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Utilisation of Fast Fourier Transform and Least-squares Modification of Stokes formula to compile approximate geoid heights over Khartoum State: Sudan

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Abstract

We use Fast Fourier Transform (FFT) and Least-squares modification (LSM) of Stokes formula to compute the approximate geoid over Khartoum State in Sudan. The two methods (FFT and LSM) have been utilised to test their efficiency with respect to EGM08 and the local GPS-levelling data. The FFT method has many advantages, it is fast and it reduces the computational complexity. The modification of Stokes formula is widely used in geoid modelling, however, its implementation based on point-wise summation requires a considerable amount of time. In FFT we combine the terrestrial gravity data and the global geopotential model (GGM) by means of a remove-compute-restore procedure and we successfully apply the modification of the Stokes formula in the least-squares sense. FFT and LSM approximate geoid solutions are evaluated against EGM2008 and the GPS-levelling data. The analysis of the undulation differences shows that the LSM solution is more compatible with EGM08 and GPS-levelling data. The discrepancies of the differences are removed using a 4-parameter model, the standard deviation (STD) of the undulation differences of LSM decreased from 0.41 m to 0.37 m and from 0.48 m to 0.39 m for FFT solution. There is no significant impact to the LSM geoid when adding the additive corrections, while the FFT geoid solution is slightly improved when terrain correction is applied.

Keywords: EGM08, FFT, GPS-levelling, geoid, least-squares modification, remove-compute-restore, Stokes formula, terrestrial gravity

1 Introduction

The geoid determination in Sudan still needs more concentration and intensive studies due to the restrictions of the existing surveying data (GPS-levelling heights and local gravity measurements). The country is vast and most of the areas are remote and difficult to access, therefore, most of the geodetic survey measurements were conducted along the banks of the Nile River. The priority of computing a precise geoid model in Khartoum State stems from the considerable need for a proper geodetic system for various engineering projects in Khartoum area.

In this paper, we utilise two different methods to compute two corresponding approximate geoid models (no corrections applied). The Fast Fourier Transform (FFT) (Bracewell, 1978) is used for the first time to compute the geoid over Khartoum State. FFT is an efficient procedure, it helps reduce the time of the computation and minimise of the memory storage compared to the familiar geoid computation methods using Stokes integral equation. FFT was broadly utilised in physical geodesy and addressed by several authors (e.g. Sideris, 1987; Schwarz *et al.*, 1990; Haagmans *et al.*, 1993; De Min, 1994; Sideris and She, 1995; Tziavos, 1996) it has a proper treatment for the complexity of the discrete numerical integral of Stokes and Vening Meinesz (Tziavos, 1996). The technique is based on the planar approximation of the Stokes kernel which helps to compute the geoid efficiently over large areas (Denker, 1990; Forsberg and Sideris, 1991).

Over the years, many improvements were added to the FFT method in order to address the drawbacks of the spherical approximation by utilising the full zero-padding technique (Tziavos, 1993; Sideris and Li, 1993). The terrestrial gravity data do not cover the Earth surface properly due to the lack of measurements, in addition the existing gravity data are available in a discrete pattern. They are scattered over certain regions on the Earth surface where measurements were conducted. This makes the use of short wavelength information incomplete. On the other hand, the far-zone contribution will be missing due to the truncation effects of Stokes formula over the gravity-covered areas (omission error). The remove-compute-restore technique can efficiently handle this problem by combining short wave-length data (existing gravity data) and the long wave-length information from the global geopotential model (GGM).

The steps of the remove-compute-restore (RCR) procedure were presented in different geoid studies (e.g. see Abdalla and Tenzer, 2012a). In short, the RCR procedure is based on the following steps: The gravity reference field is removed from the terrestrial gravity data. The high-frequency, (residual geoid) is computed from the residual gravity data using FFT. In this step the Stokes formula is reformulated into 2-D convolution format to be compatible with the planar approximation. The reference gravity field and the long wave-length geoid information are obtained from the global geopotential models. Finally the terrain correction is computed by FFT to be enclosed in RCR procedure. The near-zone contribution to gravity field can be computed by applying the discretised integral-equation approach (see Abdalla and Tenzer, 2014). While the far-zone contribution to gravity field is obtained by utilising the far-zone modified spherical harmonics (see Tenzer *et al.*, 2011).

The aim of modifying the Stokes function is to reduce the effects of the truncation errors on the geoid solution due to the lack of gravity data on the Earth's surface. This happens by combining the local and global gravity data (GGM data) mutually in a so called modified Stokes formula. Two modification methods are well known as deterministic and stochastic and have been studied and developed by a large number of scientists for a long time.

The deterministic methods intend to reduce the effects of the remote zone resultant from the truncation of the original Stokes kernel within a limited spherical cap and therefore improving its convergence using lower degree geopotential coefficients. Considering lower degree terms on the modified kernels is useful to minimise the zero-degree error of the gravity anomaly expansion. The deterministic methods were broadly studied by many scientists, for instance (Molodensky *et al.*, 1962; Wong and Gore, 1969; Meissl, 1971; Vincent and Marsh, 1974; Jekeli, 1981). Some authors employed the RCR approach using high degree coefficients of the GGM for generating a higher degree reference field and the residual field is computed from the integral formula (see e.g. Jeffreys, 1953; De Witte, 1967; Wong and Gore, 1969). More studies of the deterministic modifications to Stokes functions based on the RCR approach were discussed and investigated by Featherstone *et al.* (1998); Vaníček and Featherstone (1998); Featherstone (2003).

The stochastic modification of Stokes kernel aims to reduce the errors of the terrestrial gravity and spherical harmonic coefficients of GGM by combining both of them optimally in a least-squares sense (see e.g. Sjöberg, 1979, 1980; Wenzel, 1982; Sjöberg, 1984, 1991b, 2003b). A rigorous LSM procedure was introduced by Sjöberg (1984, 1991a, 2003b), it was properly derived in three comparable versions (biased, unbiased and optimum) each of the three versions can be adopted for obtaining the final gravimetric model, this study is limited to the biased solution.

This paper is intent to test the quality of the geoid model when modifying and approximating Stokes kernel using LSM and FFT. We investigate the FFT method by means of a remove-compute-restore (RCR) procedure and the modified Stokes integral by means of least-squares (LSM). In this paper, the comparison is conducted to evaluate the geoid solutions including the terrain correction (Forsberg and Tscherning, 1997) for RCR and the additive corrections for LSM. The main difference between the two methods stems from the way of treating the local gravity data, e.g. in RCR the gravity data are reduced to the geoid by removing the effect of the topography the effect of the reference field due to the truncation is also removed. After computing the residual geoid, the topography correction and the reference field contribution are reduced back. While in the LSM method the terrestrial gravity data are used without reduction, however, the associated additive correction are added to the later after computing the the (e.g. Abdalla, 2013a). Yildiz *et al.* (2012) conducted a comparison between LSM and FFT, least-squares collocation (LSC) based on RCR. The three methods showed comparable results when applying 1-Parameter fitting model, the differences between results were within $(\sim 5\,mm)$. Furthermore, LSM had shown best agreement with EGM08 and maintained its superiority over the mountainous areas. Abbak *et al.* (2012) conducted another comparison between LSM and RCR over a mountainous area in central Turkey where gravity measurements are a few. LSM with additive corrections again showed a significant best fit when compared with GPS-levelling data. The utilisation of LSM and associated additive corrections for the computation of the gravimetric geoid models has been widely used in literature and can be found in the publications of the geodesy division at Royal Institute of Technology (KTH) and other peer reviewed articles, among them see Kiamehr (2006); Daras (2008); Abdalla (2009); Ågren *et al.* (2009); Abdalla and Tenzer (2011); Abbak *et al.* (2012).

FFT and LSM methods are briefly presented in Section 2. The data used in this study are described and addressed in Section 3. The analysis of the numerical results obtained from the FFT and LSM methods are presented in Section 4. Finally in Section 5 a summary of the current study is given and some concluding remarks are drawn.

2 Review of FFT and LSM methods

2.1 FFT method

The geoid heights can be computed from the terrestrial gravity anomalies Δg using the well-known Stokes formula. The classical form of Bruns-Stokes formula is written as (Stokes, 1849)

$$N = \frac{R}{4\gamma_0 \pi} \iint_{\sigma} S(\psi)\, \Delta g \, \mathrm{d}\sigma \tag{2.1}$$

where R is the Earth's mean radius γ_0 is the normal gravity evaluated at the surface of the reference ellipsoid (Moritz, 1980), ψ is the geocentric angle, $S(\psi)$ denotes original Stokes function and $d\sigma$ is the infinitesimal surface element of the unit sphere σ.

Stokes original kernel can be written in a closed form as follows:

$$S(\psi) = \frac{1}{\sin\dfrac{\psi}{2}} - 6\sin\frac{\psi}{2} + 1 - 5\cos\frac{\psi}{2} - 3\cos\frac{\psi}{2}\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) \tag{2.2}$$

the Laplace spherical harmonics Δg_n^{GGM} for the gravity anomalies of degree n are defined as (Heiskanen and Moritz, 1967)

$$\Delta g_n^{GGM} = \frac{\mathrm{GM}}{R^2}\left(\frac{R}{r}\right)^{n+2}(n-1)\sum_{m=-n}^{n} C_{n,m}\mathrm{Y}_{n,m} \tag{2.3}$$

where $R = 6371 \times 10^3$ m is the Earth's mean radius, $GM = 3986005 \times 10^8$ $m^3 s^{-2}$ is the geocentric gravitational constant, r is the geocentric radius, C_{nm} are the fully-normalised harmonic coefficients which describe the disturbing potential T and Ynm are the fully-normalised surface spherical functions.

the long wavelength geoid is computed by

$$N_{GGM} = \frac{\mathrm{GM}}{R^2 \gamma_0} \left(\frac{R}{r} \right)^{n+1} \sum_{m=-n}^{n} C_{n,m} \mathrm{Y}_{n,m} \tag{2.4}$$

The form of convolution integrals of Equation 2.1 consists of a set of time-consuming point-wise summations which require massive computer memory. In addition, the truncation errors induced by confining the area of the terrestrial gravity data into the boundary of the spherical cap radius around the computation points are expected to have a significant contribution in the geoid solution. However, these can be reduced by using suitable modified kernels as will be seen in section 2.2. The FFT can efficiently handle the time problem and the numerical summation by efficacious multiplications instead.

FFT can be used in the evaluation of the Stokes integral by approximating the spherical plane by a tangent planar plane in terms of the planar coordinates. The planar distance S between the computation point and the vicinity data is obtained by

$$\ell = \left[(\mathrm{x}_k - \mathrm{x}_i)^2 + \left(\mathrm{y}_l - \mathrm{y}_j \right)^2 \right]^{\frac{1}{2}} \tag{2.5}$$

Putting $\psi = \dfrac{S}{R}$ and $\sin \psi = \psi$, the approximation of the Stokes kernel in Equation 2.2 is yielded as

$$S(\psi) = \frac{2R}{\ell} \tag{2.6}$$

The planar form of Equation 2.2 is written by putting $R\, d\sigma = \mathrm{dx}\, \mathrm{dy}$

$$N(\mathrm{x}, \mathrm{y}) = \frac{R}{2\gamma_0 \pi} \iint\limits_E \frac{\Delta \mathrm{g}}{\left[(\mathrm{x}_k - \mathrm{x}_i)^2 + \left(\mathrm{y}_l - \mathrm{y}_j \right)^2 \right]^{\frac{1}{2}}} \mathrm{dx}\, \mathrm{dy} \tag{2.7}$$

the planar form of the Stokes kernel is given as

$$l\left(\mathrm{x},\mathrm{y}\right)=\frac{1}{\left(\mathrm{x}^{2}+\mathrm{y}^{2}\right)^{\frac{1}{2}}} \tag{2.8}$$

the discrete form of the 2D convolution in Equation 2.7 is evaluated in the following spectral form

$$\begin{aligned}N\left(\mathrm{x},\mathrm{y}\right)&=\frac{R}{2\gamma_{0}\pi}\sum_{i=0}^{M-1}\sum_{j=1}^{N-1}\Delta\mathrm{g}_{\left(\mathrm{x}_{i},\mathrm{y}_{j}\right)}l_{N\left(\mathrm{x}_{k}-\mathrm{x}_{i},\mathrm{y}_{l}-\mathrm{y}_{j}\right)}\Delta\mathrm{x}\Delta\mathrm{y}\\&=\frac{R}{2\gamma_{0}\pi}\Delta\mathrm{g}\left(\mathrm{x}_{k},\mathrm{y}_{l}\right)*l_{N}\left(\mathrm{x}_{k},\mathrm{y}_{l}\right)\end{aligned} \tag{2.9}$$

where

$$l_{N\left(\mathrm{x}_{k}-\mathrm{x}_{i},\mathrm{y}_{l}-\mathrm{y}_{j}\right)}=\begin{cases}\left[\left(\mathrm{x}_{k}-\mathrm{x}_{i}\right)^{2}+\left(\mathrm{y}_{l}-\mathrm{y}_{j}\right)^{2}\right]^{-\frac{1}{2}} & \mathrm{x}_{k}\neq\mathrm{x}_{i},\mathrm{y}_{l}\neq\mathrm{y}_{j}\\0 & \mathrm{x}_{k}=\mathrm{x}_{i},\mathrm{y}_{l}=\mathrm{y}_{j}\end{cases} \tag{2.10}$$

applying 2D FFT evaluation of the geoid based on the gravity anomaly and the Stokes kernel

$$N_{FFT}=\frac{R}{2\gamma_{0}\pi}\mathcal{F}^{-1}\left\{\mathcal{F}\left[\Delta\mathrm{g}_{\left(\mathrm{x}_{k},\mathrm{y}_{l}\right)}\right].\mathcal{F}\left[l_{N\left(\mathrm{x}_{k},\mathrm{y}_{l}\right)}\right]\right\} \tag{2.11}$$

where $\mathcal{F}$ is the 2D Fourier operator, $\mathcal{F}^{-1}$is the inverse 2D Fourier operator

The big advantage when using FFT techniques is that the geoid heights will be computed over the entire gravity data grid and no need for large gravity grids to maintain the secured distance for the spherical radius cap out of the grid of the computation points.

In the RCR procedure, the effect of the topography on the gravity field is utilised in the remove step. The terrestrial gravity data are further smoothed by reducing the gravity into the geoid surface. The gravity reduction is computed by means of terrain correction as described in Heiskanen and Moritz (1967)

$$C_{p}=-\frac{G\rho R^{2}}{2}\iint_{\sigma}\frac{\left(H_{Q}-H_{P}\right)^{2}}{\ell_{0}}d\sigma \tag{2.12}$$

where G is the Newtonian gravitational constant, ρ denotes the crustal density of the Earth, ℓ_{0} is the spherical distance between the computation point P and the running point Q

The indirect effect of the topography on the geoid height is computed based on Sideris and She (1995); Abbak *et al.* (2012)

$$N_{ind} = -\frac{\pi G \rho H_P^2}{\gamma} - \frac{G\rho R^2}{6\gamma} \iint\limits_{\sigma} \frac{H_Q^3 - H_P^3}{\ell_0} d\sigma \tag{2.13}$$

2.2 LSM method

The gravimetric geoid height N is computed as a sum of the following components (Sjöberg, 2003a):

$$N = \tilde{N} + \delta N^T + \delta N^A + \delta N^{dwc} + \delta N^{ell} \tag{2.14}$$

where $\tilde{N}$ is the approximate geoid height, δN^T the combined topographic correction, δN^A the combined atmospheric correction, δN^{dwc} the downward continuation correction and δN^{ell} the ellipsoidal correction for the formulation of the Stokes formula in the spherical approximation to the problem.

For geoid modelling over Khartoum State, the approximate geoid model $\tilde{N}$ is computed by the following modified Stokes formula (Sjöberg, 1984)

$$\tilde{N} = \frac{\mathrm{R}}{4\pi\gamma_0} \iint\limits_{\sigma_0} S^{\ell}(\psi)\ \Delta g\ \sin\psi\ d\sigma_0 + \frac{\mathrm{R}}{2\gamma_0} \sum_{n=2}^{\bar{n}} b_n\ \Delta g_n^{GGM} \tag{2.15}$$

where R denotes the Earth's mean radius, ψ is the geocentric angle, $S^{\ell}(\psi)$ is the modified Stokes function, Δg is the terrestrial gravimetric data and $d\sigma_0$ denotes the surface integration element and b_n are the least-squares coefficients (cf. Sjöberg, 2003b)

The modified Stokes kernel reads

$$S^{\ell}(\psi) = S(\psi) - \sum_{n=2}^{\bar{n}} \frac{2n+1}{2} b_n\, P_n(\cos\psi) \tag{2.16}$$

where $S(\psi)$ is the (original) Stokes kernel, $P_n(\cos\psi)$ are the Legendre polynomials of degree n for the argument of cosine of the spherical distance ψ. The second constituent on the right-hand side of Equation 2.15 represents the GGM contribution to the approximate geoid heights. This contribution is computed from the GGM coefficients up to a maximum degree $\bar{n}$ of spherical harmonics and from a set of the least-squares modification parameters $\{\beta_n : n = 2, 3, ..., \bar{n}\}$.

Since LSM does not need a gravity reduction, hence, the additive corrections for the topographic, ellipsoidal, downward continuation of the gravity to the geoid surface and atmospheric effects are added to the geoid estimator. The additive corrections are extensively addressed in other literature Kiamehr (2006); Daras (2008); Abdalla (2009); Ulotu (2009); Abdalla and Tenzer (2011). A one-by-one magnitude of the additive corrections on the geoid has been investigated by Abdalla (2013b); Abdalla and Mogren (2015)

3 Data in use

3.1 Local gravity data

The terrestrial gravity data used in this study are provided by GETECH. The gravity data of the entire country (Bouguer and free-air anomalies) were evaluated against the propagation of the gross error using two cross-validation tests, more information about the refinement of the gravity data set is found in (Abdalla, 2009; Abdalla and Fairhead, 2011). The grid of gravity data used in this study was provided by Getech, it consists of a 23509 points of free-air gravity anomalies.

The Sudan gravity database was compiled by Getech in 1988 Fairhead (1988) from all available land based surveys and include academic data from e.g. the Geological Research Authority of Sudan (GRAS), Strojexport and other oil companies (see also Green and Fairhead, 1996), the distribution of the gravity data is shown in Figure 1. The age of the data ranges from 1960s to 1980s.There is a mix of altitude measurement methods used from spirit levelling, benchmark, trigonometric point to barometric all tied to bench marks and trigonometric points, least accurate was barometric at ±3 m (∼0.6 mGal error).

[Figure 1 about here]

The current data set is limited by the boundary of Khartoum State, despite the fact that some areas are not yet covered by the local gravity, we utilised information from EGM08 to fill the gaps. The distribution of the local gravity data used in this study is shown in Figure 2

[Figure 2 about here]

3.2 GGMs

The geopotential model used in this study is GOCO-TIM-R1, it was found by Abdalla *et al.* (2012) that TIM-R1 is one of the best-fit GGMs over Khartoum State. Another study by Abdalla and Tenzer (2012b) shown that GOCE time-wise GOCO-TIM-R2 has best fit with New Zealand GPS-levelling data (cf. Tenzer *et al.*, 2011, 2013) based on the newly-derived vertical offsets between the New Zealand local datums (Tenzer *et al.*, 2011). It was utilised in the determination of the long wavelength gravity and geoid at the maximum degree and order 224. The gravity field generated by this model is unbiased to any other fields, it has been improved by considering the errors of the coefficients by employing relative variance-covariance information. The satellite gravity information is taken from the satellite orbit and parametrised up to degree and order 100, while gravity data from gradients are derived from satellite gravity gradients up to degree and order 224. Regularisation is applied to near zonal coefficients and coefficients from degree and order 170 to 224, for more information the reader is referred to Pail *et al.* (2011).

The free-air gravity data were derived from the geopotential model (Equation 2.3) in order to smooth the terrestrial gravity data by removing the reference field component and obtaining the residual gravity data (remove step). The residual gravity data will be used in Equation 2.11 to generate the residual geoid model using 2D FFT (compute step). The long wave-length geoid is computed by the GGM and is added to the residual geoid solution (restore step) to recover the missing information (Equation 2.4).

3.3 GPS-levelling points

The number of the GPS-levelling points over Khartoum State is 25 points of orthometric heights and similar of co-located ellipsoidal heights (Figure 3). The data set has been collected for the sake of evaluating the vertical and horizontal control of the State (Ali, 2012). The levelling network campaign was conducted in 2010. The precise levelling measurements were conducted to determine orthometric height differences. Modern digital level instruments were employed to start from eight old benchmarks established in the early 1930s. The orthometric heights are referred to mean sea level (MSL) of the Mediterranean Sea at Alexandria tide gauge. The accuracy of the levelling network is within a tolerance of $10\sqrt{k}$ mm, where k is the length of levelling circuit in kilometers (cf. Ali, 2012). In 2010, the GPS points were co-located with the levelling points to compute the ellipsoid heights statically using the differential method. Two receivers were initially used to measure the length of the baseline (based on two known coordinate points). After that, two more receivers were attached to the measurement, considering the same configuration of the baseline receivers.

[Figure 3 about here]

4 Numerical investigations

4.1 Comparison with EGM08

The residual geoid has been computed from the residual gravity data using the 2D FFT method (Figure 4a). On the other hand, the terrestrial gravity data were employed to compute the approximate geoid height (Figure 4b) using Equation 2.15. The main difference between the two estimators is that in the LSM we use the terrestrial free-air data as is without any reduction processes. This is because we later add so called additive corrections for the effects of topography, ellipsoidal approximation, downward continuation of the gravity data to the sea level and the finally the effect of the atmosphere on the geoid. However, as we mentioned earlier in Section 1 the additive corrections are out of the study scope, thus they are not shown in this study. In this section, we firstly compare our geoidal solutions with respect to EGM08 (Pavlis *et al.*, 2012), after that another comparison with the GPS-levelling will also be given to have a complete information about the quality of the obtained results.

[Figure 4 about here]

The long wave-length geoid heights are obtained from the GGM for both methods using Equation 2.4 and the second right term in Equation 2.15. The use of the equations is straight forward, however to start using Equation 2.15, least-squares coefficients are to be derived first. We employ the direct method (no regularisation) to obtain the LS coefficients which belong to the biased solution. It is called biased because of the assumption that the gravity data the GGMs use are error-less. Therefore, the coefficients are derived directly by inverting of the design matrix, more information about the derivation of the least-squares coefficients can be found in (Ellmann, 2005).

The approximate geoid solution is obtained from the summation of two components (Stokesian and GGM solutions), see Figures 4e and 4f. The approximate geoid solutions (FFT and LSM) are compared with EGM08 model. The differences between EGM08 and FFT geoid solution start from -1 m in the south-western part and extend in the direction of the east to exceed -3 metres far east (see Figure 5a), the differences in most of the central parts are less than a metre. In some spots in the far north, south and east the differences went down to -0.50 m (see Figure 5b). It is obvious that the FFT residual solution is shifted above LSM and EGM08 solutions, while the differences of the comparison of the LSM and FFT solutions is 0.14 m (see Figures 4a, 4b, 5a and 5b).

[Figure 5 about here]

The statistical comparison of the undulation differences shown in Figure 5 reveals that the FFT solution has large discrepancies Table 1. Standard deviation of the geoid differences between EGM08 and FFT solution is 0.53 m. Analogously, LSM has a similar relation and statistics with the FFT as EGM08 model, the mean of differences of the geoid heights is up to 2.34 m, their STD is about 0.42 m. It is obvious from Table 1 that the FFT has a big portion of the discrepancies due to the differences in the FFT residual solution in Figure 4a comparing to the LSM estimator in Figure 4b.

[Table 1 about here]

The associated additive corrections (Figures 6a, 6b, 6c and 6d) are computed to be added to the LSM approximate geoid solution. For the FFT solution which based on the RCR procedure, gravity reduction (Figure 6e) has been computed by means of terrain correction (Equation 2.12) to be used in the "remove" step and the indirect effect of the topography on the geoid (Figure 6f) computed by Equation 2.13 is restored with reference field geoid and then added to the residual geoid solution obtained in the "compute" step.

[Figure 6 about here]

With respect to the correlation between the gravimetric solutions and EGM08, LSM solution shows high correlation with EGM08 ($R^2 = 0.96$). On the other hand, FFT solution is less correlated with EGM08, it shows an identical level of correlation with both EGM08 ($R^2 = 0.88$) and LSM ($R^2 = 0.89$) models as shown in Figure 7. The least-squares parameters and the modified Stokes kernel have provided a significant improvement for the quality of LSM solution.

[Figure 7 about here]

4.2 Comparison with GPS-levelling data

In order to check the absolute accuracy of our gravimetric solutions, we utilise another comparison test versus the GPS-levelling data. The values of the two gravimetric solutions were interpolated over the GPS-Levelling points to carry out our comparison tests. The mean of the geoid differences reaches up to 2 metres due to the approximation of Stokes kernel, the STD is 0.50 m in the FFT solution. On the other hand, the consistency of the LSM solution remains unchanged either with respect to STD (0.41 m), and meanwhile the mean of the anomaly differences is a decimetre level which is better than the FFT solution.

As we see that the gravimetric solutions still contain systematic errors (Table 2). In order to eliminate these errors we applied LS analytical 4-Parameter model (Kotsakis and Sideris, 1999) to reduce the systematic error found between the geometric and gravimetric geoid heights.

The observation equations were adapted to the residuals of the differences between the geometric and gravimetric geoid heights ΔN at the GPS-levelling testing network and solved using least-squares analysis. model is given as

$$\mathbf{a_i x} = \begin{bmatrix} \cos\varphi_i \cos\lambda_i \\ \cos\varphi_i \sin\lambda_i \\ \sin\varphi_i \\ 1 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{4.1}$$

The matrix system of observation is solved as

$$\mathbf{Ax} = \mathbf{\Delta N} - \varepsilon \tag{4.2}$$

applying least-square approach the parameters $\hat{\mathbf{x}}$ are obtained by

$$\hat{\mathbf{x}} = \left(\mathbf{A}^{\star}\mathbf{A}^{\mathbf{T}}\right)^{\mathbf{-1}} \mathbf{A}^{\star}\mathbf{\Delta N} \tag{4.3}$$

The accuracy after removing the biases is 0.39 m and 0.37 m in FFT and LMS, respectively.

[Table 2 about here]

The magnitude of the corrections has been tested on both FFT and LSM geoid solutions. The performance of the fitting model has also been tested with respect to the correction magnitude (see Table 3. The additive corrections have an insignificant impact on the final solution in terms of STD value, STD remains identical before and after adding the additive corrections (0.37m). On the other hand, the contribution of the topography correction have improved the STD about 1 cm compared to the uncorrected FFT solution, however, despite the proximity between the STD of the both solutions, LSM's STD remains smaller than that of the corrected FFT (0.38m).

[Table 3 about here]

5 Summary and concluding remarks

We tested the geoid accuracy over Khartoum State using two methods for geoid modelling, the Fast Fourier Transform (FFT) and the least-squares modification of Stokes formula (LSM). The use of the FFT is fast and efficient to carry out computations especially over large areas where using Stokes integrals is time consuming. The FFT solution was implemented to by means of a remove-compute-restore procedure. The idea of this paper was to test both methods without applying the associated corrections to the terrestrial gravity data.

The terrestrial gravity provided by GETECH-UK were used to compile the geoid solutions in this study. The geopotential model GOCO TIM-R1 was selected to be used in this study, the selection of this GGM was based on a previous study after testing several GOCE geopotential models against terrestrial gravity and GPS-levelling data. TIM-R1 was utilised at degree and order 224 (maximum).

For LSM, we used the same GGM degree and order (224) for the modification task, we also select the spherical cap to be 3 arc-degree around the computation points. The modification degree and the spherical cap were optimised with the terrestrial gravity error degree variance (12 mGal^2) to derive the LS coefficients. Three identical sets of LS coefficients were successfully derived, among them, the biased solution was selected for use in this study, assuming that the terrestrial gravity and the GGM are error-less.

The FFT and LSM solutions were evaluated against EGM08 and the local GPS-levelling data. Both comparisons reveal that the LSM solution is more consistent in terms of systematic errors and it is highly correlated with EGM08, the mean values of the geoid differences with respect to EGM08 and GPS-levelling data is found to be 0.14 m and 0.11 m, respectively. The approximation of Stokes kernel in FFT causes large offsets in the geoid heights when comparing with the GPS-levelling, EGM08 and LSM geoids. To remove the discrepancies between the gravimetric geoids (FFT and LSM) and GPS-levelling data, we applied the LS 4-parameter model, it significantly improved the STD of the anomaly differences between FFT solution and the GPS-levelling data from 0.48 m to 0.39 m, while a slight improvement is obtained in the consistent LSM solution. The 4-parameter model has improved the STD of anomaly heights between LSM and the GPS-levelling data to be 0.37 m instead of 0.41 m. The contribution of the assoiciated corrections to FFT and LSM solutions was checked. The additive corrections did not add any further improvements to LSM solution, while the terrain correction has changed the STD of the FFT solution from 0.39 m to 0.38 m ($\sim 1\,\mathrm{cm}$)

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References

Abbak, R. A., Erol, B., and Ustun, A. (2012). Comparison of the KTH and remove-compute-restore techniques to geoid modelling in a mountainous area. *Computers & Geosciences*, *48*, 31–40.

Abbak, R. A., Sjöberg, L. E., Ellmann, A., and Ustun, A. (2012). A precise gravimetric geoid model in a mountainous area with scarce gravity data: a case study in central Turkey. *Studia Geophysica et Geodaetica*, *56*(4).

Abdalla, A. (2009). *Determination of a gravimetric geoid model of Sudan using the KTH method.* Master's thesis, Royal Institute of Technology (KTH), Stockholm, Sweden.

Abdalla, A. (2013a). *The Combined Modelling of the Regional Quasigeoid of New Zealand Using Gravity and GPS-levelling Data.* Phd thesis, University of Otago, Dunedin, New Zealand.

Abdalla, A. (2013b). Towards a precise gravimetric geoid model in the Saudi Arabia (SAGEO13). Technical report, Independent research submitted to Dr Saad Almogren of King Saud University.

Abdalla, A. and Fairhead, D. (2011). A new gravimetric geoid model for Sudan using the KTH method. *Journal of African Earth Sciences*, *60*(4), 213–221.

Abdalla, A., Fashir, H., Ali, A., and Fairhead, D. (2012). Validation of recent GOCE/GRACE geopotential models over Khartoum state - Sudan. *Journal of Geodetic Science*, *2*(2), 88–97. 00002.

Abdalla, A. and Mogren, S. (2015). Implementation of a rigorous least-squares modification of Stokes formula to compute a gravimetric geoid model over Saudi Arabia (SAGEO13). *Canadian Journal of Earth Sciences*, (In Press).

Abdalla, A. and Tenzer, R. (2011). The evaluation of the New Zealand's geoid model using the KTH method. *Geodesy and Cartography*, *37*(1), 5–14.

Abdalla, A. and Tenzer, R. (2012a). Compilation of the regional quasigeoid model for New Zealand using the discretised integral-equation approach. *Journal of Geodetic Science*, *2*(3), 206–215.

Abdalla, A. and Tenzer, R. (2012b). The global geopotential and regional gravimetric geoid/quasigeoid models testing using the newly adjusted levelling dataset for New Zealand. *Applied Geomatics*, *4*(3), 187–195.

Abdalla, A. and Tenzer, R. (2014). The Integral-Equation-Based Approaches for Modelling the Local Gravity Field in the Remove-Restore Scheme. In C. Rizos and P. Willis (Eds.), *Earth on the Edge: Science for a Sustainable Planet*, Number 139 in International Association of Geodesy Symposia. Springer Berlin Heidelberg.

Ågren, J., Sjöberg, L. E., and Kiamehr, R. (2009). The new gravimetric quasigeoid model KTH08 over Sweden. *Journal of Applied Geodesy*, *3*(3), 143–153.

Ali, A. (2012). *Evaluation of vertical and horizontal control for Khartoum State*. Master thesis, University of Khartoum, Khartoum, Sudan.

Bracewell, R. (1978). *The Fourier Transform and Its Application* (2nd ed.). New York, USA: McGraw-Hi.

Daras, I. (2008). Determination of a gravimetric geoid model of Greece using the method of KTH. Master's thesis, Royal Institute of Technology (KTH).

De Min, E. (1994). On the numerical evaluation of Stokes' integral. *Bulletin of International Geoid Service*, *3*, 41–46.

De Witte, L. (1967). Truncation Errors in the Stokes and Vening Meinesz Formulae for Different Order Spherical Harmonic Gravity Terms. *Geophysical Journal of the Royal Astronomical Society*, *12*(5), 449–464.

Denker, H. (1990). GPS control of the 1989 gravimetric quasigeoid for the Federal Republic of Germany. In R. H. Rapp and F. Sansó (Eds.), *Determination of the Geoid, Present and Future*, 11–13. Springer-Verlag, Milan.

Ellmann, A. (2005). Computation of three stochastic modifications of Stokes's formula for regional geoid determination. *Computers & Geosciences*, *31*(6), 742–755.

Fairhead, D. (1988). African Gravity Project 1986-1988 compiled by University of Leeds. Technical report, Leeds, UK.

Featherstone, W. E. (2003). Software for computing five existing types of deterministically modified integration kernel for gravimetric geoid determination. *Comput. Geosci.*, *29*(2), 183–193.

Featherstone, W. E., Evans, J. D., and Olliver, J. G. (1998). A Meissl-modified Vaníček and Kleusberg kernel to reduce the truncation error in gravimetric geoid computations. *Journal of Geodesy*, *72*(3), 154–160.

Forsberg, R. and Sideris, M. (1991). Testing the spherical FFT formula for the geoid over large regions. In *Presented at the AGU Spring '91 meeting*, Volume Baltimore, Maryland, 11–13.

Forsberg, R. and Tscherning, C. (1997). Topographic effects in gravity field modelling for BVP. In F. Sansò and R. Rummel (Eds.), *Geodetic Boundary Value Problems in View of the One Centimeter Geoid*, Volume 65 of *Lecture Notes in Earth Sciences*, 239–272. Springer Berlin / Heidelberg.

Green, C. and Fairhead, J. (1996). New 5' x 5' Digital Gravity and Terrain Models of the Earth. In R. Rapp, A. Cazenave, and R. Nerem (Eds.), *Global Gravity Field and Its Temporal Variations*, Volume 116 of *International Association of Geodesy Symposia*, 227–235. Springer Berlin Heidelberg.

Haagmans, R., de Min, E., and Van Gelderen, M. (1993). Fast evaluation of convolution integrals on the sphere using ID FFT, and a comparison with existing methods for Stokes' integral. *Manuscripta Geodaetica*, *18*, 227–241.

Heiskanen, W. and Moritz, H. (1967). *Physical Geodesy.* Series of books in geology. San Francisco: W. H. Freeman.

Jeffreys, H. (1953). The use of Stokes's formula in the adjustment of surveys. *Bulletin Géodésique*, *30*(1), 331–338.

Jekeli, C. (1981). Modifying Stokes Function to Reduce the Error of Geoid Undulation Computations. *Journal of Geophysical Research*, *86*(Nb8), 6985–6990.

Kiamehr, R. (2006). *Precise Gravimetric Geoid Model for Iran Based on GRACE and SRTM Data and the Least-Squares Modification of Stokes Formula : with Some Geodynamic Interpretations.* Phd thesis, Royal Institute of Technology (KTH), Stockholm, Sweden.

Kotsakis, C. and Sideris, M. G. (1999). On the adjustment of combined GPS/levelling/geoid networks. *Journal of Geodesy*, *73*(8), 412–421.

Meissl, P. (1971). *Preparations for the Numerical Evaluation of Second Order Molodensky-type Formulas.* Ohio State University, Research Foundation.

Molodensky, M., Eremeev, V., and Yurkina, M. (1962). Methods for study of the external gravitational field and figure of the earth. *Jerusalem, Israel Program for Scientific Translations, 1962; [available from the Office of Technical Services, U.S. Dept. of Commerce, Washington]*, *-1.*

Moritz, H. (1980). *Advanced physical geodesy.* Wichmann ; Tunbridge, Eng. Karlsruhe: Abacus Press.

Pail, R., Goiginger, H., Mayrhofer, R., Schuh, W., Brockmann, J. M., Krasbutter, I., Hoeck, E., and Fecher, T. (2011). GOCE gravity field model derived from orbit and gradiometry data applying the time-wise method. In *The ESA Living Planet Symposium, 28 June - 2 July 2010*, Bergen, Norway. ESA.

Pavlis, N. K., Holmes, S. A., Kenyon, S. C., and Factor, J. K. (2012). The development and evaluation of the Earth Gravitational Model 2008 (EGM2008). *Journal of Geophysical Research: Solid Earth*, *117*(B4), B04406.

Schwarz, K., Sideris, M., and Forsberg, R. (1990). The use of FFT techniques in physical geodesy. *Geophys. J. Int.*, *100*, 485–514.

Sideris, M. (1987). *Spectral methods for the numerical solution of Molodensk's problem.* Phd thesis, University of Calgary, Calgary, Canada.

Sideris, M. and Li, Y. (1993). Gravity field convolutions without windowing and edge effects. *Bulletin Géodésique*, *67*, 107–118.

Sideris, M. and She, B. (1995). A new, high resolution geoid for Canada and part of the U.S. by the 1D-FFr method. *Bulletin of International Geoid Service*, *69*(2), 92–108.

Sjöberg, L. E. (1979). Integral formulas for heterogeneous data in physical geodesy. *Bulletin Géodésique*, *53*, 297–315.

Sjöberg, L. E. (1980). Least squares combination of satellite harmonics and integral formulas in physical geodesy. *Gerlands Beitraege zur Geophysik*, *89*, 371–377.

Sjöberg, L. E. (1984). *Least squares modification of Stokes' and Vening Meinesz' formulas by accounting for errors of truncation, potential coefficients and gravity data.* University of Uppsala, Institute of Geophysics, Dept. of Geodesy.

Sjöberg, L. E. (1991a). Refined least-squares modification of Stokes formula. *Manuscripta Geodaetica*, *16*, 367–375.

Sjöberg, L. E. (1991b). Some Integral Formulas for a Nonspherical Earth. *Determination of the Geoid*, *106*, 333–338 500.

Sjöberg, L. E. (2003a). A computational scheme to model the geoid by the modified Stokes formula without gravity reductions. *Journal of Geodesy*, *77*(7-8), 423–432.

Sjöberg, L. E. (2003b). A general model for modifying Stokes' formula and its least-squares solution. *Journal of Geodesy*, *77*(7-8), 459–464.

Stokes, G. (1849). On the variation of gravity and the surface of the earth. *Trans. Cambridge Phil. Soc.*, *8*, 672.

Tenzer, R., Dayoub, N., and Abdalla, A. (2013). Analysis of a relative offset between vertical datums at the North and South Islands of New Zealand. *Applied Geomatics*, *5*(2), 133–145.

Tenzer, R., Novák, P., Vajda, P., Ellmann, A., and Abdalla, A. (2011). Far-zone gravity field contributions corrected for the effect of topography by means of Molodensky's truncation coefficients. *Studia Geophysica et Geodaetica*, *55*(1), 55–71.

Tenzer, R., Vatrt, V., Abdalla, A., and Dayoub, N. (2011). Assessment of the LVD offsets for the normal-orthometric heights and different permanent tide systems-a case study of New Zealand. *Applied Geomatics*, *3*(1), 1–8.

Tenzer, R., Vatrt, V., Gan, L., Abdalla, A., and Dayoub, N. (2011). Combined approach for the unification of levelling networks in New Zealand. *Journal of Geodetic Science*, *1*(4), 324–332.

Tziavos, I. N. (1993). Gravity field modelling using the fast Hartley transform. In Forsberg and Denker (Eds.), *The European Geoid Determination*, Volume Wiesbaden, 26–33.

Tziavos, I. N. (1996). Comparisons of spectral techniques for geoid computations over large regions. *Journal of Geodesy*, *70*, 357–373.

Ulotu, P. E. (2009). *Geoid model of Tanzania from sparse and varying gravity data density by the KTH method*. Doctoral dissertation, Royal Institute of Technology (KTH), Stockholm, Sweden.

Vaníček, P. and Featherstone, W. E. (1998). Performance of three types of Stokes kernel in the combined solution for the geoid. *Journal of Geodesy*, *72*(12), 684–697.

Vincent, S. and Marsh, J. G. (1974). Global detailed gravimetric geoid. In *The Use of Artificial Satellites for Geodesy and Geodynamics*, Volume -1, 825–855.

Wenzel, H. (1982). Geoid computation by least-squares spectral combination using integral kernels. Tokyo, Japan, 438–453. Internation.

Wong, L. and Gore, R. (1969). Accuracy of Geoid Heights from Modified Stokes Kernels. *Geophysical Journal of the Royal Astronomical Society*, *18*(1), 81–91.

Yildiz, H., Forsberg, R., Ågren, J., Tscherning, C., and Sjöberg, L. E. (2012). Comparison of remove-compute-restore and least squares modification of Stokes' formula techniques to quasi-geoid determination over the Auvergne test area. *Journal of Geodetic Science*, *2*(1), 53–64.

Figures:

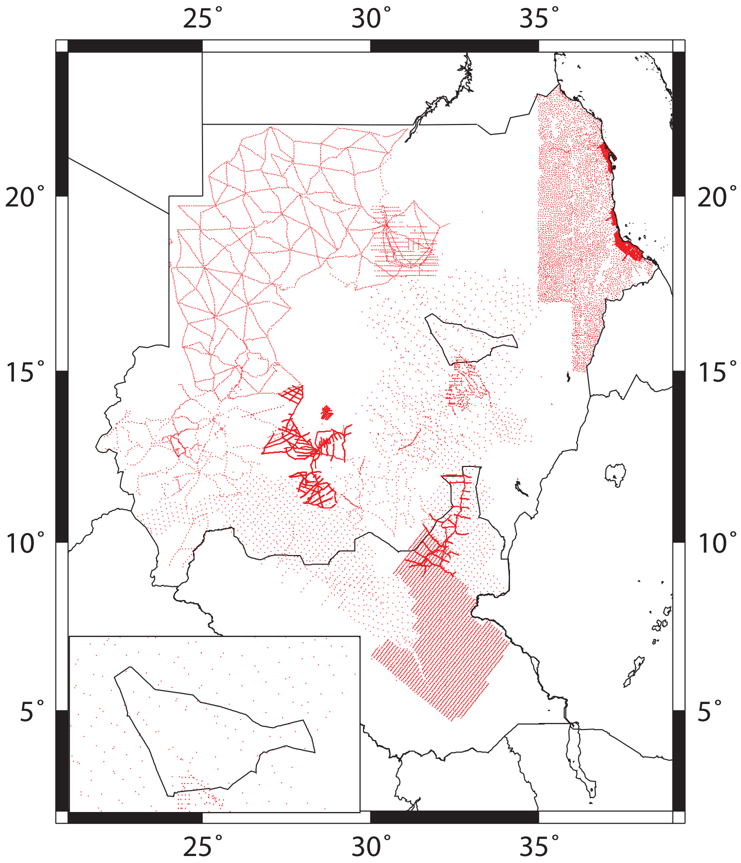


Figure 1: Local gravity data for Sudan and South Sudan. Inset focuses on the distribution of gravity stations in Khartoum State area.

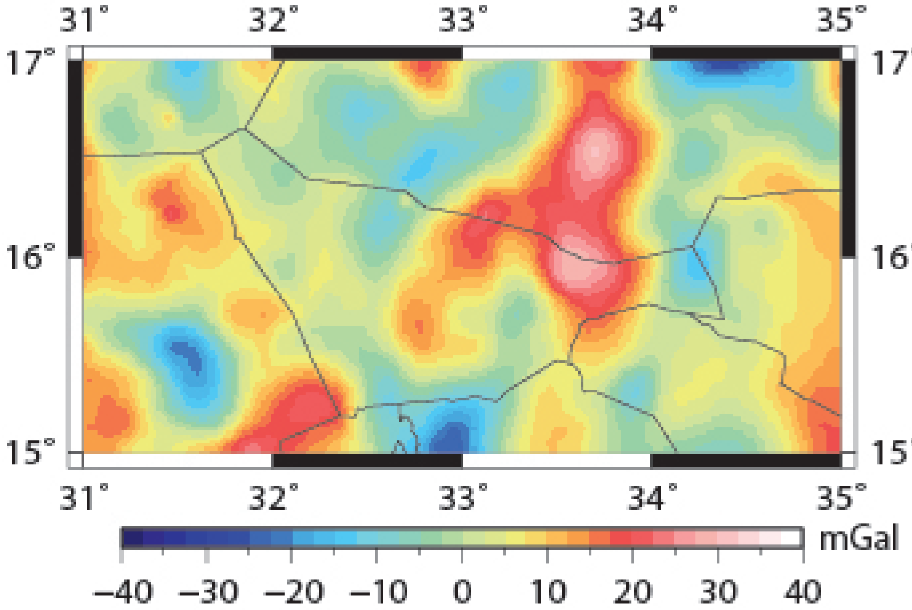


Figure 2: Free-air anomalies over Khartoum State. Unit: mGal

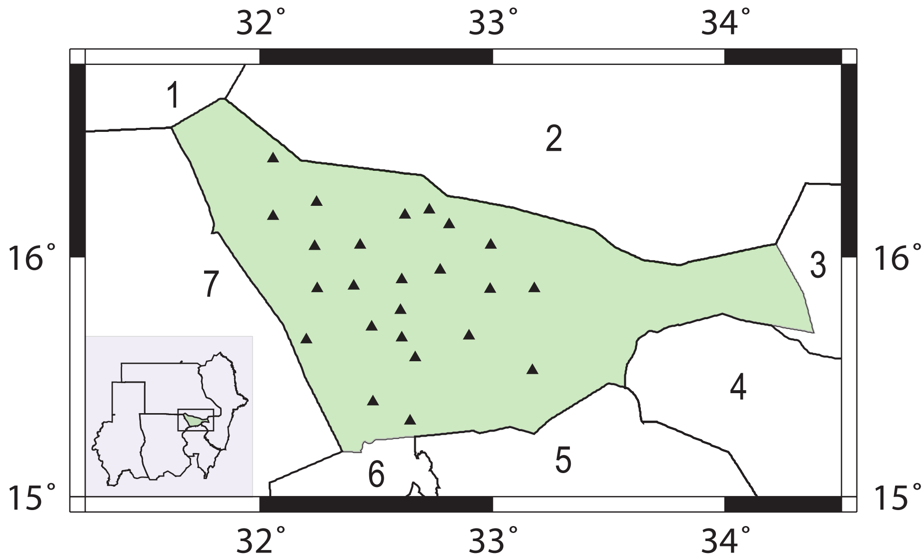
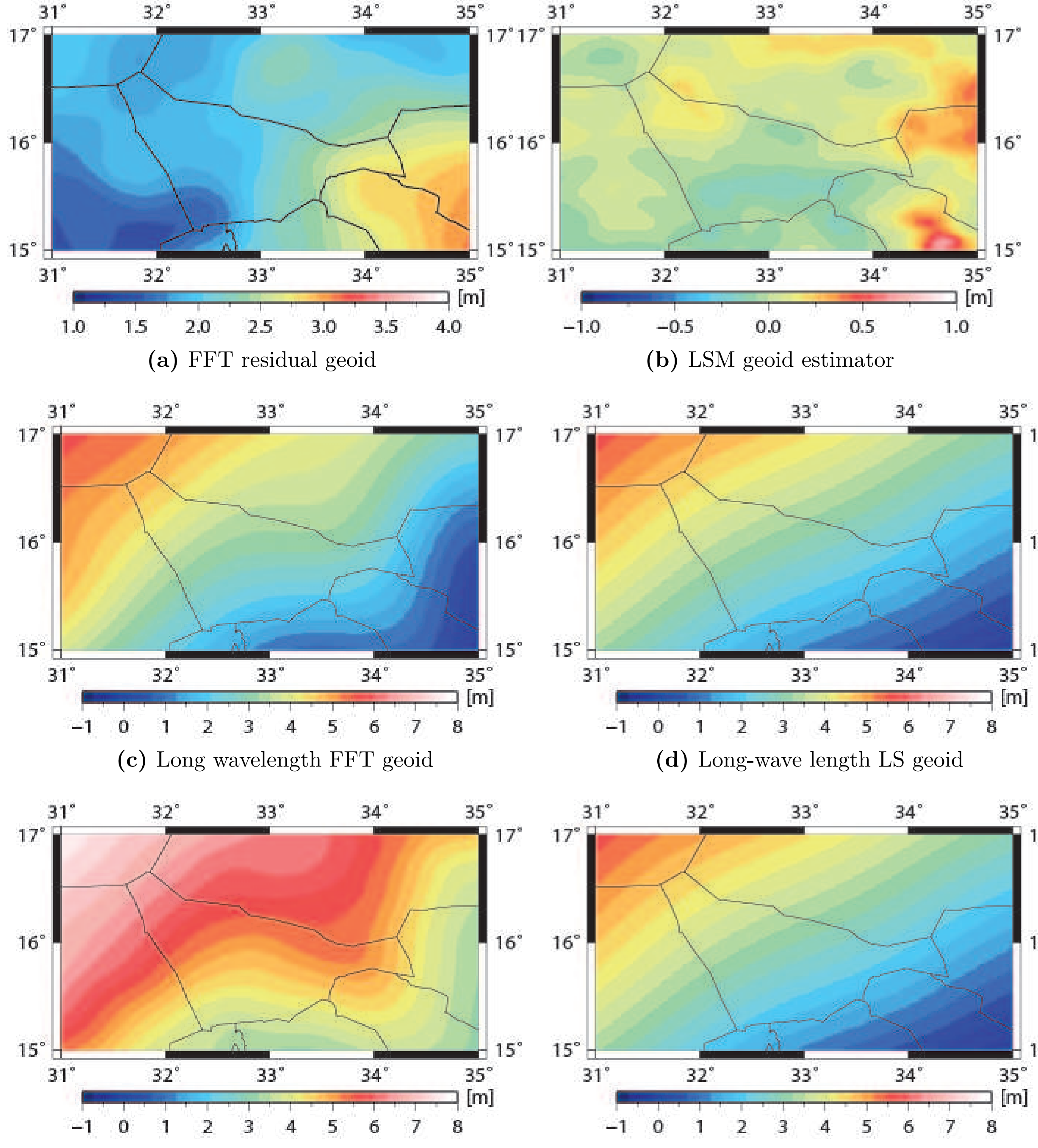


Figure 3: Location and boundary of Khartoum State (in green), GPS-levelling points and the surrounding states, 1) Northern, 2) Nile River, 3) Kassala, 4) Gadaref, 5) Gezira, 6) White Nile, 7) Northern Kordofan.



(a) FFT residual geoid

(b) LSM geoid estimator

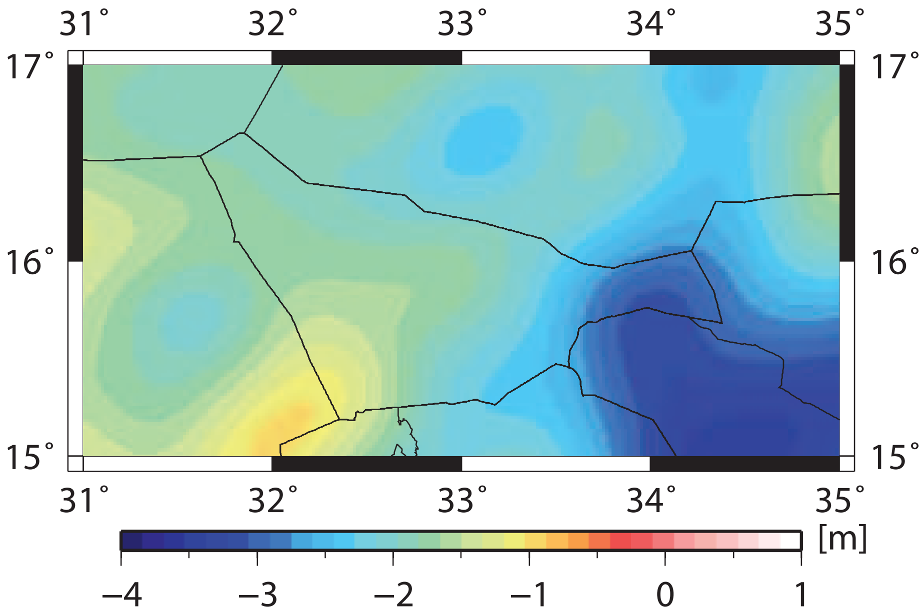
(c) Long wavelength FFT geoid

(d) Long-wave length LS geoid

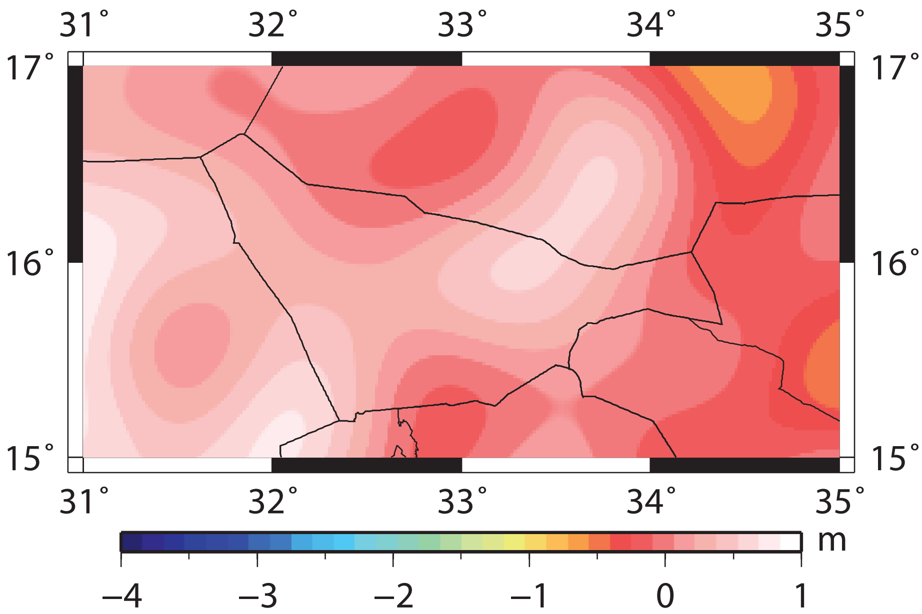
(e) FFT approximate geoid

(f) LSM approximate geoid

Figure 4: Geoid components computed from FFT (left panel) and LSM (right panel). Unit: 1m

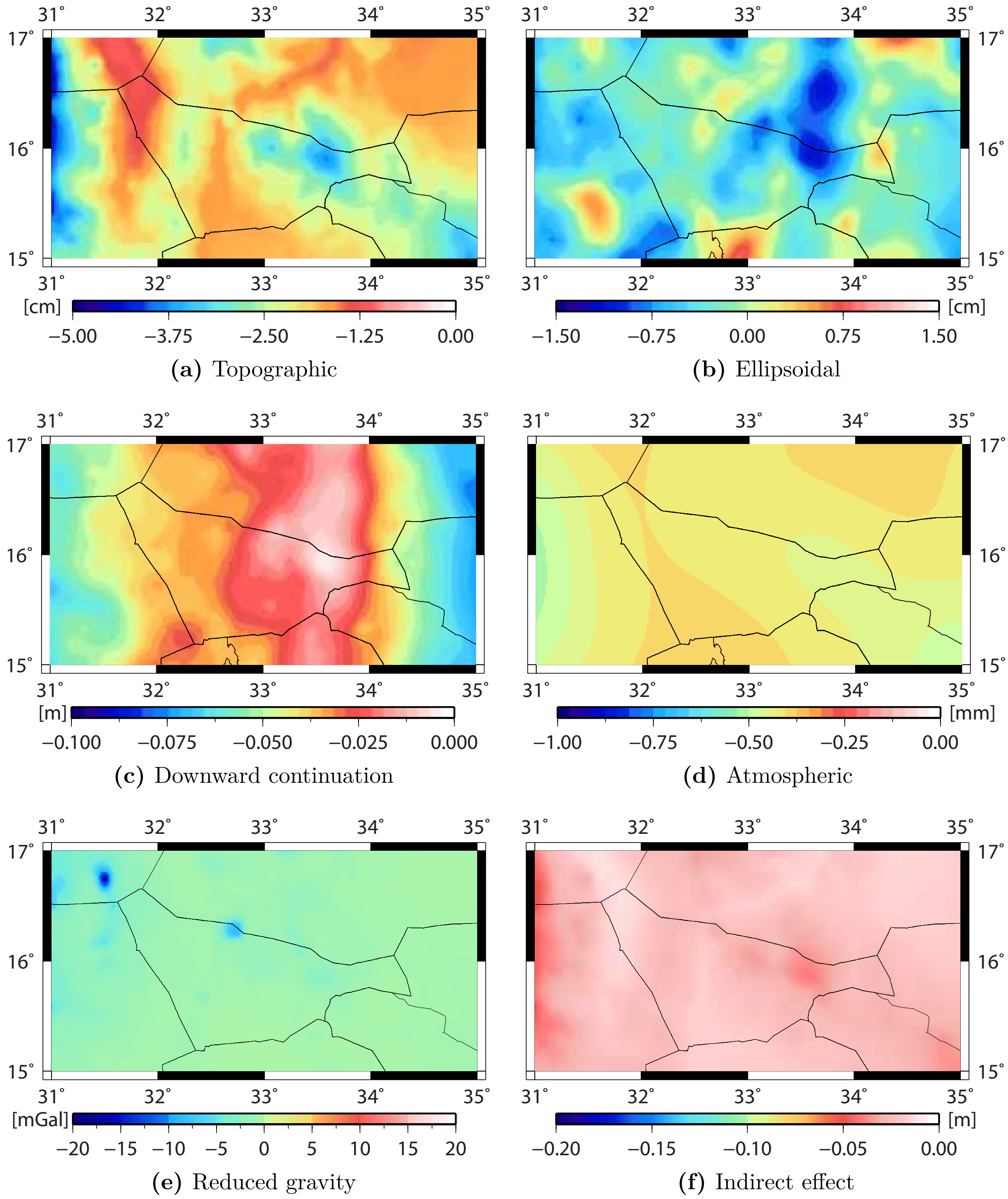


(a) EGM08 VS FFT



(b) EGM08 VS LSM

Figure 5: Undulation differences between EGM08 model to degree 360 and geoid solutions. a) FFT, b) LSM. Unit: 1m



(a) Topographic

(b) Ellipsoidal

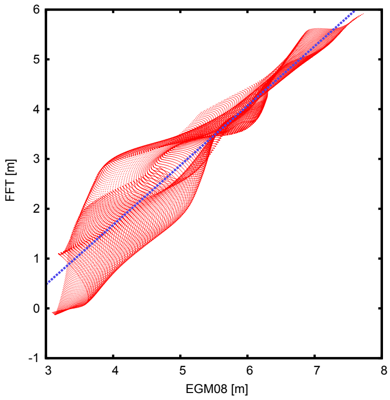
(c) Downward continuation

(d) Atmospheric

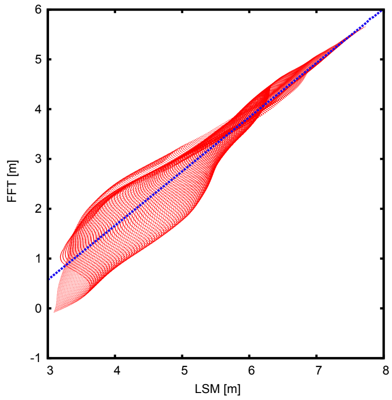
(e) Reduced gravity

(f) Indirect effect

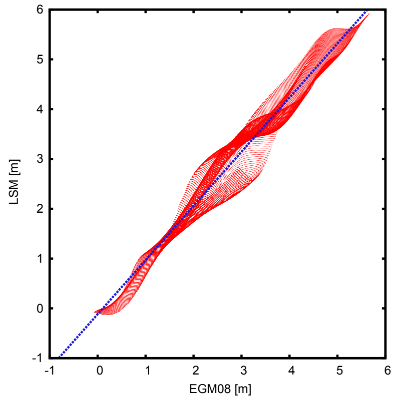
Figure 6: The additive corrections for LSM, the gravity reduction and the indirect effect of the topography on the geoid over Khartoum State



(a) FFT Vs EGM08



(b) FFT Vs LSM



(c) LSM Vs EGM08

Figure 7: Correlation between the gravimetric solutions

Tables:

Table 1: Statistics of the undulation differences between EGM08, FFT and LSM. Unit:1 m

	EGM08-FFT	**EGM08-LSM**	**LSM-FFT**
Min	-3.50	0.69	-3.30
Max	-0.98	0.82	1.50
Mean	-2.10	0.14	-2.34
STD	0.53	**0.30**	0.42

Table 2: Statistics of the undulation differences between the GPS-levelling data, FFT and LSM. Unit:1m

	FFT		**LSM**	
	Before	After	Before	After
Min	-3.10	-0.70	-1.14	-0.81
Max	-1.21	0.90	0.69	0.86
Mean	-2.07	0.00	-0.11	0.00
STD	0.48	0.39	0.41	**0.37**

Table 3: The performance of the fitting model on LSM and FFT geoid models before and after applying their associated corrections. Units: 1 m

	FFT		**LSM**	
	Before	After	Before	After
Min	-0.70	-0.74	-0.81	-0.76
Max	0.90	0.81	0.86	0.84
Mean	0.00	0.00	0.00	0.00
STD	0.39	0.38	0.37	**0.37**