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# Khovanov homotopy types and the Dold-Thom functor

Brent Everitt, Robert Lipshitz, Sucharit Sarkar and Paul Turner\*

**Abstract.** We show that the spectrum constructed by Everitt and Turner as a possible Khovanov homotopy type is a product of Eilenberg-MacLane spaces and is thus determined by Khovanov homology. By using the Dold-Thom functor it can therefore be obtained from the Khovanov homotopy type constructed by Lipshitz and Sarkar.

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A *Khovanov homotopy type* is a way of associating a (stable) space to each link  $L$  so that the classical invariants of the space yield the Khovanov homology of  $L$ . There are two recent constructions of Khovanov homotopy types, using different techniques and giving different results [3, 6]. In [3] homotopy limits were employed to build an  $\Omega$ -spectrum  $\mathbf{X}_\bullet L = \{X_k(L)\}$  with the following properties:

- (i). the homotopy type is a link invariant, and
- (ii). the homotopy groups are Khovanov homology:

$$\pi_i(\mathbf{X}_\bullet(L)) = Kh^{-i}(L).$$

The main goal of this note is to prove the following result.

**Theorem 1.** *Each of the spaces  $X_k(L)$  is homotopy equivalent to a product of Eilenberg-MacLane spaces.*

In [6] the programme of Cohen, Jones and Segal [2] was generalized to produce a suspension spectrum  $\mathcal{X}_{Kh}(L)$  with the following properties:

- (i). the homotopy type is a link invariant, and
- (ii). the reduced cohomology is Khovanov homology:

$$\tilde{H}^i(\mathcal{X}_{Kh}(L)) = Kh^i(L).$$

As a corollary we obtain that  $\mathbf{X}_\bullet(L)$  is homotopy equivalent to the infinite symmetric product of  $\mathcal{X}_{Kh}(L)$ .

To prove Theorem 1 we use the explicit model, due to McCord [8], of the Eilenberg-MacLane spaces. Given a monoid  $G$  and a based topological space  $X$ , let  $B(G, X)$  denote the set of maps

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$u: X \rightarrow G$  such that  $u(x) = 0$  for all but finitely many  $x \in X$ . Then  $B(G, X)$  is a monoid, and if  $G$  is a group (the case of interest) then  $B(G, X)$  is a group. Moreover, when  $G$  is an abelian topological group the set  $B(G, X)$  can be topologized in a natural way so that the group operation is continuous. This construction has nice functoriality: letting  $\text{Ab}$ ,  $\text{Top}_*$  and  $\text{AbTop}$  denote respectively the categories of abelian groups, based topological spaces and topological abelian groups, one has the following result.

**Proposition 1.** [8, Proposition 6.7] *McCord's construction is a bifunctor*

$$B(-, -): \text{Ab} \times \text{Top}_* \rightarrow \text{AbTop}.$$

Furthermore, as special case of [8, Theorem 11.4], for an abelian group  $G$  the space  $B(G, S^n)$  is the Eilenberg-MacLane space  $K(G, n)$ . Thus we may take as *the* Eilenberg-MacLane space functor:

$$B(-, S^n): \text{Ab} \rightarrow \text{AbTop}.$$

Conversely, the following is [4, Corollary 4K.7, p. 483] (apparently originally due to Moore; cf. [8, p. 295]):

**Proposition 2.** *A path-connected, commutative topological monoid is a product of Eilenberg-MacLane spaces.*

The spaces  $X_k(L)$  are built as homotopy limits of diagrams of spaces. Recall that given a small category  $\mathcal{C}$  and a (covariant) functor  $D: \mathcal{C} \rightarrow \text{Top}_*$  (a diagram), that  $\text{holim}_{\mathcal{C}} D$  is constructed as follows (see, e.g., [1, Section 11.5] or the concise notes [9, Section 3.7]). Consider the product

$$\prod_{\sigma \in N(\mathcal{C})} \text{Hom}(\Delta^n, D(c_n)) = \prod_{n \geq 0} \prod_{\substack{c_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} c_n \\ \alpha_i \neq \text{Id}}} \text{Hom}(\Delta^n, D(c_n)) \quad (1)$$

where  $N(\mathcal{C})$  is the subset of the nerve of  $\mathcal{C}$  consisting of all sequences of composable morphisms  $\sigma = (c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} c_n)$  in which none of the morphisms are identity maps, and  $\text{Hom}$  denotes the space of continuous maps from the standard  $n$ -simplex. The homotopy limit  $\text{holim}_{\mathcal{C}} D$  is the subspace of this product consisting of those tuples  $(f_\sigma)_{\sigma \in N(\mathcal{C})}$  such that the following diagrams commute:

$$\begin{array}{ccc} \Delta^{n-1} & \xrightarrow{f_{d_i \sigma}} & D(c_n) \\ d^i \downarrow & & \downarrow \text{Id} \\ \Delta^n & \xrightarrow{f_\sigma} & D(c_n) \end{array} \quad (2)$$

for each  $0 < i < n$ , and

$$\begin{array}{ccc} \Delta^{n-1} & \xrightarrow{f_{d_0 \sigma}} & D(c_n) \\ d^0 \downarrow & & \downarrow \text{Id} \\ \Delta^n & \xrightarrow{f_\sigma} & D(c_n) \end{array} \quad \text{and} \quad \begin{array}{ccc} \Delta^{n-1} & \xrightarrow{f_{d_n \sigma}} & D(c_{n-1}) \\ d^n \downarrow & & \downarrow D(\alpha_n) \\ \Delta^n & \xrightarrow{f_\sigma} & D(c_n) \end{array} \quad (3)$$

corresponding to the cases  $i = 0$  and  $i = n$ , respectively. Here the map  $d^i$  denotes the  $i^{\text{th}}$  face inclusion,  $d_i \sigma = (c_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{i-1}} c_{i-1} \xrightarrow{\alpha_{i+1} \alpha_i} c_{i+1} \dots \xrightarrow{\alpha_n} c_n)$  when  $0 < i < n$ , and  $d_0, d_n$  similarly.

The following is well-known, but for completeness we give its (short) proof.

**Proposition 3.** *Let  $D: \mathcal{C} \rightarrow \text{Top}_*$  be a diagram of topological abelian groups and continuous group homomorphisms. Then the homotopy limit of  $D$  is a topological abelian group.*

*Proof.* Pointwise addition makes the set  $\text{Hom}(\Delta^n, D(c_n))$  into an abelian group, and the product in formula (1) is the product (topological abelian) group. It remains to see that the diagrams (2) and (3) describe a subgroup of this product. Suppose that tuples  $(f_\sigma)$  and  $(g_\sigma)$  make these diagrams commute. Then the first two diagrams automatically commute for the pointwise sum  $(f_\sigma + g_\sigma)$ . The third diagram for the pointwise sum becomes,

$$\begin{array}{ccccccc} \Delta^{n-1} & \longrightarrow & \Delta^{n-1} \times \Delta^{n-1} & \xrightarrow{f_{d_n\sigma} \times g_{d_n\sigma}} & D(c_{n-1}) \times D(c_{n-1}) & \xrightarrow{+} & D(c_{n-1}) \\ \downarrow d^n & & \downarrow d^n \times d^n & & \downarrow D(\alpha_n) \times D(\alpha_n) & & \downarrow D(\alpha_n) \\ \Delta^n & \longrightarrow & \Delta^n \times \Delta^n & \xrightarrow{f_\sigma \times g_\sigma} & D(c_n) \times D(c_n) & \xrightarrow{+} & D(c_n) \end{array}$$

for which the first square obviously commutes, the second commutes since  $f$  and  $g$  are in the prescribed subspace and the third commutes from the fact that  $D(\alpha_n)$  is a group homomorphism. The inverse operation is similarly seen to be closed, hence the subspace defined above is a subgroup.  $\square$

*Proof (Proof of Theorem 1).* Let  $L$  be an oriented link diagram with  $c$  negative crossings. The space  $X_k(L)$  is constructed as follows. Let  $I$  denote the category with objects  $\{0, 1\}$  and a single morphism from 0 to 1, and  $I^n$  the product of  $I$  with itself  $n$  times. Let  $\bar{0}$  be the initial object in  $I^n$ , and let  $\mathbf{P}$  be the result of adjoining one more object to  $I^n$  and a single morphism from the new object to every object except  $\bar{0}$ .

In [3] it is shown that there is a functor  $F: \mathbf{P} \rightarrow \text{Ab}$  such that the  $i^{\text{th}}$  derived functor of the inverse limit,  $\varprojlim_{\mathbf{P}}^i F$ , is isomorphic to the  $i^{\text{th}}$  unreduced Khovanov homology of  $L$ . The space  $X_k(L)$  is constructed by composing this functor with the Eilenberg-MacLane space functor  $K(-, k+c)$  and taking the homotopy limit of the resulting diagram of spaces.

We may now use the explicit model for Eilenberg-MacLane spaces given by McCord. By applying Proposition 1 we define a diagram  $D: \mathbf{P} \rightarrow \text{AbTop}$  as the composition

$$\mathbf{P} \xrightarrow{F} \text{Ab} \xrightarrow{B(-, S^{k+c})} \text{AbTop}.$$

By the homotopy invariance property of the homotopy limit construction we have

$$X_k(L) \simeq \text{holim}_{\mathbf{P}} D.$$

By Proposition 3, the homotopy limit on the right is itself a topological abelian group, and hence, by Proposition 2, a product of Eilenberg-MacLane spaces.  $\square$

**Corollary 1.** *The homotopy type of  $\mathbf{X}_\bullet(L)$  is determined by  $Kh(L)$ .*

The spectrum  $\mathcal{X}_{Kh}(L) = \{\mathcal{X}_{Kh}^{(k)}(L)\}$  constructed in [6] has the additional property that the cellular cochain complex of the space  $\mathcal{X}_{Kh}^{(k)}(L)$  is isomorphic to the Khovanov complex of  $L$  (up to shift). It follows from the description of the Khovanov homology of the mirror image (see [5]) that

$$\tilde{H}_i(\mathcal{X}_{Kh}(L)) = Kh^{-i}(-L)$$

where  $-L$  denotes the mirror of  $L$ . The infinite symmetric product  $\mathrm{Sym}^\infty \mathcal{X}_{Kh}^{(k)}(L)$  is seen from the Dold-Thom theorem to be

$$\mathrm{Sym}^\infty \mathcal{X}_{Kh}^{(k)}(L) = \prod_n K(\tilde{H}_n(\mathcal{X}_{Kh}^{(k)}(L)), n)$$

from which we have the following.

**Corollary 2.** *For large enough  $k$ , the space  $X_k(-L)$  is homotopy equivalent to the infinite symmetric product  $\mathrm{Sym}^\infty \mathcal{X}_{Kh}^{(k)}(L)$ .*

We end by noting that the analogue of Theorem 1 for the spectra  $\mathcal{X}_{Kh}(L)$  is not true. For all alternating knots  $\mathcal{X}_{Kh}(L)$  is a wedge of Moore spaces [6], however there are examples of non-alternating knots for which  $\mathcal{X}_{Kh}(L)$  is not a wedge of Moore spaces (see [7]).

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