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PSEUDOFINITE STRUCTURES AND SIMPLICITY

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We explore a notion of *pseudofinite dimension*, introduced by Hrushovski and Wagner, on an infinite ultraproduct of finite structures. Certain conditions on pseudofinite dimension are identified that guarantee simplicity or supersimplicity of the underlying theory, and that a drop in pseudofinite dimension is equivalent to forking. Under a suitable assumption, a measure-theoretic condition is shown to be equivalent to local stability. Many examples are explored, including vector spaces over finite fields viewed as 2-sorted finite structures, and homocyclic groups. Connections are made to products of sets in finite groups, in particular to word maps, and a generalization of Tao's algebraic regularity lemma is noted.

Keywords: model theory, finite and pseudofinite structures, dimension, independence

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1. Introduction

We investigate a notion of pseudofinite dimension (called *quasifinite dimension* in [12]) that was introduced in [17], and applied by Hrushovski to approximate subgroups in [12] with tantalizing further directions suggested in [13]. For an ultraproduct of finite structures, the *pseudofinite dimension* $\delta(X)$ of a definable set X is defined. It takes values in a quotient of the non-standard reals, rather than in the positive integers or the ordinals, as holds for more standard model-theoretic

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dimensions and ranks. By taking an infimum in an appropriate completion, pseudofinite dimension is also defined for types. Given a definable set X , one obtains also a measure on the collection of its definable subsets which takes value 0 on strictly lower-dimensional subsets.

In the above papers, the main emphasis is on (pseudo)finite substructures of a given infinite structure such as an algebraically closed field or a simple algebraic group. A highlight is an abstract model-theoretic version of the Larsen-Pink inequality from [24], linking e.g. pseudofinite dimension to Zariski dimension.

Here, we explore general conditions on pseudofinite dimension that ensure simplicity, or stability, of the underlying theory, and yield a clear link between pseudofinite dimension and model-theoretic forking. Key conditions, introduced formally in Definition 2.1, are (A), (SA), and (DC_L) . These are all conditions on a pseudofinite structure M that is an ultraproduct of finite structures. Roughly, (A) states that for any formula $\phi(\bar{x}, \bar{y})$ the pseudofinite dimension of a consistent set of positive ϕ -instances (a partial positive ϕ -type) is obtained by a finite conjunction; it has a strengthening (A^*) , where positivity is not required. The global version of (A) is (SA), a condition on an arbitrary partial type. The condition (DC_L) roughly asserts that given an L -formula $\phi(\bar{x}, \bar{y})$, the relation $\delta(\phi(M^r, \bar{y})) < \delta(\phi(M^r, \bar{y}'))$ is definable by an L -formula $\psi(\bar{y}, \bar{y}')$. We write $\bar{a} \downarrow_C^\delta B$ if $\delta(\text{tp}(\bar{a}/B \cup C)) = \delta(\text{tp}(\bar{a}/C))$.

Our main results are as follows, all proved in Section 3.2. The assumptions are on an ultraproduct M of a class of finite structures, though the conditions (A), (SA), and (DC_L) are defined in terms also of a second sort in which counting takes place.

Theorem I. *Assume that (A) holds. Then $\text{Th}(M)$ is simple and low.*

Theorem II. *Assume (SA). Then $\text{Th}(M)$ is supersimple.*

Theorem III. *Assume (SA) and (DC_L) . Then for any \bar{a}, A, B in M with A, B countable,*

$$\bar{a} \downarrow_A B \Leftrightarrow \bar{a} \downarrow_A^\delta B.$$

Working under a local variant of (A), we also characterize in Proposition 3.3 when a formula is stable, in terms of δ and a measure μ_D defined in Section 2.1.

In addition to examples constructed specifically to delimit the conditions, some algebraically natural classes of examples are discussed in Section 4. These include ultraproducts of asymptotic classes of finite structures (from [26] and [5]); the latter include the family of finite fields, every family of finite simple groups of fixed Lie type, and, for any smoothly approximable structure, an appropriately chosen family of envelopes. Ultraproducts of asymptotic classes satisfy (DC_L) and (SA). These conditions also hold for a (2-sorted) ultraproduct of finite vector spaces over finite fields, where in the ultraproduct the vector space dimension and the field are both

infinite; unlike with asymptotic classes, the SU rank here is infinite in the vector space sort. An analogue in this setting (Theorem 4.2 below) is given of the main theorem of [3]; we view this as a first example of a multi-sorted and infinite-rank enrichment of the notion of asymptotic class. We also give a uniformity result for exact (rather than asymptotic) cardinalities of definable sets in finite homocyclic groups (direct sums of isomorphic cyclic p -groups); see Proposition 4.4. This yields an example of an abelian group satisfying (A) and (DC_L) but not (SA). The second and third author, in conjunction with W. Anscombe and D. Wood, have work in progress on a rather flexible multi-sorted generalization of asymptotic classes, which should yield many more examples.

The paper is organized as follows. The framework, basic definitions, and easy observations around them are given in Section 2. In Section 3 the main theorems stated above are proved. Section 4 focuses on examples, including the proof of Theorem 4.2. Section 5 consists of some potentially useful technical results involving our conditions. These include the transfer of the conditions to M^{eq} , one-variable criteria for the conditions, and results concerning a pregeometry defined via pseudofinite dimension (Propositions 5.3 and 5.4). Finally, in Section 6, we consider two possible lines of application of the results and framework. We note (Theorem 6.4) that the Pillay-Starchenko generalization of Tao’s Algebraic Regularity Lemma holds under the assumptions (SA), (DC_L) , and a corresponding definability condition for measure, and we also consider a possible application of our results to pseudofinite groups (Theorem 6.1). Open problems are mentioned throughout the paper.

Lastly we note that around the time that this paper was submitted for publication, the paper [7] was posted on arXiv. While the results are different, there is some common ground: it is shown there that *any* countable structure has an elementary extension with a close analogue of our dimension δ that detects dividing.

2. Preliminaries

2.1. Pseudofinite structures, dimension, and measures

We adopt the context of [12] Section 5, which extends [17]. We summarize it briefly.

We fix a countable first order language L , and consider L -formulas $\phi(\bar{x}, \bar{y})$, with the convention that $l(\bar{x}) = r$ and $l(\bar{y}) = s$. An *instance* of ϕ (in an L -structure M) is a formula $\phi(\bar{x}, \bar{a})$, for $\bar{a} \in M^s$. A *ϕ -formula* is a Boolean combination of instances of ϕ . Parameter sets, always countable in this paper, are usually denoted by A, B , possibly with subscripts. Given a parameter set B contained in a model M of a theory T , we write $S_r(B)$ for the space of r -types of T over B . A *partial positive ϕ -type* is a set of formulas $\{\phi(x, \bar{a}) : \bar{a} \in M^s\}$ that is consistent with $\text{Th}(M, m)_{m \in M}$. A *partial ϕ -type* over a parameter set is a consistent set of ϕ -formulas over that set. For a partial type p over B and $B_0 \subset B$, we let $p|_{B_0}$ denote the restriction of p to B_0 .

Let \mathcal{C} be a class of finite L -structures. It is possible to extend L to a language L^+ that includes a sort \mathbf{D} carrying the L -structure, a sort \mathbf{OF} carrying the language of

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ordered fields, and maps taking finite definable sets to their cardinalities. Formally, for each L -formula $\phi(\bar{x}, \bar{y})$ with $\bar{x} = (x_1, \dots, x_r)$ and $\bar{y} = (y_1, \dots, y_s)$, the language L^+ has a function $f_\phi : \mathbf{D}^s \rightarrow \mathbf{OF}$. Each structure $M_i \in \mathcal{C}$ gives a 2-sorted structure K_i in L^+ , the second sort being a copy of $(\mathbb{R}, <, +, \cdot, -, 0, 1, \log)$, and for $\phi(\bar{x}, \bar{y})$ as above and $\bar{a} \in M_i^s$, we put $f_\phi(\bar{a}) = |\phi(M_i^r, \bar{a})|$. Let $\mathcal{C}^+ := \{K_i : M_i \in \mathcal{C}\}$ and denote by $K^* = K^*(\mathcal{C})$ an ultraproduct of the members of \mathcal{C}^+ in L^+ , over a non-principal ultrafilter \mathcal{U} . Here K^* is 2-sorted with a sort \mathbf{OF} consisting of a non-archimedean real closed field \mathbb{R}^* , and a sort \mathbf{D} consisting of the ultraproduct of the M_i .

We fix the above context for the rest of the paper. So, for emphasis: \mathcal{C} is an infinite class of finite L -structures, \mathcal{C}^+ is the corresponding class of 2-sorted L^+ -structures in the sorts \mathbf{D} and \mathbf{OF} , \mathcal{U} is a non-principal ultrafilter on \mathcal{C}^+ , and $K^* = (M^+, \mathbb{R}^*)$ is an ultraproduct of the structures in \mathcal{C}^+ with respect to \mathcal{U} . Let M denote the induced structure of K^* in the \mathbf{D} -sort in the language L , and put $T := \text{Th}(M)$ and $T_M := \text{Th}(M, (m)_{m \in M})$. We usually refer just to the structure M and theory T , without explicit reference to the 2-sorted context. All formulas and types are with respect to the language L , unless otherwise stated. Parameter subsets of M are denoted by A, B etc., and are assumed to be countable.

This construction might be seen as an explicit construction of the Cardinality Comparison Quantifiers CCx defined by Hrushovski in [13]. Indeed, with the notation used in [13], we obtain that

$$\begin{aligned} M \models (CCx \phi)(b, b') &\Leftrightarrow |\phi(M_i, b_i)| \geq |\phi(M_i, b'_i)| \text{ for } \mathcal{U}\text{-almost all } i \\ &\Leftrightarrow K^* \models f_\phi(b) \geq f_\phi(b'). \end{aligned}$$

Let C be the convex hull of the integers \mathbb{Z} in the additive group of \mathbb{R}^* . Note that C is a convex subgroup of \mathbb{R}^* . If X is an L -definable set in M , say $X = \phi(M^r, \bar{a})$, we write $|X|$ for $f_\phi(\bar{a})$; observe that this is well-defined, that is, independent of the choice of ϕ and \bar{a} . Then define the *pseudofinite dimension of X* to be $\delta(X) := \log |X| + C \in \mathbb{R}^*/C$. For nonempty definable sets X, Y in M we have

$$\begin{aligned} \delta(X) = \delta(Y) &\Leftrightarrow \log |X| - \log |Y| \in C \Leftrightarrow \log(|X|/|Y|) \in C \\ &\Leftrightarrow -m \leq \log(|X|/|Y|) \leq m \text{ for some } m \in \mathbb{N}^{>0} \\ &\Leftrightarrow \frac{1}{n} \leq \frac{|X|}{|Y|} \leq n \text{ for some } n \in \mathbb{N}^{>0}. \end{aligned}$$

We write $\delta(\phi(\bar{x}, \bar{b}))$ for $\delta(X)$ where X is the set defined by the formula (with parameters) $\phi(\bar{x}, \bar{b})$. In general the map $\bar{a} \mapsto \delta(\phi(\bar{x}, \bar{a}))$ is not definable even in L^+ , since C and hence \mathbb{R}^*/C are not definable.

The map δ is extended to infinitely definable sets in [12]. For $\epsilon \in \mathbb{R}^*$, chosen sufficiently large and with $\epsilon > C$, put

$$V_0 = V_0(\epsilon) := \{a \in \mathbb{R}^*/C : -n\epsilon + C \leq a \leq n\epsilon + C \text{ for some } n \in \mathbb{N}\}.$$

Let $V = V(\epsilon)$ be the set of cuts in V_0 , i.e., nonempty subsets bounded above and closed downwards. Then V is a semigroup under set addition, and V_0 is identified

with its image in V . For a \wedge -definable set X , define

$$\delta(X) := \inf\{\delta(D) : D \supset X, D \text{ definable}\},$$

the infimum evaluated in $V(\epsilon)$ for sufficiently large ϵ . Given $B \subset M$ and a tuple \bar{a} from M , $\delta(\bar{a}/B)$ denotes $\delta(\text{tp}(\bar{a}/B))$, and $\delta^\phi(\bar{a}/B)$ denotes $\delta(\text{tp}^\phi(\bar{a}/B))$, that is, the dimension of the corresponding partial positive ϕ -type.

Hrushovski describes in [12, 13] different pseudofinite dimensions that can be obtained using the same construction of logarithms and taking quotients by different convex subgroups of \mathbb{R}^* . We focus in this paper on the pseudofinite dimension described above, also referred to as $\delta = \delta_{\text{fin}}$ in [13]. We call δ a *pseudofinite dimension* because its definition takes place in a pseudofinite context—an ultraproduct of finite structures—and it has some of the properties that we would expect for a dimension operator. For instance, the following are noted in [17] and [12].

Lemma 2.1.

- (i) $\delta(\emptyset) = -\infty$, and $\delta(X) = 0$ for any finite definable set X .
- (ii) If X_1, X_2 are \wedge -definable, then $\delta(X_1 \cup X_2) = \max\{\delta(X_1), \delta(X_2)\}$.
- (iii) If X_1, X_2 are \wedge -definable, then $\delta(X_1 \times X_2) = \delta(X_1) + \delta(X_2)$.
- (iv) If $(\alpha_n), (\beta_n)$ are descending sequences of cuts in V_0 , then $\inf_n(\alpha_n + \beta_n) = \inf_n \alpha_n + \inf_n \beta_n$.
- (v) If $\alpha, \alpha', \beta, \beta' \in V$ with $\alpha < \alpha'$ and $\beta < \beta'$ then $\alpha + \beta < \alpha' + \beta'$.
- (vi) If $X = \bigcap X_n$ with $X_1 \supset X_2 \supset \dots$ all \wedge -definable, then $\delta(X) = \inf_n \delta(X_n)$.
- (vii) If X is \wedge -definable, f is a definable map, $\gamma \in V_0$, and $\delta(f^{-1}(a) \cap X) \leq \gamma$ for all a , then $\delta(X) \leq \delta(f(X)) + \gamma$.

Given a fixed definable subset D of M , there is a finitely additive real-valued probability measure μ_D on definable sets X given by $\mu_D(X) := \text{st} \left(\lim_{i \rightarrow \mathcal{U}} \frac{|X(M_i) \cap D(M_i)|}{|D(M_i)|} \right)$, where $\text{st}(\cdot)$ is the standard part map. This measure can be extended to yield a countably-additive probability measure on the σ -algebra generated by the definable subsets of M , and thus is defined on the \wedge -definable subsets of M . This measure combines with δ in our characterization of when a formula is stable – see Proposition 3.3.

2.2. Conditions on the pseudofinite dimension

We investigate the following hypotheses on δ . Throughout, we work in the context described in Section 2.1, with M the L -structure induced on the sort \mathbf{D} by an infinite ultraproduct of finite L^+ -structures.

Definition 2.1.

- (1) *Attainability* (A_ϕ). There is no sequence $(p_i : i \in \omega)$ of finite partial positive ϕ -types such that $p_i \subseteq p_{i+1}$ (as sets of formulas) and $\delta(p_i) > \delta(p_{i+1})$ for each $i \in \omega$. We denote by (A_ϕ^*) the corresponding (stronger) condition, where

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the above is assumed for all increasing sequences of finite partial ϕ -types, not necessarily positive.

- (2) *Strong Attainability* (SA). For each partial type $p(\bar{x})$ over a parameter set B , there is a finite subtype p_0 of p such that $\delta(p(\bar{x})) = \delta(p_0(\bar{x}))$.
- (3) *Weak Order Definability* (WOD $_\phi$). There is $n = n_\phi \in \mathbb{N}$ such that for all $\bar{a}, \bar{b} \in M^s$,

$$\delta(\phi(\bar{x}, \bar{a})) = \delta(\phi(\bar{x}, \bar{b})) \Leftrightarrow \frac{1}{n} < \frac{|\phi(\bar{x}, \bar{a})|}{|\phi(\bar{x}, \bar{b})|} \leq n.$$

- (4) *Dimension Comparison in L^+* (DC $_{L^+}$). For all formulas $\phi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{z})$ (with $t = l(\bar{z})$), there is an L^+ -formula $\chi_{\phi, \psi}(\bar{y}, \bar{z})$ such that for all $\bar{a} \in M^s$ and $\bar{b} \in M^t$,

$$\chi_{\phi, \psi}(\bar{a}, \bar{b}) \Leftrightarrow \delta_{\bar{x}}(\phi(\bar{x}, \bar{a})) \leq \delta_{\bar{x}}(\psi(\bar{x}, \bar{b})).$$

- (5) *Dimension Comparison in L* (DC $_L$). This is as for (DC $_{L^+}$), except that the formula $\chi_{\phi, \psi}$ is chosen in L .
- (6) *Finitely Many Values* (FMV $_\phi$). There is a finite set $\{\delta_1, \dots, \delta_k\}$ such that for each $\bar{b} \in M^s$ there is $i \in \{1, \dots, k\}$ with $\delta(\phi(M^r, \bar{b})) = \delta_i$.

The conditions (A $_\phi$), (A $^*_\phi$), (WOD $_\phi$) and (FMV $_\phi$) have global versions (A), (A *), (WOD) and (FMV), where they are assumed to hold for all ϕ (with k and the δ_i in (FMV), and n in (WOD), dependent on ϕ). They can also be formulated for finite sets Δ of formulas, and it would be helpful to know to what extent they—in particular, local versions of (A)—are preserved under taking Boolean combinations of formulas.

We conclude this section with some easy observations about these conditions. Note that

$$(SA) \Rightarrow (A^*_\phi) \text{ for all } \phi \Rightarrow (A).$$

We also have

Lemma 2.2.

- (i) For every formula $\phi(\bar{x}, \bar{y})$, the conditions $((A_\phi) \wedge (A_{\neg\phi}) \wedge (A_{\phi(\bar{x}, \bar{y}_1) \wedge \neg\phi(\bar{x}, \bar{y}_2)}))$ and (A $^*_\phi$) are equivalent.
- (ii) The conditions (A) and (A *) are equivalent.

Proof. (i) The implication (\Leftarrow) is immediate. For the direction (\Rightarrow), suppose (A $^*_\phi$) fails, witnessed by an infinite increasing sequence of finite partial ϕ -types $p_1 \subset p_2 \subset \dots$ with $\delta(p_1) > \delta(p_2) > \dots$. If the p_i do not involve $\neg\phi$ then (A $_\phi$) fails, and if ϕ does not occur positively in the p_i then (A $_{\neg\phi}$) fails. Otherwise, we may suppose both ϕ and $\neg\phi$ occur in p_1 . It is now easy to construct an increasing sequence of positive $(\phi(\bar{x}, \bar{y}_1) \wedge \neg\phi(\bar{x}, \bar{y}_2))$ -types with strictly decreasing δ -values, repeating some parameters if needed.

- (ii) This is immediate from (i). □

We shall make little use of (WOD) and (DC_{L^+}) , but we note:

Lemma 2.3. *The conditions (WOD) and (DC_{L^+}) are equivalent.*

Proof. $(DC_{L^+}) \Rightarrow (WOD)$. By compactness and ω_1 -saturation of M , (DC_{L^+}) implies (WOD). Indeed, assume (DC_{L^+}) , and that for all $n \in \mathbb{N}$ there are $\bar{a}_n, \bar{b}_n \in M^s$ with

$$\delta(\phi(\bar{x}, \bar{a}_n)) = \delta(\phi(\bar{x}, \bar{b}_n)) \quad \text{and} \quad \frac{|\phi(\bar{x}, \bar{a}_n)|}{|\phi(\bar{x}, \bar{b}_n)|} < \frac{1}{n}.$$

Then the set of L^+ -formulas

$$\{\chi_{\phi, \phi}(\bar{y}_1, \bar{y}_2), \chi_{\phi, \phi}(\bar{y}_2, \bar{y}_1)\} \cup \left\{ \frac{|\phi(\bar{x}, \bar{y}_1)|}{|\phi(\bar{x}, \bar{y}_2)|} < \frac{1}{n} : n \in \omega \right\}$$

is consistent. This is impossible, by our initial remarks on pseudofinite dimension.

$(WOD) \Rightarrow (DC_{L^+})$. Fix formulas $\phi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{z})$. We may regard the union of the family of sets defined by ϕ and the family of sets defined by ψ , as a single uniformly definable family of sets, defined e.g. by the formula $\rho(\bar{x}, \bar{y}\bar{z}w_1w_2)$ which has form

$$((w = w_1 \wedge w_1 \neq w_2) \rightarrow \phi(\bar{x}, \bar{y})) \wedge (w = w_2 \wedge w_1 \neq w_2) \rightarrow \psi(\bar{x}, \bar{z})$$

$$\wedge ((w_1 = w_2 \vee (w \neq w_1 \wedge w \neq w_2)) \rightarrow (\neg\phi(\bar{x}, \bar{y}) \wedge \neg\psi(\bar{x}, \bar{z}))).$$

By (WOD), there is a number n_ρ associated with ρ . Now $\delta(\phi(\bar{x}, \bar{a})) \leq \delta(\psi(\bar{x}, \bar{b}))$ holds if and only if

$$\exists w_1, w_2 (w_1 \neq w_2 \wedge |\rho(\bar{x}, \bar{a}\bar{b}w_1w_2)| \leq n_\rho |\rho(\bar{x}, \bar{a}\bar{b}w_2w_1)|). \quad \square$$

We also note that the following strengthening of Lemma 2.1(vii) holds for definable sets. We omit the proof – it is an easy counting argument in finite structures.

Lemma 2.4.

- (i) *Let $f : X \rightarrow Y$ be a definable map in M between definable sets X, Y , and suppose that there is a positive integer n such that for all $\bar{a}, \bar{b} \in Y$, $\frac{1}{n}|f^{-1}(\bar{b})| \leq |f^{-1}(\bar{a})| \leq n|f^{-1}(\bar{b})|$ (non-standard cardinalities). Then for all $\bar{a} \in Y$, we have $\delta(X) = \delta(Y) + \delta(f^{-1}(\bar{a}))$.*
- (ii) *Assume M satisfies (WOD). Let X be a definable set in M , f a definable map, and suppose that $\delta(f^{-1}(\bar{a})) = \gamma$ for all $\bar{a} \in f(X)$. Then $\delta(X) = \delta(f(X)) + \gamma$.*

Lemma 2.5. *Assume that (A_ϕ) holds. Then there is $m = m_\phi \in \omega$ such that there do not exist $\bar{a}_1, \dots, \bar{a}_m \in M^s$ so that if $p_i := \{\phi(\bar{x}, \bar{a}_j) : j \leq i\}$ then p_i is consistent and $\delta(p_1) > \delta(p_2) > \dots > \delta(p_m)$.*

Proof. Suppose not. Then for every $N \in \omega$ there are $\bar{a}_1, \dots, \bar{a}_N$ such that

$$|\phi(\bar{x}, \bar{a}_1) \wedge \dots \wedge \phi(\bar{x}, \bar{a}_i)| > N |\phi(\bar{x}, \bar{a}_1) \wedge \dots \wedge \phi(\bar{x}, \bar{a}_{i+1})|$$

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for each $i = 1, \dots, N$. Each such statement is a partial type in L^+ . It follows by compactness and ω_1 -saturation of K^* that there are $\bar{a}_i \in M^s$ for all $i > 0$ such that for each $i, N \in \omega$, we have

$$|\phi(\bar{x}, \bar{a}_1) \wedge \dots \wedge \phi(\bar{x}, \bar{a}_i)| > N |\phi(\bar{x}, \bar{a}_1) \wedge \dots \wedge \phi(\bar{x}, \bar{a}_{i+1})|.$$

Putting $p_i := \{\phi(\bar{x}, \bar{a}_j) : j \leq i\}$, we obtain an infinite sequence $(p_i)_{i > 0}$ of finite partial positive ϕ -types with $\delta(p_i) > \delta(p_{i+1})$ for each i , contrary to (A). \square

Lemma 2.6. *Assume (SA) holds. Then there is no sequence of definable sets $(S_n : n < \omega)$ such that $S_{n+1} \subseteq S_n$ and $\delta(S_{n+1}) < \delta(S_n)$ for each $n < \omega$.*

Proof. Suppose S_i is defined by the formula $\phi_i(\bar{x}; \bar{b}_i)$ and consider the partial type

$$p := \{\phi_i(\bar{x}; \bar{b}_i) : i < \omega\}$$

By (SA), there is $m < \omega$ such that $\delta(p) = \delta(\phi_m(\bar{x}, \bar{b}_m))$, which is impossible because $\delta(p) \leq \delta(S_{m+1}) < \delta(S_m)$. \square

Lemma 2.7. *Assume (DC_L) , and suppose (FMV_ϕ) fails for some ϕ . Then T has the strict order property, so in particular is not simple.*

Proof. Under these assumptions, there is a preorder with an infinite chain definable on M^s , where we put $\bar{y} \leq \bar{y}' \Leftrightarrow \delta(\phi(\bar{x}, \bar{y})) \leq \delta(\phi(\bar{x}, \bar{y}'))$. \square

Proposition 2.1.

- (i) *Assume $\text{Th}(M)$ has the strict order property. Then (FMV) fails.*
- (ii) *If M has (DC_L) and $\text{Th}(M)$ has elimination of imaginaries (EI), then (FMV) holds, and so $\text{Th}(M)$ does not have the strict order property.*

Proof. (i) Let $\psi(\bar{u}, \bar{v})$ be a formula defining a preorder \preceq on M^t with an infinite chain. Find by ω_1 -saturation an infinite L^+ -indiscernible sequence $(\bar{a}_i : i \in \omega)$ in M^t , with $\bar{a}_i \prec \bar{a}_j$ if and only if $i < j$, for $i, j \in \omega$. Let $\chi(\bar{x}, \bar{u}\bar{v})$ express $\bar{u} \prec \bar{x} \prec \bar{v}$. Clearly, for any $n > 0$, $f_\chi(\bar{a}_0 \bar{a}_n) \geq n f_\chi(\bar{a}_0 \bar{a}_1)$, where f_χ is the L^+ -function symbol corresponding to χ ; indeed, $\chi(M^t, \bar{a}_0 \bar{a}_n) \supseteq \bigcup_{i=0}^{n-1} \chi(M^t, \bar{a}_i \bar{a}_{i+1})$ and $f_\chi(\bar{a}_i \bar{a}_{i+1}) = f_\chi(\bar{a}_0 \bar{a}_1)$ for each i . Hence by L^+ -indiscernibility $f_\chi(\bar{a}_0 \bar{a}_2) \geq n f_\chi(\bar{a}_0 \bar{a}_1)$ for each n , and so $\delta(\chi(M^t, \bar{a}_0 \bar{a}_2)) > \delta(\chi(M^t, \bar{a}_0 \bar{a}_1))$. The same argument yields that $\delta(\chi(M^t, \bar{a}_0 \bar{a}_{n+1})) > \delta(\chi(M^t, \bar{a}_0 \bar{a}_n))$, for each $n > 0$. It follows that the set $\{\delta(\chi(M^t, \bar{a}_0 \bar{a}_i)) : i > 0\}$ is infinite.

(ii) Assume (DC_L) and EI, and suppose for a contradiction that $\{\delta(\phi(M^r, \bar{a})) : \bar{a} \in M^s\}$ is infinite. Let $\psi(\bar{u}, \bar{v})$ express that $\delta(\phi(M^r, \bar{u})) \leq \delta(\phi(M^r, \bar{v}))$. Then ψ defines a total preorder on M^s . Let E be the equivalence relation defined by putting $E(\bar{u}, \bar{v}) \Leftrightarrow (\psi(\bar{u}, \bar{v}) \wedge \psi(\bar{v}, \bar{u}))$. Then ψ induces on M^s/E an \emptyset -definable total order $<$. By elimination of imaginaries, for some t there is an \emptyset -definable function $g : M^s \rightarrow M^t$ with $E(\bar{u}, \bar{v}) \Leftrightarrow (g(\bar{u}) = g(\bar{v}))$. There is then an \emptyset -definable

total order \prec on the infinite definable set $I := g(M^s)$, given by $\bar{a} \prec \bar{b}$ if and only if $g^{-1}(\bar{a}) < g^{-1}(\bar{b})$. Since I is pseudofinite we may find a sequence of subintervals $I \supset J_0 \supset J_1 \supset J_2 \supset \dots$ with $|J_i| = 2|J_{i+1}|$ (non-standard cardinalities). Since intervals are uniformly definable, this contradicts (WOD), and hence (DC_L) by Lemma 2.3. The final assertion now follows from (i). \square

In Section 5 we say more about imaginaries, in particular about the preservation of our conditions when passing to M^{eq} .

Example 2.1. It is easy to produce examples satisfying (A) or (SA) but without (FMV). Let L be a language with a single binary relation E , and let M_k be an L -structure with $\sum_{i=1}^k i^2$ elements, in which E is interpreted by an equivalence relation with a class of size i^2 for each $i = 1, \dots, k$. It can be shown that any non-principal ultraproduct M satisfies (SA) but not (DC_{L+}) or (FMV). See also Example 4.5.

The following measure-theoretic result will be applied to the measures μ_D in the proof of Proposition 3.3 in Section 3.

Proposition 2.2. *Let X be a measure space with $\mu(X) = 1$ and fix $0 < \epsilon \leq \frac{1}{2}$. Let $\langle A_i : i < \omega \rangle$ be a sequence of measurable subsets of X such that $\mu(A_i) \geq \epsilon$ for every i . Then, for every $k < \omega$ there are indices $i_1 < i_2 < \dots < i_{2^k}$ such that*

$$\mu \left(\bigcap_{j=1}^{2^k} A_{i_j} \right) \geq \epsilon^{3^k}.$$

Proof. The proof is by induction on k .

$k = 1$. Assume the conclusion is false. Then $\mu(A_i \cap A_j) < \epsilon^3$ for all choices of $2^1 = 2$ indices $i < j$. By the truncated Inclusion-Exclusion Principle we know for every $N \in \mathbb{N}$ that

$$1 \geq \mu \left(\bigcup_{i=1}^N A_i \right) \geq \sum_{i=1}^N \mu(A_i) - \sum_{1 \leq i < j \leq N} \mu(A_i \cap A_j) \geq N\epsilon - \frac{N(N-1)}{2} \epsilon^3 \quad (\dagger)$$

$$\text{Define } f(x) := x \cdot \epsilon - \frac{x(x-1)}{2} \epsilon^3 = -\frac{x^2}{2} \epsilon^3 + x \left(\epsilon + \frac{\epsilon^3}{2} \right).$$

This function achieves its maximum value at $x_0 = \frac{1}{\epsilon^2} + \frac{1}{2} > 0$, and by taking

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any integer $N \in [x_0 - 1, x_0]$ we have that

$$\begin{aligned} f(N) \geq f(x_0 - 1) &= \left(\frac{1}{\epsilon^2} - \frac{1}{2}\right) \epsilon - \frac{\left(\frac{1}{\epsilon^2} - \frac{1}{2}\right) \left(\frac{1}{\epsilon^2} - \frac{3}{2}\right)}{2} \cdot \epsilon^3 \\ &= \frac{1}{2\epsilon} + \frac{\epsilon}{2} - \frac{3}{8}\epsilon^3 \\ &\geq 1 + \epsilon \left(\frac{1}{2} - \frac{3}{8}\epsilon^2\right) \quad \left[\text{because } \epsilon \leq \frac{1}{2}\right] \\ &> 1. \end{aligned}$$

contradicting (†).

Induction Step. By induction hypothesis, we can assume that there is a tuple (i_1, \dots, i_{2^k}) satisfying

$$i_1 < \dots < i_{2^k} \text{ and } \mu \left(\bigcap_{j=1}^{2^k} A_{i_j} \right) \geq \epsilon^{3^k}. \quad (*)$$

We now claim that there are infinitely many such pairwise disjoint tuples. Indeed, if not, take $\alpha_1, \dots, \alpha_t$ be a maximal set of disjoint tuples of the form (i_1, \dots, i_{2^k}) with $i_1 < \dots < i_{2^k}$ satisfying (*) and let ℓ be the maximum of all indices appearing in these tuples. Then $\langle A_j : j \geq \ell + 1 \rangle$ is a sequence contradicting induction hypothesis.

Now let $\langle \alpha_j : j < \omega \rangle$ be an enumeration of infinitely many pairwise disjoint tuples satisfying (*) and put $B_j = \bigcap_{i \in \alpha_j} A_i$. By construction, $\mu(B_j) \geq \epsilon^{3^k}$ for all j .

By the $k = 1$ case, there are indices $j_1 \neq j_2$ such that

$$\mu(B_{j_1} \cap B_{j_2}) \geq (\epsilon^{3^k})^3 = \epsilon^{3^k \cdot 3} = \epsilon^{3^{k+1}},$$

where j_1 and j_2 correspond to the disjoint tuples α_{j_1} and α_{j_2} . In particular, there are $2^k + 2^k = 2^{k+1}$ indices $i_1 < i_2 < \dots < i_{2^{k+1}}$ such that

$$\mu \left(\bigcap_{j=1}^{2^{k+1}} A_{i_j} \right) = \mu(B_{j_1} \cap B_{j_2}) \geq \epsilon^{3^{k+1}} \quad \square$$

Remark 2.1. 1. In the $k = 1$ case of the proof above, we actually find a positive integer $N = N(\epsilon)$ such that if A_1, \dots, A_N have measure at least ϵ , there are $1 \leq i < j \leq N$ such that $\mu(A_i \cap A_j) \geq \epsilon^3$.

2. The proof of the proposition goes through with the bound ϵ^{3^k} in the conclusion replaced by ϵ^{α^k} , where it is only required that $\alpha > 2$ and $0 < \epsilon^{\alpha-2} \leq \frac{1}{2}$ (instead of $0 < \epsilon \leq \frac{1}{2}$).

3. For the measures μ_D , it is possible to provide a “pseudofinite” proof of Proposition 2.2 that uses finite combinatorics and counting transferred to the ultraproduct via the functions f_ϕ .

3. Forking independence and δ -independence

Here we define and investigate δ -independence, and prove Theorems I–III stated in the introduction. We also give a criterion for stability of a formula, Proposition 3.3. Our context is that of the whole paper, with \mathcal{C} , \mathcal{C}^+ , \mathcal{U} , K^* , M , and T as described in Section 2.1.

3.1. Properties of δ -independence.

Definition 3.1. Let \bar{a} be a tuple and A, B be countable subsets of M . We say that \bar{a} is δ -independent of B over A , written $a \downarrow_A^\delta B$, if $\delta(\bar{a}/AB) = \delta(\bar{a}/A)$; here, as usual, AB is an abbreviation for $A \cup B$. We only work with \downarrow^δ within the particular model M .

Remark 3.1. With \bar{a}, B, A as in Definition 3.1, $\bar{a} \not\downarrow_A^\delta B$ if and only if there is a formula $\theta(\bar{x}) \in \text{tp}(\bar{a}/AB)$ such that for all $\psi(\bar{x}) \in \text{tp}(\bar{a}/A)$ we have $\delta(\theta(\bar{x})) < \delta(\psi(\bar{x}))$. The direction (\Rightarrow) is immediate by Lemma 2.1(vi). The direction (\Leftarrow) requires a small compactness and saturation argument, using essentially that an element of \mathbb{R}^*/\mathcal{C} cannot have a countably infinite initiality. More generally, if p is a complete type over a countably infinite set and there is a formula θ with $\delta(p) = \delta(\theta)$, then there is $\psi \in p$ with $\delta(p) = \delta(\psi)$.

We investigate below the extent to which δ -independence satisfies standard properties of non-forking. Throughout this section, we write \downarrow for the usual forking independence relation.

Lemma 3.1. [Additivity] *Assume (DC_L) and (FMV) , and let A be a countable set of parameters from $M \models T$. Let $\bar{a} \in M^r, \bar{b} \in M^s$. Then $\delta(\bar{a}\bar{b}/A) = \delta(\bar{a}/A\bar{b}) + \delta(\bar{b}/A)$.*

Proof. Since A is countable it plays no role in the proof and we thus may suppress parameters and suppose $A = \emptyset$. Let $(\phi_n(\bar{y}))_{n \in \omega}$ enumerate the formulas in $\text{tp}(\bar{b})$ and $(\psi_n(\bar{x}, \bar{b}))_{n \in \omega}$ enumerate those in $\text{tp}(\bar{a}/\bar{b})$. We may suppose that for each $n \in \omega$, $\phi_{n+1} \rightarrow \phi_n$ and $\psi_{n+1} \rightarrow \psi_n$. Let P be the set of realizations in M of $\text{tp}(\bar{a}\bar{b})$.

Put $\epsilon_n := \delta(\phi_n(\bar{y}))$ and $\gamma_n := \delta(\psi_n(\bar{x}, \bar{b}))$. For each n there is $\rho_n(\bar{y})$ over \emptyset which expresses that $\delta(\{\bar{x} : \psi_n(\bar{x}, \bar{y})\}) = \gamma_n$. Indeed, by (FMV) , $\delta(\psi_n(M^r, \bar{b}'))$ takes finitely many values as \bar{b}' varies, and by (DC_L) we may compare these and thus express that the j^{th} such value is taken. There is $f(n) \geq n$ such that $\phi_{f(n)}(\bar{y}) \rightarrow \rho_n(\bar{y})$, and by refining the sequence $(\phi_n)_{n \in \omega}$ we may suppose that $\phi_n \rightarrow \rho_n$. Let P_n be the set defined by $\phi_n(\bar{y}) \wedge \psi_n(\bar{x}, \bar{y})$.

We claim that $\delta(P_n) = \epsilon_n + \gamma_n$ for each n . Indeed, for a fixed $N > 0$ dependent on n , by counting in finite structures we have

$$\frac{1}{N} |\psi_n(\bar{x}, \bar{b})| \cdot |\phi_n(\bar{y})| \leq |P_n| \leq N |\psi_n(\bar{x}, \bar{b})| \cdot |\phi_n(\bar{y})|.$$

The claim now follows by taking logarithms and working modulo C (see Lemma 2.4(i)).

Also, $P = \bigcap_n P_n$. By Lemma 2.1(vi),

$$\delta(P) = \inf_n \delta(P_n) = \inf_n (\epsilon_n + \gamma_n).$$

Put $\epsilon := \inf_n \epsilon_n = \delta(\text{tp}(\bar{b}))$ and $\gamma := \inf_n \gamma_n = \delta(\text{tp}(\bar{a}/\bar{b}))$. Then by Lemma 2.1(iv), $\epsilon + \gamma = \inf_n (\epsilon_n + \gamma_n)$, which equals $\delta(P)$, as required. \square

Proposition 3.1. *The following are properties of the δ -independence relation (with A, B, D all countable):*

- (i) Existence: *Given sets $A \subset B$ and $p \in S_r(A)$ (for any $r \in \mathbb{N}$) there is $\bar{a} \models p$ with $\bar{a} \downarrow_A^\delta B$.*
- (ii) Monotonicity and transitivity: *If $A \subset D \subset B$, then*

$$\bar{a} \downarrow_A^\delta B \Leftrightarrow \left(\bar{a} \downarrow_A^\delta D \text{ and } \bar{a} \downarrow_D^\delta B \right).$$

- (iii) Finite character: *If $\bar{a} \not\downarrow_A^\delta B$ then there is a finite subset $\bar{b} \subseteq B$ such that $\bar{a} \not\downarrow_A^\delta \bar{b}$.*

Proof. *Monotonicity and transitivity.* These follow immediately from the definitions.

Existence. This is easy to prove by compactness. We must show that given a partial type q over B and a formula $\phi(\bar{x}, \bar{b})$ over B , if $\delta(q) = \delta_0$ then either $\delta(q \cup \{\phi(\bar{x}, \bar{b})\}) = \delta_0$ or $\delta(q \cup \{\neg\phi(\bar{x}, \bar{b})\}) = \delta_0$. If this were to fail then there would be $\psi \in q$ such that $\delta(\psi \wedge \phi) < \delta_0$ and $\delta(\psi \wedge \neg\phi) < \delta_0$. Since $\delta(\psi) \geq \delta_0$, this contradicts Lemma 2.1(ii).

Finite character. Suppose that $\bar{a} \not\downarrow_A^\delta B$. Then $\delta(\bar{a}/AB) < \delta(\bar{a}/A)$, so there is a formula $\phi(\bar{x}, \bar{b})$ over B such that $\delta(\text{tp}(\bar{a}/A) \cup \{\phi(\bar{x}, \bar{b})\}) < \delta(\bar{a}/A)$. Then $\bar{a} \not\downarrow_A^\delta \bar{b}$. \square

Proposition 3.2. *Under the further assumptions listed below, we have (again with A, B assumed to be countable):*

- (iv) Local character: (SA) *For every \bar{a} and $B \subseteq M$, there is a finite subset $A \subseteq B$ such that*

$$\bar{a} \downarrow_A^\delta B$$

- (v) Invariance: (DC_L) *If $\alpha \in \text{Aut}(M)$, then $\bar{a} \downarrow_A^\delta B \Leftrightarrow \alpha(\bar{a}) \downarrow_{\alpha(A)}^\delta \alpha(B)$*
- (vi) Symmetry: (DC_L) and (FMV) *$\bar{a} \downarrow_A^\delta \bar{b}$ if and only if $\bar{b} \downarrow_A^\delta \bar{a}$*
- (vii) Algebraic closure: (DC_L) *If $A \subseteq B \subseteq \text{acl}^{\text{eq}}(A)$, then $\delta(\bar{a}/A) = \delta(\bar{a}/B)$, where δ is defined in the natural way for formulas with parameters in M^{eq} .*

Proof. *Local character.* Let $p := \text{tp}(\bar{a}/B)$. By (SA), there is a single formula $\phi(\bar{x}, \bar{b}) \in \text{tp}(\bar{a}/B)$ such that $\delta(\bar{a}/B) = \delta(\phi(\bar{x}, \bar{b}))$, and we may take A to be the set of elements of \bar{b} .

Invariance. Suppose $\bar{a} \downarrow_A^\delta B$ and $\alpha \in \text{Aut}(M)$. For every $\phi(\bar{x}, \bar{b}) \in \text{tp}(\bar{a}/AB)$ there is $\psi(\bar{x}, \bar{c}) \in \text{tp}(\bar{a}/A)$ such that $\delta(\psi(\bar{x}, \bar{c})) \leq \delta(\phi(\bar{x}, \bar{b}))$. By (DC_L), we have $\chi_{\psi, \phi}(\bar{c}, \bar{b})$, so $\chi_{\psi, \phi}(\alpha(\bar{c}), \alpha(\bar{b}))$, noting that $\chi_{\psi, \phi}$ is an L -formula so is preserved by α . Hence, $\delta(\psi(\bar{x}, \alpha(\bar{c}))) \leq \delta(\phi(\bar{x}, \alpha(\bar{b})))$, and it follows that $\alpha(\bar{a}) \downarrow_{\alpha(A)}^\delta \alpha(B)$.

Symmetry. It suffices to show that $\bar{a} \downarrow_A^\delta \bar{b} \Rightarrow \bar{b} \downarrow_A^\delta \bar{a}$, so we suppose $\bar{a} \downarrow_A^\delta \bar{b}$, that is, $\delta(\bar{a}/\bar{b}A) = \delta(\bar{a}/A)$. By Lemma 3.1, using (DC_L) and (FMV),

$$\delta(\bar{b}/A) + \delta(\bar{a}/\bar{b}A) = \delta(\bar{a}/A) + \delta(\bar{b}/\bar{a}A).$$

It follows that $\delta(\bar{b}/\bar{a}A) = \delta(\bar{b}/A)$, as required.

Algebraic closure. Suppose $\chi(\bar{a}, \bar{b})$ holds, where \bar{b} is possibly an imaginary tuple in $\text{acl}^{\text{eq}}(A)$, and let $\bar{b} = \bar{b}_1, \dots, \bar{b}_k$ be the conjugates of \bar{b} over A . Then $\delta(\chi(\bar{x}/\bar{b})) = \delta(\chi(\bar{x}/\bar{b}_i))$ for each i by (DC_L). There is a formula $\rho(\bar{x}) \in \text{tp}(\bar{a}/A)$ which is equivalent to $\bigvee_{i=1}^k \chi(\bar{x}, \bar{b}_i)$, and by Lemma 2.1(ii) we have $\delta(\rho(\bar{x})) = \delta(\chi(\bar{x}, \bar{b}))$. \square

3.2. Simplicity and forking

Here we prove our main results, Theorems I, II, and III, linking (A), (SA), and (DC_L) to simplicity and forking, and related notions; these statements are included, respectively, in Theorems 3.3, 3.4, and 3.5. Examples are given in Section 4.1 showing that natural strengthenings of the main theorems do not hold.

We first fix some terminology for simple theories, taken from [39].

Definition 3.2. Let T be a complete theory, and M an ω_1 -saturated model of T from which the parameters below are taken.

- (1) A formula $\phi(\bar{x}, \bar{y})$ has the *tree property* (with respect to T) if there are $k < \omega$ and a sequence $\langle \bar{a}_\mu : \mu \in {}^{<\omega}\omega \rangle$ such that:
 - (a) for every $\mu \in {}^{<\omega}\omega$, the set $\{\phi(\bar{x}, \bar{a}_{\mu \smallfrown i}) : i < \omega\}$ is k -inconsistent, and
 - (b) for every $\sigma \in {}^\omega\omega$, the set $\{\phi(\bar{x}, \bar{a}_{\sigma \upharpoonright i}) : i < \omega\}$ is consistent.
- (2) The theory T is *simple* if no formula ϕ has the tree property with respect to T .
- (3) A *dividing chain of length α* for ϕ , or *dividing ϕ -chain of length α* , is a sequence $(\bar{a}_i : i \in \alpha)$ such that $\bigcup_{i < \alpha} \phi(\bar{x}, \bar{a}_i)$ is consistent and $\phi(\bar{x}, \bar{a}_i)$ divides over $\{\bar{a}_j : j < i\}$ for all $i < \alpha$.
- (4) A simple theory T is *low* if for every formula ϕ there is $n_\phi < \omega$ such that there is no dividing ϕ -chain of length n_ϕ .

Theorem 3.3.

- (i) Assume that (A) holds. Then T is simple and low.
- (ii) If (A) and (DC_L) hold then (FMV) holds.

We first note the following lemma, based on Proposition 2.2 and ultimately on Inclusion-Exclusion.

Lemma 3.2.

- (i) Let D be an A -definable subset of M^r in L , and let $\phi(\bar{x}, \bar{y})$ be an L -formula with $l(\bar{x}) = r$ and $l(\bar{y}) = s$. Let $(\bar{a}_i : i \in I)$ be an infinite L^+ -indiscernible sequence over A of elements of M^s . Put $D_i := \phi(M^r, \bar{a}_i)$ for each $i \in I$, and suppose that $D_i \subset D$ and $(D_i : i \in I)$ is inconsistent. Then $\delta(D_i) < \delta(D)$ holds for some (and hence all) $i \in I$.
- (ii) Let $D = \psi(M^r, \bar{a})$ be a definable subset of M^r and $\phi(\bar{x}, \bar{b})$ a formula implying $\psi(\bar{x}, \bar{a})$. If $\phi(\bar{x}, \bar{b})$ divides over \bar{a} , then there exists a tuple $\bar{b}' \in M^r$ with $\bar{b}' \models \text{tp}(\bar{b}/\bar{a})$ such that $\delta(\phi(\bar{x}, \bar{b}')) < \delta(D)$.

Proof. (i) Suppose for a contradiction that $\delta(D_i) = \delta(D)$ for every $i \in I$. Since I is infinite, we may assume without loss of generality that $I = \omega$.

By indiscernibility there is $k \in \omega$ such that $(D_i : i \in \omega)$ is k -inconsistent. By indiscernibility in L^+ and our assumption that $\delta(D_i) = \delta(D)$ for all i , there is some non-zero $m \in \mathbb{N}$ such that $|D_i| > \frac{|D|}{m}$ for each i . To ensure that Proposition 2.2 is applicable below, we may assume $m \geq 2$.

Consider now the measure μ_D on M , as defined in Section 2.1. For every $i < \omega$ we have that $\mu_D(D_i) \geq \frac{1}{m}$, and, by Proposition 2.2, there are i_1, \dots, i_k such that

$$\mu_D \left(\bigcap_{j=1}^k D_{i_j} \right) \geq \frac{1}{m^{3^{k-1}}} > 0.$$

Thus, $\delta(\bigcap_{j=1}^k D_{i_j}) = \delta(D)$, contradicting k -inconsistency.

(ii) Since $\phi(\bar{x}, \bar{b})$ divides over \bar{a} , there is an indiscernible sequence $(\bar{b}_i : i < \omega)$ such that $\bar{b}_i \models \text{tp}(\bar{b}/\bar{a})$ and the set $\{\phi(\bar{x}, \bar{b}_i) : i < \omega\}$ is k -inconsistent for some $k < \omega$. By saturation, we may suppose that this sequence is L^+ -indiscernible. Since $\bar{b}_i \models \text{tp}(\bar{b}/\bar{a})$, we have $\phi(\bar{x}, \bar{b}_i) \subseteq D$. The result now follows from (i). \square

Remark 3.2. By the last paragraph of the proof of (i), there is a strengthening of Lemma 3.2(i), in which the assumption that $(D_i : i \in I)$ is inconsistent is weakened to an assumption that for some $k \in \omega$ and all $i_1 < \dots < i_k \in I$, we have $\delta(D_{i_1} \cap \dots \cap D_{i_k}) < \delta(D)$.

Part (ii) above is close to Lemma 2.9 of [12].

Proof of Theorem 3.3. (i) We show for every L -formula $\phi(\bar{x}, \bar{y})$ that every dividing ϕ -chain has length at most $m := m_\phi$, where m_ϕ is provided by Lemma 2.5; this suffices, by [39] Proposition 2.8.6. Suppose for a contradiction that there is a sequence $(\bar{a}_j : 1 \leq j \leq m+1)$ from M^s such that each $\phi(\bar{x}, \bar{a}_j)$ divides over $\{\bar{a}_i : i < j\}$. We show by induction that there is a sequence $(\bar{b}_j : 1 \leq j \leq m+1)$ such that $\text{tp}_L(\bar{b}_j : 1 \leq j \leq m+1) = \text{tp}_L(\bar{a}_j : 1 \leq j \leq m+1)$ and

$\delta(\bigwedge_{1 \leq i \leq k+1} \phi(x, \bar{b}_i)) < \delta(\bigwedge_{1 \leq i \leq k} \phi(x, \bar{b}_i))$ for each $k = 1, \dots, m$, contradicting the choice of m .

To start the induction, put $\bar{b}_1 = \bar{a}_1$. For the induction step, suppose that $\bar{b}_1, \dots, \bar{b}_k$ have been constructed as above. As $\text{tp}_L(\bar{b}_j : 1 \leq j \leq k) = \text{tp}_L(\bar{a}_j : 1 \leq j \leq k)$, there is \bar{c} such that $\text{tp}_L(\bar{b}_1, \dots, \bar{b}_k, \bar{c}) = \text{tp}_L(\bar{a}_1, \dots, \bar{a}_{k+1})$. We apply Lemma 3.2 with A the union of the elements in $\{\bar{b}_i : i \leq k\}$ and D the set defined by $\phi(\bar{x}, \bar{b}_1) \wedge \dots \wedge \phi(\bar{x}, \bar{b}_k)$. Let $(\bar{d}_i : i \in \omega)$ be an indiscernible sequence over A witnessing the dividing of $\phi(\bar{x}, \bar{c})$ over A ; that is, $\text{tp}(\bar{d}_i/A) = \text{tp}(\bar{c}/A)$ for each i and $\{\phi(\bar{x}, \bar{d}_i) : i \in \omega\}$ is inconsistent. By Ramsey's Theorem, compactness, and ω_1 -saturation we may suppose that the tuples \bar{d}_i all lie in M and that the indiscernibility is with respect to L^+ . Let D_i be the solution set in M^r of $\phi(\bar{x}, \bar{b}_1) \wedge \dots \wedge \phi(\bar{x}, \bar{b}_k) \wedge \phi(\bar{x}, \bar{d}_i)$ for each $i \in \omega$. By Lemma 3.2, there is $i \in \omega$ such that $\delta(D_i) < \delta(D)$. Then put $\bar{b}_{k+1} := \bar{d}_i$.

(ii) This follows from Lemma 2.7 and the fact that simplicity implies there is no formula with the strict order property. \square

Theorem 3.4. *Let A, B be countable subsets of M , and \bar{a} a tuple from M .*

(i) *Assume (A) and (DC_L) . Then*

$$\bar{a} \downarrow_A B \Rightarrow \bar{a} \downarrow_A^\delta B.$$

(ii) *Assume (SA) and (DC_L) . Then*

$$\bar{a} \downarrow_A^\delta B \Rightarrow \bar{a} \downarrow_A B.$$

In particular, under (SA) and (DC_L) we have

$$\bar{a} \downarrow_A B \Leftrightarrow \bar{a} \downarrow_A^\delta B.$$

We emphasize that this is a statement about forking in the particular (ω_1 -saturated) model M . The proof of (ii) uses the following result, which is close to Lemma 2.9 of [12].

Proof of Theorem 3.4. (i) We follow the proof of Claim 1 in [21] Theorem 4.2, which roughly speaking states that any independence relation satisfying the properties from Propositions 3.1 and 3.2 is implied by forking independence. Note that by (A) and Theorem 3.3, T is simple and (FMV) holds. Hence, by our assumptions (DC_L) and (A), the properties in the Propositions 3.1 and 3.2 (except possibly (iv)) all hold for \downarrow^δ .

Suppose $\bar{a} \not\downarrow_A^\delta B$. Hence $\delta(\bar{a}/AB) < \delta(\bar{a}/A)$ so there is a formula $\phi(\bar{x}, \bar{b}) \in \text{tp}(\bar{a}/AB)$ such that $\delta(\phi(\bar{x}, \bar{b})) < \delta(\bar{a}/A)$, so $\bar{a} \not\downarrow_A^\delta \bar{b}$. We must show $\bar{a} \not\downarrow_A B$, for which it suffices to show $\bar{a} \not\downarrow_A \bar{b}$; in fact, our argument shows $\phi(\bar{x}, \bar{b})$ forks over A . Suppose for a contradiction that $\bar{a} \downarrow_A \bar{b}$. By Proposition 3.1(i) and ω_1 -saturation,

there is in M a sequence $(\bar{b}_i : i \in \omega_1)$ of realizations of $\text{tp}(\bar{b}/A)$ such that $\bar{b}_i \downarrow_A^\delta \{\bar{b}_j : j < i\}$ for all i . By ω_1 -saturation and Ramsey's Theorem, using DC_L , we may suppose $(\bar{b}_i : i \in \omega_1)$ is A -indiscernible in L^+ . Let $p = p(\bar{x}, \bar{b}) := \text{tp}(\bar{a}/\bar{b}A)$. By our assumption that $\bar{a} \downarrow_A \bar{b}$, the set $\bigcup\{\phi(\bar{x}, \bar{b}_i) : i \in \omega_1\}$ is consistent, realized by \bar{a}' , say. It follows from (DC_L) that $\bar{a}' \not\downarrow_A^\delta \bar{b}_i$ for all i ; indeed, by (DC_L) and (FMV) we have $\delta(\phi(\bar{x}, \bar{b}_i)) = \delta(\phi(\bar{x}, \bar{b}))$ for each $i \in \omega_1$. Hence, using symmetry and transitivity of \downarrow^δ , it follows that $\bar{a}' \not\downarrow_{A\{\bar{b}_j : j < i\}}^\delta \bar{b}_i$ for all $i \in \omega_1$: indeed, otherwise $\bar{a}' \downarrow_{A\{\bar{b}_j : j < i\}}^\delta \bar{b}_i$, so $\bar{b}_i \downarrow_{A\{\bar{b}_j : j < i\}}^\delta \bar{a}'$ and $\bar{b}_i \downarrow_A^\delta \{\bar{b}_j : j < i\}$, so $\bar{b}_i \downarrow_A^\delta \bar{a}'$ and hence $\bar{a}' \downarrow_A^\delta \bar{b}_i$, a contradiction.

Let $B^* := \bigcup\{\text{rng}(\bar{b}_i) : i < \omega_1\}$, where $\text{rng}(\bar{b})$ is the set of entries of \bar{b} . For each formula $\psi(\bar{x}, \bar{z})$ (where \bar{z} has any length), there is by (A) a finite set $B_\psi \subset A \cup B^*$ such that $\delta^\psi(\bar{a}/B_\psi) = \delta^\psi(\bar{a}/B')$ for any countable B' with $B_\psi \subset B' \subset B^*$ (with δ^ψ as defined in Section 2.1). Let B_1 be the union of the sets B_ψ over all such ψ . Then B_1 is countable, so there is $i < \omega_1$ such that $B_1 \subset A \cup \bigcup\{\text{rng}(\bar{b}_j) : j < i\}$. Then $\bar{a}' \downarrow_{A\{\bar{b}_j : j < i\}}^\delta \{\bar{b}_j : j \in \omega_1\}$, contradicting the last paragraph.

(ii) Suppose $\bar{a} \not\downarrow_A B$. We must show $\delta(\bar{a}/BA) < \delta(\bar{a}/A)$. By (SA), there is $\psi(\bar{x}) \in p := \text{tp}(\bar{a}/A)$ such that $\delta(\psi(\bar{x})) = \delta(p)$. Since $\bar{a} \not\downarrow_A B$, there is a formula $\phi(\bar{x}, \bar{b}) \in \text{tp}(\bar{a}/BA)$ that forks over A , and we may suppose that $\phi(M^r, \bar{b}) \subseteq \psi(M^r)$. Since (SA) implies (A), by Theorem 3.3 T is simple, so forking and dividing agree. Hence, by Lemma 3.2(ii), there is some $\bar{b}' \models \text{tp}(\bar{b}/A)$ such that $\delta(\phi(\bar{x}, \bar{b}')) < \delta(\psi(\bar{x}))$. By (DC_L) this inequality is implied by $\text{tp}(\bar{b}'/A)$, so $\delta(\phi(\bar{x}, \bar{b})) < \delta(\psi(\bar{x}))$, and so $\bar{a} \not\downarrow_A^\delta \bar{b}$. \square

Remark 3.3. (1) The above proof shows, assuming (SA) and (DC_L) , that a formula $\phi(\bar{x}, \bar{b})$ forks over A if and only if, for every $\psi(\bar{x}) \in L_A$ that is consistent with ϕ , we have $\delta(\phi \wedge \psi) < \delta(\psi)$. The direction (\Leftarrow) again just requires (A) and (DC_L) .

(2) The proof of (ii) yields the following, just under the assumption (SA), so without (DC_L) . Assume (SA) and suppose $\bar{a} \not\downarrow_A B$. Then there is $\bar{a}'B'$ such that $\text{tp}_L(\bar{a}B/A) = \text{tp}_L(\bar{a}'B'/A)$ and $\bar{a}' \not\downarrow_A^\delta B'$. Indeed, in the proof above and with $\phi(\bar{x}, \bar{b})$ as above, we obtain \bar{b}' such that $\delta(\psi(\bar{x}) \wedge \phi(\bar{x}, \bar{b}')) < \delta(\psi(\bar{x}))$. Choose $\bar{a}'B'$ by ω_1 -saturation so that $\text{tp}_L(\bar{a}\bar{b}B/A) = \text{tp}_L(\bar{a}'\bar{b}'B'/A)$. Then $\bar{a}' \not\downarrow_A^\delta B'$. The point to note here is that δ -dimension is not part of the L -type, and is not preserved in general by automorphisms of the ultraproduct.

(3) Example 4.4 shows that (ii) is not true if either of the assumptions (SA) or (DC_L) is dropped.

Question 3.1. Is there a local version of Theorem 3.4? For example, in the setting (DC_L) and (FMV) , along with (A_ϕ^*) , is it true that for all \bar{a}, \bar{b} and countable A , some Boolean combination $\psi(\bar{x})$ of ϕ -formulas in $\text{tp}(\bar{a}/A\bar{b})$ forks over A if and only if $\delta(\psi(\bar{x})) < \delta(\chi(\bar{x}))$ for every Boolean combination χ of ϕ -formulas in $\text{tp}(\bar{a}/A)$?

Most parts of the proof of Theorem 3.4 localize easily, to Boolean combinations of instances of a finite set Γ of formulas $\phi(\bar{x}, \bar{y})$. The problem occurs in the proof

that δ -forking implies forking, when symmetry of \downarrow^δ is applied. This rests on Lemma 3.1, which we have not been able to localize, essentially because of the additional formulas used to witness (DC_L).

Theorem 3.5. *Assume (SA). Then T is supersimple.*

Proof. Suppose for a contradiction that T is not supersimple. Then there are countable (in fact, finite) sets $B_0 \subset B_1 \subset \dots$ and a type p over $B = \bigcup_{i < \omega} B_i$ such

that for all $i < \omega$, $p|_{B_{i+1}}$ forks over B_i . Let \bar{a} realise p .

For every n we build sets B'_n , with $B'_0 \subset \dots \subset B'_n$, along with tuples \bar{a}_n , such that $\text{tp}_L(\bar{a}_n B'_n) = \text{tp}_L(\bar{a} B_n)$ and $\delta(\bar{a}_n/B'_{i+1}) < \delta(\bar{a}_n/B'_i)$ for each $i < n$. Suppose that these have been found for some given n . Then there is B^*_{n+1} such that $\text{tp}_L(\bar{a}_n B^*_{n+1}) = \text{tp}_L(\bar{a} B_{n+1})$. Thus $\bar{a}_n \not\downarrow_{B'_n} B^*_{n+1}$, so by Remark 3.3(2), there is $\bar{a}_{n+1} B'_{n+1}$ such that

$$\text{tp}_L(\bar{a}_{n+1} B'_{n+1}/B'_n) = \text{tp}_L(\bar{a}_n B^*_{n+1}/B'_n)$$

and $\delta(\bar{a}_{n+1}/B'_{n+1}) < \delta(\bar{a}_{n+1}/B'_n)$.

Finally, let $B' := \bigcup (B'_i : i \in \omega)$ and put $p_n := \text{tp}(\bar{a}_n/B'_n)$ for each n , and $p' := \bigcup (p_n : n \in \omega)$. Then $p' \in S(B')$ and $\delta(p'|_{B'_{n+1}}) < \delta(p'|_{B'_n})$ for each n , contradicting (SA). \square

3.3. Pseudofinite dimension and stability

In Proposition 3.3 below we characterize, among structures M satisfying (A) — so among simple structures M — when M is stable. The statement involves the measure μ_D on definable subsets of D defined in Section 2.1 as well as dimension, and is presented as a local result. Note that by Lemma 2.2, the assumption (A $^*_\phi$) for all ϕ is equivalent to (A). In Example 4.2 we present an example demonstrating that stability does not necessarily imply (A). Below, condition (iii) was pointed out by the referee, who also shortened the proof.

Proposition 3.3. *Assume (A $^*_\phi$). Then the following are equivalent:*

- (i) $\phi(\bar{x}, \bar{y})$ is unstable.
- (ii) there is for some \bar{d} a \bar{d} -definable set $D \subseteq M^r$ and a sequence $(\bar{a}_i : i \in \omega)$, L^+ -indiscernible over \bar{d} , such that $\delta(D) = \delta(D \wedge \bigwedge_{i \in \omega} \phi(\bar{x}, \bar{a}_i))$, and

$$\mu_D(\phi(\bar{x}, \bar{a}_i) \wedge \phi(\bar{x}, \bar{a}_j)) < \mu_D(\phi(\bar{x}, \bar{a}_i)) \text{ for all } i < j.$$

- (iii) $\phi(\bar{x}, \bar{y})$ has the Independence Property.

Proof. (iii) \Rightarrow (i). This is immediate.

(i) \Rightarrow (ii). Assume (i), and find a set D , defined by a finite partial ϕ -type, such that there are $\{\bar{a}_i : i \in \omega\}$ and $\{\bar{b}_i : i \in \omega\} \subset D$ with $\phi(\bar{b}_i, \bar{a}_j)$ holding if and only

if $i > j$, and such that there is no such set $D' \subset D$ with $\delta(D') < \delta(D)$. This is possible by (A_ϕ^*) .

Suppose that (ii) is false. Using compactness and saturation, after relabelling there is a sequence $(\bar{a}_i \bar{b}_i : i \in \omega + 1)$ which is L^+ -indiscernible over parameters used to define D , with $\bar{b}_i \in D$ for all $i \in \omega + 1$, and such that $\phi(\bar{b}_i, \bar{a}_j)$ holds if and only if $i > j$. It follows from indiscernibility that $\mu_D(\phi(\bar{x}, \bar{a}_i))$ is constant. By the minimality in the choice of D , we have $\delta(D) = \delta(D \cap \bigwedge_{j < i} \phi(M^r, \bar{a}_i))$ for each $i < \omega$. It follows by Lemma 2.1(vi) that $\delta(D) = \delta(D \wedge \bigwedge_{i \in \omega} \phi(\bar{x}, \bar{a}_i))$.

As we have assumed that (ii) is false, by L^+ -indiscernibility for all $i < j < \omega + 1$ we have $\mu_D(\phi(\bar{x}, \bar{a}_i) \wedge \phi(\bar{x}, \bar{a}_j)) = \mu_D(\phi(\bar{x}, \bar{a}_i))$. Hence $\mu_D(\phi(\bar{x}, \bar{a}_i) \wedge \neg\phi(\bar{x}, \bar{a}_j)) = 0$, so $\delta(D \cap \phi(\bar{x}, \bar{a}_i) \wedge \neg\phi(\bar{x}, \bar{a}_j)) < \delta(D)$. Now put $D' = D \cap (\phi(\bar{x}, \bar{a}_0) \wedge \neg\phi(\bar{x}, \bar{a}_\omega))$. Then $\delta(D') < \delta(D)$. Since $\bar{b}_i \in D'$ for each $i \in \omega$ with $i > 0$, this contradicts the assumption of minimality of $\delta(D)$ in the choice of D .

(ii) \Rightarrow (iii). With D and $(\bar{a}_i : i \in \omega)$ as in (ii), put $D_i := D \cap (\phi(\bar{x}, \bar{a}_{2i}) \wedge \neg\phi(\bar{x}, \bar{a}_{2i+1}))$ for each i . Then as $\delta(\phi(\bar{x}, \bar{a}_{2i}) \wedge \phi(\bar{x}, \bar{a}_{2i+1})) < \delta$, we have $\mu_D(D_i) > 0$. Hence, by indiscernibility and Proposition 2.2, we have $\mu_D(D_{i_1} \cap \dots \cap D_{i_k}) > 0$ for any $i_1 < \dots < i_k$. In particular each set $D_{i_1} \cap \dots \cap D_{i_k}$ is non-empty, so ϕ has the Independence Property. \square

Remark 3.4. Proposition 4.1 below shows that ultraproducts of asymptotic classes (in the sense of [26] and [5]) provide natural examples satisfying (SA) and (DC_L) . It was shown in [5] Proposition 6.5 (with a point of confusion concerning unimodularity clarified in [18]) that any *stable* ultraproduct of an asymptotic class — in fact, any stable structure which is measurable in the sense of [26] — is one-based. It would be interesting to generalize this to the assumptions of this paper.

It is known that any pseudofinite superstable structure of finite U-rank is one-based. Indeed, as in [5] (see also the final remarks in [18]) it is sufficient to prove that any pseudofinite strongly minimal set is one-based. The latter is shown by Pillay in [33].

We therefore may give conditions in the language of this paper which ensure that M is one-based. Indeed, if M satisfies (SA), (DC_L) , and condition (ii) of Proposition 3.3 for all formulas, and there is a finite upper bound on the length of chains $p_0 \subset \dots \subset p_t$ of complete types over countable sets such that $\delta(p_{i+1}) < \delta(p_i)$ for each i , then M is one-based. We do not know if this holds without the last assumption, which ensures finiteness of rank.

It is not true that every pseudofinite superstable structure is one-based. For example, in [27] Section 5, an example is sketched of a pseudofinite ω -stable group G that is not nilpotent-by-finite. It has the form $(\mathbb{C}, +, \cdot, T)$ where T is an infinite subgroup of (\mathbb{C}^*, \cdot) , and its construction is due, independently, to Chapuis, Khelif, Simonetta, and Zilber. By the main theorem of [15], one-based stable groups are abelian-by-finite, so G is not one-based. We have not checked if it satisfies conditions such as (A) or (DC_L) . There is also an example in [27] Section 5 of a nilpotent class 2 but not abelian-by-finite ω -stable pseudofinite group G , based on the Mekler

construction that codes graphs into groups; again G cannot be one-based.

4. Examples

This section has two aims. We first present examples designed to show that the obvious strengthenings of the principal results in Section 3.2 fail. Then in 4.2, 4.3, and 4.4 we investigate the conditions (A), (SA), and (DC_L) in the context of rather natural examples: asymptotic classes of finite structures; pseudofinite 2-sorted infinite-dimensional vector spaces over pseudofinite fields; and, ultraproducts of homocyclic groups.

4.1. Counterexamples

Example 4.1. We show that the converse to Theorem 3.5 is false. Let L consist of infinitely many unary predicates $(P_i : i \in \omega)$. For each n , let M_n be a finite structure with domain of size n^n such that $P_i(M_n) \supseteq P_{i+1}(M_n)$ for all i , and $|P_i(M_n)| = n^{n-i}$ for $i \leq n$ and $P_i(M_n) = \emptyset$ for $i > n$. Let M be a non-principal ultraproduct. Then $\text{Th}(M)$ is superstable of U-rank 1. However, $\delta(P_i(M)) > \delta(P_{i+1}(M))$ for all i , so (SA) fails.

We further note that supersimplicity does not follow from (A) + (DC_L) (as distinct from (SA) as in Theorem 3.5): Proposition 4.3 below provides a counterexample.

Problem 4.1. Find natural conditions on pseudofinite dimension in M that are equivalent to supersimplicity of $\text{Th}(M)$.

Example 4.2. From the definitions one might believe that stability of a formula $\phi(x, y)$ implies (A_ϕ) , but we present here a counterexample showing that not even the superstability of $\text{Th}(M)$ implies attainability.

For each $n < \omega$, let (M_n, E) be the structure where M_n has $\sum_{i=1}^n n^i$ elements and E is interpreted as an equivalence relation with a class of size n^i for each i with $1 \leq i \leq n$. Let $a_{n,i} \in M_n$ be an element in the class of size n^{n-i} . The theory of $M = \prod_{\mathcal{U}} M_n$ is just the theory of an equivalence relation with infinitely many infinite classes, so $\text{Th}(M)$ is stable of U -rank 2 (and low). By taking the formula $\phi(x, y) := \neg(xEy)$ and the sequence of elements $\langle a_i := [(a_{n,i})_{n < \omega}]_{\mathcal{U}} : i < \omega \rangle$ in M , we show that attainability fails for the positive ϕ -type $p := \{\phi(x, a_i) : i < \omega\}$.

First note for every $t < \omega$ that $\delta(xEa_t) > \delta(p)$. Indeed, otherwise, we have

$$\delta(xEa_t) \leq \delta(p) = \inf_{k \in \omega} (\neg xEa_0 \wedge \dots \wedge \neg xEa_k).$$

In particular, it follows that there is $N \in \omega \setminus \{0\}$ such that for \mathcal{U} many n , we have

$$\log(Nn^{n-t}) \leq \log(n + \dots + n^{n-(t+1)}) = \log\left(\frac{n(n^{n-t-1} - 1)}{n - 1}\right) \leq \log(2n^{n-t-1}),$$

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and so

$$\log N + (n - t) \log n \leq \log 2 + (n - t - 1) \log n,$$

which is clearly false for sufficiently large n .

Therefore, given finitely many a_{i_1}, \dots, a_{i_k} the set defined by $\bigwedge_{j \leq k} \phi(x; a_{i_j})$ contains $x E a_t$ for $t > i_1, \dots, i_k$, and we have

$$\delta \left(\bigwedge_{j=1}^k \phi(x; a_{i_j}) \right) \geq \delta(x E a_t) > \delta(p).$$

Thus, (A_ϕ) does not hold.

This example also shows that the converse to Theorem 3.3(i) fails.

The next example is close in spirit to that discussed in Theorem 4.2, and will be developed more fully in subsequent work.

Example 4.3. Fix a prime p , and let \mathcal{C} be the collection of all 2-sorted structures consisting of a finite field F of characteristic p and an even-dimensional vector space V over F , with a function $F \times V \rightarrow V$ for scalar multiplication and also a function symbol $\beta : V \times V \rightarrow F$ interpreted by a non-degenerate alternating bilinear form, that is, a symplectic form. Let $M = (V^*, F^*)$ be an ultraproduct of members of \mathcal{C} with both F^* and $\dim V^*$ infinite. By Proposition 7.4.1 of [9], $\text{Th}(V^*, F^*)$ is not simple.

It is easy to see directly that M does not satisfy (A). Indeed, let $\phi(x, y)$ be the formula $\beta(x, y) = 0$. For $(V, F) \in \mathcal{C}$ and $a_1, \dots, a_n \in V$ that are linearly independent with $\beta(a_i, a_j) = 0$ for all $i, j \in \{1, \dots, n\}$, let $X_n := \{x : \phi(x, a_1) \wedge \dots \wedge \phi(x, a_n)\}$. Then $|X_n| = \frac{|X_{n-1}|}{|F|}$. Thus, if sets X'_n are defined in the same way in the ultraproduct, we have $\delta(X_n) > \delta(X_{n+1})$ for all n .

In work in preparation by the second and third authors with W. Anscombe and D. Wood, it is shown that M has (DC_L) ; in fact, an asymptotic result analogous to Theorem 4.2 below is proved. It can be seen that if $a \in V^* \setminus \{0\}$ then the formula $\beta(x, a) = 0$ does not fork over \emptyset . Thus, in Theorem 3.4(i), the condition (A) cannot be omitted.

Example 4.4. We now show that the assumption (SA) cannot be omitted in Theorem 3.4(ii). Let L be a language with binary relations $(E_i)_{i \in \omega}$, all to be interpreted by equivalence relations, a binary relation F , and a unary relation P . In the ultraproduct M of a class $\mathcal{C} = \{M_j : j < \omega\}$ of finite L structures we require for each $i < \omega$ that each E_i -class is a union of infinitely many E_{i+1} -classes. Let E be the intersection of the relations E_i in M , a \bigwedge -definable equivalence relation. The predicate P in M will have infinite coinfinite intersection with each E -class. Lastly, the interpretation of F is an equivalence relation on the complement of P with infinitely many classes, such that each F -class has infinite intersection with each E_i -class for each $i < \omega$. We can arrange that the finite L -structures M_j are built in a uniform way, so that, for example, if j is sufficiently large relative to i then each

E_i -class of M_j is a union of E_{i+1} -classes all of the same size, with a corresponding uniformity for P and F . We also can ensure for each i that all E_i -classes have the same pseudofinite dimension δ_i , and that if Q is an E_i -class and D an F -class then $\delta_i > \delta(Q \cap D) > \delta_{i+1}$.

The theory of M is stable, has quantifier elimination, but is not superstable, so (SA) fails. The finite structures can be chosen so that M satisfies (A) and (DC_L) . Let c realize P , let q be the type over $\{c\}$ containing $\{\neg P(x), E_i(x, c) : i \in \omega\}$, and let b realize q . Then clearly there is a type q' over $\{c, b\}$ containing $F(x, b)$ and extending q that forks over c but satisfies $\delta(q') = \delta(q)$.

The condition (DC_L) cannot be dropped in Theorem 3.4(ii) either; that is, without DC_L the conclusion of Theorem 3.3 fails in general. Indeed, consider the following variant of Example 2.1 (see also the second example in 4.5 below). The language L consists of a single binary relation E , and the class \mathcal{C} consists of finite structures M_n for $n > 0$ in which E is interpreted by an equivalence relation with n classes of size n and one of size n^2 . The ultraproduct M is superstable of rank 2 and has (SA). There is a unique equivalence class B of M such that $\delta(B) = \delta(M)$. If $b \in B$, then the formula $E(x, b)$ forks over \emptyset , and if $b' \in B \setminus \{b\}$ then $b' \not\downarrow_{\emptyset} b$ but $b' \downarrow_{\emptyset}^{\delta} b$. This example shows also that in Lemma 3.2(ii), we cannot expect the conclusion $\delta(\phi(\bar{x}, \bar{b})) < \delta(D)$.

Example 4.5. While Theorem 3.3(ii) asserts that (A) + (DC_L) implies (FMV), the conditions (SA) + (DC_L) do not imply that the set $\{\delta(X) : X \subset M, X \text{ definable}\}$ is finite. Theorem 4.2, below, provides an example: using the notation there, in the vector space sort, for each k there is a k -dimensional definable subspace V_k , and for $k < l$ we have $\delta(V_k) < \delta(V_l)$. See also Example 2.1.

We also have an example of a structure with finite SU-rank that satisfies (SA) but not (DC_L) in which δ takes infinitely many values on a uniformly definable family. Let L be a language with a single binary relation E , and for each $n < \omega$ let M_n be an L -structure with $n \sum_{i=1}^n n^i$ elements, with E interpreted by an equivalence relation having, for each $i \in \{1, \dots, n\}$, exactly n equivalence classes of size n^i . In the ultraproduct M , the equivalence relation E has infinitely many classes, all infinite, so $\text{Th}(M)$ is ω -stable of Morley rank 2. However the *uniformly* definable family of sets $\{x : E(x, b)\}$ of equivalence classes takes infinitely many δ -values as b varies. It is routine to check using quantifier elimination that (SA) holds. Thus, (DC_L) fails (this can be seen directly, or by Theorem 3.3 (ii)). Unlike the similar Example 2.1, (DC_{L+}) does hold in this example.

Example 4.6. Under the assumptions (SA) and (DC_L) , we may have a definable set X in M and a definable subset $Y \subseteq X$ such that $\delta(Y) = \delta(X)$ and $SU(Y) < SU(X)$. To see this, let L have a unary predicate P and a binary predicate E . For $n < \omega$ let M_n be a finite structure in which approximately half the elements satisfy P , and E is interpreted by an equivalence relation on P with classes all of

the same size, with both the size and number of E -classes increasing as $n \rightarrow \infty$. If M is a non-principal ultraproduct of $(M_n : n < \omega)$ then $\delta(-P(x)) = \delta(M)$ but $SU(-P(x)) = 1 < 2 = SU(M)$.

We note here that Lemma 6.1 shows that this phenomenon is not possible if X has a definable group structure.

Remark 4.1. The elimination of imaginaries assumption is required in Proposition 2.1(ii). Consider the class \mathcal{C} of finite structures M_n equipped with binary relations E and $<$ such that $<$ is a total preorder on M_n and E is an equivalence relation with n classes, with E the natural equivalence relation induced by the preorder. Furthermore, as n increases, the size of all classes should increase without bound, and for $k < k+1 \leq n$, the size of the $(k+1)$ st $<$ equivalence class should be much larger than the n -fold cartesian power of the k th class. The ultraproduct M of \mathcal{C} is a discretely ordered preorder whose quotient is infinite and has first and last elements—and thus has the strict order property—such that each equivalence class is infinite. Moreover, M admits elimination of quantifiers in a suitable enlargement of the language and evidently has (DC_L) .

4.2. *Asymptotic classes*

As usual, we consider a class \mathcal{C} of finite structures in a language L , with a corresponding class \mathcal{C}^+ in the 2-sorted language L^+ of which we form an ultraproduct K^* (often denoted $K^*(\mathcal{C})$) in the 2-sorted language L^+ , and then consider the reduct M to L of the structure induced by K^* on the sort \mathbf{D} .

Recall from [5] (see also [26] and [6]) the definition of an *asymptotic class* of finite structures, and the corresponding notion of *measurable structure*. Every non-principal ultraproduct of an asymptotic class is measurable—but not conversely—and every measurable structure has supersimple finite rank theory. The motivating example of an asymptotic class, by the main theorem of [3], is the class of finite fields. Likewise, by [37] Theorem 3.5.8 of Ryten, if p is a prime and m, n are coprime natural numbers with $m > 1$ and $n \geq 1$, then the class of all finite difference fields $(\mathbb{F}_{p^{kn+m}}, \text{Frob}^k)$, where Frob is the Frobenius automorphism $x \mapsto x^p$, is a 1-dimensional asymptotic class. Using this, Ryten showed that the class of all finite simple groups of any fixed Lie type is an asymptotic class. Likewise, Elwes [5] showed that any smoothly approximable structure has an approximating sequence of envelopes which forms an asymptotic class.

Our first result shows that asymptotic classes and their ultraproducts fall under the framework of Section 1. If \mathcal{C} is an N -dimensional asymptotic class, $M_n \in \mathcal{C}$, and X is a definable set in M_n of cardinality approximately $\mu|M_n|^d$ (in the sense of asymptotic classes), we shall say that X has dimension Nd . This notion is well-defined, provided M_n is sufficiently large, relative to the formula defining X . This dimension induces a notion of dimension, denoted $\dim(X)$, on any definable set X in an ultraproduct of \mathcal{C} ; the latter is exactly the dimension in the sense of measurable structures.

Proposition 4.1. *Let $\mathcal{C} = \{M_j : j \in \omega\}$ be an asymptotic class of finite structures in a language L . Then $K^*(\mathcal{C})$ satisfies (SA), (DC_L) and (FMV), and for all definable subsets $X, Y \subseteq M^r$, $\dim(X) = \dim(Y)$ if and only if $\delta(X) = \delta(Y)$.*

Proof. As usual, let M be the reduct of the \mathbf{D} -sort of $K^*(\mathcal{C})$ to L . We claim that if X, Y are definable sets in M and $|X|$ denotes the non-standard cardinality of X , then

$$\dim(X) \leq \dim(Y) \Leftrightarrow (\exists n \in \omega) \frac{|X|}{|Y|} \leq n \Leftrightarrow \delta(X) \leq \delta(Y).$$

The second inequality is immediate, and for the first, note that $\frac{|X|}{|Y|} \leq n$ if and only if there is $U \in \mathcal{U}$ such that $\frac{|X_j|}{|Y_j|} \leq n$ for all $j \in U$, where X_j, Y_j are the corresponding definable sets in M_j . The latter condition implies $\dim(X) \leq \dim(Y)$. Conversely, $\dim(X) \leq \dim(Y)$ implies that there is $n \in \omega$ and $U \in \mathcal{U}$ such that for all $j \in U$, we have $\frac{|X_j|}{|Y_j|} \leq n$ in M_j .

Since dimension takes values in ω , it follows immediately that M satisfies (SA). Furthermore, (DC_L) follows immediately from definability of dimension in measurable structures, and (FMV) follows from the fact that in a measurable structure, any uniformly definable family of sets takes finitely many dimensions. The final assertion in the proposition also follows from the above equivalence. \square

Remark 4.2. It is known (I. Ben-Yaacov, personal communication) that in a measurable structure M , change in dimension corresponds to forking. That is, $\dim(a/B \cup C) = \dim(a/C)$ if and only if $SU(a/B \cup C) = SU(a/C)$. However, we do not claim that in every measurable structure the dimension can be adjusted so that it coincides with SU-rank. As an example, in a language L with a single unary predicate P , consider the class \mathcal{C} of finite L -structures M_n of size n^2 for $n < \omega$ such that $P(M_n)$ has size n . This is a 2-dimensional asymptotic class, and taking an ultraproduct, the universe has dimension 2 but the SU-rank is 1. Although the dimension of $x = x$ could be changed to 1, this could not be done preserving the relation ‘has the same dimension’.

Question 4.1. If \mathcal{C} is an asymptotic class, we may consider the class \mathcal{C} of corresponding 2-sorted structures in the language L^+ , and the ultraproducts $K^*(\mathcal{C})$. In key examples such as if \mathcal{C} is the class of finite fields, does $K^*(\mathcal{C})$ have an NTP₂ theory?

4.3. Pseudofinite vector spaces

In this and the next section, we consider two examples that belong to the framework of ‘multi-dimensional asymptotic classes’ being developed by the second and third authors along with W. Anscombe and D. Wood. These are classes of finite structures, typically in a multi-sorted language, in which there is a strong uniformity in the asymptotic cardinalities of definable sets, in terms of the cardinalities

of certain sorts. The example in Theorem 4.2 below is prototypical, and, unlike ultraproducts of asymptotic classes, has infinite SU-rank. Example 4.3 is similar, but has ultraproducts whose theory is not simple. The example in Proposition 4.3 also belongs to this framework, but has the additional property that the cardinalities of definable sets are given exactly rather than asymptotically. We also take the opportunity in this section to include some further observations on the model theory of infinite-dimensional vector spaces over pseudofinite fields in Propositions 4.2 and 4.3.

Let L be a 2-sorted language containing: a sort \mathcal{V} , the vector space sort, equipped with a binary function symbol $+$, a unary function symbol $-$, and a constant symbol 0 ; a sort \mathcal{K} , the field sort, equipped with the language L_R of rings; and, a function symbol for scalar multiplication $\mathcal{K} \times \mathcal{V} \rightarrow \mathcal{V}$. Let L_{vs} be obtained from L by adding, for each $n > 0$, an n -ary relation symbol θ_n . In a vector space V over K we interpret $\theta_n(v_1, \dots, v_n)$ as expressing that the vectors v_1, \dots, v_n are linearly independent. Let T_{vs} be the theory of infinite-dimensional vector spaces, in this language L_{vs} , and for a field F let $T_{vs}(F)$ be the theory of infinite-dimensional vector spaces over models of $\text{Th}(F)$. It is clear that each theory $T_{vs}(F)$ is complete (see also Granger [9] Corollary 11.1.6). Furthermore, $T_{vs}(F)$ eliminates quantifiers in the sort \mathcal{V} . More formally, by Proposition 11.1.7 of [9], which itself is a slight elaboration of the main theorem of Kuzichev [22] (see also [31]), we have the following, where a θ -formula is an instance of some θ_n .

Lemma 4.1. *Let ϕ be a formula of L_{vs} . Then ϕ is equivalent modulo $T_{vs}(F)$ to a boolean combination of θ -formulas, quantifier-free formulas, and L_R -formulas in the field sort.*

Our main result in this section is the following.

Theorem 4.2. *Let \mathcal{C} be the class of all L_{vs} -structures (V, F) where V is a finite-dimensional vector space over the finite field F . Let $\phi(x_1, \dots, x_r, y_1, \dots, y_s)$ be a formula, and \mathbf{V}, \mathbf{F} be indeterminates. Then there is a finite set E of polynomials $p(\mathbf{V}, \mathbf{F}) \in \mathbb{Q}[\mathbf{V}, \mathbf{F}]$, such that for every $M = (V, F) \in \mathcal{C}$ and all $a_1, \dots, a_s \in M$, there is some $p(\mathbf{V}, \mathbf{F}) \in E$ such that*

$$|\phi(M^r, a_1, \dots, a_r)| - p(|V|, |F|) = o(p(|V|, |F|)) \quad (*)$$

Furthermore—the definability condition—for all $p \in E$, there is a formula $\phi_p(y_1, \dots, y_s)$ such that if $M \in \mathcal{C}$ is sufficiently large and $a_1, \dots, a_s \in M$, then $(*)$ holds if and only if $M \models \phi_p(a_1, \dots, a_s)$.

The following corollary is almost immediate, as in Proposition 4.1.

Corollary 4.1. *Let M be an infinite ultraproduct of members of \mathcal{C} , viewed as usual as the structure induced on the sort \mathbf{D} (itself formally a pair of sorts) by an ultraproduct $K^*(\mathcal{C})$. Then M satisfies (SA) and (DC_L) .*

Proof. Put $M = (V, F)$. We may suppose that $\dim_F(V)$ is infinite. Indeed, otherwise, M is definable in F (over a basis of V) and we may apply Proposition 4.1, using the definability clause in Theorem 4.2 to ensure DC_L .

Let X be a definable set in M , arising as an ultraproduct of definable sets X_j in M_j for $M_j \in \mathcal{C}$. There is $U \in \mathcal{U}$ such that the same polynomial $p(\mathbf{V}, \mathbf{F})$ is associated with X_j for each $j \in U$. Let d be the largest exponent of \mathbf{V} in a monomial of p , and e the largest exponent of \mathbf{F} in a monomial of p of form $c\mathbf{V}^d\mathbf{F}^e$. We define the ‘dimension’ of X to be the pair $(d, e) \in \omega \times \omega$ (ordered lexicographically), and then argue as in the proof of Proposition 4.1. \square

The proof of Theorem 4.2 is by induction on r , using a fibering argument in spirit like the proof of the o-minimal Cell Decomposition Theorem. The main work, which rests on [3] and hence ultimately on the Lang-Weil estimates, is contained in the following lemma that starts the induction.

Lemma 4.2. *The conclusion of Theorem 4.2 holds when $r = 1$, that is, for formulas $\phi(x, y_1, \dots, y_s)$.*

Proof. We consider separately the two cases where x lies in the sort \mathcal{V} or the sort \mathcal{K} . We work in a large finite structure $M = (V, F) \in \mathcal{C}$.

Case 1. Suppose first that x lies in the sort \mathcal{V} , and, replacing x by u , write ϕ as $\phi(u, \bar{v}\bar{y})$, where the parameter variables $\bar{v}\bar{y}$ consist of \bar{v} from the sort \mathcal{V} and \bar{y} from the sort \mathcal{K} . By Lemma 4.1, we can write ϕ as a disjunction of contradictory conjunctions of (possibly negated) θ -formulas, quantifier-free formulas, and field formulas. Since we can sum the cardinalities of disjoint definable sets and add the corresponding polynomials, and since the definability condition lifts to the disjunction, we may suppose that ϕ itself is such a conjunction.

Thus we assume that ϕ is $\bigwedge_{i=1}^n \psi_i$ where each ψ_i is a possibly negated θ -formula, a possibly negated formula of form $t(u, \bar{v}, \bar{w}) = 0$, or a field formula. Since u does not occur in a field formula, we may ignore these. Likewise, a term equality is satisfied by one or all elements of V , so we may ignore these as well. Hence, we may suppose that each ψ_i is a θ -formula or its negation.

We focus on one such formula, say $\theta_n \left(z_1 u + \sum_{i=1}^r z_{1i} v_i, \dots, z_n u + \sum_{i=1}^n z_{ni} v_i \right)$, where the z_i and z_{ji} are L_R -terms, that is, polynomials in the field variables \bar{y} . Dividing out by non-zero z_i , which we may do by increasing the initial set of disjuncts, and collecting terms, we can write this formula in the form $\theta_n(u + w_1, \dots, u + w_m, w'_1, \dots, w'_m)$, where $m + m' = n$ and the w_i and w'_i are terms in \bar{v} and \bar{y} . In what follows, the notation $\langle v_1, \dots, v_k \rangle$ for vectors v_1, \dots, v_k denotes the F -span of v_1, \dots, v_k . We then have

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$$\begin{aligned}
 M & \models \theta_n(u + w_1, \dots, u + w_m, w'_1, \dots, w'_{m'}) \\
 & \Leftrightarrow (\forall \bar{a} \in F^m)(\forall \bar{b} \in F^{m'}) \\
 & \quad \left[\left(\sum_{i=1}^m a_i(u + w_i) + \sum_{i=1}^{m'} b_i w'_i = 0 \right) \rightarrow \left(\bigwedge_{i=1}^m a_i = 0 \wedge \bigwedge_{i=1}^{m'} b_i = 0 \right) \right] \\
 & \Leftrightarrow \left[u \notin \langle \bar{w}, \bar{w}' \rangle_F \right. \\
 & \quad \left. \wedge \left(\text{there is no non-trivial relation } \sum_{i=1}^m c_i w_i + \sum_{i=1}^{m'} d_i w'_i = 0 \text{ with } \sum_{i=1}^m c_i = 0 \right) \right] \\
 & \qquad \qquad \qquad \text{or} \\
 & \quad \left(\bar{w}, \bar{w}' \text{ are linearly independent } \wedge u = \sum_{i=1}^m c_i w_i + \sum_{i=1}^{m'} d_i w'_i \text{ with } \sum_{i=1}^m c_i \neq -1 \right).
 \end{aligned}$$

The first disjunct in the last equivalence yields a definable set that has cardinality $|V| - |F|^{\dim(\bar{w}\bar{w}')}$ or 0. The second disjunct, which follows by linear algebra (we omit the details), gives a definable set of size $|F|^{m+m'} - |F|^{m+m'-1}$. In both cases, the conditions on $\bar{w}\bar{w}'$ determine the size of the definable set, which in turn are determined entirely by the *parameters* $\bar{v}\bar{y}$.

By breaking up the formula ϕ according to conditions on the parameters, we reduce to the case where ϕ is a conjunction of θ -formulas (possibly negated) in terms in u, \bar{v} , and \bar{y} , expressing that $u \in (U_1 \cap \dots \cap U_l) \setminus (V_1 \cup \dots \cup V_k)$ where the U_i and V_i are cosets of subspaces spanned by terms in the parameters, possibly with $l = 0$. Such a set has cardinality given by one of finitely many polynomials in $|V|, |F|$, of the form $|V| + p(|F|)$, or $p(|F|)$. The parameters corresponding to each polynomial are uniformly definable in the parameters of the formula.

Case 2. We now suppose that x lies in the field sort. Again, by breaking ϕ into a disjunction of conjunctions, we may assume that ϕ is a conjunction of field formulas, along with possibly negated equations and θ -formulas in the vector space sort. By breaking ϕ into more disjunctions, we shall reduce to the case where each conjunct is a field formula. In this case, the fact that finite fields form a 1-dimensional asymptotic class (the content of the main theorem of [3]) is applicable, with the polynomials being monomials of the form μ or $\mu|F|$, where $\mu \in \mathbb{Q}$. The definability clause (concerning the formulas ϕ_p) is also easily checked.

Equations in the vector space sort take the form $\sum_{i=1}^t p_i(x, \bar{y})v_i = 0$, where each p_i is a polynomial. If the v_i are linearly independent—a condition on the parameters—

this is equivalent to $\bigwedge_{i=1}^t p_i(x, \bar{y}) = 0$, a field formula. If $v_1, \dots, v_{t'}$ are linearly independent, and $v_{t'+1}, \dots, v_t$ are in the span of $v_1, \dots, v_{t'}$, then the corresponding scalars exhibiting this are definable in \bar{v} , and the original equation becomes equivalent to a field formula in x, \bar{y} , and these scalars.

Lastly, we consider θ -formulas, which (possibly negated) have the form

$$\theta_n \left(\sum_{j=1}^t p_{1j}(x, \bar{y})v_j, \dots, \sum_{j=1}^t p_{nj}(x, \bar{y})v_j \right).$$

If v_1, \dots, v_t are linearly independent, this formula is equivalent to a field condition on $x\bar{y}$ (that the matrix $(p_{ij}(x, \bar{y}))$ has rank n). And again, if $v_1, \dots, v_{t'}$ are linearly independent, and $v_{t'+1}, \dots, v_t$ lie in the span of $v_1, \dots, v_{t'}$, then θ is equivalent to a field condition in x, \bar{y} , and the scalars exhibiting linear dependence, which, as before, are definable in the parameters \bar{v} . Lemma 4.2 is now proved. \square

Proof of Theorem 4.2. We prove Theorem 4.2 by induction on r , mimicking the proof of [5] Lemma 2.2, with many details omitted. By Lemma 4.2 the induction starts. Assume the proposition holds for r , and consider a formula $\phi(\bar{x}, \bar{y})$ where $l(\bar{x}) = r + 1$. Put $\bar{x} = z\bar{x}'$. By the case $r = 1$, there is a finite set E of polynomials $p(\mathbf{V}, \mathbf{F}) \in \mathbb{Q}[\mathbf{V}, \mathbf{F}]$ such that Theorem 4.2 holds for the formula $\phi(z, \bar{x}'\bar{y})$, and there are corresponding formulas ϕ_p for each $p \in E$. Put $E = \{p_1, \dots, p_t\}$, and $\phi_i = \phi_{p_i}$ for each $i = 1, \dots, t$.

By the induction assumption, for each $p_i \in E$, there is a finite set E_i of polynomials $q(\mathbf{V}, \mathbf{F}) \in \mathbb{Q}[\mathbf{V}, \mathbf{F}]$ such that for all $M = (V, F) \in \mathcal{C}$ and $\bar{a} \in M^s$, there is $q \in E_i$ such that

$$|\phi_i(M^r, \bar{a})| - q(|V|, |F|) = o(q(|V|, |F|)),$$

along with a corresponding formula $(\phi_i)_q(\bar{y})$. Put $E_i = \{q_{ij} : j = 1, \dots, r_i\}$, and $\phi_{ij} := (\phi_i)_{q_{ij}}$ for each i, j . Observe for each $M \in \mathcal{C}$ and $\bar{a} \in M^s$ that there is a unique function $h : \{1, \dots, t\} \rightarrow \omega$ such that for each $i = 1, \dots, t$, we have $M \models \phi_{ih(i)}(\bar{a})$. Also, for each i , we have $h(i) \in \{1, \dots, r_i\}$, so the set of all such h is finite.

Now fix $M \in \mathcal{C}$ and $\bar{a} \in M^s$, and let $h : \{1, \dots, t\} \rightarrow \omega$ be the corresponding function as above, so that $M \models \phi_{ih(i)}(\bar{a})$ for each $i = 1, \dots, t$. Then for each $i = 1, \dots, t$ we have

$$|\phi_i(M^r, \bar{a}) - q_{ih(i)}(|V|, |F|)| = o(q_{ih(i)}(|V|, |F|)),$$

and for each $\bar{b} \in \phi_i(M^r, \bar{a})$, we have

$$|\phi(M, \bar{b}, \bar{a}) - p_i(|V|, |F|)| = o(p_i(|V|, |F|)).$$

Let $P(\mathbf{V}, \mathbf{F}) = \sum_{i=1}^t p_i(\mathbf{V}, \mathbf{F}) \cdot q_{ih(i)}(\mathbf{V}, \mathbf{F})$. Then

$$|\phi(M^{r+1}, \bar{a}) - P(|V|, |F|)| = o(P(|V|, |F|)).$$

This completes the proof of Theorem 4.2 □

Remark 4.3. The proof of Theorem 4.2 provides somewhat more information than stated. First, for a formula $\phi(u, \bar{y})$ where u ranges through the vector space sort, there is a finite set E of pairs $(k, p(\mathbf{F}))$ where $k \in \{0, 1\}$ and $p(\mathbf{F}) \in \mathbb{Q}[\mathbf{F}]$ such that for all $M = (V, F) \in \mathcal{C}$ and $\bar{a} \in M^s$, there is $(k, p(\mathbf{F})) \in E$ such that $\phi(M, \bar{a})$ has size *exactly* $k|V| + p(|F|)$. Likewise, for any such $\phi(x, \bar{y})$ where x ranges through the field sort, there is a constant C and a finite set $E \subset \mathbb{Q}^{>0}$ such that for any $M = (V, F) \in \mathcal{C}$ and $\bar{a} \in M^s$, either $|\phi(M, \bar{a})| \leq C$ or there is $\mu \in E$ such that $||\phi(M, \bar{a})| - \mu|F|| \leq C|F|^{\frac{1}{2}}$.

More generally, for any formula $\phi(\bar{x}, \bar{y})$, there is a finite set E of polynomials $P(\mathbf{V}, \mathbf{F}) \in \mathbb{Q}[\mathbf{V}, \mathbf{F}]$, each of the form $\prod_{i=1}^d (k_i \mathbf{V} + p_i(\mathbf{F})) \cdot \mu \mathbf{F}^e$, with $k_i \in \{0, 1\}$, $d, e < \omega$ and $p_i(\mathbf{F}) \in \mathbb{Q}[\mathbf{F}]$, and a constant $C > 0$, such that for all $M = (V, F) \in \mathcal{C}$ and $\bar{a} \in M^s$ there is $P(\mathbf{V}, \mathbf{F})$ as above such that

$$||\phi(M^r, \bar{a})| - P(|V|, |F|)| \leq C \prod_{i=1}^d (k_i |V| + p_i(|F|)) \cdot |F|^{e - \frac{1}{2}}.$$

The corresponding definability clauses also hold. We omit the details; compare the proof of [26] Theorem 2.1.

Now, let T_{pvf} be the theory of all finite-dimensional vector spaces over finite fields, and let K be a pseudofinite field. Then $T_{vs}(K)$ is a completion of T_{pvf} . Recall that a complete theory T is said to be *near model complete* if, modulo the theory, every formula is equivalent to a Boolean combination of existential formulas in the same variables.

Proposition 4.2.

- (i) *The theory $T_{vs}(K)$ is near model complete.*
- (ii) *$T_{vs}(K)$ is a supersimple theory such that the vector space sort has rank ω and the field sort has rank 1. If $(V_0, K_0) \models T_{vs}(K)$ then K_0 is stably embedded, with the \emptyset -definable subsets of K_0^n given just by the structure of K_0^n in the language of rings.*

Proof. Assertion (i) follows immediately from Lemma 4.1 together with the near model-completeness of any complete theory of pseudofinite fields; the latter follows from Kiefe [19] – see also Chatzidakis [1]. In fact, the near model completeness assertion can be strengthened, and is uniform across all theories of pseudofinite fields.

For (ii), observe that $T_{vs}(F)$ is definable in the SU-rank ω theory ACFA. For if $(K, \sigma) \models \text{ACFA}$, then $\text{Fix}(\sigma)$ is a rank 1 pseudofinite field, and K is an infinite degree extension of $\text{Fix}(\sigma)$ so may be viewed as an infinite dimensional vector space over $\text{Fix}(\sigma)$. Furthermore, any theory of pseudofinite fields occurs as the fixed field theory in some completion of ACFA; for example, if F is the ultraproduct $\prod_{i \in \omega} \mathbb{F}_q / \mathcal{U}$, then by the main theorem of [14], the difference field $\prod_{i \in \omega} (\mathbb{F}_q^{\text{alg}}, x \mapsto x^q) / \mathcal{U}$ is a model of

ACFA and has fixed field F . The assertions about the induced structure on K_0 can be derived directly from (i), or from the corresponding statements about $\text{Fix}(\sigma)$ in $(K, \sigma) \models \text{ACFA}$ – see e.g. [2] Proposition 5.3. \square

For interest, we make some further observations on $T_{v,s}$. Recall [16] Definition 2.7 that a global A -invariant type $p(\bar{x})$ over a large saturated model \mathbb{U} is said to have *NIP* if every Morley sequence $(\bar{b}_i : i < \omega)$ in p over A —which has a uniquely determined complete type over A —has the property that for every formula $\phi(\bar{x}, \bar{y})$ there is $n_\phi < \omega$ such that for any \bar{c} , there are at most n_ϕ alternations of truth values of $\phi(\bar{b}_i, \bar{c})$ as i increases. Also, following [34], a global type $p \in S(\mathbb{U})$ is *generically stable* if it is invariant over some small set $A \subset \mathbb{U}$ and if for some (every) Morley sequence $(\bar{a}_i : i < \lambda)$ in p over A and every formula $\phi(\bar{x})$ (not necessarily over A), the set $\{i \in \lambda : \phi(\bar{a}_i)\}$ is finite or cofinite in λ . This definition differs slightly from that in [16], but agrees for types with NIP.

Proposition 4.3. *There is a unique complete 1-type p over \mathbb{U} containing the formula $x \in V$ and all formulas of the form $v \notin \langle u_1, \dots, u_r \rangle$, where $u_1, \dots, u_r \in \mathbb{U}$. The type p is invariant over \emptyset , is NIP in the sense of [16] Remark 2.7, and is generically stable.*

Proof. The existence and invariance are clear. For uniqueness, suppose that u, u' both satisfy the prescribed formulas over \mathbb{U} , and let \mathbb{U}' be an elementary extension of \mathbb{U} containing u, u' . Then there is $g \in \text{Aut}(\mathbb{U}')$ fixing pointwise \mathbb{U} and $K(\mathbb{U}')$ (so g is linear over $K(\mathbb{U}')$) with $g(u) = u'$.

To see that p is NIP, let $(v_i : i \in \omega)$ be a Morley sequence over \emptyset , that is, a sequence of linearly independent vectors, and let $\phi(x, \bar{y}\bar{z})$ be a formula. Let \bar{c}, \bar{a} be tuples from $\mathbb{U}(V)$ and $\mathbb{U}(K)$, respectively. At most $l(\bar{c})$ of the v_i are in $\langle \bar{c} \rangle$, and it follows that $\phi(v_i, \bar{c}, \bar{a})$ has at most $l(\bar{c}) + 1$ alternations of truth value.

To see that p is generically stable, by [16] Proposition 3.1, Remark 3.3(iii), we must show that any Morley sequence in p over any parameter set A is totally indiscernible. This is immediate. \square

4.4. Ultraproducts of finite homocyclic groups

We give here an exact—rather than just asymptotic—uniformity result on the cardinalities of definable sets in homocyclic p -groups. Here, a *homocyclic group* is a direct sum of isomorphic cyclic p -groups. In this subsection, for a prime p we denote by \mathcal{C}_p the set of all finite groups $(\mathbb{Z}/p^n\mathbb{Z})^m$ (for $m, n \in \mathbb{N}$). Put $\mathcal{C} := \bigcup (\mathcal{C}_p : p \text{ prime})$.

Theorem 4.3. *Let p be prime. Then any infinite ultraproduct G of groups $((\mathbb{Z}/p^n\mathbb{Z})^n : n < \omega)$ is stable but not superstable, and satisfies (A) and (DC_L).*

We first prove the following more general result, Proposition 4.4, analogous to Theorem 4.2. By the classical elimination theory for abelian groups of Szmielew (see [10] [Theorem A.2.2]), modulo the theory of abelian groups, every formula $\phi(\bar{x}, \bar{y})$ is

equivalent to a boolean combination of formulas of form $t(\bar{x}, \bar{y}) = 0$ and $p^\ell | t(\bar{x}, \bar{y})$, where t is a term in the language of groups, and p is a prime; we shall say that such a formula is in *standard form*. For an abelian group G , we write $G[p^k]$ for the subgroup $\{g \in G : p^k g = 0\}$. For nonnegative integers d, k , let $S(d, k)$ be the set of functions of the form $P(X, u, v) = \sum_{i=0}^k \sum_{j=-kd}^{kd} c_{ij} X^{u(iv+j)}$, where $c_{ij} \in \mathbb{Z}$ for all $0 \leq i \leq k$ and $-kd \leq j \leq kd$.

Proposition 4.4. *Let $\phi(\bar{x}, \bar{y})$ be a formula in the language of groups in standard form. Let d be the greatest integer l such that for some prime p , either some subformula $p^l | t(\bar{x}, \bar{y})$ occurs in ϕ or some term $t(\bar{x}, \bar{y})$ occurring in ϕ has a coefficient divisible by p^l . Then*

- (i) *There is a finite subset $F = F(\phi)$ of $S(d, r)$ (where $r = l(\bar{x})$) such that for each $G = (\mathbb{Z}/p^n \mathbb{Z})^m \in \mathcal{C}$ and $\bar{a} \in G^s$, there is $P(X, u, v) = \sum_{i=0}^k \sum_{j=-kd}^{kd} c_{ij} X^{u(iv+j)} \in F$ with $c_{ij} = 0$ whenever $in + j < 0$, such that $|\phi(G^r, \bar{a})| = P(p, m, n)$.*
- (ii) *For each such function $P \in F$ there is a formula ϕ_P such that for each $G = (\mathbb{Z}/p^n \mathbb{Z})^m \in \mathcal{C}$ and $\bar{a} \in G^s$, we have $G \models \phi_P(\bar{a})$ if and only if $|\phi(G^r, \bar{a})| = P(p, m, n)$.*

In addition, the same value of d suffices if $\phi(\bar{x}, \bar{y})$ is replaced by a formula of form $\bigwedge_{i=1}^f \phi(\bar{x}, \bar{y}_i)$, for a fixed f .

Proof. The proof is by induction on $r = l(\bar{x})$, using a fibering argument like that in the proof of Theorem 4.2. To start the induction consider a formula $\phi(x, \bar{y})$ in standard form in the group $G = (\mathbb{Z}/p^n \mathbb{Z})^m$. This formula is a boolean combination of formulas of form $t(x, \bar{y}) = 0$ and $q^\ell | t(x, \bar{y})$, where t is a term in the language of groups, and q is a prime. Clearly, we may assume $q = p$, since if $(q, p) = 1$ then every element of G is q^l -divisible. Also, as in the proof of Lemma 4.2, we may assume ϕ is a conjunction of such formulas and their negations. A formula of form $t(x, \bar{a}) = 0$ either defines \emptyset or G , or has the same solution set as a formula of the form $p^l x = a'$ (where $l \leq d$), and so defines in G a coset of the subgroup $G[p^l]$ of order p^{lm} .

Consider now a formula $p^\ell | t(x, \bar{a})$. First observe that the formula $p^\ell | x$ has exactly $(p^{n-\ell})^m$ solutions in G . Now suppose $t(x, \bar{y})$ has form $kx + \sum_{i=1}^s n_i y_i$. Let $k = p^j \cdot k'$ where $(p, k') = 1$. As the map $z \mapsto p^j z$ has kernel of size $(p^j)^m$, the formula $p^\ell | t(x, \bar{a})$ has solution set of size $p^{jm} \cdot (p^{n-\ell})^m = p^{m(n+j-\ell)}$ with $\ell \geq j$, or no solutions. In particular, it defines \emptyset or a coset of the subgroup $p^i G$ of G of order $p^{(n-i)m}$ for some $i \leq d$. Thus, it has exactly $p^{(n-i)m}$ solutions, with $n \geq i$, or no solutions, where i is determined just by the original formula $p^\ell | t(x, \bar{y})$.

To complete the proof for $r = 1$, consider an arbitrary conjunction of such formulas or their negations. A finite conjunction of such formulas again has solution set \emptyset or a coset of a subgroup of order p^{im} or $p^{(n-i)m}$ for some $i \leq d$ determined by the conjunctions. We use here that $G[p] < G[p^2] < \dots G[p^u] = p^{n-u} G < \dots pG < G$

and that each conjunction, if consistent, defines a coset of some group in this chain. It follows, using Inclusion-Exclusion, that for a formula $\phi(x, \bar{y})$ in standard form, there is a finite set E of tuples $\bar{e}\bar{e}'$ of integers, where $\bar{e} = (e_0, \dots, e_d)$ and $\bar{e}' = (e'_0, \dots, e'_d)$

such that for all $\bar{a} \in G^s$, there is $\bar{e}\bar{e}' \in E$ with $|\phi(G, \bar{a})| = \sum_{j=0}^d e_j p^{jm} + e'_j p^{(n-j)m}$.

Furthermore, for each $\bar{e}\bar{e}' \in E$ there is a formula $\phi_{\bar{e}\bar{e}'}$ defining the corresponding set of $\bar{a} \in G^s$, uniformly as G ranges through \mathcal{C}' . Putting $c_{0j} = e_j$ for $0 \leq j \leq d$, $c_{0j} = 0$ for $-d \leq j < 0$, and $c_{1j} = e'_{-j}$ for j with $-d < j \leq 0$ and $c_{1j} = 0$ for $0 < j \leq d$, we see that $|\phi(G, \bar{a})| = \sum_{i=0}^1 \sum_{j=-d}^d c_{ij} p^{m(in+j)}$, as required for $r = 1$.

The proposition for formulas $\phi(\bar{x}, \bar{y})$ now follows by a standard fibering argument as in the proof of Theorem 4.2. As the notation is intricate, we provide details.

Let $\phi(\bar{x}, \bar{y})$ be a formula with $l(\bar{x}) = r + 1$, and put $\bar{x} = z\bar{x}'$. By the case $r = 1$, there are $t < \omega$ and a finite subset $D = \{P_1, \dots, P_t\}$ of $S(d, 1)$, such that for any $\bar{b}\bar{a} \in M^{r+s}$ there is $i \in \{1, \dots, t\}$ such that $|\phi(G, \bar{b}, \bar{a})| = P_i(p, m, n)$. Furthermore, there is for each $i = 1, \dots, t$ a further formula $\phi_i(\bar{x}', \bar{y})$ defining the set of such $\bar{b}\bar{a}$.

By induction, for each $i \in \{1, \dots, t\}$ there is a finite subset D_i of $S(d, r)$ such that for any $G \in \mathcal{C}$ and $\bar{a} \in G^s$, there is $Q(X, u, v) \in D_i$ such that $|\phi_i(G^r, \bar{a})| = Q(p, m, n)$. Let $D_i = \{Q_{i1}, \dots, Q_{i,r_i}\}$ for each i , and let $\phi_{ij}(\bar{y})$ be the corresponding formula defining the set of such \bar{a} . As with Theorem 4.2, for each $G \in \mathcal{C}$ and $\bar{a} \in G^r$, there is a unique function $h : \{1, \dots, t\} \rightarrow \omega$ such that $G \models \phi_{ih(i)}(\bar{a})$ for each i . We then have, for each $i = 1, \dots, t$,

$$|\phi_i(G^r, \bar{a})| = Q_{ih(i)}(p, m, n),$$

and, for each $\bar{b} \in \phi_i(G^r, \bar{a})$,

$$|\phi(G, \bar{b}, \bar{a})| = P_i(p, m, n).$$

Put $R(X, u, v) = \sum_{i=1}^t P_i(X, u, v) Q_{ih(i)}(X, u, v)$. Then $|\phi(G^{r+1}, \bar{a})| = R(p, m, n)$, and $R \in S(d, r + 1)$, that is, R has the required form. As there are finitely many such functions h , the set F of all possible functions R is also finite.

The final assertion follows immediately from the way that d is defined. \square

Proof of Theorem 4.3. The fact that the ultraproduct is stable but not superstable follows immediately from the fact that it is an abelian group with a descending chain of definable subgroups (of the form $G > pG > p^2G > \dots$) each of infinite index in its predecessor.

The condition (DC_L) follows easily from Proposition 4.4(ii); indeed, two definable sets X and Y satisfy $\delta(X) = \delta(Y)$ precisely if, on a set in the ultrafilter, the corresponding definable sets have cardinalities which are polynomials of the same degree in p , and this is a definable condition. It follows from Theorem 3.5 that (SA) does not hold. Finally, condition (A) follows from the final assertion of Proposition 4.4. For given $G = (\mathbb{Z}/p^n\mathbb{Z})^n$, any positive ϕ -formula defines a set of size $\sum_{i=0}^r \sum_{j=-rd}^{rd} c'_{ij} p^{n(in+j)}$, where r, d depend just on ϕ and the c'_{ij} just on the number of conjuncts, not on n . \square

5. Further Properties of (A), (SA), (DC_L), and (FMV)

We first consider a number of technical questions around our conditions (A), (SA), (FMV), and (DC_L): the independence theorem and stable formulas; 1-variable criteria for the conditions; transferrability to M^{eq} . We also explore consequences of assuming (FMV) and (DC_L), obtaining a pregeometry under an extra hypothesis.

5.1. The Independence Theorem

We first observe that, under the strong hypotheses (DC_L) and (SA), the Independence Theorem has the following translation.

Proposition 5.1. *Assume that M satisfies (DC_L) and (SA), let $E = \text{acl}^{\text{eq}}(E)$ be a countable parameter set, and let P_1, P_2, P_3 be the solution sets in M of 1-types p_1, p_2, p_3 over E . For $i < j$ with $i, j \in \{1, 2, 3\}$ let $q_{ij}(x_i, x_j)$ be a 2-type over E extending $p_i(x_i) \cup p_j(x_j)$, and let Q_{ij} be the set of realizations in M of q_{ij} . Let $\gamma_i := \delta(P_i)$ and suppose that $\delta(Q_{ij}) = \gamma_i + \gamma_j$ for each $i < j$. Then there is a 3-type $r(x_1, x_2, x_3)$ over E extending $q_{12}(x_1, x_2) \cup q_{13}(x_1, x_3) \cup q_{23}(x_2, x_3)$ with $\delta(r) = \gamma_1 + \gamma_2 + \gamma_3$.*

Proof. By Theorem 3.3(i) and [39] Theorem 6.4.6, strong type and Lascar strong type coincide in $T = \text{Th}(M)$. The result thus follows directly from the usual Independence Theorem for Lascar strong types (see [21] Theorem 5.8, or [39] Theorem 2.5.20), via Theorem 3.4 and Lemma 3.1. \square

We would like to prove a version of the Independence Theorem in the manner of the proof of Proposition 8.4.3 of [4], that is, based directly on counting arguments rather than quoting results for simple theories. In particular, we ask:

Question 5.1. Does a version of Proposition 5.1 hold just under the assumptions (DC_L) and (FMV)?

In this direction, we make two observations—Lemmas 5.1 and 5.2—both of which are standard.

Lemma 5.1. *Assume that M satisfies (DC_L), let $Z \subset M^t$ be definable with $\delta(Z) = \delta_0$, and let $\phi(\bar{x}, \bar{z})$ and $\psi(\bar{y}, \bar{z})$ be formulas implying $\bar{z} \in Z$, with $l(\bar{x}) = r$ and $l(\bar{y}) = s$. Let $\theta(\bar{x}, \bar{y})$ be the formula, given by (DC_L), which expresses that $\delta(\phi(\bar{x}, Z) \wedge \psi(\bar{y}, Z)) < \delta_0$. Then θ is stable.*

Proof. We essentially repeat the argument of Lemma 8.4.2 of [4]. Suppose $(\bar{a}_i, \bar{b}_i)_{i < \omega}$ is an L^+ -indiscernible sequence from M^{r+s} , with $\theta(\bar{a}_i, \bar{b}_j)$ holding whenever $i < j$. It suffices to show that $\theta(\bar{a}_i, \bar{b}_i)$ holds for each i . Let $Z_i := \{\bar{z} : \phi(\bar{a}_i, \bar{z}) \wedge \psi(\bar{b}_i, \bar{z})\}$. For $i < j$, as $Z_i \cap Z_j \subseteq \phi(\bar{a}_i, Z) \cap \psi(\bar{b}_j, Z)$ and $\theta(\bar{a}_i, \bar{b}_j)$ holds, we have $\delta(Z_i \cap Z_j) < \delta_0$. It follows by Remark 3.2 (or can be derived from Proposition 2.2) that $\delta(Z_i) < \delta_0$ for each i , as required. \square

In part for the next section, we recall from Section 2.1 the natural notion of measure on definable subsets of a given definable set in M . Given L -formulas $\phi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{z})$ with $l(\bar{x}) = r$, $l(\bar{y}) = s$ and $l(\bar{z}) = t$, and $\bar{a} \in M^s$ and $\bar{b} \in M^t$, we define the normalized measure $\mu_{\psi(\bar{x}, \bar{b})}(\phi(\bar{x}, \bar{a}))$ with respect to $\psi(\bar{x}, \bar{b})$. This gives a finitely additive real-valued probability measure on definable subsets of a given definable set.

Definition 5.1. We say that M satisfies (MD_L) if for each pair of formulas $\phi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{z})$, there is a formula $\chi_\phi^\psi(\bar{y}_1, \bar{y}_2, \bar{z})$, with $l(\bar{y}_1) = l(\bar{y}_2) = s$ and $l(\bar{z}) = t$, such that for all $\bar{a}_1, \bar{a}_2 \in M^s$ and $\bar{b} \in M^t$, we have

$$\chi_\phi^\psi(\phi(\bar{a}_1, \bar{a}_2, \bar{b})) \Leftrightarrow [\mu_{\psi(\bar{x}, \bar{b})}(\phi(\bar{x}, \bar{a}_1)) \leq \mu_{\psi(\bar{x}, \bar{b})}(\phi(\bar{x}, \bar{a}_2))].$$

Note that this is an \wedge -definable condition in the language L^+ . We do not consider the corresponding notion (MD_{L^+}) . It is easily seen that if (A) and (MD_L) both hold, then for all formulas $\psi(\bar{x}, \bar{z})$ and $\phi(\bar{x}, \bar{y})$, the quotient total ordering of the above preordering is finite; that is, the collection of ϕ -definable subsets of $\psi(\bar{x}, \bar{b})$ assumes finitely many measures as \bar{y} varies, cf., Remark 2.7.

For future reference, we explicitly record the following measure-theoretic version of Lemma 5.1, used in the proof of Theorem 6.4 below. It follows immediately from [12] Proposition 2.25.

Lemma 5.2. *Assume that M satisfies (DC_L) and (MD_L) , and that D is a definable set in M^t . Let $\phi(\bar{x}, \bar{z})$ and $\psi(\bar{y}, \bar{z})$ be formulas which imply $\bar{z} \in D$. For some $\mu > 0$, let $\theta(\bar{x}, \bar{y})$ define the set of all (\bar{a}, \bar{b}) such that $\mu_D(\phi(\bar{a}, M^t) \wedge \psi(\bar{b}, M^t)) = \mu$. Then θ is stable.*

5.2. 1-variable criteria

By Lemma 2.2 of [5], to show that a collection \mathcal{C} of finite structures is an asymptotic class, it suffices to verify the conditions for formulas $\phi(x, \bar{y})$ where x is a single variable. This is analogous to combinatorial conditions on stable and NIP formulas, and our use above of Lemma 4.2. Below, we give a clean result for (DC_L) and (FMV), but have not obtained analogues for (A) or (SA).

Definition 5.2. The conditions $(DC_L)(k)$, $(FMV)(k)$ are defined like (DC_L) , (FMV) respectively, but only for $r = l(\bar{x}) \leq k$.

Lemma 5.3. *Assume $(FMV)(1)$ and $(DC_L)(1)$. Then (DC_L) and (FMV) hold.*

Proof. We show by induction on k simultaneously that $(DC_L)(k)$ and $(FMV)(k)$ both hold. By our assumptions, both assertions hold for $k = 1$.

Assume both statements hold for some k and consider the formula $\phi(\bar{x}, \bar{y})$ where $l(\bar{x}) = k + 1$. Put $\bar{x} = z\bar{x}'$. Define $Q = \{\delta(\phi(M, \bar{b}\bar{a})) : \bar{b}\bar{a} \in M^{k+s}\}$; this set is finite by $(FMV)(1)$, so we may put $Q = \{\gamma_1, \dots, \gamma_t\}$. For each $\gamma \in Q$, let $\psi_\gamma(\bar{x}', \bar{y})$

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hold if and only if $\delta(\phi(M, \bar{x}', \bar{y})) = \gamma$ —the formula ψ_γ exists by $(DC_L)(1)$ and the finiteness of Q . Then put

$$Q(\gamma) := \{\delta(\psi_\gamma(M^k, \bar{a})) : \bar{a} \in M^s\}.$$

By the induction assumption $(FMV)(k)$, this set is finite. Then for all $\bar{a} \in M^s$ we have

$$\delta(\phi(M^{k+1}, \bar{a})) = \sum_{i=1}^t \gamma_i + \delta(\psi_{\gamma_i}(M^k, \bar{a})). \quad (**)$$

(We use here that by $(DC_L)(1)$ we have $(WOD)_{\phi(z, \bar{x}'\bar{y})}$ – cf. Lemma 2.3 – and then argue as in Lemma 2.4.) Also, the set of all such values $\delta(\phi(M^{k+1}, \bar{a}))$ is finite as each $Q(\gamma_i)$ is finite. Thus $(FMV)(k+1)$ holds. Using the induction assumption $(DC_L)(k)$ and $(**)$, it is easy to see $(DC_L)(k+1)$ also holds. \square

Question 5.2. Are there analogues of Lemma 5.3 for the conditions (A) and (SA), possibly local for (A)?

5.3. *Transferring conditions to M^{eq}*

We consider here the extent to which the conditions on which this paper has focused, (A), (SA), and (DC_L) , extend to M^{eq} . Analogously, it is shown in [5] that, essentially, if \mathcal{C} is an asymptotic class and \mathcal{C}' is obtained from \mathcal{C} by adding finitely many sorts from M^{eq} , then \mathcal{C}' is an asymptotic class.

Proposition 5.2.

- (i) *If M has (DC_L) and (FMV) , then M^{eq} satisfies (DC_L) and (FMV) .*
- (ii) *If M has (DC_L) and $\text{Th}(M)$ does not have the strict order property, then M^{eq} satisfies (DC_L) and (FMV) .*
- (iii) *If M satisfies (A) and (DC_L) , then M^{eq} satisfies (A) and (DC_L) .*
- (iv) *If M satisfies (SA) and (DC_L) , then M^{eq} satisfies (SA) and (DC_L) .*

Proof. (i) Let E and F be \emptyset -definable equivalence relations on M^n and M^m respectively, and let $\phi(x, \bar{y})$ and $\psi(u, \bar{v})$ be L^{eq} -formulas such that x ranges through M^n/E and u through M^m/F . Using Lemma 5.3, it suffices to show that the relation $\delta(\phi(x, \bar{a})) \leq \delta(\psi(u, \bar{b}))$ is defined by some formula $\chi(\bar{a}, \bar{b})$. By (FMV) , the E and F -classes take just finitely many δ -values. Hence, the set of E -classes or F -classes taking any given δ -value is \emptyset -definable. The result now follows easily, using Lemmas 2.3 and 2.4(i).

(ii) This is immediate from Lemma 2.7, in conjunction with (i).

(iii) First observe that M^{eq} satisfies (DC_L) , by (i) and Theorem 3.3. To see that M^{eq} satisfies (A), let E be an \emptyset -definable equivalence relation on M^n , and suppose that there is a sequence of subsets $(X_i : i \in \omega)$ of M^n/E such that $X_i \supset X_{i+1}$ and each X_i is a conjunction of ϕ -instances for some L^{eq} -formula ϕ . (For simplicity, we

are handling a special case where $X_i \subset M^n/E$ for each i ; the general argument when $X_i \subseteq (M^n/E)^r$ is similar.) For each $i < \omega$ let X'_i be the union of the E -classes lying in X_i . Then $X'_i \supset X'_{i+1}$ for all i , and it is easily checked that there is an L -formula ϕ' such that each X'_i is a conjunction of ϕ' -instances. By (A), there is t such that $\delta(X'_i) = \delta(X'_t)$ for all $i \geq t$. Also, the set of E -classes is uniformly definable, so by (A) and Lemma 2.7, these take finitely many δ -values. It follows, again using Lemma 2.4(i), that the sequence $(\delta(X_i) : i \in \omega)$ takes finitely many values, as required.

(iv) This is proved essentially as in (iii). □

Remark 5.1. It would be helpful to clarify what hypotheses are needed in Proposition 5.2. In (i) above, it seems we require some assumption in addition to (DC_L) for M . Likewise, in (iii), we probably cannot deduce that M^{eq} satisfies (A) just from the assumption that M satisfies (A). In these cases we have not constructed counterexamples.

In (iv), (DC_L) is required, that is, we cannot lift (SA) on its own from M to M^{eq} . Consider a language L with a binary relation E and unary relations $\{P_i : i \in \omega\}$. We can choose an increasing sequence $\delta_0 < \delta_1 < \dots$ and build a family of finite structures with ultraproduct M such that: E is an equivalence relation on M ; each P_i is a union of E -classes with $P_0 \supset P_1 \supset \dots$; the structure M satisfies (SA); and, $\delta(P_i)$ takes a fixed value ϵ for all i , but $\delta(P_i/E) > \delta(P_{i+1}/E)$ for all $i < \omega$. For this, we arrange that the E -classes in $P_i \setminus P_{i+1}$ all have δ -value δ_i . We omit the details.

5.4. Consequences of (FMV) and (DC_L)

We assume that both (FMV) and (DC_L) hold throughout this subsection. Note that these assumptions hold for the examples considered in Section 4: asymptotic classes, the 2-sorted vector space structures with theory T_{vs} of Theorem 4.2, and the ultraproducts of homocyclic groups of Theorem 4.3. These all satisfy (A) and so have a simple theory, unlike the expansion of T_{vs} by a symplectic bilinear form considered in Example 4.3, which is not simple but does satisfy (FMV) and (DC_L) .

Under (FMV) and (DC_L) we have additivity of δ -dimension given by Lemma 3.1 and all properties of \downarrow^δ considered in Propositions 3.1 and 3.2 except for local character (which fails in the symplectic bilinear form example). Also, by Proposition 5.2(i), properties (FMV) and (DC_L) transfer to M^{eq} , and by Lemma 5.3 it suffices to verify them for formulas of form $\phi(x, \bar{y})$.

Our main additional observation is

Proposition 5.3. *Assume that M satisfies (FMV) and (DC_L) , let D be an infinite interpretable set in M^{eq} over parameters \bar{e} , and put $\delta_0 := \delta(D)$. Suppose there is a proper convex subsemigroup S of \mathbb{R}^*/\mathbb{C} with $\delta_0 \notin S$, such that $\delta(D') \in S$ for every definable subset D' of D with $\delta(D') < \delta_0$. For $a \in D$ and $B \subset D$, define $a \in \text{cl}(B)$ if and only if there is a $B\bar{e}$ -definable subset D' of D containing a with $\delta(D') < \delta_0$. Then cl defines a pregeometry on D .*

Proof. For ease of notation we suppose that $D \subset M$ and that D is \emptyset -definable. We must verify, for $A, B \subset D$, that:

- (i) if $A \subseteq B$ then $A \subseteq \text{cl}(A) \subseteq \text{cl}(B)$;
- (ii) if $a \in \text{cl}(B)$ then $a \in \text{cl}(F)$ for some finite $F \subseteq B$;
- (iii) $\text{cl}(\text{cl}(A)) = \text{cl}(A)$;
- (iv) for all $a_1, a_2 \in D$, we have $a_1 \in \text{cl}(a_2B) \setminus \text{cl}(B) \Rightarrow a_2 \in \text{cl}(a_1B)$.

Properties (i) and (ii) are immediate. To prove (iv) we first note that we may assume B to be countable, in which case $a \in \text{cl}(B) \setminus \text{cl}(\emptyset)$ if and only if $a \not\downarrow^\delta B$. Then, if $a_1 \notin \text{cl}(B)$ and $a_2 \notin \text{cl}(Ba_1)$ we have $a_1 \not\downarrow^\delta B$ and $a_2 \not\downarrow^\delta Ba_1$. Thus $a_2 \not\downarrow_B^\delta a_1$, whence by Proposition 3.2(vi) we have $a_1 \not\downarrow_B^\delta a_2$, giving $a_1 \not\downarrow^\delta Ba_2$ and finally $a_1 \notin \text{cl}(B, a_2)$.

For (iii), first observe that for A countable, $a \in \text{cl}(A)$ if and only if $\delta(a/A) < \delta_0$. Suppose that $a \in \text{cl}(\text{cl}(A))$. We may suppose that A is countable. There are $d_1, \dots, d_k \in \text{cl}(A)$ such that $a \in \text{cl}(A, d_1, \dots, d_k)$. Thus,

$$\delta(a/A) \leq \delta(ad_1 \dots d_k/A) \leq \delta(a/Ad_1 \dots d_k) + \sum_{i=1}^k \delta(d_i/A).$$

The latter sum is bounded above by an element of S , so is less than δ_0 . □

We shall call a set interpretable in M *geometric* if it satisfies the assumptions on D in Proposition 5.3. Examples of geometric sets include ultraproducts of one-dimensional asymptotic classes (where the subsemigroup S in Proposition 5.3 is trivial), and both the vector space sort and the field sort in T_{vs} . It would be interesting to investigate the pregeometry from the viewpoint of Zilber Trichotomy, and show, for example, that in the locally modular non-trivial case there is an infinite definable group.

We conclude this section with the proposition below for geometric sets which are groups; the analogous results for asymptotic classes and measurable structures are [26] Theorems 3.12, 5.15. The conclusion cannot be strengthened to ‘abelian-by-finite’ since for an odd prime p the class of finite extraspecial p -groups of exponent p is a one-dimensional asymptotic class ([26] Proposition 3.11), and has finite-by-abelian but not abelian-by-finite ultraproducts. The result suggests that if a geometric set is a pure group, it should be one-based in a Zilber Trichotomy.

Proposition 5.4. *Assume that M satisfies (FMV) and (DC_L) , and let G be an infinite group interpretable in M such that the domain of G is a geometric set. Then G is finite-by-abelian-by-finite.*

Proof. By a theorem of Landau [23], for every $k < \omega$ there are just finitely many finite groups with k conjugacy classes. Hence, as G is a pseudofinite group, G has infinitely many conjugacy classes, say $\{C_i : i \in I\}$. The conjugacy classes are uniformly definable, so the set $\{\delta(C_i) : i \in I\}$ is finite. For a conjugacy class C_i and $a \in C_i$, we have $|G| = |C_i| \cdot |C_G(a)|$ (non-standard cardinality), so $\delta_0 := \delta(G) =$

$\delta(C_i) + \delta(C_G(a))$. If a conjugacy class C_i is infinite then $|G : C_G(a)|$ is infinite. In this case $\delta(C_G(a)) < \delta(G)$, and thus, as G is geometric, $\delta(C_i) = \delta_0$. Hence, by counting, G has just finitely many infinite conjugacy classes. By (DC_L) the finite conjugacy classes have bounded size, which yields that the set of finite conjugacy classes of G is definable and hence its union is a definable non-trivial normal subgroup N of G . As G/N has finitely many conjugacy classes, it is finite.

The group N is a so-called BFC group, that is, a group whose conjugacy classes have finite bounded size. It follows by [29] Theorem 3.1 that its derived subgroup N' is finite, that is, G is finite-by-abelian-by-finite. \square

6. Applications

We consider here two potential routes for applications of pseudofinite dimension. The first is to pseudofinite groups, and the second to possible generalizations of Tao's 'Algebraic Regularity Lemma' [38]. Many other possible lines of application, from a different viewpoint, are described in [13].

6.1. Pseudofinite dimension and groups

We here assume that there is a group G definable in M . Under the assumption (A), the entire theory of groups with simple theory is applicable. We first give a small adaptation of some observations from [28] Section 4, where applications to finite simple groups of fixed Lie rank are described. These are the subject of 6.1-6.3.

Below, if $U, V \subset G$ then $UV := \{uv : u \in U, v \in V\}$.

Theorem 6.1. *Assume that M satisfies (SA) and (DC_L). Suppose that G is a definable group in M that has no proper definable subgroups of finite index, and let X_1, X_2, X_3 be definable subsets of G with $\delta(X_i) = \delta(G) = \delta_0$ for each i . Then*

- (i) $X_1X_2X_3 = G$, and
- (ii) $\delta(G \setminus X_1X_2) < \delta_0$.

Before giving the proof, we collect some basic facts about generic types in simple theories, taken from [39] Section 4.3 (and originally in [32]). First, following [39], by a *type-definable group* G we mean a type-definable set together with a definable binary operation which induces a group operation on the domain. By [39] Theorem 5.5.4, if G is a type-definable group in a supersimple theory, then G is an intersection of definable groups. In particular, if G is a type-definable subgroup of the definable group H , then G is an intersection of definable subgroups of H .

Given an \emptyset -type-definable group G and a countable set A of parameters, by $S_G(A)$ we denote the set of complete types over A which contain the formula $x \in G$. For such a group G in an ambient simple theory, given a countable set A of parameters, a type $p \in S_G(A)$ is called *left generic* if for all b realizing a type in $S_G(A)$, and all $a \models p$ with $a \downarrow_A b$, we have $ba \downarrow_A b$. There is an analogous definition of 'right-generic', but the two notions coincide ([39] Lemma 4.3.4), so we

just call such a type *generic*. It is shown in [39] p.168, that if the ambient theory is supersimple, then p is a generic type of G if and only if $SU(p) = SU(G)$. Recall also that if X is an A -definable set, then $SU(X)$ is the supremum of the ranks $SU(p)$, as p ranges over types over A concentrating on X . In a general supersimple theory, this supremum may not be realized, but if X is a group, then it is realized by any generic type (see [39] Section 5.4).

Lemma 6.1. *Assume (SA) and (DC_L). Suppose that G is a group definable in M and let $X \subset G$ be definable with $\delta(X) = \delta(G)$. Then $SU(X) = SU(G)$, and X realizes a generic type of G (over any small parameter set).*

Proof. Adding constants if necessary, we may assume that X and G are \emptyset -definable. Let p be a generic type of G over \emptyset . Choose $b \in G$ realizing p and $c \in Xb^{-1}$ with $c \perp^{\delta} b$; this is possible as $\delta(Xb^{-1}) = \delta(X) = \delta(G)$, as per Proposition 3.1(i). Then $c \perp b$ by Theorem 3.4, and $cb \in X$. Also, $cb \perp c$ as b is generic. Hence, as cb and b are interdefinable over c , cb is generic over c and so is generic over \emptyset , by [39] Lemma 4.1.2(1), (3). In particular, $SU(X) = SU(G)$. \square

Proof of Theorem 6.1. By Theorem 3.5, $\text{Th}(M)$ is supersimple.

(i) We work over a countable elementary submodel $M_0 \prec M$. Observe that $G_{M_0}^o$, the smallest M_0 -type-definable subgroup of G of bounded index, is equal to G . Indeed, as noted above, $G_{M_0}^o$ is an intersection of definable subgroups of G of bounded index, and by compactness and saturation such subgroups have finite index in G , so equal G by assumption.

By Lemma 6.1 and our assumption for each i that $\delta(X_i) = \delta(G)$, we have $SU(X_i) = SU(G)$ for $i = 1, 2, 3$. Since $G = G_{M_0}^o$, every generic type of G over M_0 is *principal* (see [39] Definition 4.4.6). Hence, by Proposition 4.7 (ii) of [28]—a small translation of [35] Proposition 2.2—if r_1, r_2, r_3 are generic types of G over M_0 and r is any type of G over M_0 , there are $a_i \in G$ realizing r_i for $i = 1, 2, 3$ such that $a_1 a_2 a_3 \models r$.

For a contradiction suppose that $X_1 X_2 X_3 \neq G$. Let r be any type over M_0 containing the formula $x \in G \setminus X_1 X_2 X_3$. By Lemma 6.1, for each i there is a generic type p_i of G over M_0 containing the formula $x \in X_i$. By the conclusion of last paragraph, there are $a_i \models p_i$ in G , for each i , with $a_1 a_2 a_3 \models r$. This, however, contradicts the assumption that r contains the formula $x \in G \setminus X_1 X_2 X_3$.

(ii) The proof is a small adaptation of that of [28] Theorem 4.8(ii). If the conclusion were false, then (arguing as in (i)) there would be a generic type q of G (over M_0) containing the formula $x \in G \setminus X_1 X_2$. Choosing p_1, p_2 as in (i), we find by [35] Proposition 2.2, realizations $a_1 \models p_1$ and $a_2 \models p_2$ with $a_1 a_2 \models q$, a contradiction. \square

Theorem 6.1(i) has consequences for finite groups. For example we have

Corollary 6.1. *Let \mathcal{C} be a class of finite groups (possibly with extra structure) such that all ultraproducts of \mathcal{C}^+ satisfy (SA) and (DC_L). Assume for each positive*

integer d and formula $\psi(x, \bar{y})$ that there are only finitely many $G \in \mathcal{C}$ containing a tuple \bar{a} such that $\psi(G, \bar{a})$ is a proper subgroup of G of index at most d . Let $N < \omega$ and let $\chi_i(x, \bar{z}_i)$ for $i = 1, 2, 3$ be formulas. Then there is $K < \omega$ such that if $G \in \mathcal{C}$ with $|G| > K$ and $\bar{a}_i \in G^{l(\bar{z}_i)}$ with $|\chi_i(G, \bar{a}_i)| \geq \frac{1}{N}|G|$ for $i = 1, 2, 3$, then $\chi_1(G, \bar{a}_1) \cdot \chi_2(G, \bar{a}_2) \cdot \chi_3(G, \bar{a}_3) = G$.

Corollary 6.1 is analogous to Corollary 1 of Nikolov-Pyber [30] (see also [13] Remark 3.3, which concerns Hrushovski’s ‘coarse pseudofinite dimension’). The latter, which uses a result of Gowers [8], has no model-theoretic assumptions, but assumes that the groups have no non-trivial representations of bounded finite degree. The Nikolov-Pyber theorem has the following pseudofinite consequence, noted also in [13]. Here ‘internal’ has the usual meaning from non-standard analysis; an internal representation would arise as an ultraproduct of representations of the finite groups.

Theorem 6.2. *Let G be an infinite ultraproduct of finite groups, with no non-trivial internal finite degree representation. Let X_1, X_2, X_3 be definable subsets of G with $\delta(X_i) = \delta(G)$ for all i . Then $X_1 X_2 X_3 = G$.*

Since any family of finite simple groups of fixed Lie type is an asymptotic class, Theorem 6.1 (via Proposition 4.1) has the following consequence for finite simple groups, already noted in [28] and derivable also from the Nikolov-Pyber theorem. There is a much stronger statement in [25], where the result is proved with two words rather than three, and without the restriction on Lie type. If $w(x_1, \dots, x_d)$ is a non-trivial word in the free group on x_1, \dots, x_d and G is a group, then $w(G) := \{w(g_1, \dots, g_d) : g_1, \dots, g_d \in G\}$.

Theorem 6.3. *Let w_1, w_2, w_3 be non-trivial group words. Then for any fixed Lie type τ there is $N = N(w_1, w_2, w_3, \tau)$ such that if G is a finite simple group of Lie type τ and $|G| \geq N$, then $G = w_1(G) \cdot w_2(G) \cdot w_3(G)$.*

6.2. Tao’s Algebraic Regularity Lemma

We here give a generalization of Tao’s Algebraic Regularity Lemma, proved in [38] with a remarkable application to expansion properties for polynomials. No new ideas are involved in our treatment—it is a routine application of methods of Pillay and Starchenko [36], combined with the argument from [38] to deduce Tao’s Lemma 5 from his Proposition 27. In unpublished work, Hrushovski gives a rather stronger generalization. We omit the details. Below, ‘complexity’ refers to the length of a formula and (MD_L) is as in Definition 5.1.

Theorem 6.4. *Let \mathcal{C} be a class of finite L -structures, and assume that every infinite ultraproduct of members of \mathcal{C} satisfies (SA), (DC_L) , and (MD_L) . Then for every $N \in \mathbb{N}^{>0}$ there is $C = C_N \in \mathbb{N}^{>0}$ such that: whenever $M \in \mathcal{C}$ has cardinality greater than C , the sets V and W are non-empty subsets of cartesian powers of M , and $E \subseteq V \times W$, with V, W and E all definable of complexity at most N , there are*

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partitions $V = V_1 \cup \dots \cup V_a$ and $W = W_1 \cup \dots \cup W_b$ into definable sets of complexity at most C , with:

- (1) for all $i = 1, \dots, a$ and $j = 1, \dots, b$, we have $|V_i| \geq |V|/C$ and $|W_j| \geq |W|/C$, and
- (2) for all i, j , and sets $A \subset V_i$ and $B \subset W_j$, we have

$$||E \cap (A \times B)| - d_{ij}|A||B|| = o(|V_i||W_j|),$$

where $d_{ij} = |E \cap (V_i \times W_j)|/|V_i||W_j|$.

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