasible parallel-update distributed MPC

In this section, the main distributed MPC algorithm is presented, including a distributed algorithm for the initialization step, with guaranteed convergence to a feasible solution. Finally, robust feasibility and stability results are established.

5.1. Feasible parallel-update distributed MPC algorithm

The revised DOCP, with on-line computation of $\tilde{q}_{ci}(j)$, is used in the following algorithm.

Algorithm 1 (Feasible parallel-update DMPC for subsystem *i*).

Offline: Compute K_i and κ_i^{f}, sets $\mathcal{R}_i$, $\mathcal{S}_i$, $\mathcal{T}_c$, $\mathcal{X}_i^{\mathrm{f}}$. Tighten local constraint sets $\mathbb{X}_i$, $\mathbb{U}_i$ and determine the vector, t_c, of support functions to $\mathcal{T}_c$.

Online:

1. Set $k = 0$. Obtain an initial feasible solution using Algorithm 2.
2. Sample current state $x_i(k)$.
3. Update plan:
   If $i \in \mathcal{I}^{\mathrm{opt}}(k)$
   (a) Extract $\bar{z}_c^*(k+j), \forall c \in C_i, j \in \mathbb{N}_{N-1}$, from $\mathbf{z}_i^*(k)$.
   (b) Set $\tilde{q}_{ci}(j) = \dfrac{q_c - t_c + (\beta_{ci} - 1)P_c\bar{z}_c^*(k+j)}{\beta_{ci}}$, $\forall c \in C_i$, $j \in \mathbb{N}_{N-1}$, with $\beta_{ci} \geq n(\mathcal{I}_c^{\mathrm{opt}}(k))$.
   (c) Obtain $\mathbf{u}_i^0(k)$ as solution to MDOCP-i.
   (d) Transmit coupling information $\bar{z}_{ci}^0(k+j|k)$, $j \in \mathbb{N}_{N-1}$, to coupled $q \in Q_i$.
   (e) Set $\mathbf{u}_i^*(k) = \mathbf{u}_i^0(k)$.
   Else renew current plan via (8): $\mathbf{u}_i^*(k) = \tilde{\mathbf{u}}_i(k)$.
4. Build $\mathbf{z}_i^*(k+1)$, via (18), using new information received from coupled updating subsystems $q \in Q_i \cap \mathcal{I}^{\mathrm{opt}}(k)$ and previous information from coupled non-updating subsystems $r \in Q_i \setminus \mathcal{I}^{\mathrm{opt}}(k)$.
5. Apply $u_i(k) = \bar{u}_i^*(k|k) + K_i(x_i(k) - \bar{x}_i^*(k|k))$. Wait one time step, increment k, go to step 2.

Details of Algorithm 1 are now described. The algorithm begins with the off-line computation of feedback laws and constraint sets. Following this, Algorithm 2, which will be described in Section 5.2, is employed at the initial $k = 0$ step. At a subsequent time step k, a subset of subsystems, $\mathcal{I}^{\mathrm{opt}}(k)$, the choice of which is unrestricted, optimize plans by solving their respective MDOCPs. Subsystems not in $\mathcal{I}^{\mathrm{opt}}(k)$ renew their current plans via (8). The on-line calculation of $\tilde{q}_{ci}(j)$ for use in the MDOCP-i requires knowledge of q_c, t_c, P_c, $\bar{z}_c^*(\cdot)$, $n(\mathcal{I}_c^{\mathrm{opt}}(k))$. The former three are computed off-line, while $\bar{z}_c^*(\cdot)$ contains coupling output information transmitted by other subsystems, as described below. The final term, $n(\mathcal{I}_c^{\mathrm{opt}}(k))$, is the number of subsystems sharing constraint c and updating at time k. While it could be assumed that each subsystem knows how many other coupled subsystems will optimize at time k, this assumption may be too strong and inflexible in some cases. Instead, it is sufficient to set $\beta_{ci} = n(\mathcal{I}_c)$—where this is the total number of subsystems sharing constraint c—and since $n(\mathcal{I}_c) \geq n(\mathcal{I}_c^{\text{opt}}(k))$ by definition, then this allows all subsystems to optimize in parallel, at any time step, without the need for further communication or *a-priori* arrangement. Though such an approach may add unnecessary tightening, hence conservatism, in many applications sparsity exists in the coupling constraints (a constraint c does not couple all subsystems) and $n(\mathcal{I}_c)$ may be significantly smaller than the number of subsystems.

Following optimization, subsystems $i \in \mathcal{I}^{\text{opt}}$ exchange information with coupled neighbours, as per step 3d. The received information is used, in step 4, to build the coupling information $\mathbf{z}_i^*(k+1)$ for use at the next time step, $k+1$. For subsystem i considering the coupling output of subsystem r, this is done as

$$\bar{z}_{cr}^*(k+j) = \begin{cases} \bar{z}_{cr}^0(k+j|k), & r = i \\ \bar{z}_{cr}^0(k+j|k), & r \in Q_i \cap \mathcal{I}^{\text{opt}}(k), \\ \bar{z}_{cr}^0(k+j|\hat{k}_r), & r \in Q_i \setminus \mathcal{I}^{\text{opt}}(k), \end{cases} \tag{18}$$

for $j \in \mathbb{N}_{[1:N]}$, where $\hat{k}_r$ is the last time at which subsystem r solved its MDOCP.

5.2. A distributed algorithm for initialization

The following algorithm is employed as the initialization step of Algorithm 1. For clarity of notation, we denote the original coupled constraint set $\mathbb{Z}_c(q_c) \ominus \mathcal{T}_c$, *i.e.*, that in (6g), as $\bar{\mathbb{Z}}_c$.

Algorithm 2 (Initialization for a subsystem i).

1. For all $c \in C_i$, obtain $\bar{\mathbb{Z}}_c^i$ as the projection of set $\bar{\mathbb{Z}}_c$ onto the subspace $\mathbb{R}^{r_{ci}}$.
2. Measure $x_i(0)$, set $p = 0$, and obtain $\mathbf{u}_i^{[p]}$ as solution to

$$\begin{aligned} &\min_{\mathbf{u}_i} J_i(\mathbf{u}_i) \\ &\text{subject to (6a)–(6f)}, \ \bar{z}_{ci}(j) \in \bar{\mathbb{Z}}_c^i, \forall j \in \mathbb{N}_{N-1}, c \in C_i. \end{aligned} \tag{19}$$

3. Transmit coupling information $\bar{z}_{ci}^{[p]}(j)$, $j \in \mathbb{N}_{N-1}$ to coupled subsystems $q \in Q_i$.
4. **If** (6g) is satisfied by $\bar{z}_{ci}^{[p]}(j)$ together with $\bar{z}_{c(-i)}^{[p]}(j), \forall j \in \mathbb{N}_{N-1}, c \in C_i$, terminate.
   Else
   (a) Obtain $\mathbf{u}_i^{[p^+]}$ as solution to

$$D_i\left(\mathbf{u}_i^{[p^+]}, \mathbf{u}_{-i}^{[p]}\right) = \min_{\mathbf{u}_i} \sum_{j=0}^{N-1} \sum_{c \in C_i} \frac{1}{n(\mathcal{I}_c)} d\left(\left(\bar{z}_{ci}(j), \bar{z}_{c(-i)}^{[p]}(j)\right), \bar{\mathbb{Z}}_c\right)$$
$$\text{subject to (6a)–(6f)} \tag{20}$$

   (b) Set $\mathbf{u}_i^{[p+1]} = w_i \mathbf{u}_i^{[p^+]} + (1 - w_i)\mathbf{u}_i^{[p]}$, where $w_i > 0$ and $\sum_{i \in \mathcal{I}} w_i = 1$.
   (c) Increment p and go to step 3.

In this algorithm, subsystems begin by decoupling the coupled constraint sets, via a projection onto the subspace corresponding to the local subsystem's coupling outputs. Consequently, the subsystems obtain initial solutions satisfying local constraints, but not necessarily coupled constraints. To work towards coupled constraint satisfaction, the subsystems follow the iterative procedure of steps 3 and 4. The following result, the proof of which may be found in Appendix A, establishes convergence to an initial feasible solution satisfying all coupled constraints.

Proposition 4. *(Convergence of Algorithm 2) Suppose that $\mathcal{U}(x(0)) \neq \emptyset$ and let $\{\mathbf{u}_i^{[p]}\}$ be the sequence generated, for each $i \in \mathcal{I}$, by Algorithm 2. Then, for all $i \in \mathcal{I}$, (i) problem* (19) *is feasible; (ii) problem* (20) *is feasible at every iteration p; (iii) the cost function $D\left(\mathbf{u}^{[p]}\right)$, where*

$$D(\mathbf{u}) \triangleq \sum_{j=0}^{N-1} \sum_{c \in C} d\left(\left(\bar{z}_c(j), \bar{\mathbb{Z}}_c\right)\right)$$

and $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_{N_i})$, is non-increasing with iteration p; (iv) the cost sequence $\left\{D\left(\mathbf{u}^{[p]}\right)\right\}$ converges to 0 *and the solutions $\{\mathbf{u}^p\}$ converge to the feasible set $\mathcal{U}(x(0))$.*

Remark 2. The optimality of obtained solutions, and hence closed-loop performance of the proposed DMPC, with respect to the system-wide objective and tube-based CMPC, will depend on (i) the optimality of the solutions obtained at initialization, and (ii) the size and description of the coupled constraint sets following the on-line extra tightening. The former is influenced by the weights $w_i, i \in \mathcal{I}$, and has been well studied in the literature. The latter depends on the β_{ci} parameter used in (17), and is a topic of current research.

5.3. Robust feasibility and stability

The remainder of this section shows that system-wide robust feasibility and stability are guaranteed for any update sequence $\{\mathcal{I}^{\text{opt}}(k)\}$.

Theorem 2 (Robust feasibility and stability)**.** *Suppose that, for each $i \in \mathcal{I}$, $\mathbf{u}_i^*(k)$ exists and is a feasible (but not necessarily optimal) solution to DOCP-i at time k. Consider some optimizing set of subsystems, $\mathcal{I}^{\text{opt}}(k) \subseteq \mathcal{I}$. Then, (i) $\mathbf{u}_i^*(k)$ is a feasible solution to MDOCP-i for $i \in \mathcal{I}^{\text{opt}}(k)$; (ii) any feasible (but not necessarily optimal) solution, $\mathbf{u}_i^0(k)$, to problem MDOCP-i for each $i \in \mathcal{I}^{\text{opt}}(k)$ satisfies*

$$\left((\mathbf{u}_i^0(k))_{i \in \mathcal{I}^{\text{opt}}}, (\mathbf{u}_r^*(k))_{r \notin \mathcal{I}^{\text{opt}}}\right) \in \mathcal{U}(x(k));$$

(iii) for all $x_i(k+1) \in A_i x_i(k) + B_i u_i(k) \oplus \mathbb{W}_i$, where

$$u_i(k) = \begin{cases} \bar{u}_i^0(k|k) + K_i(x_i(k) - \bar{x}_i^0(k|k)) & i \in \mathcal{I}^{\text{opt}}(k) \\ \bar{u}_i^*(k|k) + K_i(x_i(k) - \bar{x}_i^*(k|k)) & i \notin \mathcal{I}^{\text{opt}}(k), \end{cases} \tag{21}$$

the candidate solution $\tilde{\mathbf{u}}_i(k+1)$ is a feasible solution to DOCP-i for all $i \in \mathcal{I}$, and MDOCP-i for all $i \in \mathcal{I}^{\text{opt}}(k+1) \subseteq \mathcal{I}$; (iv) each cost function is monotonically decreasing:

$$J_i(\mathbf{u}_i^*(k+1)) \leq J_i(\mathbf{u}_i^*(k)) - l_i(\bar{x}_i^*(k|k), \bar{u}_i^*(k|k)),$$

where $\mathbf{u}_i^*(k+1)$ is the solution adopted by i at time $k+1$. Subsequently, (v) the closed-loop system controlled by Algorithm 1 is robustly feasible and $x_i(k) \to \mathcal{R}_i$ and $u_i(k) \to K_i\mathcal{R}_i$ as $k \to \infty$, for each $i \in \mathcal{I}$, for any choice of update sequence $\{\mathcal{I}^{\text{opt}}(k)\}_{k\geq 0}$.

Proof. Parts (i) and (ii) follow directly from Theorem 1. For part (iii), since $\mathbf{u}_i^*(k) \in \mathcal{U}_i(x_i(k), \mathbf{z}_i^*(k)), \forall i$, then $(\mathbf{u}_i^*(k))_{i\in\mathcal{I}} \in \mathcal{U}(x(k))$ (Lemma 1). From [17], it follows that

$$\tilde{\mathbf{u}}_i(k+1) \in \mathcal{U}_i(x_i(k+1), \mathbf{z}_i^*(k+1)), \forall i \in \mathcal{I},$$
$$(\tilde{\mathbf{u}}_i(k+1))_{i\in\mathcal{I}} \in \mathcal{U}(x(k+1)).$$

and from Theorem 1 that $\tilde{\mathbf{u}}_i(k+1) \in \tilde{\mathcal{U}}_i(x_i(k+1), \mathbf{z}_i^*(k+1), \tilde{\mathbf{q}}_i(k+1)), \forall i \in \mathcal{I}^{\text{opt}}(k+1) \subseteq \mathcal{I}$. For (iv), given $\mathbf{u}_i^*(k)$, with cost

$$J_i(\mathbf{u}_i^*(k)) = F_i(\bar{x}_i^*(k+N|k)) + \sum_{j=0}^{N-1} l_i(\bar{x}_i^*(k+j|k), \bar{u}_i^*(k+j|k))$$

the solution $\tilde{\mathbf{u}}_i(k+1)$ is a feasible solution at $k+1$, with cost

$$\begin{aligned} J_i(\tilde{\mathbf{u}}_i(k+1)) &= F_i\Big(A_i\bar{x}_i^*(k+N|k) + B_i\kappa_i^{\text{f}}(\bar{x}_i^*(k+N|k))\Big) \\ &\quad + l_i\Big(\bar{x}_i^*(k+N|k), \kappa_i^{\text{f}}(\bar{x}_i^*(k+N|k))\Big) \\ &\quad + \sum_{j=1}^{N-1} l_i(\bar{x}_i^*(k+j|k), \bar{u}_i^*(k+j|k)) \\ &\leq J_i(\mathbf{u}_i^*(k)), \end{aligned}$$

where the inequality follows from Assumption 7. Furthermore, an optimizing subsystem $i \in \mathcal{I}^{\text{opt}}(k+1)$ at step $k+1$ obtains a solution $\mathbf{u}_i^0(k+1)$, with cost $J_i(\mathbf{u}_i^0(k+1)) \leq J_i(\tilde{\mathbf{u}}_i(k+1))$. All $r \notin \mathcal{I}^{\text{opt}}$ adopt $\tilde{\mathbf{u}}(k+1)$, cost $J_r(\tilde{\mathbf{u}}_r(k+1))$. Thus, $J_i(\mathbf{u}_i^*(k+1)) \leq J_i(\tilde{\mathbf{u}}_i(k+1))$, where $\mathbf{u}_i^*(k+1)$ is the adopted solution.

Part (v) follows by recursion: an initial feasible collection $\mathbf{u}_i^*(0) \in \mathcal{U}_i(x_i(0), \mathbf{z}_i^*(0))$ implies all subsequent optimizations are feasible, and $(\mathbf{u}_i^*(k))_{i\in\mathcal{I}} \in \mathcal{U}(x(k))$ regardless of update sequence $\{\mathcal{I}^{\text{opt}}(k)\}_{k\geq 0}$. Convergence of each $x_i(k)$ to $\mathcal{R}_i$ and $u_i(k) \to K_i\mathcal{R}_i$ follows from the monotonicity of J and the standard arguments [3]. □

6. Numerical example

Consider four identical point masses with

$$A_i = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},$$

and local constraint sets $\mathbb{X}_i = \{x_i \in \mathbb{R}^2 : -[10,5]^\top \leq x_i \leq [10,5]^\top\}$, $\mathbb{U}_i = \{u_i \in \mathbb{R} : -1 \leq u_i \leq 1\}$. A single coupled constraint restricts the local control inputs across all subsystems to a value less than the sum of the local limits:

$$z_i = \begin{bmatrix} 0 & 0 \end{bmatrix} x_i + 1u_i, \qquad \sum_{i=1}^{4} |z_i| \leq 2.5$$

The local objectives are $l_i(x_i, u_i) = x_i^\top Q_i x_i + u_i^\top R_i u_i$, with Q_i, R_i to be defined, and a zero terminal cost. The disturbance set is $\mathbb{W}_i = \{w_i \in \mathbb{R}^2 : \|w_i\|_\infty \leq 0.05\}$. For simplicity, the local controller is nilpotent, *i.e.,* $K_i = -[1 \quad 1.5]$, the terminal law is $\kappa_i^{\text{f}} = K_i$, and together with $\mathcal{X}_i^{\text{f}} = \{0\}$, robust asymptotic convergence to $\mathcal{R}_i = \mathbb{W}_i \oplus (A_i + B_iK_i)\mathbb{W}_i$ is assured. Initial conditions are $x_i = [5, -2]^\top, \forall i$, and the prediction horizon is $N = 8$.

Table 1: Comparison of DMPC schemes.

	SU-DMPC	S-DMPC	P-DMPC	FP-DMPC	CMPC
Updates	Single	All	All	All	All
Timing	–	Sequential	Parallel	Parallel	–
Exchanges per step	1	N_i	N_i	N_i	$2N_i$

Five different control schemes are used:

1. 'SU-DMPC': single-update DMPC [17], wherein a single, different subsystem optimizes per time step;
2. 'S-DMPC': sequential DMPC, similar to [15], wherein all optimize *within* a time step, in a sequence. Feasibility is guaranteed by each subsystem sharing its new plan before the next-in-line subsystem updates;
3. 'P-DMPC': parallel DMPC, wherein all optimize in parallel, but with no extra tightening of coupled constraints;
4. 'FP-DMPC': the proposed feasible-parallel DMPC;
5. 'CMPC': centralized MPC.

To allow direct comparisons, each of the distributed controllers is initialized using Algorithm 2, even though the published SU-DMPC and S-DMPC schemes, [17] and [15] respectively, assume a centralized initialization. Note that for each scheme, a subsystem shares its new plan immediately after updating. Owing to the different updating arrangements (parallel versus sequential; single versus all), this leads to different levels of communication, as shown in Table 1.

Figure 1 shows, for $Q_i = I$, $R_i = 1$, the total control effort used at each time step. The in-parallel optimizations of P-DMPC lead to a sustained constraint violation. All other schemes satisfy the coupled constraint, and FP-DMPC can be seen to use the full range. Note that although S-DMPC and FP-DMPC are apparently similar, there is more variation in the individual u_i for the former, which is not perceptable in the figure.

Table 2 shows the closed-loop costs obtained for each controller. Two scenarios are shown: scenario 1, with identical cost matrices $Q_i = I$, $R_i = 1$, and scenario 2, with differing costs, $Q_i = iI$, $R_i = 1/i$. In each scenario, P-DMPC obtains the lowest cost, lower even than CMPC, but only because the coupled constraint is violated by the parallel decision making (Fig. 1). SU-DMPC performs the worst, owing to its restrictive, single-update nature. Remarkably, FP-DMPC performs best among the DMPC controllers, out-performing even S-DMPC, which has sharing of up-to-date plans within a time step. S-DMPC leads to inequitable sharing of the control effort; the leading subsystems in the update sequence use more of the available control, leaving less for subsystems later in the sequence. The extra tightening in FP-DMPC not only guarantees feasibility, but in this example discourages "greedy" behaviour by restricting the control available to each subsystem.

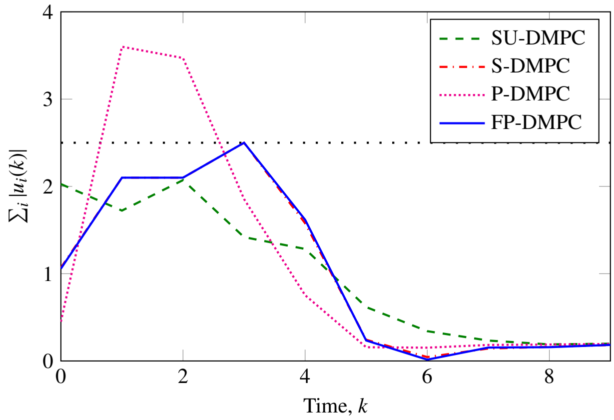

Figure 1: Total control effort of the four point masses.

Table 2: Comparison of closed-loop costs. The asterisk denotes costs obtained while violating coupling constraints.

Scenario	SU-DMPC	S-DMPC	P-DMPC	FP-DMPC	CMPC
1	198.39	192.49	189.06*	192.04	191.99
2	484.24	475.41	458.81*	470.57	467.67

7. Conclusions

A distributed MPC approach has been presented for uncertain linear, dynamically decoupled subsystems sharing convex constraints. The distributed controllers optimize in parallel at each time step, and no iteration is required. Robust feasibility and stability in the presence of additive, bounded disturbances is guaranteed. Extra constraint tightening in local optimization problems guarantees robust coupled constraint satisfaction, despite the local optimization problems being solved in parallel. The proposed method has been demonstrated by numerical examples.

Appendix A. Proof of Proposition 4

For (i) and (ii), by construction, $\prod_{i\in\mathcal{I}} \hat{\hat{\mathcal{U}}}_i \supset \prod_{i\in\mathcal{I}} \hat{\mathcal{U}}_i \supset \mathcal{U}$, where $\hat{\mathcal{U}}_i$ and $\hat{\hat{\mathcal{U}}}_i$ are subsystem i's feasible sets for the problems (19) and (20), respectively, and for brevity the initial-state dependence of each of these sets has been omitted. Non-emptiness of $\mathcal{U}$ implies non-emptiness of $\hat{\mathcal{U}}_i$ and $\hat{\hat{\mathcal{U}}}_i$ for all $i \in \mathcal{I}$. For (iii), consider some iteration p, at which some subsystem $i \in \mathcal{I}$ has $\mathbf{u}_i^{[p]}$, with cost

$$D_i\left(\mathbf{u}_i^{[p]}, \mathbf{u}_{-i}^{[p]}\right) = \sum_{j=0}^{N-1} \sum_{c\in C_i} \frac{1}{n(\mathcal{I}_c)} d\Big(\left(\bar{z}_{ci}^{[p]}(j), \bar{z}_{c(-i)}^{[p]}(j)\right), \bar{\mathbb{Z}}_c\Big)$$

Then there exists a solution $\mathbf{u}_i^{[p^+]}$ to problem (20), with cost

$$\begin{aligned} D_i\left(\mathbf{u}_i^{[p^+]}, \mathbf{u}_{-i}^{[p]}\right) &= \sum_{j=0}^{N-1} \sum_{c\in C_i} \frac{1}{n(\mathcal{I}_c)} d\Big(\left(\bar{z}_{ci}^{[p^+]}(j), \bar{z}_{c(-i)}^{[p]}(j)\right), \bar{\mathbb{Z}}_c\Big) \\ &\leq D_i\left(\mathbf{u}_i, \mathbf{u}_{-i}^{[p]}\right), \forall \mathbf{u}_i \in \hat{\hat{\mathcal{U}}}_i. \end{aligned} \tag{A.1}$$

The subsystem $i \in \mathcal{I}$ adopts the solution $\mathbf{u}_i^{[p+1]} = w_i \mathbf{u}_i^{[p^+]} + (1 - w_i)\mathbf{u}_i^{[p]}$, where $w_i \in (0, 1)$, with cost

$$D_i\left(\mathbf{u}_i^{[p+1]}, \mathbf{u}_i^{[p]}\right) = \sum_{j=0}^{N-1} \sum_{c\in C_i} \frac{1}{n(\mathcal{I}_c)} d\Big(\left(\bar{z}_{ci}^{[p+1]}(j), \bar{z}_{c(-i)}^{[p]}(j)\right), \bar{\mathbb{Z}}_c\Big)$$

By linearity, $\left(\bar{z}_{ci}^{[p+1]}(j), \bar{z}_{c(-i)}^{[p]}(j)\right) = w_i \left(\bar{z}_{ci}^{[p^+]}(j), \bar{z}_{c(-i)}^{[p]}(j)\right) + (1 - w_i)\left(\bar{z}_{ci}^{[p]}(j), \bar{z}_{c(-i)}^{[p]}(j)\right)$. It follows that

$$0 \leq D_i\left(\mathbf{u}_i^{[p+1]}, \mathbf{u}_{-i}^{[p]}\right) \leq D_i\left(\mathbf{u}_i^{[p]}, \mathbf{u}_{-i}^{[p]}\right). \tag{A.2}$$

for any $i \in \mathcal{I}$. Therefore, for any $i \in \mathcal{I}$, the cost $D_i\left(\mathbf{u}_i, \mathbf{u}_{-i}^{[p]}\right)$ is non-increasing and bounded below when i iterates from $\mathbf{u}_i = \mathbf{u}_i^{[p]} \to \mathbf{u}_i^{[p+1]}$ while $\mathbf{u}_{-i}^{[p]}$ are held constant. Now consider the cost when *all* subsystems iterate from p to $p + 1$.

$$D_i\left(\mathbf{u}_i^{[p+1]}, \mathbf{u}_i^{[p+1]}\right) = \sum_{j=0}^{N-1} \sum_{c\in C_i} \frac{1}{n(\mathcal{I}_c)} d\Big(\left(\bar{z}_{ci}^{[p+1]}(j), \bar{z}_{c(-i)}^{[p+1]}(j)\right), \bar{\mathbb{Z}}_c\Big) \tag{A.3}$$

The summands in (A.3) satisfy

$$\begin{aligned} & d\Big(\left(\bar{z}_{ci}^{[p+1]}(j), \bar{z}_{c(-i)}^{[p+1]}(j)\right), \bar{\mathbb{Z}}_c\Big) \\ &\leq d\left(\sum_{l\in\mathcal{I}_c} w_l \left(\bar{z}_{cl}^{[p^+]}(j), \bar{z}_{c(-l)}^{[p]}(j)\right), \bar{\mathbb{Z}}_c\right) \\ &\quad + d\left(\sum_{m\in\mathcal{I}\setminus\mathcal{I}_c} w_m \left(\bar{z}_{cm}^{[p]}(j), \bar{z}_{c(-m)}^{[p]}(j)\right), \bar{\mathbb{Z}}_c\right) \\ &\leq \sum_{l\in\mathcal{I}_c} w_l d\Big(\left(\bar{z}_{cl}^{[p^+]}(j), \bar{z}_{c(-l)}^{[p]}(j)\right), \bar{\mathbb{Z}}_c\Big) \\ &\quad + \sum_{m\in\mathcal{I}\setminus\mathcal{I}_c} w_m d\Big(\left(\bar{z}_{cm}^{[p]}(j), \bar{z}_{c(-m)}^{[p]}(j)\right), \bar{\mathbb{Z}}_c\Big), \end{aligned} \tag{A.4}$$

because, for all $i \in \mathcal{I}$, $\mathbf{u}_i^{[p+1]} = w_i \mathbf{u}_i^{[p^+]} + (1 - w_i)\mathbf{u}_i^{[p]}$ and so

$$\begin{aligned} (\bar{z}_{ci}^{[p+1]}(j), \bar{z}_{c(-i)}^{[p+1]}(j)) = &\sum_{l\in\mathcal{I}_c} w_l (\bar{z}_{cl}^{[p^+]}(j), \bar{z}_{c(-l)}^{[p]}(j)) \\ &+ \sum_{m\in\mathcal{I}\setminus\mathcal{I}_c} w_m (\bar{z}_{cm}^{[p]}(j), \bar{z}_{c(-m)}^{[p]}(j)). \end{aligned}$$

for all $i \in \mathcal{I}, c \in C_i$. The first inequality in (A.4) follows from the triangle inequality, and the second follows from convexity of $d(\cdot, \cdot)$. Combining (A.1), (A.3), (A.4)—and using the facts that $d(\cdot, \cdot) \geq 0$ and $\sum_{i\in\mathcal{I}} w_i = 1$—we conclude that

$$0 \leq D_i\left(\mathbf{u}_i^{[p+1]}, \mathbf{u}_{-i}^{[p+1]}\right) \leq D_i\left(\mathbf{u}_i^{[p+1]}, \mathbf{u}_{-i}^{[p]}\right) \leq D_i\left(\mathbf{u}_i^{[p]}, \mathbf{u}_{-i}^{[p]}\right), \forall i \in \mathcal{I}. \tag{A.5}$$

Moreover, by definition of D and D_i, $D(\mathbf{u}) = \sum_{i\in\mathcal{I}} D_i(\mathbf{u}_i, \mathbf{u}_{-i})$ and so

$$0 \leq D\left(\mathbf{u}^{[p+1]}\right) \leq D\left(\left(\mathbf{u}_i^{[p+1]}, \mathbf{u}_{-i}^{[p]}\right)\right) \leq D\left(\mathbf{u}^{[p]}\right)$$

where the middle inequalities hold for any $i \in \mathcal{I}$.

For (iv), since each $D_i\left(\mathbf{u}_i^{[p]}, \mathbf{u}_{-i}^{[p]}\right)$, and $D\left(\mathbf{u}^{[p]}\right)$, are non-increasing and bounded below, the sequences $\left\{D_i\left(\mathbf{u}_i^{[p]}, \mathbf{u}_i^{[p]}\right)\right\}$ and $\left\{D\left(\mathbf{u}^{[p]}\right)\right\}$ converge to some limits, say D_i^* and $D^* = \sum_{i\in\mathcal{I}} D_i^*$. It remains to show that D^*, and each D_i^*, are equal to zero. For this, note that (a) $\mathcal{U}$ and each $\hat{\hat{\mathcal{U}}}_i$ is convex and compact (closed and bounded); (b) the cost function $D(\mathbf{u})$ is, by definition, convex for $\mathbf{u} \in \prod_{i\in\mathcal{I}} \hat{\hat{\mathcal{U}}}_i \setminus \mathcal{U}$ and equal to 0 for $\mathbf{u} \in \mathcal{U}$. Therefore, the sequence $\left\{\mathbf{u}^{[p]}\right\}$ has at least one accumulation point. Consider a subsequence of iterations, $\mathcal{P} \subset \{1, 2, \ldots\}$, so that $\left\{\mathbf{u}^{[p']}\right\}$ (where $p' \in \mathcal{P}$) converges to an accumulation point $\mathbf{u}^*$. By continuity, $\left\{D(\mathbf{u}^{[p']}\right\} \to D(\mathbf{u}^*) = D^*$. By the assumptions on D, its minimum value (0) is attained for $\mathbf{u} \in \mathcal{U}$. Suppose that $D^* > 0$, so that $\mathbf{u}^* \notin \mathcal{U}$. It follows that

$$D(\mathbf{u}^0) - D(\mathbf{u}^*) < 0, \forall \mathbf{u}^= \in \mathcal{U}.$$

Taking limits of (A.1) as the iterations $p' \in \mathcal{P}$ tend to infinity,

$$D(\mathbf{u}_i^*, \mathbf{u}_{-i}^*) \leq D(\mathbf{u}_i, \mathbf{u}_{-i}^*), \forall \mathbf{u}_i \in \hat{\hat{\mathcal{U}}}_i. \tag{A.6}$$

Inequality (A.6) holds for all $i \in \mathcal{I}$. Thus,

$$D(\mathbf{u}^*) \leq D(\mathbf{u}), \forall \mathbf{u} \in \prod_{i\in\mathcal{I}} \hat{\hat{\mathcal{U}}}_i.$$

But $0 = D(\mathbf{u}^0) \leq D(\mathbf{u}), \forall \mathbf{u}^0 \in \mathcal{U}, \mathbf{u} \in \mathbb{R}^{\sum_i m}$ and $\mathcal{U} \subset \prod_{i\in\mathcal{I}} \hat{\hat{\mathcal{U}}}_i$, so we have a contradiction. Hence, $D^* = 0$ and $\mathbf{u}^* \in \mathcal{U}$. Finally, because $\mathbf{u}^*$ was an arbitrary accumulation point, it follows that all accumulation points lie in $\mathcal{U}$. Therefore, the whole sequence $\{\mathbf{u}^p\}$ converges to $\mathcal{U}$. □

References

[1] J. M. Maciejowski, Predictive Control with Constraints, Prentice Hall, 2002.

[2] J. B. Rawlings, B. T. Stewart, Coordinating multiple optimization-based controllers: New opportunities and challenges, Journal of Process Control 18 (2008) 839–845.

[3] D. Q. Mayne, J. B. Rawlings, C. V. Rao, P. O. M. Scokaert, Constrained model predictive control: Stability and optimality, Automatica 36 (2000) 789–814.

[4] E. Camponogara, D. Jia, B. H. Krogh, S. Talukdar, Distributed model predictive control, IEEE Control Systems Magazine 22 (1) (2002) 44–52.

[5] J. M. Maestre, R. R. Negenborn (Eds.), Distributed Model Predictive Control Made Easy, Springer, 2014.

[6] R. Scattolini, Architectures for distributed and hierarchical model predictive control – a review, Journal of Process Control 19 (2009) 723–731.

[7] P. D. Christofides, R. Scattolini, D. Muñoz del la Peña, J. Liu, Distributed model predictive control: A tutorial review and future research directions, Computers and Chemical Engineering 51 (2013) 21–41.

[8] S. L. Waslander, G. Inalhan, C. J. Tomlin, Theory and Algorithms for Cooperative Systems, Vol. 4 of Series on Computers and Operations Research, World Scientific, 2004, Ch. Decentralized Optimization via Nash Bargaining, pp. 565–585, chapter 25.

[9] F. Valencia, J. Espinosa, B. De Schutter, K. Stavnková, Feasible-cooperation distributed model predictive control scheme based on game theory, in: 18th IFAC world congress, 2011.

[10] E. Camponogara, M. de Lima, Distributed optimization for MPC of linear networks with uncertain dynamics, IEEE Transactions on Automatic Control 57 (3) (2012) 804–809.

[11] R. L. Raffard, C. J. Tomlin, S. P. Boyd, Distributed optimization for cooperative agents: Application to formation flight, in: Proceedings of the 43rd IEEE Conference on Decision and Control, 2004, pp. 2453–2459.

[12] I. Necoara, D. Clipici, An efficient parallel coordinate descent algorithm for distributed MPC, Journal of Process Control 23 (2013) 243–253.

[13] P. Giselsson, M. D. Doan, T. Keviczky, B. D. Schutter, A. Rantzer, Accelerated gradient methods and dual decomposition in distributed model predictive control, Automatica 49 (3) (2013) 829–833.

[14] B. T. Stewart, A. N. Venkat, J. B. Rawlings, S. J. Wright, G. Pannocchia, Cooperative distributed model predictive control, Systems & Control Letters 59 (2010) 460–469.

[15] A. Richards, J. P. How, Robust distributed model predictive control, International Journal of Control 80 (9) (2007) 1517–1531.

[16] M. A. Müller, M. Reble, F. Allgöwer, Cooperative control of dynamically decoupled systems via distributed model predictive control, International Journal of Robust and Nonlinear Control 22 (2012) 1376–1397.

[17] P. Trodden, A. Richards, Distributed model predictive control of linear systems with persistent disturbances, International Journal of Control 83 (8) (2010) 1653–1663.

[18] D. Q. Mayne, M. M. Seron, S. V. Raković, Robust model predictive control of constrained linear systems with bounded disturbances, Automatica 41 (2) (2005) 219–224.

[19] M. Farina, R. Scattolini, Distributed predictive control: A non-cooperative algorithm with neighbor-to-neighbor communication for linear systems, Automatica 48 (2012) 1088–1096.

[20] S. Riverso, G. Ferrari-Trecate, Tube-based distributed control of linear constrained systems, Automatica 48 (2012) 2860–2865.

[21] M. D. Doan, T. Keviczky, B. De Schutter, A dual decomposition-based optimization method with guaranteed feasibility for hierarchical MPC problems, in: 18th IFAC world congress, 2011.

[22] M. Herceg, M. Kvasnica, C. N. Jones, M. Morari, Multi-Parametric Toolbox 3.0, in: Proc. of the European Control Conference, Zürich, Switzerland, 2013, pp. 502–510, `http://control.ee.ethz.ch/ mpt`.

[23] S. V. Raković, E. C. Kerrigan, K. I. Kouramas, D. Q. Mayne, Invariant approximations of the minimal robust positively invariant set, IEEE Transactions on Automatic Control 50 (4) (2005) 406–410.

[24] S. V. Raković, B. Kern, R. Findeisen, Practical robust positive invariance for large-scale discrete time systems, in: IFAC World Congress, 2011, pp. 6425–6430.

[25] R. T. Rockafellar, Convex Analysis, Princeton Mathematical Series, Princeton University Press, 1970.